



HAL
open science

Automorphisms and symplectic leaves of Calogero-Moser spaces

Cédric Bonnafé

► **To cite this version:**

Cédric Bonnafé. Automorphisms and symplectic leaves of Calogero-Moser spaces. *Journal of the Australian Mathematical Society*, 2023, 115, pp.26-57. 10.1017/S1446788722000180 . hal-03565121v2

HAL Id: hal-03565121

<https://hal.science/hal-03565121v2>

Submitted on 5 Oct 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

AUTOMORPHISMS AND SYMPLECTIC LEAVES OF CALOGERO-MOSER SPACES

by

CÉDRIC BONNAFÉ

Abstract. — We study the symplectic leaves of the subvariety of fixed points of an automorphism of a Calogero-Moser space induced by an element of finite order of the normalizer of the associated complex reflection group. We give a parametrization *à la Harish-Chandra* of its symplectic leaves (generalizing earlier works of Bellamy and Losev). This result is inspired by the mysterious relations between the geometry of Calogero-Moser spaces and unipotent representations of finite reductive groups, which is the theme of another paper [Bon2].

Let V be a finite dimensional vector space and let W be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$ generated by reflections. To a class function k on W supported on the set of reflections, Etingof and Ginzburg [EtGi] have associated a normal irreducible affine complex variety $\mathcal{X}_k(V, W)$ called a (generalized) *Calogero-Moser space*. If τ is an element of finite order of the normalizer of W in $\mathbf{GL}_{\mathbb{C}}(V)$ stabilizing the class function k , it induces an automorphism of $\mathcal{X}_k(V, W)$. The main theme of this paper is the study of the symplectic leaves of the variety $\mathcal{X}_k(V, W)^{\tau}$ of its fixed points in $\mathcal{X}_k(V, W)$ (endowed with its reduced closed subscheme structure).

Note that W acts trivially on $\mathcal{X}_k(V, W)$ so, by replacing τ by $w\tau$ for some $w \in W$ if necessary, we may assume that the natural morphism $V^{\tau} \rightarrow (V/W)^{\tau}$ is onto (the argument is due to Springer [Spr] and will be recalled in §3): this will be assumed throughout this paper and will simplify the statements.

The Poisson bracket on $\mathcal{X}_k(V, W)$ induces a Poisson bracket on $\mathcal{X}_k(V, W)^{\tau}$ and we are interested in parametrizing the symplectic leaves of this fixed points subvariety. For this, we define a τ -*cuspidal* symplectic leaf (or a τ -*cuspidal* point) to be a zero-dimensional symplectic leaf of $\mathcal{X}_k(V, W)^{\tau}$, and we define a τ -*split* parabolic subgroup of W to be the stabilizer of some point in V^{τ} . We also denote by W_{τ} the quotient of the setwise stabilizer of V^{τ} in W by the pointwise stabilizer. For its action on V^{τ} , the group W_{τ} is a reflection group [LeSp]. Our result is as follows:

Theorem A. *Assume that the natural morphism $V^{\tau} \rightarrow (V/W)^{\tau}$ is onto. Then there is a natural bijection between the set of symplectic leaves of $\mathcal{X}_k(V, W)^{\tau}$ and the set of W_{τ} -orbits of pairs (P, p) , where P is a τ -split parabolic subgroup and p is a τ -cuspidal point of $\mathcal{X}_{k_P}(V_P, P)^{\tau}$.*

Moreover, the dimension of the symplectic leaf associated with (P, p) through this bijection is equal to $2 \dim(V^P)^{\tau}$.

Here, k_P is the restriction of k to P . We will give in Section 8 an explicit description of the bijection. If $\tau = 1$, this result was proved by Bellamy [Bel1] and Losev [Los] and might be viewed as a *Harish-Chandra theory* of symplectic leaves. So Theorem A can be thought as a τ -*Harish-Chandra theory*, inspired by Broué-Malle-Michel d -*Harish-Chandra theory* of unipotent representations of finite reductive groups [BMM] (see [Bon2] for a further discussion of this analogy and applications of Theorem A). The main point is to combine Springer/Lehrer-Springer theory (which describes the action of the setwise stabilizer of V^τ on V^τ) with work of Bellamy/Losev. We propose the following conjecture about the geometry of symplectic leaves of $\mathfrak{X}_k(V, W)^\tau$.

Conjecture B. *Let (P, p) be as in Theorem A and let \mathcal{S} denote the corresponding symplectic leaf of $\mathfrak{X}_k(V, W)^\tau$. Then there exists a parameter l for the pair $((V^P)^\tau, N_{W_\tau}(P_\tau)/P_\tau)$ and a \mathbb{C}^\times -equivariant isomorphism of Poisson varieties*

$$\overline{\mathcal{S}}^{\text{nor}} \simeq \mathfrak{X}_l((V^P)^\tau, N_{W_\tau}(P_\tau)/P_\tau).$$

Here, $\overline{\mathcal{S}}^{\text{nor}}$ denotes the normalization of the closure of \mathcal{S} .

Note that this conjecture is not known even in the case where $\tau = \text{Id}_V$ (in which case $W_\tau = W$ and $P_\tau = P$). It has been proved by Maksimau and the author [BoMa] whenever $\mathfrak{X}_k(V, W)$ is smooth and $\tau \in W \cdot \mathbb{C}^\times$.

The paper is organized as follows. We recall the set-up (reflection groups, Cherednik algebras, Calogero-Moser spaces...) in the first section and the second section is a recollection of useful results on Poisson structures and symplectic leaves. In the third section, we recall the main results of Lehrer-Springer on the group W_τ and some of its consequences. In the fourth section, we restate Theorem A and Conjecture B in more precise terms. The proof of Theorem A is given in Sections 5 to 8 (see the end of Section 4 for the description of the different steps). We propose in the ninth section an overview of the known cases for Conjecture B. A short appendix summarizes easy results about completions of rings that are needed in Section 8 to conclude the proof of Theorem A.

Acknowledgements. We thank warmly Gwyn Bellamy for sharing his files containing preliminary versions of his upcoming papers with Chalykh [BeCh] and with Maksimau and Schedler [BeSc], and for his help for understanding his work on symplectic leaves of Calogero-Moser spaces.

We also wish to thank warmly Jean Michel for very long and fruitful discussions about d -Harish-Chandra theory of unipotent representations of finite reductive groups: even if it does not appear in this text (but will be explained in [Bon2]), all what I learned from him was a great source of inspiration for this paper.

1. Set-up

1.A. Complex numbers. — Throughout this paper, we will abbreviate $\otimes_{\mathbb{C}}$ as \otimes and all varieties will be algebraic, complex, quasi-projective and reduced. If \mathcal{X} is an irreducible variety, we denote by \mathcal{X}^{nor} its normalization. If \mathcal{X} is an affine variety, we denote by $\mathbb{C}[\mathcal{X}]$ its coordinate ring: if moreover \mathcal{X} is irreducible, then \mathcal{X}^{nor} is also affine and $\mathbb{C}[\mathcal{X}^{\text{nor}}]$ is the integral closure of $\mathbb{C}[\mathcal{X}]$ in its fraction field (which will be denoted by $\mathbb{C}(\mathcal{X})$).

We fix in this paper a complex vector space V of finite dimension n . If X is a subset of V (or V^*), and if G is a subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$, we denote by G_X^{set} (resp. G_X^{pt}) the setwise (resp. pointwise) stabilizer of X and we set $G[X] = G_X^{\text{set}}/G_X^{\text{pt}}$. Then $G[X]$ acts faithfully on X (and on the vector space spanned by X). If $X = \{v\}$ is a singleton, then $G_X^{\text{set}} = G_X^{\text{pt}}$ (and we will denote both simply by G_X or G_v) and $G[X] = 1$. If H is a subgroup of G , we set $\bar{N}_G(H) = N_G(H)/H$.

If moreover G is finite, we identify $(V^G)^*$ and $(V^*)^G$, and we denote by V_G the unique G -stable subspace of V such that $V = V_G \oplus V^G$.

1.B. Reflections. — Let W be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$. We set

$$\text{Ref}(W) = \{s \in W \mid \dim_{\mathbb{C}} V^s = n - 1\}$$

and note that, for the moment, we *do not* assume that W is generated by $\text{Ref}(W)$. We set $\varepsilon : W \rightarrow \mathbb{C}^\times$, $w \mapsto \det(w)$. We identify $\mathbb{C}[V]$ (resp. $\mathbb{C}[V^*]$) with the symmetric algebra $S(V^*)$ (resp. $S(V)$).

We denote by \mathcal{A} the set of *reflecting hyperplanes* of W , namely

$$\mathcal{A} = \{V^s \mid s \in \text{Ref}(W)\}.$$

If $H \in \mathcal{A}$, we denote by α_H an element of V^* such that $H = \text{Ker}(\alpha_H)$ and by α_H^\vee an element of V such that $V = H \oplus \mathbb{C}\alpha_H^\vee$ and the line $\mathbb{C}\alpha_H^\vee$ is W_H^{pt} -stable. We set $e_H = |W_H^{\text{pt}}|$. Note that W_H^{pt} is cyclic of order e_H and that $\text{Irr}(W_H^{\text{pt}}) = \{\text{Res}_{W_H^{\text{pt}}}^W \varepsilon^j \mid 0 \leq j \leq e - 1\}$. We denote by $\varepsilon_{H,j}$ the (central) primitive idempotent of $\mathbb{C}W_H^{\text{pt}}$ associated with the character $\text{Res}_{W_H^{\text{pt}}}^W \varepsilon^{-j}$, namely

$$\varepsilon_{H,j} = \frac{1}{e_H} \sum_{w \in W_H^{\text{pt}}} \varepsilon(w)^j w \in \mathbb{C}W_H^{\text{pt}}.$$

If Ω is a W -orbit of reflecting hyperplanes, we write e_Ω for the common value of all the e_H , where $H \in \Omega$. We denote by ∇ the set of pairs (Ω, j) where $\Omega \in \mathcal{A}/W$ and $0 \leq j \leq e_\Omega - 1$. The vector space of families of complex numbers indexed by ∇ will be denoted by \mathbb{C}^∇ : elements of \mathbb{C}^∇ will be called *parameters*. If $k = (k_{\Omega,j})_{(\Omega,j) \in \nabla} \in \mathbb{C}^\nabla$, we define $k_{H,j}$ for all $H \in \Omega$ and $j \in \mathbb{Z}$ by $k_{H,j} = k_{\Omega,j_0}$ where Ω is the W -orbit of H and j_0 is the unique element of $\{0, 1, \dots, e_H - 1\}$ such that $j \equiv j_0 \pmod{e_H}$.

1.C. Parabolic subgroups. — We denote by $\text{Parab}(W)$ the set of parabolic subgroups of W (i.e. the set of subgroups of W which are stabilizers of some point of V) and by $\text{Parab}(W)/W$ the set of conjugacy classes of parabolic subgroups of W . If $P \in \text{Parab}(W)$, we denote by $\mathcal{V}(P)$ the set of elements $v \in V$ such that $W_v = P$: it is a non-empty open subset of V^P . By definition, $W_{V^P}^{\text{pt}} = P$ and $W_{V^P}^{\text{set}} = N_W(P)$, so that $W[V^P] = \bar{N}_W(P)$. The family $(\mathcal{V}(P))_{P \in \text{Parab}(W)}$ is a stratification of V (the order between strata corresponds to the reverse order of the inclusion of parabolic subgroups).

This stratification is stable under the action of the group W . If $\mathfrak{P} \in \text{Parab}(W)/W$, we denote by $\mathcal{U}(\mathfrak{P})$ the image of $\mathcal{V}(P)$ in V/W , where P is any element of \mathfrak{P} . Then $(\mathcal{U}(\mathfrak{P}))_{\mathfrak{P} \in \text{Parab}(W)/W}$ is a stratification of V/W (the order between strata corresponds to the reverse order of the inclusion, up to conjugacy, of parabolic subgroups). Replacing V by V^* , we define similarly $\mathcal{V}^*(P)$ and $\mathcal{U}^*(\mathfrak{P})$ for $P \in \text{Parab}(W)$ and $\mathfrak{P} \in \text{Parab}(W)/W$.

By definition, $\overline{N}_W(P)$ acts freely on $\mathcal{V}(P)$ or $\mathcal{V}^*(P)$. Moreover, for $P \in \mathfrak{P}$, the natural map $\mathcal{V}(P) \rightarrow \mathcal{U}(\mathfrak{P})$ induces an isomorphism of varieties

$$(1.1) \quad \mathcal{V}(P)/\overline{N}_W(P) \xrightarrow{\sim} \mathcal{U}(\mathfrak{P}).$$

In particular, $\mathcal{U}(\mathfrak{P})$ is smooth.

1.D. Rational Cherednik algebra at $t = 0$. — Let $k \in \mathbb{C}^\nabla$. We define the *rational Cherednik algebra* \mathbf{H}_k (at $t = 0$) to be the quotient of the algebra $T(V \oplus V^*) \rtimes W$ (the semi-direct product of the tensor algebra $T(V \oplus V^*)$ with the group W) by the relations

$$(1.2) \quad \begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \sum_{H \in \mathfrak{A}} \sum_{j=0}^{e_H-1} e_H(k_{H,j} - k_{H,j+1}) \frac{\langle y, \alpha_H \rangle \cdot \langle \alpha_H^\vee, x \rangle}{\langle \alpha_H^\vee, \alpha_H \rangle} \varepsilon_{H,j}, \end{cases}$$

for all $x, x' \in V^*$, $y, y' \in V$. Here $\langle \cdot, \cdot \rangle : V \times V^* \rightarrow \mathbb{C}$ is the standard pairing. The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \rightarrow \mathbf{H}_k$ and $\mathbb{C}[V^*] \rightarrow \mathbf{H}_k$. Recall [EtGi, Theo. 1.3] that we have an isomorphism of \mathbb{C} -vector spaces

$$(1.3) \quad \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_k$$

induced by multiplication (this is the so-called *PBW-decomposition*).

Remark 1.4. — Let $(l_\Omega)_{\Omega \in \mathfrak{A}/W}$ be a family of complex numbers and let $k' \in \mathbb{C}^\nabla$ be defined by $k'_{\Omega,j} = k_{\Omega,j} + l_\Omega$. Then $\mathbf{H}_k = \mathbf{H}_{k'}$. This means that there is no restriction to generality if we consider for instance only parameters k such that $k_{\Omega,0} = 0$ for all Ω , or only parameters k such that $k_{\Omega,0} + k_{\Omega,1} + \cdots + k_{\Omega,e_\Omega-1} = 0$ for all Ω (as in [BoRo]). ■

1.E. Calogero-Moser space. — We denote by \mathbf{Z}_k the center of the algebra \mathbf{H}_k : it is well-known [EtGi, Theo 3.3 and Lem. 3.5] that \mathbf{Z}_k is an integral domain, which is integrally closed. Moreover, it contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ as subalgebras [Gor, Prop. 3.6] (so it contains $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$). If $W = \langle \text{Ref}(W) \rangle$, then \mathbf{Z}_k is a free \mathbf{P} -module of rank $|W|$ (see [EtGi, Prop. 4.15]). We denote by \mathcal{X}_k the affine algebraic variety whose ring of regular functions $\mathbb{C}[\mathcal{X}_k]$ is \mathbf{Z}_k : this is the *Calogero-Moser space* associated with the datum (V, W, k) . It is irreducible and normal.

We set $\mathcal{P} = V/W \times V^*/W$, so that $\mathbb{C}[\mathcal{P}] = \mathbf{P}$ and the inclusion $\mathbf{P} \hookrightarrow \mathbf{Z}_k$ induces a morphism of varieties

$$\Upsilon_k : \mathcal{X}_k \longrightarrow \mathcal{P}$$

which is finite (and flat if $W = \langle \text{Ref}(W) \rangle$).

1.F. Extra-structures on the Calogero-Moser space. — The Calogero-Moser space \mathcal{X}_k is endowed with extra-structures (a \mathbb{C}^\times -action, a Poisson bracket, a filtration, an action of $N_{\text{GL}_\mathbb{C}(V)}(W)$...) which are described below.

