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A family of second-order dissipative finite volume schemes for hyperbolic systems of conservation laws

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Abstract. We propose and study a family of formally second-order accurate schemes to approximate weak solutions of hyperbolic systems of conservation laws. These schemes are based on a dissipative property satisfied by the second-order discretization in space. They are proven to satisfy a global entropy inequality for a generic strictly convex entropy. These schemes do not involve limitation techniques. Numerical results are provided to illustrate their accuracy and stability.

Keywords. Systems of conservation laws, Second-order finite Volume schemes, Explicit schemes, Global entropy inequality.

Math. classification. 65N08, 35L65, 35L67.

1. Introduction

This work is concerned with the numerical approximation of weak solutions of systems made of $d \geq 1$ conservation laws in one space dimension given by the following Cauchy problem

$$\begin{cases} \partial_t w + \partial_x f(w) &= 0, & t > 0, & x \in \mathbb{R}, \\ w(x, t = 0) &= w_0(x). \end{cases} \quad (1.1)$$

The unknown state vector $w(x, t)$ is assumed to belong to Ω , a non-empty convex open subset of \mathbb{R}^d . Here, $f : \Omega \rightarrow \mathbb{R}^d$ is a given smooth flux function. For all $w \in \Omega$, the $d \times d$ Jacobian matrix $\nabla f(w)$ is assumed to be diagonalizable in \mathbb{R} . The system (1.1) is then a hyperbolic system of conservation laws. The initial data $w_0 : \mathbb{R} \rightarrow \Omega$ is a given measurable function in $L^1_{\text{loc}}(\mathbb{R})$. Even though the initial data is smooth, it is well-known that the solutions to (1.1) may develop, in a finite time, discontinuities [27, 28, 34, 29]. Weak solutions are in general non unique and one has to select a physically admissible solution among many others. In this regard, the system (1.1) is usually endowed with entropy inequalities. We consider in this work to be given a strictly convex function $\eta \in C^2(\Omega, \mathbb{R})$, called entropy function, and an entropy flux function $G \in C^1(\Omega, \mathbb{R})$ such that we have

$$\forall w \in \Omega, \quad \nabla \eta(w)^T \nabla f(w) = \nabla G(w)^T. \quad (1.2)$$

It is then required that weak solutions verify an entropy inequality given in the weak sense by

$$\partial_t \eta(w) + \partial_x G(w) \leq 0. \quad (1.3)$$

Instead of (1.3), we shall focus in this work on the weaker inequality

$$\frac{d}{dt} \int_{\mathbb{R}} \eta(w(x, t)) dx \leq 0, \quad (1.4)$$

which results from an integration in space of the inequality (1.3) for compactly supported solutions. This inequality will be called throughout this work a global entropy inequality. Our main concern is to design a family of schemes that are formally second-order accurate in space and that verifies a discrete analogue to (1.4). We consider uniform meshes in space and time $(x_{i+\frac{1}{2}})_{i \in \mathbb{Z}} \subset \mathbb{R}$ and $(t^n)_{n \in \mathbb{N}} \subset [0, +\infty)$ of respectively constant size $\Delta x > 0$ and $\Delta t > 0$. We have $x_{i+\frac{1}{2}} = x_{i-\frac{1}{2}} + \Delta x$ for all $i \in \mathbb{Z}$ and $t^{n+1} = t^n + \Delta t$ for all $n \in \mathbb{N}$. Weak solutions to (1.1) are approximated within the standard finite volume framework. At time t^n , we consider the following piecewise constant function

$$w_{\Delta}(x, t^n) = w_i^n \quad \text{if } x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}), \quad (1.5)$$

where the quantities w_i^n are approximations of the average of the solution over the cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$,

$$w_i^n \approx \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x, t^n) dx,$$

where $w(x, t^n)$ naturally belongs to $L^1_{\text{loc}}(\mathbb{R})$.

Many strategies are known to define an updated sequence $(w_i^{n+1})_{i \in \mathbb{Z}}$ starting from $(w_i^n)_{i \in \mathbb{Z}}$ (for instance, see [39, 21, 33, 10] and references therein). The crux of the problem lies in the discrete analogue to (1.3) which is known to ensure the stability of the scheme and to avoid non-admissible discontinuous waves (for instance, see [8, 25]). Several first-order accurate finite volume schemes are known to satisfy (1.3): Godunov [16, 21], the kinetic schemes [26, 3], the HLL scheme [20, 21], the HLLC scheme [40], Suliciu relaxation approaches [2, 4, 9, 12, 24, 23, 7] or many other schemes [22, 15, 14, 12, 12, 24, 23, 7] based on the strategy introduced by Tadmor [38, 37]. As far as second-order accurate in space schemes are concerned, the literature is less exhaustive. Several results have nevertheless been obtained in the semi discrete case [6, 37] or in the fully discrete case using an entropy averaging technique [1, 5, 31, 32]. Both strategies are however not sufficient to rule out non-admissible discontinuities in the converged solutions. Other strategies such as MUSCL technique [41] or ENO/WENO schemes [30, 35, 36] or DG schemes [36, 11, 23] which require limitation techniques are also of common use. However efficient, in this work we shall not use limitation techniques.