1.F.1. *Grading, \mathbb{C}^\times -action.* — The algebra $\mathbb{T}(V \oplus V^*) \rtimes W$ can be \mathbb{Z} -graded in such a way that the generators have the following degrees

$$\begin{cases} \deg(y) = -1 & \text{if } y \in V, \\ \deg(x) = 1 & \text{if } x \in V^*, \\ \deg(w) = 0 & \text{if } w \in W. \end{cases}$$

This descends to a \mathbb{Z} -grading on \mathbf{H}_k , because the defining relations (1.2) are homogeneous. Since the center of a graded algebra is always graded, the subalgebra \mathbf{Z}_k is also \mathbb{Z} -graded. So the Calogero-Moser space \mathcal{X}_k inherits a regular \mathbb{C}^\times -action. Note also that by definition $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ is clearly a graded subalgebra of \mathbf{Z}_k .

1.F.2. *Poisson structure.* — Let $t \in \mathbb{C}$. One can define a deformation $\mathbf{H}_{t,k}$ of \mathbf{H}_k as follows: $\mathbf{H}_{t,k}$ is the quotient of the algebra $\mathbb{T}(V \oplus V^*) \rtimes W$ by the relations

$$(1.5) \quad \begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = t\langle y, x \rangle + \sum_{H \in \mathcal{A}} \sum_{i=0}^{e_H-1} e_H(k_{H,i} - k_{H,i+1}) \frac{\langle y, \alpha_H \rangle \cdot \langle \alpha_H^\vee, x \rangle}{\langle \alpha_H^\vee, \alpha_H \rangle} \varepsilon_{H,i}, \end{cases}$$

for all $x, x' \in V^*$, $y, y' \in V$. It is well-known [EtGi] that the PBW decomposition still holds so that the family $(\mathbf{H}_{t,k})_{t \in \mathbb{C}}$ is a flat deformation of $\mathbf{H}_k = \mathbf{H}_{0,k}$. This allows us to define a Poisson bracket $\{, \}$ on \mathbf{Z}_k as follows: if $z_1, z_2 \in \mathbf{Z}_k$, we denote by z_1^t, z_2^t the corresponding element of $\mathbf{H}_{t,k}$ through the PBW decomposition and we define

$$\{z_1, z_2\} = \lim_{t \rightarrow 0} \frac{[z_1^t, z_2^t]}{t}.$$

Finally, note that

$$(1.6) \quad \text{The Poisson bracket is } \mathbb{C}^\times\text{-equivariant.}$$

1.F.3. *Filtration.* — The tensor algebra $\mathbb{T}(V \oplus V^*)$ is naturally filtered by the subspaces $(\bigoplus_{j=0}^d (V \oplus V^*)^{\otimes j})$. This induces a filtration of $\mathbb{T}(V \oplus V^*) \rtimes W$ by putting W in degree 0 and so induces a filtration $(\mathcal{F}_j \mathbf{H}_k)_{j \geq 0}$ of the rational Cherednik algebra. By convention, we set $\mathcal{F}_{-1} \mathbf{H}_k = 0$. If M is any subspace of \mathbf{H}_k , we set $\mathcal{F}_j M = M \cap \mathcal{F}_j \mathbf{H}_k$, so that M also inherits a filtration and we denote by $\text{Rees}_{\mathcal{F}} M$ the Rees module of M (associated with the filtration $(\mathcal{F}_j X)_{j \geq 0}$), namely the $\mathbb{C}[\hbar]$ -submodule of $\mathbb{C}[\hbar] \otimes M$ equal to

$$\text{Rees}_{\mathcal{F}} M = \bigoplus_{j \geq 0} \hbar^j \mathcal{F}_j M.$$

Recall that, if $\lambda \in \mathbb{C}$, then

$$(1.7) \quad \mathbb{C}[\hbar]/\langle \hbar - \lambda \rangle \otimes_{\mathbb{C}[\hbar]} \text{Rees}_{\mathcal{F}} M \simeq \begin{cases} M & \text{if } \lambda \neq 0, \\ \text{gr}_{\mathcal{F}}(M) & \text{if } \lambda = 0, \end{cases}$$

where $\text{gr}_{\mathcal{F}}(M) = \bigoplus_{j \geq 0} \mathcal{F}_j M / \mathcal{F}_{j-1} M$ is the graded vector space associated with M and its filtration.

If A is a subalgebra of \mathbf{H}_k and J is an ideal of A , then $\text{Rees}_{\mathcal{F}}(A)$ is a subalgebra of $\mathbb{C}[\hbar] \otimes A$ (called the *Rees algebra* of A) and $\text{Rees}_{\mathcal{F}}(J)$ is an ideal of $\text{Rees}_{\mathcal{F}}(A)$. Recall [EtGi, Theo. 1.3] that

$$(1.8) \quad \text{gr}_{\mathcal{F}} \mathbf{H}_k \simeq \mathbf{H}_0 = \mathbb{C}[V \times V^*] \rtimes W \quad \text{and} \quad \text{gr}_{\mathcal{F}} \mathbf{Z}_k \simeq \mathbf{Z}_0 = \mathbb{C}[V \times V^*]^W.$$

1.F.4. Action of the normalizer. — The group $N_{\text{GL}_{\mathbb{C}}(V)}(W)$ acts on the set ∇ and so on the space of parameters \mathbb{C}^{∇} . If $\tau \in N_{\text{GL}_{\mathbb{C}}(V)}(W)$, then τ induces an isomorphism of algebras $\mathbf{H}_k \rightarrow \mathbf{H}_{\tau(k)}$. So, if $\tau(k) = k$, then it induces an action on the algebra \mathbf{H}_k , and so on its center \mathbf{Z}_k and on the Calogero-Moser space \mathcal{X}_k , which preserves the \mathbb{C}^{\times} -action and the Poisson bracket. We set

$$\delta(\tau) = \max_{w \in W} \dim V^{w\tau}.$$

Of course, $\delta(\tau)$ depends only on the coset $W\tau$ and not on τ . We say that τ is *W-full* if $\delta(\tau) = \dim V^{\tau}$. Since W acts trivially on \mathcal{X}_k , the study of the action of τ on \mathcal{X}_k is equivalent to the study of the action of $w\tau$. So, by replacing τ by $w\tau$ if necessary, we may assume that τ is *W-full*.

Example 1.9. — An element $\tau \in N_{\text{GL}_{\mathbb{C}}(V)}(W)$ is called *W-regular* (or simply *regular* if W is clear from the context) if $V^{\tau} \cap V_{\text{reg}} \neq \emptyset$. A *W-regular* element of $N_{\text{GL}_{\mathbb{C}}(V)}(W)$ is *W-full* [Spr]. ■

Hypothesis and notation. *From now on, and until the end of this paper, we assume that*

$$W = \langle \text{Ref}(W) \rangle,$$

*we fix a parameter $k \in \mathbb{C}^{\nabla}$ and an element τ of finite order of $N_{\text{GL}_{\mathbb{C}}(V)}(W)$ such that $\tau(k) = k$. We also assume that τ is *W-full*.*

If \star is one of the objects defined in the previous sections ($\mathbf{H}_k, \mathcal{X}_k, \nabla, \mathcal{A}, \dots$), we will sometimes denote it by $\star(W)$ or $\star(V, W)$ if we need to emphasize the context.

2. Recollection about Poisson structures and symplectic leaves

Notation. *We fix in this section, and only in this section, a commutative noetherian Poisson \mathbb{C} -algebra R , whose Poisson bracket is denoted by $\{, \}$.*

2.A. Poisson ideals. — An ideal I of R is called a *Poisson ideal* of R if $\{r, I\} \subset I$ for all $r \in R$. The following facts may be found in [Dix, Lem. 3.3.3]:

Proposition 2.1. — *Let I be a Poisson ideal of R . Then:*

- (a) *Every minimal prime ideal containing I is Poisson.*
- (b) *The radical of I is Poisson.*

2.B. Normalisation. — The next result is due to Kaledin [Kal]:

Theorem 2.2 (Kaledin). — *Assume that R is a domain. Then there is a unique Poisson bracket on the normalisation of R extending $\{, \}$.*

2.C. Action of a finite group. — We assume in this subsection that we are given a finite group G acting on the \mathbb{C} -algebra R in such a way that the Poisson bracket is G -equivariant (i.e. $\{g(r), g(r')\} = g(\{r, r'\})$ for all $g \in G$ and $r, r' \in R$). Let I denote the ideal of R generated by the family $(g(r) - r)_{\substack{g \in G \\ r \in R}}$. Then R/I is the biggest quotient algebra of R on which G acts trivially.

Since G is finite and \mathbb{C} has characteristic 0, the natural map

$$R^G \longrightarrow (R/I)^G = R/I$$

is surjective and its kernel is I^G . Moreover, R^G is a Poisson subalgebra of R (because the Poisson bracket is G -equivariant). Note that I is not in general a Poisson ideal of R , but it is easily checked that

$$(2.3) \quad I^G \text{ is a Poisson ideal of } R^G.$$

Therefore, $R/I = R^G/I^G$ can be naturally endowed with a Poisson bracket. And, by Proposition 2.1(b), $R/\sqrt{I} = R^G/\sqrt{I^G}$ also inherits a Poisson bracket.

Remark 2.4. — If $R = \mathbb{C}[\mathcal{X}]$ is the coordinate ring of an affine variety \mathcal{X} , then R/I is the coordinate ring of the G -fixed points scheme of \mathcal{X} (which will be denoted by $\mathcal{X}^{(G)}$), while R/\sqrt{I} is the coordinate ring of its reduced subscheme (which will be denoted by \mathcal{X}^G). The above construction shows that the closed subvariety \mathcal{X}^G of \mathcal{X} inherits a Poisson structure from the one on \mathcal{X} , even though it is not in general a Poisson subvariety of \mathcal{X} (i.e. the natural map $\mathcal{X}^G \hookrightarrow \mathcal{X}$ is not Poisson). However, \mathcal{X}/G is also a Poisson variety and the natural map $\mathcal{X}^G \hookrightarrow \mathcal{X}/G$ is Poisson, i.e. \mathcal{X}^G is a closed Poisson subvariety of \mathcal{X}/G .

If moreover \mathcal{X} is smooth, then $\mathcal{X}^{(G)} = \mathcal{X}^G$ is also smooth, and if the Poisson structure on \mathcal{X} makes it into a symplectic variety, then \mathcal{X}^G is also symplectic for the induced Poisson structure. ■

Example 2.5. — Let E be a \mathbb{C} -vector space endowed with a symplectic form ω and assume here that $R = \mathbb{C}[E]$ and that $G \subset \mathbf{Sp}(E, \omega)$. Then the restriction of ω to E^G is non-degenerate, so this endows E^G with a structure of Poisson (even more, symplectic) variety. On the other hand, via the above Remark 2.4, the variety E^G also inherits from E a structure of Poisson variety. It is easily checked that both structures coincide. ■

2.D. Symplectic leaves. — Assume in this subsection that $R = \mathbb{C}[\mathcal{X}]$ is the coordinate ring of an affine variety \mathcal{X} . Then Brown-Gordon [BrGo] have defined a stratification of \mathcal{X} by *symplectic leaves*, which are in general not algebraic subvarieties of \mathcal{X} . We denote by $\text{Sympl}(\mathcal{X})$ the set of symplectic leaves of \mathcal{X} .

When \mathcal{X} has finitely many symplectic leaves, then the symplectic leaves are algebraic [BrGo, Prop. 3.7] and the stratification of \mathcal{X} into symplectic leaves is given as follows. Let $(\mathcal{S}_j)_{j \geq 0}$ be the sequence of closed subvarieties of \mathcal{X} defined by

$$\begin{cases} \mathcal{S}_0 = \mathcal{X}, \\ \text{If } j \geq 0, \text{ then } \mathcal{S}_{j+1} \text{ is the reduced singular locus of } \mathcal{S}_j. \end{cases}$$

Then the symplectic leaves of \mathcal{X} are the irreducible components of the locally closed subvarieties $(\mathcal{S}_j \setminus \mathcal{S}_{j+1})_{j \geq 0}$. Let $\text{PSpec}(\mathbb{C}[\mathcal{X}])$ denote the set of prime ideals that are Poisson. If \mathcal{S} is a symplectic leaf of \mathcal{X} , we denote by $\mathfrak{p}_{\mathcal{S}}$ the defining ideal of \mathcal{S} in $\mathbb{C}[\mathcal{X}]$: it belongs to $\text{PSpec}(\mathbb{C}[\mathcal{X}])$. If \mathcal{X} has finitely many symplectic leaves, then the map

$$(2.6) \quad \begin{array}{ccc} \text{Sympl}(\mathcal{X}) & \longrightarrow & \text{PSpec}(\mathbb{C}[\mathcal{X}]) \\ \mathcal{S} & \longmapsto & \mathfrak{p}_{\mathcal{S}} \end{array}$$

is bijective [BrGo, Lem. 3.4]. The inverse is given as follows: if $\mathfrak{p} \in \text{PSpec}(\mathbb{C}[\mathcal{X}])$ corresponds to \mathcal{S} through this bijection, then \mathcal{S} is the smooth locus of the closed irreducible subvariety of \mathcal{X} defined by \mathfrak{p} .

Lemma 2.7. — *Assume that \mathcal{X} has finitely many symplectic leaves and that \mathcal{Y} is a locally closed Poisson subvariety of \mathcal{X} . Then \mathcal{Y} has finitely many symplectic leaves.*

Proof. — Taking the closure of \mathcal{Y} , which is also Poisson, allows to assume that \mathcal{Y} is closed. Let $(\mathcal{S}_i)_{i \in L}$ be the family of symplectic leaves of \mathcal{X} (for some finite indexing set L). Let \mathcal{F} be an irreducible component of \mathcal{Y} . Then \mathcal{F} is also Poisson by Proposition 2.1(a) so it is the closure of a symplectic leaf thanks to the bijection (2.6). In particular, there exists a subset I of L such that \mathcal{F} is the union of the \mathcal{S}_i , for $i \in I$. This proves that \mathcal{Y} is a union of symplectic leaves of \mathcal{X} , each of which being also a symplectic leaf of \mathcal{Y} . \square

Now, let G be a finite group acting on \mathcal{X} and preserving the Poisson bracket. Then \mathcal{X}/G is an affine Poisson variety (because $\mathbb{C}[\mathcal{X}/G] = \mathbb{C}[\mathcal{X}]^G$ is a Poisson subalgebra of $\mathbb{C}[\mathcal{X}]$, see Remark 2.4). If H is a subgroup of G , we denote by $\mathcal{X}(H)$ the set of elements $x \in \mathcal{X}$ whose stabilizer is exactly H . Then $\mathcal{X}(H)$ is a locally closed subvariety of \mathcal{X} (it is open in \mathcal{X}^H). The subgroup H is called *parabolic* if $\mathcal{X}(H) \neq \emptyset$. Let $\text{Parab}(G)$ denote the set of parabolic subgroups of G .

If \mathfrak{h} is a conjugacy class of parabolic subgroups of G , we denote by $(\mathcal{X}/G)(\mathfrak{h})$ the image of $\mathcal{X}(H)$ in \mathcal{X}/G for some (or any) $H \in \mathfrak{h}$. Then the group $\overline{N}_G(H)/H$ acts freely on $\mathcal{X}(H)$ and the natural map $\mathcal{X}(H) \rightarrow (\mathcal{X}/G)(\mathfrak{h})$ induces an isomorphism

$$(2.8) \quad \mathcal{X}(H)/G(H) \xrightarrow{\sim} (\mathcal{X}/G)(\mathfrak{h}).$$

Indeed, if $g \in G$ and $x, x' \in \mathcal{X}(H)$ are such that $g \cdot x = x'$, then $H = G_{x'} = {}^g G_x = {}^g H$ and so $g \in N_G(H)$. The next results generalizes slightly [BrGo, Prop. 7.4]:

Proposition 2.9. — *Assume that \mathcal{X} is smooth and symplectic. Then the symplectic leaves of \mathcal{X}/G are the irreducible components of the locally closed subvarieties $(\mathcal{X}/G)(\mathfrak{h})$ where \mathfrak{h} runs over $\text{Parab}(G)/G$.*

In particular, if all the subvarieties $(\mathcal{X}/G)(\mathfrak{h})$ are irreducible, then

$$\mathcal{X}/G = \bigcup_{\mathfrak{h} \in \text{Parab}(G)/G} (\mathcal{X}/G)(\mathfrak{h})$$

is the stratification of \mathcal{X}/G into symplectic leaves.

Proof. — Let $\mathfrak{h} \in \text{Parab}(G)$ and let $H \in \mathfrak{h}$. Since \mathcal{X} is smooth and symplectic, the subvariety \mathcal{X}^H is also smooth and symplectic. So its open subset $\mathcal{X}(H)$ is also smooth and symplectic as well as $(\mathcal{X}/G)(\mathfrak{h})$ thanks to the isomorphism (2.8). And the morphism $\mathcal{X}^H \rightarrow \mathcal{X}/G$ is Poisson: this proves that any irreducible component of $(\mathcal{X}/G)(\mathfrak{h})$ is contained in a unique symplectic leaf. In particular, \mathcal{X}/G has finitely many symplectic leaves.

It remains to show that any irreducible component \mathcal{F} of $(\mathcal{X}/G)(\mathfrak{h})$ is a symplectic leaf. But $\overline{\mathcal{F}}$ is a closed Poisson subvariety of \mathcal{X}/G , so its smooth locus is a symplectic leaf of \mathcal{X}/G by the bijection (2.6). Since \mathcal{F} is smooth, it remains to show that \mathcal{F} is the smooth locus of $\overline{\mathcal{F}}$. Note that $\overline{\mathcal{F}}$ has finitely many symplectic leaves so, by the discussion at the beginning of this subsection, it is sufficient to show that $\overline{\mathcal{F}} \setminus \mathcal{F}$ is a (closed) Poisson subvariety.

But $\overline{(\mathcal{X}/G)(\mathfrak{h})} \setminus (\mathcal{X}/G)(\mathfrak{h})$ is the union of the $\overline{(\mathcal{X}/G)(\mathfrak{h}')}$, where \mathfrak{h}' runs over the set of conjugacy classes of parabolic subgroups of G containing strictly at least one element of \mathfrak{h} , so it is Poisson. Since $\overline{\mathcal{F}} \setminus \mathcal{F} = \overline{\mathcal{F}} \cap \left(\overline{(\mathcal{X}/G)(\mathfrak{h})} \setminus (\mathcal{X}/G)(\mathfrak{h}) \right)$, we get that $\overline{\mathcal{F}} \setminus \mathcal{F}$ is Poisson, as desired. \square

Corollary 2.10. — *Assume that \mathcal{X} has finitely many symplectic leaves. Then \mathcal{X}/G and \mathcal{X}^G have finitely many symplectic leaves.*

Proof. — Let \mathcal{S} denote a symplectic leaf of \mathcal{X} and let $H = G_{\mathcal{S}}^{\text{set}}$. As symplectic leaves form a partition of \mathcal{X} , and since $g(\mathcal{S})$ is a symplectic leaf of \mathcal{X} for any $g \in G$, we get that

$$g(\mathcal{S}) \cap \mathcal{S} = \emptyset$$

for all $g \notin H$. So, the image \mathcal{S} in \mathcal{X}/G is isomorphic to \mathcal{S}/H and is a locally closed Poisson subvariety of \mathcal{X}/G . But, by Proposition 2.9, \mathcal{S}/H has finitely many symplectic leaves.

As \mathcal{X} has finitely many symplectic leaves, this shows that \mathcal{X}/G also has finitely many symplectic leaves. Now, \mathcal{X}^G is a closed Poisson subvariety of \mathcal{X}/G , so it also admits finitely many symplectic leaves by Lemma 2.7. \square

As a consequence of the above proof, we get:

Corollary 2.11. — *Assume that \mathcal{X} has finitely many symplectic leaves and that G acts freely on \mathcal{X} . Then the map $\text{Symp}(\mathcal{X})/G \rightarrow \text{Symp}(\mathcal{X}/G)$ sending the G -orbit of a symplectic leaf of \mathcal{X} to its image in \mathcal{X}/G is well-defined and bijective.*

3. Recollection of Lehrer-Springer theory

3.A. Reflection groups. — Recall that $\tau \in N_{\mathrm{GL}_{\mathbb{C}}(V)}(W)$ is assumed to be W -full. This implies that [Spr]

$$(3.1) \quad \delta(\tau) = \dim(V/W)^\tau.$$

Therefore, since $(V/W)^\tau$ is irreducible (it is isomorphic to an affine space [Spr]), we get that

$$(3.2) \quad \text{the natural map } V^\tau \rightarrow (V/W)^\tau \text{ is onto.}$$

To simplify notation, we set $W_\tau = W_{V^\tau}^{\mathrm{set}}/W_{V^\tau}^{\mathrm{pt}}$. Note that $W^\tau \subset W_{V^\tau}^{\mathrm{set}}$. Moreover, W_τ acts faithfully on V^τ , so

$$(3.3) \quad \tau \text{ acts trivially on } W_\tau.$$

Lehrer-Springer theory [LeSp, Theo. 2.5 and Cor. 2.7] gives the following result:

Theorem 3.4 (Springer, Lehrer-Springer). — Recall that τ is W -full. Then:

- (a) The group W_τ is a reflection group for its action on V^τ .
- (b) The natural map

$$i_\tau : V^\tau/W_\tau \longrightarrow (V/W)^\tau$$

is an isomorphism of varieties.

- (c) The reflecting hyperplanes of W_τ are exactly the intersections with V^τ of the reflecting hyperplanes of W which do not contain V^τ .

Similarly, the natural map $i_\tau^\vee : V^{*\tau}/W_\tau \rightarrow (V^*/W)^\tau$ is an isomorphism of varieties.

Example 3.5. — If τ is W -regular (as defined in Example 1.9), then $W_\tau = W^\tau$ by [Spr]. ■

3.B. τ -split parabolic subgroups. — A parabolic subgroup P of W is called τ -split if it is the stabilizer of some point of V^τ (i.e. if $P = W_{V^P \cap V^\tau}^{\mathrm{pt}}$ or, in other words, if $\mathcal{V}(P) \cap V^\tau \neq \emptyset$). This is equivalent to say that P is the stabilizer of some point of $V^{*\tau}$. Note the following easy fact:

$$(3.6) \quad \text{The intersection of } \tau\text{-split parabolic subgroups is } \tau\text{-split.}$$

In this case, P is normalized by τ and τ is P -full, and we define the τ -rank of P to be the number $\dim(V^P)^\tau$. We denote by $\mathrm{Parab}_\tau(W)$ the set of τ -split parabolic subgroups of W . If $P \in \mathrm{Parab}_\tau(W)$, then $W_{V^\tau}^{\mathrm{pt}} \subset P$ and $P_\tau = (P \cap W_{V^\tau}^{\mathrm{set}})/W_{V^\tau}^{\mathrm{pt}}$ is a parabolic subgroup of W_τ . This shows that the map

$$\begin{array}{ccc} \mathrm{Parab}_\tau(W) & \longrightarrow & \mathrm{Parab}(W_\tau) \\ P & \longmapsto & P_\tau \end{array}$$

is well-defined.

Lemma 3.7. — The map

$$\begin{array}{ccc} \mathrm{Parab}_\tau(W) & \longrightarrow & \mathrm{Parab}(W_\tau) \\ P & \longmapsto & P_\tau \end{array}$$

is bijective.

Proof. — First, if Q is a parabolic subgroup of W_τ , then there exists $v \in V^\tau$ such that $Q = (W_\tau)_v$ and so, if we set $P = W_v$, then P is τ -split and $P_\tau = Q$. This shows that the map is surjective.

Now, if P is a τ -split parabolic subgroups of W , then

$$(3.8) \quad (V^P)^\tau = (V^\tau)^{P_\tau}.$$

Proof of (3.8). — Since P is τ -split, there exists $v \in (V^P)^\tau$ such that $P = W_v$ (and so $P_\tau = (W_\tau)_v$). Therefore, by Theorem 3.4(a),

$$(V^\tau)^{P_\tau} = \bigcap_{\substack{H \in \mathcal{A}(V^\tau, W_\tau) \\ v \in H}} H,$$

and so, by Theorem 3.4(c),

$$(V^\tau)^{P_\tau} = V^\tau \cap \left(\bigcap_{\substack{H \in \mathcal{A}(V, W) \\ v \in H}} H \right) = V^\tau \cap V^P,$$

as expected. \square

Since $P = W_{(V^P)^\tau}^{\text{pt}}$, the group P_τ determines P . This means that the map of the lemma is injective. \square

If $\Omega \in \text{Parab}(W_\tau)/W_\tau$ and $Q \in \Omega$, we denote by $\mathcal{V}_\tau(Q)$ and $\mathcal{U}_\tau(\Omega)$ the analogues of $\mathcal{V}(P)$ and $\mathcal{U}(\mathfrak{P})$ for $\mathfrak{P} \in \text{Parab}(W)/W$ and $P \in \mathfrak{P}$. We also define similarly $\mathcal{V}_\tau^*(Q)$ and $\mathcal{U}_\tau^*(\Omega)$. The same argument as in the above proof (using Theorem 3.4(c)) shows that, if P is τ -split, then

$$(3.9) \quad \mathcal{V}_\tau(P_\tau) = \mathcal{V}(P)^\tau.$$

3.C. Normalizers. — Fix a τ -split parabolic subgroup P of W . If $w \in W_\tau$, then wP does not depend on the representative of w in $W_{V^\tau}^{\text{set}}$, because $W_{V^\tau}^{\text{pt}} \subset P$ by definition. So we can define the *normalizer* $N_{W_\tau}(P)$ of P in W_τ and, by the bijectivity proved in Lemma 3.7, it coincides with the normalizer $N_{W_\tau}(P_\tau)$. The kernel of the well-defined composition

$$N_{W_\tau}(P_\tau) = N_{W_\tau}(P) \longrightarrow \overline{N}_W(P)/P$$

is equal to P_τ , so we get a natural injective map

$$(3.10) \quad \overline{N}_{W_\tau}(P_\tau) \hookrightarrow \overline{N}_W(P).$$

Now, τ acts on $\overline{N}_W(P)$. The next result describes the image of the above injective map:

Lemma 3.11. — *The image of the morphism (3.10) is equal to $\overline{N}_W(P)^\tau$.*

Proof. — Let G denote the image of the morphism (3.10) and let $v \in \mathcal{V}(P)^\tau$ (so that $P = W_v$).

Let $w \in N_{W_\tau}(P_\tau)$ and let \dot{w} be a representative of w in $W_{V^\tau}^{\text{set}}$. Then $\dot{w}(v) \in \mathcal{V}(P)^\tau$ by (3.9). So $\tau(\dot{w}(v)) = \dot{w}(v)$, i.e. $\dot{w}^{-1}\tau(\dot{w}) \in P$. So the image of w in $\overline{N}_W(P)$ is τ -invariant. This proves that $G \subset \overline{N}_W(P)^\tau$.

Conversely, let $w \in \overline{N}_W(P)^\tau$ and let \dot{w} denote a representative of w in $N_W(P)$. Then $\tau(\dot{w}(v)) = \dot{w}(\dot{w}^{-1}\tau(\dot{w}))(v)$. But $(\dot{w}^{-1}\tau(\dot{w}))(v) = v$ since $\dot{w}^{-1}\tau(\dot{w}) \in P$ by hypothesis. So v and $\dot{w}(v)$ belong to V^τ so, by Lehrer-Springer Theorem 3.4(b), there exists $x \in W_\tau^{\text{set}}$ such

that $\dot{w}(v) = x(v)$. In other words, $x^{-1}\dot{w} \in P$. Moreover, v and $\dot{w}(v)$ both belong to $\mathcal{V}(P)$, so x normalizes P (and P_τ), so w is the image of x under the morphism (3.10). In other words, $\bar{N}_W(P)^\tau \subset G$. \square

Thanks to Lemma 3.11, we will identify $\bar{N}_{W_\tau}(P_\tau)$ with $\bar{N}_W(P)^\tau$. Note that $\bar{N}_{W_\tau}(P_\tau) = \bar{N}_W(P)^\tau$ is the stabilizer of the set $\mathcal{V}(P)^\tau$ in $\bar{N}_W(P)$.

3.D. Orbits of τ -split parabolic subgroups. — We denote by $\text{Parab}(W)_{\text{spl}}^\tau$ the set of τ -split parabolic subgroups of W and by $(\text{Parab}(W)/W)_{\text{spl}}^\tau$ the set of W -orbits of parabolic subgroups of W containing a τ -split one. The group $W_{V^\tau}^{\text{set}}$ acts on $\text{Parab}(W)_{\text{spl}}^\tau$ by conjugacy and, since any τ -split parabolic subgroup of W contains $W_{V^\tau}^{\text{pt}}$, this action factorizes through an action of W_τ . If $\mathfrak{P} \in (\text{Parab}(W)/W)_{\text{spl}}^\tau$, we set $\mathfrak{P}_{\text{spl}}^\tau = \mathfrak{P} \cap \text{Parab}(W)_{\text{spl}}^\tau$. Now, let \mathcal{E}_P (resp. $\tilde{\mathcal{E}}_P$) denote the set of elements $w \in N_W(P)$ (resp. $\bar{N}_W(P)$) such that $\mathcal{V}(P)^{w\tau} \neq \emptyset$. Then $\bar{N}_W(P)$ acts by conjugacy on the set $\tilde{\mathcal{E}}_P\tau$. If $w \in \mathcal{E}_P$, we denote by $[w\tau]$ the $\bar{N}_W(P)$ -orbit of the image of $w\tau$ in $\bar{N}_W(P)\tau$.

Proposition 3.12. — *Let P be a τ -split parabolic subgroup and let \mathfrak{P} denote its W -orbit. Then:*

- (a) *Let $x \in W$. Then xP is τ -split if and only if $x^{-1}\tau(x) \in \mathcal{E}_P$.*
- (b) *The map $\mathfrak{P}_{\text{spl}}^\tau \rightarrow \tilde{\mathcal{E}}_P\tau/\bar{N}_W(P)$, ${}^xP \mapsto [x^{-1}\tau x]$ is well-defined and induces a bijection*

$$\mathfrak{P}_{\text{spl}}^\tau/W_\tau \xrightarrow{\sim} \tilde{\mathcal{E}}_P\tau/\bar{N}_W(P).$$

Proof. — (a) Assume that xP is τ -split. In other words, there exists $v \in V^\tau$ such that ${}^xP = W_v$. Now, let $w = x^{-1}\tau(x)$: then $x^{-1}(v) \in \mathcal{V}(P)^{w\tau}$ and so $w \in \mathcal{E}_P$.

Conversely, assume that $w = x^{-1}\tau(x) \in \mathcal{E}_P$. Then there exists $v \in \mathcal{V}(P)$ such that $v \in V^{w\tau}$. Therefore, $P = W_v$ and so ${}^xP = W_{x(v)}$. But $\tau(x(v)) = wx^{-1}\tau(x)\tau(v) = xw\tau(v) = x(v)$, so $x(v) \in V^\tau$. This implies that xP is τ -split by definition.

(b) Let us first show that the map (let us denote it by ϕ) is well-defined. For this, let x and y be two elements of W such that ${}^xP = {}^yP$ is τ -split. Then there exists $u \in N_W(P)$ such that $y = xu$. So $y^{-1}\tau y = u^{-1}x^{-1}\tau x u$ and so $[y^{-1}\tau y] = [x^{-1}\tau x]$, as expected.

Let us now prove that ϕ is constant on W_τ -orbits. For this, let $w \in W_\tau$ and $x \in W$ be such that $\xi^{-1}\tau(x) \in \mathcal{E}_P$. Then $(xw)^{-1}\tau(wx) = x^{-1}w^{-1}\tau(w)xx^{-1}\tau(x)$. But $w^{-1}\tau(w) \in W_{V^\tau}^{\text{pt}} \subset P$ by (3.3), so the images of $x^{-1}\tau(x)$ and $(xw)^{-1}\tau(wx)$ in $\tilde{\mathcal{E}}_P$ coincide. Therefore, ϕ factorizes through a map

$$\tilde{\phi} : \mathfrak{P}_{\text{spl}}^\tau/W_\tau \longrightarrow \tilde{\mathcal{E}}_P\tau/\bar{N}_W(P).$$

Let us prove that $\tilde{\phi}$ is injective. So let x and y be two elements of W such that xP and yP are τ -split and $[x^{-1}\tau x] = [y^{-1}\tau y]$. Then there exists $u \in N_W(P)$ and $p \in P$ such that $y^{-1}\tau y = u^{-1}x^{-1}\tau x u p$. In particular, $\mathcal{V}(P)^{y^{-1}\tau y} = \mathcal{V}(P)^{u^{-1}x^{-1}\tau x u}$. Since ${}^xP = {}^{xu}P$, we may (and we will) assume that $u = 1$. As xP is τ -split, the set $\mathcal{V}({}^xP)^\tau$ is non-empty, so we may pick an element $v \in \mathcal{V}({}^xP)^\tau$. Then

$$\tau y x^{-1}(v) = y y^{-1} \tau y x^{-1}(v) = y x^{-1} \tau x p x^{-1}(v).$$

But $x^{-1}(v) \in \mathcal{V}(P)$ so $yx^{-1}(v) = x^{-1}(v)$. Consequently, $\tau yx^{-1}(v) = yx^{-1}\tau(v) = yx^{-1}(v)$. In other words, $yx^{-1}(v) \in \mathcal{V}(yP)^\tau \subset V^\tau$. By Lehrer-Springer Theorem 3.4(b), there exists $a \in W_\tau$ such that $yx^{-1}(v) = a(v)$. Then

$$yP = yx^{-1}(xP) = yx^{-1}W_v = W_{yx^{-1}(v)} = W_{a(v)} = {}^aW_v = {}^a(xP),$$

which shows that yP and xP are W_τ -conjugate.

Let us now prove that $\tilde{\phi}$ is surjective. So let $w \in \mathcal{E}_P$. Then there exists $v \in \mathcal{V}(P)^{w\tau}$. So $W \cdot v \in (V/W)^\tau$. By Lehrer-Springer Theorem 3.4, there exists $x \in W$ such that $x(v) \in V^\tau$. Therefore, $xP = W_{x(v)}$ is τ -split and $v \in V^{x^{-1}\tau x}$. So, if we set $p = w^{-1}x^{-1}\tau(x)$, then $p(v) = v$ so $p \in P$ and $\phi(xP) = [x^{-1}\tau(x)] = [wp] = [w]$, as desired. \square

3.E. Stratification of $(V/W)^\tau$. — Applying 1.C to the pair (V^τ, W_τ) , the variety V^τ/W_τ admits a stratification $(\mathcal{U}_\tau(\Omega))_{\Omega \in \text{Parab}(W_\tau)/W_\tau}$ while the variety $(V/W)^\tau$ admits a stratification $(\mathcal{U}(\mathfrak{P})^\tau)_{\mathfrak{P} \in \text{Parab}(W)/W}$. Both varieties are isomorphic and so both stratifications can be compared: through this isomorphism, the first one is a refinement of the second one, as will be shown in Corollary 3.15 below by using Proposition 3.12.

Proposition 3.13. — *Let $\mathfrak{P} \in \text{Parab}(W)/W$. Then $\mathcal{U}(\mathfrak{P})^\tau$ is non-empty if and only if \mathfrak{P} contains a τ -split parabolic subgroup.*

Proof. — If \mathfrak{P} contains a τ -split parabolic subgroup P and if $v \in V^\tau$ is such that $P = W_v$, then the W -orbit of v belongs to $\mathcal{U}(\mathfrak{P})^\tau$ which is therefore non-empty. Conversely, if $\mathcal{U}(\mathfrak{P})^\tau$ is non-empty, it then follows from Theorem 3.4(b) that there exists $v \in V^\tau$ whose W -orbit belongs to $\mathcal{U}(\mathfrak{P})^\tau$. By construction, $W_v \in \mathfrak{P}$ and is τ -split. \square

After eliminating the empty pieces, Proposition 3.13 shows that $(V/W)^\tau$ admits a stratification $(\mathcal{U}(\mathfrak{P})^\tau)_{\mathfrak{P} \in (\text{Parab}(W)/W)_{\text{spl}}^\tau}$. Let us decompose the pieces of this stratification into irreducible components. For this, fix a τ -split parabolic subgroup P and let \mathfrak{P} denote its conjugacy class. Then $\mathcal{U}(\mathfrak{P})^\tau$ is smooth since $\mathcal{U}(\mathfrak{P})$ is smooth and τ has finite order, and we have

$$\mathcal{U}(\mathfrak{P})^\tau = (\mathcal{V}(P)/\overline{N}_W(P))^\tau = \left(\bigcup_{w \in \tilde{\mathcal{E}}_P} \mathcal{V}(P)^{w\tau} \right) / \overline{N}_W(P).$$

By definition of $\mathcal{V}(P)$, $\mathcal{V}(P)^{w\tau} \cap \mathcal{V}(P)^{w'\tau} = \emptyset$ if $w \neq w'$. If E is a subset of $\tilde{\mathcal{E}}_P\tau$, we denote by $\mathcal{V}(P)^E$ the (disjoint) union of the $\mathcal{V}(P)^g$ for $g \in E$. Then

$$(3.14) \quad \mathcal{U}(\mathfrak{P})^\tau = \bigcup_{E \in \tilde{\mathcal{E}}_P\tau / \overline{N}_W(P)} \mathcal{V}(P)^E / \overline{N}_W(P).$$

Then $\mathcal{V}(P)^E / \overline{N}_W(P)$ is the image of some $\mathcal{V}(P)^g$ for some $g \in E$ and so $\mathcal{V}(P)^E / \overline{N}_W(P)$ is closed (in $\mathcal{U}(\mathfrak{P})^\tau$) and irreducible. So the decomposition (3.14) is the decomposition of $\mathcal{U}(\mathfrak{P})^\tau$ into irreducible (i.e. connected because they are disjoint) components.

So the stratification $(\mathcal{U}(\mathfrak{P})^\tau)_{\mathfrak{P} \in (\text{Parab}(W)/W)_{\text{spl}}^\tau}$ of $(V/W)^\tau$ together with the decomposition (3.14) provides a finer stratification of $(V/W)^\tau$, indexed by the W_τ -orbits of τ -split parabolic subgroups (by using the bijection of Proposition 3.12(b)). On the other hand, V^τ/W_τ admits a stratification $(\mathcal{U}_\tau(\Omega))_{\Omega \in \text{Parab}(W_\tau)/W_\tau}$. Both stratifications coincide through the isomorphism i_τ , as shown by the next result:

Corollary 3.15. — Let $\mathfrak{P} \in (\text{Parab}(W)/W)_{\text{spl}}^\tau$, let $P \in \mathfrak{P}$ and let $E \in \tilde{\mathcal{E}}_{P\tau}/\overline{N}_W(P)$. Let \mathfrak{P}_E denote the W_τ -orbit of τ -split parabolic subgroups of W associated with E through the bijection of Proposition 3.12(b). Let \mathfrak{Q}_E denote the W_τ -orbit of parabolic subgroups of W_τ of the form Q_τ for $Q \in \mathfrak{P}_E$ (see Lemma 3.7). Then

$$i_\tau(\mathcal{U}_\tau(\mathfrak{Q}_E)) = \mathcal{V}(P)^E/\overline{N}_W(P).$$

Proof. — Let $g \in E$ and let $x \in W$ be such that $[x^{-1}\tau x] = [g]$ (the existence of such an x is guaranteed by Proposition 3.12(b)). We set $Q = {}^xP$. Then Q is τ -split by Proposition 3.12(a) and \mathfrak{P}_E (resp. \mathfrak{Q}_E) is the W_τ -orbit of Q (resp. Q_τ) by construction.

Now, $i_\tau(\mathcal{U}_\tau(\mathfrak{Q}_E))$ is the image of $\mathcal{V}({}^xP)^\tau$ in $(V/W)^P$ and, through the isomorphism $\mathcal{V}(P)/\overline{N}_W(P) \simeq \mathcal{U}(\mathfrak{P})$, the result comes from the fact that x^{-1} induces an isomorphism between $\mathcal{V}({}^xP)^\tau$ and $\mathcal{V}(P)^{x^{-1}\tau x}$. \square

4. The problem, the main result

4.A. Symplectic leaves. — Let I_k denote the ideal of \mathbf{Z}_k generated by $(\tau(z) - z)_{z \in \mathbf{Z}_k}$. It is τ -stable. Recall from Remark 2.4 that $\mathbb{C}[\mathcal{X}_k^\tau] = \mathbf{Z}_k/\sqrt{I_k} = \mathbf{Z}_k^\tau/\sqrt{I_k^\tau}$, and that \mathbf{Z}_k/I_k inherits a Poisson bracket which makes \mathcal{X}_k^τ into an affine Poisson variety. Therefore, \mathcal{X}_k^τ admits a stratification into symplectic leaves [BrGo, §3.5]. We denote by $\text{Sym}\rho(\mathcal{X}_k^\tau)$ the set of symplectic leaves of \mathcal{X}_k^τ .

Remark 4.1. — Note that \mathcal{X}_k^τ is generally not irreducible, not connected, not equidimensional and that its irreducible components might not coincide with its connected components. \blacksquare

Since \mathcal{X}_k has finitely many symplectic leaves [BrGo, Prop. 7.4], it follows from Corollary 2.10 that \mathcal{X}_k^τ has finitely many symplectic leaves too. They are obtained as in §2.D.

Remark 4.2. — This description shows that the symplectic leaves of \mathcal{X}_k^τ are \mathbb{C}^\times -stable. \blacksquare

If \mathcal{S} is a symplectic leaf of \mathcal{X}_k^τ , we denote by $\mathfrak{p}_\mathcal{S}$ the defining ideal of $\overline{\mathcal{S}}$ in \mathbf{Z}_k/I_k : it belongs to $\text{PSpec}(\mathbf{Z}_k/I_k)$. Since \mathcal{X}_k^τ has finitely many symplectic leaves, the map

$$(4.3) \quad \begin{array}{ccc} \text{Sym}\rho(\mathcal{X}_k^\tau) & \longrightarrow & \text{PSpec}(\mathbf{Z}_k/I_k) \\ \mathcal{S} & \longmapsto & \mathfrak{p}_\mathcal{S} \end{array}$$

is bijective (see (2.6)).

4.B. τ -cuspidality. — We define a τ -cuspidal symplectic leaf^(*) to be a zero-dimensional symplectic leaf of \mathcal{X}_k^τ . We will therefore also call it a τ -cuspidal point. Through the bijection (4.3), the set of τ -cuspidal points is naturally in bijection with the set $\text{PMax}(\mathbf{Z}_k/\sqrt{I_k})$ of maximal ideals of the algebra $\mathbb{C}[\mathcal{X}_k^\tau] = \mathbf{Z}_k/\sqrt{I_k}$ which are also Poisson ideals (note that $\text{PMax}(\mathbf{Z}_k/I_k) = \text{PMax}(\mathbf{Z}_k/\sqrt{I_k}) \subset \text{PSpec}(\mathbf{Z}_k/\sqrt{I_k})$).

Remark 4.4. — It follows from Remark 4.2 that τ -cuspidal points are fixed under the action of \mathbb{C}^\times . ■

We denote by $\text{Cus}_k^\tau(V, W)$ the set of pairs (P, p) where P is a τ -split parabolic subgroup of W and p is a τ -cuspidal point of $\mathcal{X}_{k_P}(V_P, P)^\tau$, where k_P denotes the restriction of k to the parabolic subgroup P . The group W_τ acts on $\text{Cus}_k^\tau(V, W)$ and we denote by $\text{Cus}_k^\tau(V, W)/W_\tau$ the set of its orbits in $\text{Cus}_k^\tau(V, W)$. If $(P, p) \in \text{Cus}_k^\tau(V, W)$, we denote by $[P, p]$ its W_τ -orbit.

4.C. Main result. — With the above notation, Theorem A can be restated (and made more precise) as follows:

Theorem A. *There is a natural bijection (which will be explicitly constructed in Section 8)*

$$\begin{array}{ccc} \text{Cus}_k^\tau(V, W)/W_\tau & \longrightarrow & \text{Symp}(\mathcal{X}_k^\tau) \\ [P, p] & \longmapsto & \mathcal{S}_{P,p}. \end{array}$$

*It satisfies that $\Upsilon_k(\overline{\mathcal{S}}_{P,p})$ is the image of $(V^P)^\tau \times (V^{*P})^\tau$ in $V/W \times V^*/W$. In particular,*

$$\dim \mathcal{S}_{P,p} = 2 \dim(V^P)^\tau.$$

We will prove Theorem A in the next sections. First, in Section 5, we will recall the proof, essentially due to Brown-Gordon [BrGo, Prop. 7.4] of Theorem A whenever $k = 0$ and $\tau = 1$. In Section 6, we will use Lehrer-Springer Theorem 3.4 to prove Theorem A whenever $k = 0$. In Section 7, we will use a deformation argument to attach to each symplectic leaf a W_τ -orbit of τ -split parabolic subgroups: in some sense, this is half of the construction of the above bijection. The second half will be constructed in Section 8, where the proof of Theorem A will be completed.

Let us also restate Conjecture B:

Conjecture B. *Let $(P, p) \in \text{Cus}_k^\tau(V, W)$. Then there exists $l \in \nabla((V^P)^\tau, \overline{N}_{W_\tau}(P_\tau))$ and a \mathbb{C}^\times -equivariant isomorphism of Poisson varieties*

$$\overline{\mathcal{S}}_{P,p}^{\text{nor}} \simeq \mathcal{X}_l((V^P)^\tau, \overline{N}_{W_\tau}(P_\tau)).$$

^(*)This definition coincides with the notion of *cuspidal leaf* of \mathcal{X}_k introduced by Bellamy [Bell1, §5] in the case where $\tau = 1$.

5. Symplectic leaves of $\mathfrak{X}_0 = (V \times V^*)/W$

The Poisson bracket on $\mathbf{Z}_0 = \mathbb{C}[V \times V^*]^W$ is the one obtained by restriction from the usual Poisson bracket on $\mathbb{C}[V \times V^*]$. The symplectic leaves of \mathfrak{X}_0 have been described in [BrGo, Prop. 7.4]: we recall their description in this section, and give some more precise details about the structure of their closure.

If $P \in \text{Parab}(W)$, let $\mathcal{V}\mathcal{V}^*(P)$ denote the set of elements $(v, v^*) \in V \times V^*$ such that $W_v \cap W_{v^*} = P$. Again, the family $(\mathcal{V}\mathcal{V}^*(P))_{P \in \text{Parab}(W)}$ is a stratification of $V \times V^*$ (the order between strata corresponds to the reverse order of the inclusion of parabolic subgroups). If $\mathfrak{P} \in \text{Parab}(W)/W$, we let $\mathcal{U}\mathcal{U}^*(\mathfrak{P})$ denote the image of $\mathcal{V}\mathcal{V}^*(P)$ in $(V \times V^*)/W$, where P is any element of \mathfrak{P} . Then $(\mathcal{U}\mathcal{U}^*(\mathfrak{P}))_{\mathfrak{P} \in \text{Parab}(W)/W}$ is a stratification of $(V \times V^*)/W$ (the order between strata corresponds to the reverse order of the inclusion, up to conjugacy, of parabolic subgroups).

Fix now $\mathfrak{P} \in \text{Parab}(W)/W$ and $P \in \mathfrak{P}$. Then

$$(5.1) \quad \mathcal{V}(P) \times \mathcal{V}^*(P) \subset \mathcal{V}\mathcal{V}^*(P) \subset V^P \times V^{*P}.$$

Note that $\overline{N}_W(P)$ acts on $V^P \times V^{*P}$ and that $\mathcal{V}\mathcal{V}^*(P)$ is the open subset of $V^P \times V^{*P}$ on which it acts freely. The image of $\overline{\mathcal{V}\mathcal{V}^*(P)} = V^P \times V^{*P}$ is equal to $\overline{\mathcal{U}\mathcal{U}^*(\mathfrak{P})}$.

Recall from §2.C that $V^P \times V^{*P}$ is not a Poisson subvariety of $V \times V^*$ but that it inherits from $V \times V^*$ a Poisson structure. This Poisson structure is the natural one endowed by the product of a vector space with its dual: it is $\overline{N}_W(P)$ -equivariant, so $(V^P \times V^{*P})/\overline{N}_W(P)$ is also a Poisson variety. By definition, $\overline{N}_W(P)$ acts freely on the open subset $\mathcal{V}\mathcal{V}^*(P)$, so the variety $\mathcal{V}\mathcal{V}^*(P)/\overline{N}_W(P)$ is smooth and its Poisson bracket makes it a symplectic variety. The next proposition is a particular case of the discussion preceding Proposition 2.9:

Lemma 5.2. — *Let $\mathfrak{P} \in \text{Parab}(W)/W$ and let $P \in \mathfrak{P}$. Then:*

- (a) *The closed subvariety $\overline{\mathcal{U}\mathcal{U}^*(\mathfrak{P})}$ is a Poisson subvariety of $(V \times V^*)/W$.*
- (b) *The map $\mathcal{V}\mathcal{V}^*(P) \rightarrow \mathcal{U}\mathcal{U}^*(\mathfrak{P})$ induces an isomorphism*

$$\mathcal{V}\mathcal{V}^*(P)/\overline{N}_W(P) \xrightarrow{\sim} \mathcal{U}\mathcal{U}^*(\mathfrak{P})$$

of Poisson varieties.

Corollary 5.3. — *Let $\mathfrak{P} \in \text{Parab}(W)/W$ and let $P \in \mathfrak{P}$. Then the above isomorphism $\mathcal{V}\mathcal{V}^*(P)/\overline{N}_W(P) \xrightarrow{\sim} \mathcal{U}\mathcal{U}^*(\mathfrak{P})$ extends to an isomorphism of Poisson varieties*

$$(V^P \times V^{*P})/\overline{N}_W(P) \xrightarrow{\sim} \overline{\mathcal{U}\mathcal{U}^*(\mathfrak{P})}^{\text{nor}}.$$

Proof. — The surjective map $\varphi : (V^P \times V^{*P})/\overline{N}_W(P) \rightarrow \overline{\mathcal{U}\mathcal{U}^*(\mathfrak{P})}$ induces an injection $\mathbb{C}[\overline{\mathcal{U}\mathcal{U}^*(\mathfrak{P})}] \subset \mathbb{C}[V^P \times V^{*P}]^{\overline{N}_W(P)}$ between algebra of regular functions, and both algebras have the same fraction fields by Lemma 5.2(b). But φ is finite and $\mathbb{C}[V^P \times V^{*P}]^{\overline{N}_W(P)}$ is integrally closed, so $\mathbb{C}[V^P \times V^{*P}]^{\overline{N}_W(P)}$ is the integral closure of $\mathbb{C}[\overline{\mathcal{U}\mathcal{U}^*(\mathfrak{P})}]$ in its fraction field. This completes the proof of the corollary. \square

The next result follows immediately from Lemma 5.2 and is a particular case of Proposition 2.9 (see also [BrGo, Prop. 7.4]):

Proposition 5.4. — *The family $(\mathcal{U}^*(\mathfrak{P}))_{\mathfrak{P} \in \text{Parab}(W)/W}$ of locally closed subvarieties is the stratification of $\mathcal{X}_0 = (V \times V^*)/W$ by symplectic leaves.*

Let us interpret the results of this section in terms of Theorem A and Conjecture B for $k = 0$ and $\tau = \text{Id}_V$. First, it follows from Corollary 5.3 that, if $\mathfrak{P} \in \text{Parab}(W)/W$ and if $P \in \mathfrak{P}$, then $\dim \mathcal{U}^*(\mathfrak{P}) = 2 \dim V^P$. Therefore, $\mathcal{U}^*(\mathfrak{P})$ is Id_V -cuspidal (we will say *cuspidal* for simplification) if and only if $V^P = 0$. Therefore, there is at most one cuspidal leaf of \mathcal{X}_0 and there is actually one if and only if $V^W = 0$ (in this case, this cuspidal leaf will be simply denoted by 0 , as it is the W -orbit of $0 \in V \times V^*$). This shows that

$$\text{Cus}_0^{\text{Id}_V}(V, W) = \{(P, 0) \mid P \in \text{Parab}(W)\} \xrightarrow{\sim} \text{Parab}(W).$$

Consequently, the bijection $\text{Cus}_0^{\text{Id}_V}(V, W)/W \xrightarrow{\sim} \text{Sympl}(\mathcal{X}_0)$ predicted by Theorem A in the case where $k = 0$ and $\tau = \text{Id}_V$ is simply given by the formula

$$\mathcal{S}_{P,0} = \mathcal{U}^*(\mathfrak{P})$$

for all $\mathfrak{P} \in \text{Parab}(W)/W$ and all $P \in \mathfrak{P}$: this is the content of Proposition 5.4. Moreover, Corollary 5.3 proves Conjecture B in this case:

Proposition 5.5. — *Theorem A and Conjecture B hold if $k = 0$ and $\tau = \text{Id}_V$.*

6. Symplectic leaves of \mathcal{X}_0^τ

We have $\mathcal{X}_0^\tau = ((V \times V^*)/W)^\tau$. For studying its symplectic leaves, the next consequence of Lehrer-Springer Theorem 3.4 will be crucial:

Proposition 6.1. — *The natural map*

$$\dot{i}_\tau : (V^\tau \times V^{*\tau})/W_\tau \longrightarrow ((V \times V^*)/W)^\tau = \mathcal{X}_0^\tau$$

is a finite bijective morphism of Poisson varieties: it is the normalization of the variety \mathcal{X}_0^τ .

Proof. — Only the statement on the bijectivity needs to be proved, the others being obvious or immediate consequences. Let us first prove that \dot{i}_τ is injective. Let (v_1, v_1^*) and $(v_2, v_2^*) \in V^\tau \times V^{*\tau}$ be such that (v_2, v_2^*) belong to the W -orbit of (v_1, v_1^*) . Then there exists $a \in W$ such that $(v_2, v_2^*) = a(v_1, v_1^*)$. By Theorem 3.4(b), there exists $b \in W_{V^\tau}^{\text{set}}$ such that $v_2 = b(v_1)$. Therefore, $b^{-1}a(v_1) = v_1$ and $b^{-1}(v_2^*) = b^{-1}a(v_1^*)$. In other words, $b^{-1}a$ belongs to the stabilizer W_{v_1} of v_1 in W (it is a parabolic subgroup). Since $\tau(v_1) = v_1$, τ normalizes W_{v_1} . Hence, since τ is W_{v_1} -full by (3.2), we may apply Theorem 3.4(b) to the pair (W_{v_1}, τ) so that, by dualizing, there exists $c \in (W_{V^\tau}^{\text{set}})_{v_1}$ such that $b^{-1}(v_2^*) = c(v_1^*)$. Therefore, $bc \in W_{V^\tau}^{\text{set}}$ and $bc(v_1, v_1^*) = (v_2, v_2^*)$, as desired.

Let us now prove that \dot{i}_τ is surjective. Let $(v, v^*) \in V \times V^*$ be such that its W -orbit is τ -stable. By Theorem 3.4(b), there exists $x \in W$ such that $\tau(x(v)) = x(v)$. So, by replacing (v, v^*) by $x(v, v^*)$ if necessary, we may, and we will, assume that $\tau(v) = v$. Therefore, there exists $a \in W$ such that $(\tau(v), \tau(v^*)) = (a(v), a(v^*))$. In other words, $a(v) = v$ and

$\tau(v^*) = a(v^*)$. So a belongs to the parabolic subgroup W_v , which is τ -stable. Applying again Theorem 3.4(b) to (W_v, τ) (since τ is W_v -full by (3.2)), and dualizing, one gets that there exists $b \in W_v$ such that $\tau(b(v^*)) = b(v^*)$. Therefore, $ab(v, v^*) \in V^\tau \times V^{*\tau}$, as desired. \square

Remark 6.2. — We do not know if there are examples of pairs (W, τ) such that the variety \mathcal{X}_0^τ is not normal. By the above proposition, saying that \mathcal{X}_0^τ is normal is equivalent to saying that any W_τ -invariant polynomial function on $V^\tau \times V^{*\tau}$ extends to a W -invariant polynomial function on $V \times V^*$. \blacksquare

A bijective morphism of Poisson varieties does not necessarily induce a bijection between symplectic leaves, but it turns out that this holds for our map ii_τ , as shown by the Corollary 6.6 below. Before proving it, let us introduce some notation. If Q is a parabolic subgroup of W_τ , we denote by $\mathcal{V}\mathcal{V}_\tau^*(Q)$ the set of pairs $(v, v^*) \in V^\tau \times V^{*\tau}$ such that $Q = W_v \cap W_{v^*}$. If Ω denotes the W_τ -orbit of Q , we denote by $\mathcal{U}\mathcal{U}_\tau^*(\Omega)$ the image of $\mathcal{V}\mathcal{V}_\tau^*(Q)$ in $(V^\tau \times V^{*\tau})/W_\tau$. By Proposition 5.4 applied to the pair (V^τ, W_τ) , the locally closed subvariety $\mathcal{U}\mathcal{U}_\tau^*(\Omega)$ is a symplectic leaf of $(V^\tau \times V^{*\tau})/W_\tau$ and all the symplectic leaves are obtained in this way. Note first the following easy fact:

Lemma 6.3. — Let $\mathfrak{P} \in \text{Parab}(W)/W$, let $P \in \mathfrak{P}$ and let $w \in \overline{N}_W(P)$. Then:

- (a) $\mathcal{V}\mathcal{V}^*(P)^{w\tau} \neq \emptyset$ if and only if $\mathcal{V}(P)^{w\tau} \neq \emptyset$.
- (b) $\mathcal{U}\mathcal{U}^*(\mathfrak{P})^\tau \neq \emptyset$ if and only if $\mathcal{U}(\mathfrak{P})^\tau \neq \emptyset$.

Proof. — Note that (a) implies (b) by Lemma 5.2(b). On the other hand, if $\mathcal{V}(P)^{w\tau} \neq \emptyset$, then $\mathcal{V}^*(P)^{w\tau} \neq \emptyset$. So, if we pick $v \in \mathcal{V}(P)^{w\tau}$ and $v^* \in \mathcal{V}^*(P)^{w\tau}$, then $(v, v^*) \in \mathcal{V}\mathcal{V}^*(P)^{w\tau}$. This proves the “if” part of (a).

Conversely, if $\mathcal{V}\mathcal{V}^*(P)^{w\tau} \neq \emptyset$, pick $(v, v^*) \in \mathcal{V}\mathcal{V}^*(P)^{w\tau}$. Then $\mathcal{V}^*(W_{v^*})^{w\tau} \neq \emptyset$ so $\mathcal{V}(W_{v^*})^{w\tau} \neq \emptyset$. Pick $v' \in \mathcal{V}(W_{v^*})^{w\tau}$ and let S denote the subspace of V generated by v and v' . Then $P = W_S^{\text{pt}}$, so there exists $v'' \in S$ such that $W_{v''} = P$. But $v'' \in V^{w\tau} \cap \mathcal{V}(P)$, which proves the “only if” part of (a). \square

The above Lemma allows to apply to the bijective morphism of varieties $ii_\tau : (V^\tau \times V^{*\tau})/W_\tau \rightarrow \mathcal{X}_0^\tau$ the same arguments as in §3.E. For instance, if $\mathfrak{P} \in \text{Parab}(W)/W$, then it follows from Lemma 6.3 and Proposition 3.13 that $\mathcal{U}\mathcal{U}^*(\mathfrak{P})^\tau \neq \emptyset$ if and only if \mathfrak{P} contains a τ -split parabolic subgroup.

Moreover, if $\mathfrak{P} \in (\text{Parab}(W)/W)_{\text{spl}}^\tau$ and if $P \in \mathfrak{P}$ is τ -split, then the τ -equivariant isomorphism $\mathcal{U}\mathcal{U}^*(\mathfrak{P}) \simeq \mathcal{V}\mathcal{V}^*(P)/\overline{N}_W(P)$ induces a decomposition into irreducible components

$$(6.4) \quad \mathcal{U}\mathcal{U}^*(\mathfrak{P})^\tau = \bigcup_{E \in \tilde{\mathcal{E}}_{P\tau}/\overline{N}_W(P)} \mathcal{V}\mathcal{V}^*(P)^E/\overline{N}_W(P),$$

where $\mathcal{V}\mathcal{V}^*(P)^E$ is defined in the same way as $\mathcal{V}(P)^E$. Similarly, the analogue of Corollary 6.6 is given as follows:

Proposition 6.5. — Let $\mathfrak{P} \in (\text{Parab}(W)/W)_{\text{spl}}^\tau$, let $P \in \mathfrak{P}$ and let $E \in \tilde{\mathcal{E}}_{P\tau}/\overline{N}_W(P)$. Let \mathfrak{P}_E denote the W_τ -orbit of τ -split parabolic subgroups of W associated with E through the bijection of Proposition 3.12(b). Let \mathfrak{Q}_E denote the W_τ -orbit of parabolic subgroups of W_τ of the form Q_τ for $Q \in \mathfrak{P}_E$ (see Lemma 3.7). Then

$$ii_\tau(\mathcal{U}\mathcal{U}^*(\mathfrak{Q}_E)) = \mathcal{V}\mathcal{V}^*(P)^E/\overline{N}_W(P).$$

Corollary 6.6. — The bijective morphism of varieties $ii_\tau : (V^\tau \times V^{*\tau})/W_\tau \longrightarrow \mathcal{X}_0^\tau$ induces a bijection between symplectic leaves.

Proof. — Both varieties admits finitely many symplectic leaves so, by taking the closure, these leaves are, in both cases, in bijection with the set of irreducible closed Poisson subvarieties.

Now, let \mathcal{S} be an irreducible closed Poisson subvariety of $(V^\tau \times V^{*\tau})/W_\tau$. Since ii_τ respects the Poisson bracket, $ii_\tau(\mathcal{S})$ is also an irreducible closed Poisson subvariety of \mathcal{X}_0^τ : this shows that ii_τ induces an injective map between the symplectic leaves of $(V^\tau \times V^{*\tau})/W_\tau$ and those of \mathcal{X}_0^τ .

Let us now show that this map is surjective. For this, let \mathcal{S} be a symplectic leaf of \mathcal{X}_0^τ . Then there exists $\mathfrak{P} \in \text{Parab}(W)/W$ such that $\mathcal{S} \cap \mathcal{U}\mathcal{U}^*(\mathfrak{P})^\tau$ is open and dense in \mathcal{S} . So \mathfrak{P} contains a τ -split parabolic subgroup P and the decomposition of $\mathcal{U}\mathcal{U}^*(\mathfrak{P})^\tau \neq \emptyset$ into irreducible components is given by (6.4). But $\mathcal{U}\mathcal{U}^*(\mathfrak{P})$ is smooth and symplectic, so $\mathcal{U}\mathcal{U}^*(\mathfrak{P})^\tau$ is also smooth and symplectic, so $\mathcal{S} \cap \mathcal{U}\mathcal{U}^*(\mathfrak{P})^\tau$ is equal to one of these irreducible components. The result then follows from Proposition 6.5. \square

Proposition 6.7. — If $k = 0$, then Theorem A and Conjecture B hold.

Proof. — By Corollary 6.6, \mathcal{X}_0^τ admits a τ -cuspidal point if and only if $(V^\tau)^{W_\tau} = 0$ and, in this case, there is only one τ -cuspidal point, namely the orbit of 0. So, still by Corollary 6.6 (and (3.8)), $\text{Cus}_0^\tau(V, W)$ is in bijection with conjugacy classes of parabolic subgroups of W_τ and $\text{Sym}\rho(\mathcal{X}_0^\tau)$ is also in bijection with conjugacy classes of parabolic subgroups of W_τ . This provides a natural bijection between $\text{Cus}_0^\tau(V, W)$ and $\text{Sym}\rho(\mathcal{X}_0^\tau)$, which satisfies the required properties of Theorem A.

Let us now prove Conjecture B in this case. So let \mathcal{S} be a symplectic leaf of \mathcal{X}_0^τ . Let $\mathcal{L} = ii_\tau^{-1}(\mathcal{S})$: it is a symplectic leaf of $(V^\tau \times V^{*\tau})^{W_\tau}$ and

$$\overline{\mathcal{M}} = ii_\tau^{-1}(\overline{\mathcal{S}}).$$

Since ii_τ is bijective, we have $\overline{\mathcal{L}}^{\text{nor}} = \overline{\mathcal{S}}^{\text{nor}}$, so Conjecture B now follows from Lemma 5.2 applied to the pair (V^τ, W_τ) instead of (V, W) . \square

Remark 6.8. — If $k = 0$, then the parameter l involved in Conjecture B is equal to 0. \blacksquare

7. Parabolic subgroups attached to symplectic leaves

7.A. Définition. — Let \mathcal{S} be a symplectic leaf of \mathcal{X}_k^τ . We denote by $\mathfrak{p}_\mathcal{S}$ the prime ideal of \mathbf{Z}_k defining $\overline{\mathcal{S}}$: then $\mathfrak{p}_\mathcal{S}^\tau \in \text{PSpec}(\mathbf{Z}_k^\tau)$. Now, the isomorphism $\text{gr}_{\mathcal{F}} \mathbf{Z}_k \simeq \mathbf{Z}_0$ is τ -equivariant and $(\text{gr}_{\mathcal{F}} \mathbf{Z}_k)^\tau = \text{gr}_{\mathcal{F}}(\mathbf{Z}_k^\tau)$. So $\text{gr}_{\mathcal{F}}(\mathfrak{p}_\mathcal{S}^\tau)$ is an ideal of \mathbf{Z}_0^τ . The next important result follows mainly from [Mar1, Theo. 2.8].

Lemma 7.1. — *The ideal $\sqrt{\text{gr}_{\mathcal{F}}(\mathfrak{p}_\mathcal{S}^\tau)}$ of \mathbf{Z}_0^τ is prime, Poisson and contains I_0^τ .*

Proof. — First, the Poisson bracket $\{, \}$ on \mathbf{Z}_k is a *proto-Poisson bracket* of degree -2 in the sense of [Mar1, Def. 2.4] and its associated graded Poisson bracket on \mathbf{Z}_0 is also the natural Poisson bracket on \mathbf{Z}_0 (for a proof of both facts, see [EtGi, Lem. 2.26]).

The same facts also holds by taking fixed points under the τ -action and so, the fact that $\sqrt{\text{gr}_{\mathcal{F}}(\mathfrak{p}_\mathcal{S}^\tau)}$ is a prime ideal of \mathbf{Z}_0^τ which is Poisson is an application of [Mar1, Theo. 2.8] (because \mathcal{X}_k^τ has finitely many symplectic leaves by Corollary 2.10).

Finally, τ acts trivially on $\mathbf{Z}_k/\mathfrak{p}_\mathcal{S}$, so it acts trivially on $\text{gr}_{\mathcal{F}}(\mathbf{Z}_k/\mathfrak{p}_\mathcal{S}) = \text{gr}_{\mathcal{F}}(\mathbf{Z}_k)/\text{gr}_{\mathcal{F}}(\mathfrak{p}_\mathcal{S}) = \mathbf{Z}_0/\text{gr}_{\mathcal{F}}(\mathfrak{p}_\mathcal{S})$. This shows that $\text{gr}_{\mathcal{F}}(\mathfrak{p}_\mathcal{S})$ contains I_0 and so $\text{gr}_{\mathcal{F}}(\mathfrak{p}_\mathcal{S}^\tau)$ contains I_0^τ . \square

Lemma 7.1 shows that $\sqrt{\text{gr}_{\mathcal{F}}(\mathfrak{p}_\mathcal{S}^\tau)}$ defines a symplectic leaf \mathcal{S}_0 of \mathcal{X}_0^τ so, by Corollary 6.6, there exists a unique W_τ -orbit $\mathcal{Q}_\mathcal{S}$ of parabolic subgroups of W_τ such that $\sqrt{\text{gr}_{\mathcal{F}}(\mathfrak{p}_\mathcal{S}^\tau)}$ is the defining ideal of $i\iota_\tau(\mathcal{U}_\tau^*(\mathcal{Q}_\mathcal{S}))$. Through the bijection of Lemma 3.7, there exists a unique W_τ -orbit $\mathfrak{P}_\mathcal{S}$ of τ -split parabolic subgroups of W such that $\mathcal{Q}_\mathcal{S} = \{P_\tau \mid P \in \mathfrak{P}_\mathcal{S}\}$.

Definition 7.2. — *Let \mathcal{S} be a symplectic leaf of \mathcal{X}_k^τ . The W_τ -orbit of τ -split parabolic subgroups $\mathfrak{P}_\mathcal{S}$ is called the W_τ -orbit **associated with \mathcal{S}** . Any element of $\mathfrak{P}_\mathcal{S}$ is called an **associated τ -split parabolic subgroup** (with \mathcal{S}).*

If P is a τ -split parabolic subgroup of W associated with \mathcal{S} , then

$$(7.3) \quad \dim \mathcal{S} = 2 \dim(V^P)^\tau.$$

Indeed, $\dim \mathcal{S} = \dim \mathcal{S}_0$.

7.B. Geometric construction. — Let $\pi : V/W \times V^*/W \rightarrow V/W$ and $\pi^\vee : V/W \times V^*/W \rightarrow V^*/W$ denote the first and second projection respectively. The next proposition gives another characterization of the W_τ -orbit of τ -split parabolic subgroups associated with a symplectic leaf:

Proposition 7.4. — *Let \mathcal{S} be a symplectic leaf of \mathcal{X}_0^τ . Then:*

- (a) $\Upsilon_k(\overline{\mathcal{S}}) = \iota_\tau(\overline{\mathcal{U}_\tau(\mathcal{Q}_\mathcal{S})}) \times \iota_\tau^\vee(\overline{\mathcal{U}_\tau^*(\mathcal{Q}_\mathcal{S})})$.
- (b) $\pi(\Upsilon_k(\overline{\mathcal{S}})) = \iota_\tau(\overline{\mathcal{U}_\tau(\mathcal{Q}_\mathcal{S})})$.
- (c) $\pi^\vee(\Upsilon_k(\overline{\mathcal{S}})) = \iota_\tau^\vee(\overline{\mathcal{U}_\tau^*(\mathcal{Q}_\mathcal{S})})$.

Proof. — Let $\mathbf{H}_k^\#$ denote the $\mathbb{C}[\hbar]$ -algebra obtained as the quotient of the algebra $\mathbb{C}[\hbar] \otimes (\mathbb{T}(V \oplus V^*) \rtimes W)$ by the relations

$$(7.5) \quad \begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \hbar^2 \sum_{H \in \mathfrak{A}} \sum_{j=0}^{e_H-1} e_H(k_{H,i} - k_{H,i+1}) \frac{\langle y, \alpha_H \rangle \cdot \langle \alpha_H^\vee, x \rangle}{\langle \alpha_H^\vee, \alpha_H \rangle} \varepsilon_{H,i}, \end{cases}$$

for all $y, y' \in V$ and $x, x' \in V^*$. It follows from the comparison of the relations (1.2) and (7.5) that there is a well-defined morphism of $\mathbb{C}[\hbar]$ -algebras $\theta : \mathbf{H}_k^\# \rightarrow \text{Rees}_{\mathfrak{F}} \mathbf{H}_k$ such that

$$\theta(y) = \hbar y, \quad \theta(x) = \hbar x \quad \text{and} \quad \theta(w) = w$$

for all $y \in V, x \in V^*$ and $w \in W$. In fact,

$$(7.6) \quad \theta \text{ is an isomorphism of algebras.}$$

Indeed, the surjectivity is immediate while the injectivity follows from the PBW decomposition (1.3), which also holds for $\mathbf{H}_k^\#$, namely, the map

$$\mathbb{C}[\hbar] \otimes \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \longrightarrow \mathbf{H}_k^\#$$

induced by the multiplication is an isomorphism of $\mathbb{C}[\hbar]$ -modules.

Let $\mathbf{Z}_k^\#$ denote the centre of $\mathbf{H}_k^\#$. Then it follows from (7.6) that θ induces an isomorphism of algebras

$$(7.7) \quad \mathbf{Z}_k^\# \longrightarrow \text{Rees}_{\mathfrak{F}} \mathbf{Z}_k.$$

Again, $\mathbf{Z}_k^\#$ is a flat family of deformations of $\mathbf{Z}_0 = \mathbb{C}[V \times V^*]^W$. We denote by $\mathfrak{X}_k^\#$ the affine variety such that $\mathbb{C}[\mathfrak{X}_k^\#] = \mathbf{Z}_k^\#$. The inclusion $\mathbf{P} \hookrightarrow \mathbf{Z}_k^\#$ induces a morphism $\Upsilon_k^\# : \mathfrak{X}_k^\# \rightarrow \mathcal{P} = V/W \times V^*/W$.

The action of τ and \mathbb{C}^\times extends easily to $\mathbf{H}_k^\#$, by letting them act trivially on the indeterminate \hbar . However, $\mathbf{H}_k^\#$ (and so $\mathbf{Z}_k^\#$) inherits an extra-action of \mathbb{C}^\times . Namely, there is an action of $\mathbb{C}^\times \times \mathbb{C}^\times$ given by

$$\begin{cases} (\xi, \xi') \cdot x = \xi \xi' x & \text{if } x \in V^*, \\ (\xi, \xi') \cdot y = \xi^{-1} \xi' y & \text{if } y \in V, \\ (\xi, \xi') \cdot w = w & \text{if } w \in W, \\ (\xi, \xi') \cdot \hbar = \xi' \hbar. \end{cases}$$

The action of the first copy of \mathbb{C}^\times extends the one which has been already defined in §1.F.1. Through the isomorphism θ , this action on $\text{Rees}_{\mathfrak{F}} \mathbf{H}_k$ is just the restriction of the action on $\mathbb{C}[\hbar] \otimes \mathbf{H}_k$ given by

$$(\xi, \xi') \cdot (P(\hbar) \otimes h) = P(\xi' \hbar) \otimes (\xi \cdot h).$$

Then (1.8) can be retrieved thanks to (1.7) and the isomorphism θ , by specializing \hbar to 0.

Specializing \hbar to $\lambda \in \mathbb{C}$ gives the algebras $\mathbf{H}_{\lambda^2 k}$ and $\mathbf{Z}_{\lambda^2 k}$. Geometrically, the inclusion $\mathbb{C}[\hbar] \hookrightarrow \mathbf{Z}_k^\#$ induces a flat morphism $\mathfrak{X}_k^\# \rightarrow \mathbb{C}$ whose fiber at λ is the Calogero-Moser space $\mathfrak{X}_{\lambda^2 k}$.

Now, view \mathcal{S} as a subvariety of $\mathfrak{X}_k^\#$ and let $\bar{\mathcal{S}}_0 = \overline{(1 \times \mathbb{C}^\times) \cdot \mathcal{S} \cap \mathfrak{X}_0}$, endowed with its reduced structure. Then, using the isomorphism θ , it follows from the definition of the action of the second copy of \mathbb{C}^\times on $\text{Rees}_{\mathfrak{F}}(\mathbf{H}_k)$ that the defining ideal of $\bar{\mathcal{S}}_0$ is $\sqrt{\text{gr}_{\mathfrak{F}}(\mathfrak{p}_{\mathcal{S}})}$.

Since \mathcal{S} is $(\mathbb{C}^\times \times 1)$ -stable, we have

$$(1 \times \mathbb{C}^\times) \cdot \mathcal{S} = (\mathbb{C}^\times \times \mathbb{C}^\times) \cdot \mathcal{S} = \Delta(\mathbb{C}^\times) \cdot \mathcal{S},$$

where $\Delta(\mathbb{C}^\times)$ is the diagonal in $\mathbb{C}^\times \times \mathbb{C}^\times$. Note also that, if $\xi \in \mathbb{C}^\times$ and $z \in \mathcal{X}_k^\#$, then $\pi \circ \Upsilon_k^\#((\xi, \xi) \cdot z) = \pi \circ \Upsilon_k^\#(z)$. Moreover, if $z \in \overline{\mathcal{S}}$, then $z_0 = \lim_{\xi \rightarrow 0} (\xi, \xi) \cdot z$ exists (because the action $\Delta(\mathbb{C}^\times)$ has non-negative weights) and $z_0 \in \overline{\mathcal{S}}_0$. So $(\pi \circ \Upsilon_k)(z) = (\pi \circ \Upsilon_0)(z_0)$ belongs to $\iota_\tau(\overline{\mathcal{U}_\tau(\mathcal{Q}_\mathcal{S})})$, as expected. This shows that

$$\pi(\Upsilon_k(\overline{\mathcal{S}})) \subset \iota_\tau(\overline{\mathcal{U}_\tau(\mathcal{Q}_\mathcal{S})}).$$

By exchanging the role of V and V^* , we have

$$\pi^\vee(\Upsilon_k(\overline{\mathcal{S}})) \subset \iota_\tau^\vee(\overline{\mathcal{U}_\tau^*(\mathcal{Q}_\mathcal{S})}).$$

Therefore,

$$\Upsilon_k(\overline{\mathcal{S}}) \subset \iota_\tau(\overline{\mathcal{U}_\tau(\mathcal{Q}_\mathcal{S})}) \times \iota_\tau^\vee(\overline{\mathcal{U}_\tau^*(\mathcal{Q}_\mathcal{S})}).$$

Since $\Upsilon_k(\overline{\mathcal{S}})$ is closed irreducible of dimension $2 \dim(V^P)^\tau$ (by (7.3) and the finiteness of the morphism Υ_k), we get that

$$\Upsilon_k(\overline{\mathcal{S}}) = \iota_\tau(\overline{\mathcal{U}_\tau(\mathcal{Q}_\mathcal{S})}) \times \iota_\tau^\vee(\overline{\mathcal{U}_\tau^*(\mathcal{Q}_\mathcal{S})}).$$

In other words, this proves (a). Now, (b) and (c) follow from (a). \square

Keep the notation introduced in the above proof ($\mathbf{H}_k^\#, \mathbf{Z}_k^\#, \mathcal{X}_k^\#, \dots$) and let us explain how this proof provides a justification of Conjecture B as well as a possible strategy for proving it. Indeed, if \mathcal{S} is a symplectic leaf of \mathcal{X}_k , let $\overline{\mathcal{S}}^\# = \overline{(1 \times \mathbb{C}^\times) \cdot \mathcal{S}}$. Then $\overline{\mathcal{S}}^\#$ comes equipped with a morphism $\varpi : \overline{\mathcal{S}}^\# \rightarrow \mathbb{C}$ and we denote by $\nu : (\overline{\mathcal{S}}^\#)^{\text{nor}} \rightarrow \mathbb{C}$ the composition of the normalization morphism $(\overline{\mathcal{S}}^\#)^{\text{nor}} \rightarrow \overline{\mathcal{S}}^\#$ with ϖ . Then ν is flat [Har, Chap. III, Prop. 9.7]. Since $\varpi^{-1}(\mathbb{C}^\times) \simeq \mathbb{C}^\times \times \overline{\mathcal{S}}$, we have

$$\nu^{-1}(\mathbb{C}^\times) \simeq \mathbb{C}^\times \times \overline{\mathcal{S}}^{\text{nor}}.$$

Let $\overline{\mathcal{S}}_0^\star$ denote the scheme-theoretic fiber of ν at 0. Assume that we are able to show the following two facts:

- (1) The reduced subscheme of $\overline{\mathcal{S}}_0^\star$ is the normalization of $\overline{\mathcal{S}}_0$.
- (2) The scheme $\overline{\mathcal{S}}_0^\star$ is generically reduced.

Then a Theorem of Hironaka [Har, Chap. III, Theo. 9.11] would show that ν is a flat family of schemes, all of whose scheme-theoretic fibers are reduced, irreducible and normal varieties. As $\overline{\mathcal{S}}_0^\star = \overline{\mathcal{S}}_0^{\text{nor}}$ is the normalization of $\overline{\mathcal{S}}_0$ by (1), this would imply that $\overline{\mathcal{S}}^{\text{nor}}$ is a Poisson deformation of $\overline{\mathcal{S}}_0^{\text{nor}}$. So Conjecture B would then follow from Propositions 6.5, 6.7 and a result of Bellamy [Bel2, Theo. 1.4] (which follows works of Ginzburg-Kaledin [GiKa] and Namikawa [Nam1], [Nam2]).

8. τ -Harish-Chandra theory of symplectic leaves

Let P be a parabolic subgroup of W and let \mathfrak{P} denote its conjugacy class. Let k_P denote the restriction of k to the hyperplane arrangement of P and let k_P° denote its “extension by zero” to the hyperplane arrangement of $N_W(P)$: in other words, if $H \in \mathcal{A}(V, N_W(P))$ and $0 \leq i \leq e_H - 1$, we set

$$(k_P^\circ)_{H,i} = \begin{cases} k_{H,i} & \text{if } H \in \mathcal{A}(V, P), \\ 0 & \text{otherwise.} \end{cases}$$

If \mathcal{X} is a locally closed subvariety of V/W , we denote by $\widehat{\mathcal{F}_k(V, W)}_{\mathcal{X}}$ the scheme equal to the completion of $\mathcal{F}_k(V, W)$ at its locally closed subvariety $(\pi \circ \Upsilon_k)^{-1}(\mathcal{X})$. Note that it inherits a Poisson structure from the one of $\mathcal{F}_k(V, W)$ (see for instance [Bel1, Lem. 3.5]).

Our construction of the τ -Harish-Chandra theory of symplectic leaves will follow from an upcoming result of Bellamy-Chalykh [BeCh] which says that there is a natural isomorphism of Poisson schemes^(†)

$$(8.1) \quad \widehat{\mathcal{F}_k(V, W)}_{\mathcal{U}(\mathfrak{P})} \simeq \widehat{\mathcal{F}_{k_P^\circ}(V, N_W(P))}_{\mathcal{U}(\mathfrak{P})}.$$

Note that $\mathcal{U}(\mathfrak{P}) \simeq \mathcal{V}(P)/N_W(P)$ maybe be viewed as a locally closed subvariety of both V/W and $V/N_W(P)$. The construction of this isomorphism implies that it is τ -equivariant.

The sheafified version of Proposition A.4 given in Remark A.6 implies that we can take fixed points under the action of τ in the above isomorphism and get an isomorphism

$$(8.2) \quad \widehat{\mathcal{F}_k(V, W)}_{\mathcal{U}(\mathfrak{P})^\tau} \simeq \widehat{\mathcal{F}_{k_P^\circ}(V, N_W(P))}_{\mathcal{U}(\mathfrak{P})^\tau},$$

with obvious notation. Moreover, this isomorphism is also Poisson: indeed, the Poisson structure on the left-hand side comes from the Poisson structure on the quotient scheme $(\widehat{\mathcal{F}_k(V, W)}_{\mathcal{U}(\mathfrak{P})})/\langle \tau \rangle$ and one can use Corollary A.3 (and similiary for the right-hand side).

Now, the irreducible (i.e. connected in this case) components of $\mathcal{U}(\mathfrak{P})^\tau$ have been described in Corollary 3.15: this leads to a decomposition of the two schemes involved in isomorphism (8.2). We focus on the irreducible component $\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\text{spl}}^\tau))$ of $\mathcal{U}(\mathfrak{P})^\tau$ and get an isomorphism of Poisson schemes

$$(8.3) \quad \widehat{\mathcal{F}_k(V, W)}_{\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\text{spl}}^\tau))} \simeq \widehat{\mathcal{F}_{k_P^\circ}(V, N_W(P))}_{\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\text{spl}}^\tau))}.$$

A sheafified version of [Bel1, Lem. 3.3, 3.4, 3.5] provides a natural bijection between Poisson reduced irreducible subschemes of $\mathcal{F}_k(V, W)^\tau$ of dimension $2 \dim(V^P)^\tau$ meeting $(\pi \circ \Upsilon_k)^{-1}(\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\text{spl}}^\tau)))$ and Poisson reduced irreducible subschemes of $\widehat{\mathcal{F}_k(V, W)}_{\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\text{spl}}^\tau))}^\tau$ of dimension $2 \dim(V^P)^\tau$. A similar bijection is obtained with the right-hand side of (8.3). Using the isomorphism (8.3), one gets a bijection between the following two sets:

- The set $\mathcal{S}_{\mathfrak{P}_{\text{spl}}^\tau}$ of symplectic leaves of $\mathcal{F}_k(V, W)^\tau$ of dimension $2 \dim(V^P)^\tau$ meeting $(\pi \circ \Upsilon_k)^{-1}(\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\text{spl}}^\tau)))$;
- The set $\mathcal{S}'_{\mathfrak{P}_{\text{spl}}^\tau}$ of symplectic leaves of $\mathcal{F}_{k_P^\circ}(V, N_W(P))^\tau$ of dimension $2 \dim(V^P)^\tau$ meeting $(\pi' \circ \Upsilon'_k)^{-1}(\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\text{spl}}^\tau)))$.

^(†)As [BeCh] is still not published, we just mention here that it is based on *Bezrukavnikov-Etingof like* constructions of isomorphisms when completing at a single point of V/W .

Here, the maps π' and Υ'_k are the analogues of π and Υ_k for the Calogero-Moser space $\mathcal{X}_{k_P^\circ}(V, \mathbb{N}_W(P))$.

But it follows from Proposition 7.4 that $\mathcal{S}_{\mathfrak{P}_{\text{spl}}^\tau}$ is exactly the set of symplectic leaves \mathcal{S} of $\mathcal{X}_k(V, W)^\tau$ such that $\mathfrak{P}_\mathcal{S} = \mathfrak{P}_{\text{spl}}^\tau$. So Theorem A will follow from the next lemma:

Lemma 8.4. — *The set $\mathcal{S}'_{\mathfrak{P}_{\text{spl}}^\tau}$ is in natural bijection with the set of $\mathbb{N}_{W_\tau}(P_\tau)$ -orbits of cuspidal points of $\mathcal{X}_{k_P}(V_P, P)^\tau$.*

Proof. — Since k_P° is the extension by zero of k_P , we have

$$\mathcal{X}_{k_P^\circ}(V, \mathbb{N}_W(P)) = \mathcal{X}_{k_P}(V, P)/\overline{\mathbb{N}}_W(P) = (V^P \times V^{*P} \times \mathcal{X}_{k_P}(V_P, P))/\overline{\mathbb{N}}_W(P).$$

Consequently,

$$\mathcal{U}(\mathfrak{P}) \times_{V/\mathbb{N}_W(P)} \mathcal{X}_{k_P^\circ}(V, \mathbb{N}_W(P)) = (\mathcal{V}(P) \times V^{*P} \times (0 \times_{V_P/P} \mathcal{X}_{k_P}(V_P, P)))/\overline{\mathbb{N}}_W(P).$$

As in (3.14) and (6.4), the τ -fixed points of $(\mathcal{V}(P) \times V^{*P} \times (0 \times_{V_P/P} \mathcal{X}_{k_P}(V_P, P)))/\overline{\mathbb{N}}_W(P)$ decomposes into pieces indexed by $\tilde{\mathcal{E}}_{P\tau}/\overline{\mathbb{N}}_W(P)$ as follows:

$$\begin{aligned} & \left((\mathcal{V}(P) \times V^{*P} \times (0 \times_{V_P/P} \mathcal{X}_{k_P}(V_P, P)))/\overline{\mathbb{N}}_W(P) \right)^\tau = \\ & \bigcup_{w \in [\tilde{\mathcal{E}}_{P\tau}/\overline{\mathbb{N}}_W(P)]} (\mathcal{V}(P)^{w\tau} \times (V^{*P})^{w\tau} \times \mathcal{X}_{k_P}(V_P, P)^{w\tau})/\overline{\mathbb{N}}_W(P)^{w\tau}. \end{aligned}$$

Here, $[\tilde{\mathcal{E}}_{P\tau}/\overline{\mathbb{N}}_W(P)]$ is a set of representatives of $\tilde{\mathcal{E}}_{P\tau}/\overline{\mathbb{N}}_W(P)$. We may, and we will, assume that $1 \in [\tilde{\mathcal{E}}_{P\tau}/\overline{\mathbb{N}}_W(P)]$. Then, by construction, only the piece indexed by 1 meets $(\pi' \circ \Upsilon'_k)^{-1}(\iota_\tau(\mathcal{U}_\tau(\mathfrak{P}_{\text{spl}}^\tau)))$. Therefore, $\mathcal{S}'_{\mathfrak{P}_{\text{spl}}^\tau}$ is in natural bijection with the set of symplectic leaves of

$$\mathcal{X} = (\mathcal{V}(P)^\tau \times (V^{*P})^\tau \times \mathcal{X}_{k_P}(V_P, P)^\tau)/\overline{\mathbb{N}}_W(P)^\tau$$

of dimension $2 \dim(V^P)^\tau$. But $\overline{\mathbb{N}}_W(P)^\tau = \overline{\mathbb{N}}_{W_\tau}(P_\tau)$ by Lemma 3.11, and it acts freely on $\mathcal{V}(P)^\tau \times (V^{*P})^\tau \times \mathcal{X}_{k_P}(V_P, P)^\tau$. So it follows from Corollary 2.11 that the set of symplectic leaves of \mathcal{X} is in natural bijection with the set of $\overline{\mathbb{N}}_{W_\tau}(P_\tau)$ -orbits of symplectic leaves of

$$\mathcal{Y} = \mathcal{V}(P)^\tau \times (V^{*P})^\tau \times \mathcal{X}_{k_P}(V_P, P)^\tau.$$

But any symplectic leaf of \mathcal{Y} is of the form $\mathcal{V}(P)^\tau \times (V^{*P})^\tau \times \mathcal{S}$, where \mathcal{S} is a symplectic leaf of $\mathcal{X}_{k_P}(V_P, P)^\tau$. For dimension reasons, $\mathcal{S}'_{\mathfrak{P}_{\text{spl}}^\tau}$ is in natural bijection with the set of $\overline{\mathbb{N}}_{W_\tau}(P_\tau)$ -orbits of symplectic leaves of $\mathcal{X}_{k_P}(V_P, P)^\tau$ of dimension 0, which is exactly the desired statement. \square

9. Examples

9.A. Smooth case. — Assume in this subsection, and only in this subsection, that \mathcal{X}_k is smooth and that τ is of the form ζw for some root of unity ζ and $w \in W$. We denote by d the order of ζ . Then

$$(9.1) \quad \mathcal{X}_k^\tau = \mathcal{X}_k^{\mu^d}.$$

Since \mathcal{X}_k is smooth, it is symplectic by (4.3), and so \mathcal{X}_k^τ is also smooth and symplectic: its symplectic leaves are exactly its irreducible (i.e. connected) components.

In [BoMa], Maksimau and the author have described the irreducible components of $\mathcal{X}_k^{\mu_d}$ as particular Calogero-Moser spaces, and the reader can check that this description is compatible with Conjecture B: however, it is not proved that the isomorphism preserves the Poisson structure. In other words [BoMa, Theo. 2.13 and 5.1]:

Theorem 9.2. — *If \mathcal{X}_k is smooth and $\tau \in \mathbb{C}^\times W$, then Conjecture B holds, possibly up to the Poisson structure.*

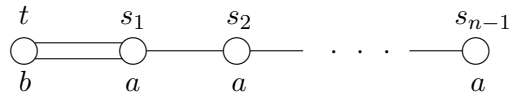
9.B. Type G_4 . — Thiel and the author [BoTh] have developed algorithms for computing presentations of \mathbf{Z}_k that have been implemented in MAGMA [Mag] (more precisely, in the CHAMP package for MAGMA written by Thiel [Thi]). This allows computations for (very) small groups.

We assume in this subsection, and only in this subsection, that W is the group G_4 , in Shephard-Todd classification [ShTo]. Then a presentation of \mathbf{Z}_k can be obtained with MAGMA (see for instance [BoMa, §5] or [BoTh, Theo. 5.2]) and it has been checked in [BoTh, Theo 4.7] that Conjecture B holds in this case:

Theorem 9.3. — *If $W = G_4$, then Conjecture B holds.*

9.C. Type B . — Assume in this subsection, and only in this subsection, that $W = W_n$ is a Coxeter group of type B_n for some $n \geq 2$ (i.e. we may assume that $W = G(2, 1, n)$ in Shephard-Todd classification) and that $\tau = \text{Id}_V$. Let $t = \text{diag}(-1, 1, \dots, 1) \in W_n$ and, for $1 \leq j \leq n - 1$, let s_j denote the permutation matrix corresponding to the transposition $(j, j + 1)$.

There are two conjugacy classes of reflections: the class of t (which generates an elementary abelian normal subgroup of order 2^n) and the one of s_1 (which generates a normal subgroup $W'_n = G(2, 2, n)$ of index 2 isomorphic to a Coxeter group of type D_n). We set $b = c_k(t)$ and $a = c_k(s_1)$ and we denote by I_n the set of $m \in \mathbb{Z}$ such that $|m| \leq n - 1$. The Dynkin diagram, together with the values of the parameter function c_k , is given as follows:



The case where $a = 0$ is somewhat uninteresting, as then $\mathcal{X}_k \simeq (C_b)^n / \mathfrak{S}_n$, where C_b is the Calogero-Moser associated with the cyclic group of order 2 whose equation is given by $C_b = \{(x, y, z) \in \mathbb{C}^3 \mid z^2 = xy + 4b^2\}$. So we assume throughout this subsection that $a \neq 0$. This implies that

$$(9.4) \quad \mathcal{X}_k \text{ is smooth if and only if } b/a \notin I_n.$$

As $\tau = \text{Id}_V$, the smooth case is uninteresting so we assume that $b/a = m \in I_n$. As the cases $b/a = m$ and $b/a = -m$ are equivalent, we also may assume that $m \geq 0$. The Calogero-Moser space \mathcal{X}_k will then be denoted by $\mathcal{X}_{a,ma}(n)$. Symplectic leaves have been parametrized by Martino in his PhD Thesis [Mar2, §5.4]. Bellamy and Thiel have then reinterpreted his result in terms of Bellamy parametrization à la Harish-Chandra [BeTh, Lem. 6.5]. This can be summarized as follows:

- $\mathcal{X}_{a,ma}(n)$ admits a cuspidal point if and only if there exists $r \in \mathbb{Z}_{\geq 0}$ such that $n = r(r+m)$: if so, there is only one cuspidal point that we denote by p_n .
- Therefore, $\text{Cus}_{\text{Id}_V}(\mathcal{X}_{a,ma}(n)) = \{(W_{r(r+m)}, p_{r(r+m)}) \mid r(r+m) \leq n\}$, with the convention that $W_0 = 1$ and $W_1 = \langle t \rangle$.
- If $r(r+m) \leq n$, we denote by $\mathcal{S}_r^m(n)$ the symplectic leaf of $\mathcal{X}_{a,ma}(n)$ associated with $(W_{r(r+m)}, p_{r(r+m)})$ through the bijection of Theorem A (since we are in the case where $\tau = \text{Id}_V$, this bijection was established by Bellamy [Bel1] and Losev [Los]). We have

$$\dim \mathcal{S}_r^m(n) = 2(n - r(r+m)).$$

If $r(r+m) \leq n$, then

$$\overline{N}_{W_n}(W_{r(r+m)}) \simeq W_{n-r(r+m)}.$$

Using the description of $\mathcal{X}_{a,ma}(n)$ in terms of quiver varieties, Bellamy, Maksimau and Schedler proved the following result [BeSc]:

Theorem 9.5 (Bellamy-Maksimau-Schedler). — *If $r(r+m) \leq n$, then there is a \mathbb{C}^\times -equivariant isomorphism of Poisson varieties*

$$\overline{\mathcal{S}_r^m(n)}^{\text{nor}} \simeq \mathcal{X}_{a,(m+2r)a}(n - r(r+m)).$$

Corollary 9.6. — *Conjecture B holds if W is a Coxeter group of type B_n and $\tau = \text{Id}_V$.*

9.D. Type D. — Assume in this subsection, and only in this subsection, that $W = W'_n$ is a Coxeter group of type D_n for some $n \geq 4$ (i.e. we may assume that $W = G(2, 2, n)$ in Shephard-Todd classification). We set $a = c_k(s_1)$, as in the previous subsection, and the Calogero-Moser space \mathcal{X}_k will be denoted by $\mathcal{X}'_a(n)$. The case where $a = 0$ being treated in Section 6, we assume throughout this subsection that $a \neq 0$. The following facts are proved in [BeTh, Theo. 7.2]^(†):

- $\mathcal{X}'_a(n)$ admits a cuspidal point if and only if there exists $r \in \mathbb{Z}_{\geq 0} \setminus \{1\}$ such that $n = r^2$: if so, there is only one cuspidal point that we denote by p'_n .
- Therefore, $\text{Cus}_{\text{Id}_V}(\mathcal{X}'_a(n)) = \{(W'_{r^2}, p'_{r^2}) \mid 0 \leq r^2 \leq n \text{ and } r \neq 1\}$, with the convention that $W'_0 = 1$.
- If $0 \leq r^2 \leq n$ and $r \neq 1$, we denote by $\mathcal{S}'_r(n)$ the symplectic leaf of $\mathcal{X}'_a(n)$ associated with (W'_{r^2}, p'_{r^2}) . We have

$$\dim \mathcal{S}'_r(n) = 2(n - r^2).$$

Let us give another description, coming from the link between $\mathcal{X}'_a(n)$ and the Calogero-Moser space $\mathcal{X}_{a,ma}(n)$ of type B_n defined in the previous section for the special value $m = 0$. Indeed, $\mathcal{X}'_a(n)$ admits an action of the element $t \in W_n \subset N_{\text{GL}_C(V)}(W'_n)$ defined in the previous subsection and [BeTh, Prop. 4.17]

$$\mathcal{X}'_a(n)/\langle t \rangle \simeq \mathcal{X}_{a,0}(n),$$

^(†)Note that there is a little mistake in [BeTh, Theo. 7.2], which can be easily corrected to give the statement written here. More precisely, and keeping the notation of [BeTh, §6, 7], the statement of [BeTh, Lem. 7.1] is false for $k = 1$ (but true for $k = 0$) because [BeTh, Theo. 6.24] cannot be applied to the case $(k, m) = (1, 0)$ for going from type B_1 to $D_1 = B_1$ (!). So the correct statement of [BeTh, Theo. 7.2] is to replace the set $\{k \geq 1 \mid k^2 \leq n\}$ by the set $\{0\} \cup \{k \geq 2 \mid k^2 \leq n\}$: then, for instance, \mathcal{L}_0 is the smooth symplectic leaf.

as Poisson varieties endowed with a \mathbb{C}^\times -action. Denote by $\gamma_n : \mathcal{X}'_a(n) \rightarrow \mathcal{X}_{a,0}(n)$ the quotient morphism.

Proposition 9.7. — *We have*

$$\mathcal{S}'_0(n) = \gamma_n^{-1}(\mathcal{S}_0^0(n) \cup \mathcal{S}_1^0(n)) \quad \text{and} \quad \mathcal{S}'_r(n) = \gamma_n^{-1}(\mathcal{S}_r^0(n))$$

for all $r \geq 2$ such that $r^2 \leq n$.

Proof. — The symplectic leaves of $\mathcal{X}'_a(n)$ are characterized by their dimension, so every symplectic leaf is t -stable (so is the inverse image, under γ_n , of its image in $\mathcal{X}_{a,0}(n)$). But if $0 \leq r^2 \leq n$ and $r \neq 1$, then $\gamma_n(\overline{\mathcal{S}'_r(n)})$ is a closed irreducible Poisson subvariety of $\mathcal{X}_{a,0}(n)$, so it is the closure of a symplectic leaf. For dimension reason, it must be equal to $\overline{\mathcal{S}_r^0(n)}$. The result follows. \square

Corollary 9.8. — *We have*

$$\mathcal{X}'_a(n)^t = \gamma_n^{-1}(\overline{\mathcal{S}_1^0(n)}).$$

In particular, if $4 \leq r^2 \leq n$, then t acts trivially on $\mathcal{S}'_r(n)$.

Proof. — First, t does not act trivially on $\mathcal{X}'_a(n)$ so it does not act trivially on the open leaf $\mathcal{S}'_0(n)$. Since $\mathcal{S}'_0(n)$ is smooth and symplectic, the description of the symplectic leaves of $\mathcal{S}'_0(n)/\langle t \rangle$ is given by Proposition 2.9. But $\mathcal{S}'_0(n)/\langle t \rangle = \mathcal{S}_0^0(n) \cup \mathcal{S}_1^0(n)$ by Proposition 9.7. Comparing both descriptions shows that t acts freely on $\mathcal{S}_0^0(n)$ and trivially on $\mathcal{S}_1^0(n)$.

Therefore, t acts trivially on the closure of $\gamma_n^{-1}(\mathcal{S}_1^0(n))$ and freely on $\mathcal{S}'_0(n)$. But the closure of $\mathcal{S}'_0(n)$ is the union of the $\mathcal{S}'_r(n)$ for $r \geq 1$ (see [BeTh, Lem. 6.5]). So the corollary follows now directly from Proposition 9.7. \square

Assume now that $4 \leq r^2 \leq n$. Then Corollary 9.8 shows that $\mathcal{S}'_r(n) \simeq \mathcal{S}_r^0(n)$. Moreover, $\overline{\mathbb{N}}_{W'_n}(W'_{r^2}) \simeq W_{n-r^2}$. So Theorem 9.5 shows the following result:

Corollary 9.9. — *Conjecture B holds if W is a Coxeter group of type D_n and $\tau \in \{\text{Id}_V, t\}$.*

Proof. — For the case $\tau = \text{Id}_V$, the work has already been done. For the case where $\tau = t$, one must notice that τ is W'_n -regular so it is W'_n -full (see Example 1.9), that $(W'_n)_\tau = (W'_n)^\tau \simeq W_{n-1}$ (see Example 3.5) and that

$$\overline{\mathbb{N}}_{(W'_n)_\tau}((W'_{r^2})_\tau) = \overline{\mathbb{N}}_{W_{n-1}}(W_{r^2-1}) \simeq W_{n-r^2}.$$

Then the result follows from Theorem 9.5 and Corollary 9.8. \square

9.E. Dihedral groups at equal parameters. — Let d be a natural number and let ξ denote a primitive $2d$ -th root of unity. For $j \in \mathbb{Z}/2d\mathbb{Z}$, we set

$$s_j = \begin{pmatrix} 0 & \xi^j \\ \xi^{-j} & 0 \end{pmatrix}$$

We assume in this section, and only in this section, that $W = \langle s_0, s_2 \rangle$ is dihedral of order $2d$ and that $\tau = s_1$: note that $\tau^2 = \text{Id}_V$, that $\tau s_0 \tau^{-1} = s_2$, that $\tau s_2 \tau^{-1} = s_0$ and that τ is W -full. We set $a = c_k(s_0)$ and, since k is τ -stable by hypothesis, we have $c_k(s_2) = a$. In other words, we are in the equal parameter case studied by the author in [Bon1]. Moreover, in [Bon1, §4], the author determined the structure of \mathcal{X}_k^τ . This gives:

Proposition 9.10. — *If W is dihedral of order $2d$ and if τ is as above, then Conjecture B holds.*

Appendix A

Completion and finite group actions

Hypothesis and notation. We fix in this appendix a commutative noetherian \mathbb{C} -algebra R , an ideal I of R , a finite group G acting on the \mathbb{C} -algebra R and we assume that I is G -stable. We set $J = \langle I^G \rangle_R$.

Let \mathfrak{v}_G be the ideal of R generated by the family $(r - g(r))_{r \in R, g \in G}$ and set $R(G) = R/\sqrt{\mathfrak{v}_G}$. We denote by $I(G)$ the image of I in $R(G)$. Note that $R(G)$ is the biggest quotient algebra of R which is reduced and on which G acts trivially.

Finally, we denote by \hat{R}_I the I -adic completion of R , i.e.

$$\hat{R}_I = \varprojlim_j R/I^j,$$

and by $\iota : R \rightarrow \hat{R}_I$ the canonical map.

The results of this Appendix do not pretend to any originality, and might certainly be written in greater generality. We nevertheless cannot find appropriate references containing all of them, and decided to state them in terms which are suitable for our purpose.

Lemma A.1. — *There exists an integer m such that $I^m \subset J$.*

Proof. — Let \mathfrak{p} be a prime ideal of R containing I^G . We first wish to prove that \mathfrak{p} contains I . For this, let $r \in I$. Then $\prod_{g \in G} g(r) \in I^G$, and so there exists $g_r \in G$ such that $g_r(r) \in \mathfrak{p}$ because \mathfrak{p} is prime. This shows that $I \subset \cup_{g \in G} g(\mathfrak{p})$. By the Prime Avoidance Lemma, we get that there exists $g \in G$ such that $I \subset g(\mathfrak{p})$. Since moreover I is G -stable, we get that $I \subset \mathfrak{p}$. In other words, I is contained in any prime ideal containing J . So $I \subset \sqrt{J}$. As R is noetherian, the result follows from Levitsky's Theorem [Lam, Theo. 10.30]. \square

Lemma A.2. — *Let $j \geq 0$. Then $(J^j)^G = (I^G)^j$.*

Proof. — The inclusion $(I^G)^j \subset (J^j)^G$ is obvious. Conversely, let $r \in (J^j)^G$. Then there exists a finite set E , a family $(r_e)_{e \in E}$ of elements of R and a family $(i_e^{(1)}, \dots, i_e^{(j)})$ of j -uples of elements of I^G such that

$$r = \sum_{e \in E} r_e i_e^{(1)} \cdots i_e^{(j)}.$$

Since r is G -invariant, we have $r = (1/|G|) \sum_{g \in G} g(r)$, so

$$r = \sum_{e \in E} \left(\frac{1}{|G|} \sum_{g \in G} g(r_e) \right) i_e^{(1)} \cdots i_e^{(j)}.$$

Hence, $r \in (I^G)^j$. □

Since I and J are G -stable, the completions \hat{R}_I and \hat{R}_J inherit a G -action.

Corollary A.3. — *The \mathbb{C} -algebras $(\hat{R}_I)^G$ and $(\widehat{R^G})_{I^G}$ are canonically isomorphic.*

Proof. — As $I^m \subset J \subset I$ for some m by Lemma A.1, the completions \hat{R}_I and \hat{R}_J are canonically isomorphic, and the isomorphism is G -equivariant. This gives an isomorphism $(\hat{R}_I)^G \simeq (\hat{R}_J)^G$. So the result follows directly from Lemma A.2, because $(R/J^j)^G = R^G/(J^j)^G$ since we work in characteristic zero. □

Proposition A.4. — *Assume that R is Nagata. Then*

$$\widehat{R(G)}_I = \hat{R}_I(G).$$

Proof. — Let $\hat{\tau}_G$ denote the completion of τ_G at I . Since R is noetherian, $\hat{\tau}_G$ is the ideal of \hat{R}_I generated by τ_G and

$$(\widehat{R/\tau_G})_I = \hat{R}_I/\hat{\tau}_G$$

(see for instance [GrSa, §4]). This shows that G acts trivially on $\hat{R}_I/\hat{\tau}_G$ and so $\hat{\tau}_G$ is the ideal of \hat{R}_I generated by $(g(r) - r)_{r \in \hat{R}_I, g \in G}$.

Moreover, as R is Nagata, we have that $\sqrt{\hat{\tau}_G} = \widehat{\sqrt{\tau_G}}$ by [GrSa, Coro. 14.8]. The proposition follows. □

Example A.5. — *Assume that R is a localization or a completion of a finitely generated algebra. Then R is Nagata. ■*

Remark A.6. — *Let \mathcal{X} be a quasi-projective variety acted on by the finite group G and let \mathcal{U} be a G -stable locally closed subvariety. Then we have an isomorphism of formal schemes*

$$(A.7) \quad (\hat{\mathcal{X}}_{\mathcal{U}})^G \simeq (\widehat{\mathcal{X}^G})_{\mathcal{U}^G}.$$

Indeed, by replacing \mathcal{X} by the G -stable open subset $\mathcal{X} \setminus (\overline{\mathcal{U}} \setminus \mathcal{U})$, we may assume that \mathcal{U} is closed in \mathcal{X} and we denote by \mathcal{F} its sheaf of ideals in $\mathcal{O}_{\mathcal{X}}$. The underlying topological

space of both sides of (A.7) is equal to \mathcal{U}^G and there is a natural morphism of sheaf of algebras

$$\widehat{\mathcal{O}}_{(\mathcal{X}^G)_{\mathcal{U}^G}} \longrightarrow \widehat{\mathcal{O}}_{(\hat{\mathcal{X}}_{\mathcal{U}})^G}.$$

We need to prove that it is an isomorphism and this can be checked locally. Since \mathcal{X} is quasi-projective, it can be covered by G -stable open affine subsets and for each of these open affine subsets, the expected isomorphism follows from Proposition A.4. ■

References

- [Bel1] G. BELLAMY, *Cuspidal representations of rational Cherednik algebras at $t = 0$* , Math. Z. **269** (2011), 609-627.
- [Bel2] G. BELLAMY, *Counting resolutions of symplectic quotient singularities*, Compos. Math. **152** (2016), 99-114.
- [BeCh] G. BELLAMY & O. CHALYKH, *Motivic invariants of Calogero-Moser spaces*, in preparation.
- [BeSc] G. BELLAMY, R. MAKSIMAU & T. SCHEDLER, in preparation.
- [BeTh] G. BELLAMY & U. THIEL, *Cuspidal Calogero-Moser and Lusztig families for Coxeter groups*, J. Algebra **462** (2016), 197-252.
- [Bon1] C. BONNAFÉ, *On the Calogero-Moser space associated with dihedral groups II: the equal parameter case*, preprint (2021), arXiv:2112.12401.
- [Bon2] C. BONNAFÉ, *Calogero-Moser spaces vs unipotent representations*, preprint (2021), arXiv:2112.13684. To appear in Pure and App. Math. Quart.
- [BoMa] C. BONNAFÉ & R. MAKSIMAU *Fixed points in smooth Calogero-Moser spaces*, Ann. Inst. Fourier, **71** (2021), 643-678. .
- [BoRo] C. BONNAFÉ & R. ROUQUIER, *Cherednik algebras and Calogero-Moser cells*, preprint (2017), arXiv:1708.09764.
- [BoTh] C. BONNAFÉ & U. THIEL, *Computational aspects of Calogero-Moser spaces*, preprint (2021), arXiv:2112.15495.
- [BrGo] K. BROWN & I. GORDON, *Poisson orders, symplectic reflection algebras and representation theory*, J. Reine Angew. Math. **559** (2003), 193-216.
- [BMM] M. BROUÉ, G. MALLE & J. MICHEL, *Generic blocks of finite reductive groups*, in *Représentations unipotentes génériques et blocs des groupes réductifs finis*, Astérisque **212** (1993), 7-92.
- [BrGo] K. A. BROWN & I. GORDON, *Poisson orders, symplectic reflection algebras and representation theory*, J. Reine Angew. Math. **559** (2003), 193-216.
- [Dix] J. DIXMIER, *Enveloping algebras*, North-Holland Math. Lib. **14**, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977, xvi+375 pp.
- [EtGi] P. ETINGOF & V. GINZBURG, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), 243-348.
- [Gor] I. GORDON, *Baby Verma modules for rational Cherednik algebras*, Bull. London Math. Soc. **35** (2003), 321-336.
- [GiKa] V. GINZBURG & D. KALEDIN, *Poisson deformations of symplectic quotient singularities*, Adv. Math. **186** (2004), 1-57.
- [GrSa] S. GRECO & P. SALMON, *Topics in m-adic topology*, Ergeb. der Math. und ihrer Grenz. **58**, Springer-Verlag, New York-Berlin, 1971, vii+74 pp.
- [Har] R. HARTSHORNE, *Algebraic geometry*, Graduate Texts in Mathematics **52**. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.
- [Kal] D. KALEDIN, *Normalization of a Poisson algebra is Poisson*, Proc. Steklov Inst. Math. **264** (2009), 70-73.
- [Lam] T.Y. LAM, *A First Course in Noncommutative Rings*, Grad. Texts in Math. **131**, 1991, Springer-Verlag.

- [LeSp] G.I. LEHRER & T.A. SPRINGER, *Intersection multiplicities and reflection subquotients of unitary reflection groups I*, Geometric group theory down under (Canberra, 1996), 181-193, de Gruyter, Berlin, 1999.
- [Los] I. LOSEV, *Completions of symplectic reflection algebras*, Selecta Math. **18** (2012), 179-251.
- [Mag] W. BOSMA, J. CANNON & C. PLAYOUST, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235-265.
- [Mar1] M. MARTINO, *The associated variety of a Poisson prime ideal*, J. London Math. Soc. **72** (2005), 110-120.
- [Mar2] M. MARTINO, *Symplectic reflection algebras and Poisson geometry*, PhD thesis, University of Glasgow, 2006.
- [Nam1] Y. NAMIKAWA, *Poisson deformations of affine symplectic varieties*, Duke Math. J. **156** (2011), 51-85.
- [Nam2] Y. NAMIKAWA, *Poisson deformations of affine symplectic varieties, II*, Kyoto J. Math. **50** (2010), 727-752.
- [Spr] T.A. SPRINGER, *Regular elements of finite reflection groups*, Invent. Math. **25** (1974), 159-198.
- [ShTo] G. C. SHEPHARD & J. A. TODD, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274-304.
- [Thi] U. THIEL, *Champ: a Cherednik algebra Magma package*, LMS J. Comput. Math. **18** (2015), 266-307.

August 4, 2022

CÉDRIC BONNAFÉ, IMAG, Université de Montpellier, CNRS, Montpellier, France
E-mail: cedric.bonnafe@umontpellier.fr