We propose here to study a family of formally second-order accurate schemes (both in space and time) that verifies a discrete analogue to (1.4). It is derived starting from the HLL scheme [21] complemented with a second-order correction in space which is asked to verify a dissipative property. In Section 3, a quadratic stability study in the simpler case of the scalar linear transport equation, using both algebraic and Fourier approaches is performed. The purpose of this section is twofold: to explicit the dissipative property in a simple setting, and then to justify the use of a second-order in time discretization to recover the quadratic stability under a hyperbolic CFL (Courant, Friedrichs, Levy) condition. In Section 4, we establish the global entropy inequality (1.4) in the general case of non linear systems (1.1) for a given single entropy-entropy-flux pair (η, G) . The proof relies on the following ingredients: a second-order Taylor expansion of the entropy, a reformulation of the global entropy dissipation, the use of the dissipative property granted by the second-order discretization in space and the choice of a large enough viscosity coefficient. Because the proof is also intended to be made in the general case of an open convex subset $\Omega \subset \mathbb{R}^d$, a topological restriction on the approximate solution is also needed. Namely, it is required to belong to a compact subset of Ω . It is somehow restrictive but it is up to our knowledge very standard in the literature, notably in the case of the Euler equations where it consists in assuming the solutions to be bounded and uniformly far away from the vacuum

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[34]. In Section 5, we perform numerical experiments that assess the stability and the accuracy of our schemes. Eventually, a short conclusion is given at the very end of this paper.

2. Unlimited second-order space and time HLL type schemes

The starting point is the original first-order HLL scheme [21] that reads as follows:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} (\mathcal{F}_\lambda^{O1}(w_i^n, w_{i+1}^n) - \mathcal{F}_\lambda^{O1}(w_{i-1}^n, w_i^n)), \quad (2.1)$$

where the numerical flux function is given by

$$\mathcal{F}_\lambda^{O1}(w_i^n, w_{i+1}^n) = \frac{1}{2} (f(w_i^n) + f(w_{i+1}^n)) - \frac{\lambda}{2} (w_{i+1}^n - w_i^n). \quad (2.2)$$

Here, $\lambda > 0$ stands for the numerical viscosity coefficient. Under the following CFL condition:

$$\frac{\Delta t}{\Delta x} \lambda \leq \frac{1}{2} \quad \text{with} \quad \lambda \geq \max_{i \in \mathbb{Z}} (|\mu(w_i^n)|),$$

where $\mu(w)$ denotes the spectral radius of the jacobian matrix $\nabla f(w)$, the scheme (2.1) is first-order in space, entropy preserving and convergent (see [21]). Equipped with this first-order scheme, we are now going to increase the order of accuracy in space. For the sake of clarity of this work, we remind of the notion of second-order consistency in space (some details can be found in [4] and [13]). We consider here only five-points finite volume schemes.

Definition 2.1 (Second-order weak consistency). Let $\mathcal{F} : (\mathbb{R}^d)^4 \rightarrow \mathbb{R}^d$ be a continuous function which verifies the consistency relation:

$$\forall u \in \Omega, \quad \mathcal{F}(u, u, u, u) = f(u). \quad (2.3)$$

Consider the semi-discrete numerical scheme defined for $t \in [0, T)$ with $T > 0$ and for $i \in \mathbb{Z}$ by:

$$\frac{du_i}{dt}(t) = -\frac{1}{\Delta x} (\mathcal{F}(u_{i-1}, \dots, u_{i+2}) - \mathcal{F}(u_{i-2}, \dots, u_{i+1})). \quad (2.4)$$

Assume for all $i \in \mathbb{Z}$, $u_i(t) = w_i(t) = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x, t) dx$ where w is a smooth solution to (1.1)

defined on $\mathbb{R} \times [0, T)$. The semi-discrete scheme is second-order consistent in the weak sense, if for all times $t \in [0, T)$ and for any smooth compactly supported function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \frac{dw_\Delta(t)}{dt} \varphi(x) dx = \int_{\mathbb{R}} (\partial_t w(t, x)) \varphi(x) dx + \mathcal{O}(\Delta x^2), \quad (2.5)$$

where $w_\Delta(t) = \sum_{i \in \mathbb{Z}} \mathbf{1}_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})} w_i(t)$.

This definition is based on the notion of weak convergence in space and does not prevent a semi-discrete scheme from being at a lower order of space accuracy when assessing the error in a strong topology. We now give the following result which gives a sufficient condition for the second-order consistency in the weak sense. It is somehow a standard result but we give a proof for the sake of completeness of this work.

Lemma 2.2 (Weak consistency). Let $\mathcal{F} : (\mathbb{R}^d)^4 \rightarrow \mathbb{R}^d$ be a continuous function which verifies the consistency relation:

$$\forall u \in \Omega, \quad \mathcal{F}(u, u, u, u) = f(u).$$

Consider a semi-discrete numerical scheme defined for $t \in [0, T)$ with $T > 0$ and for $i \in \mathbb{Z}$ by:

$$\frac{du_i}{dt}(t) = -\frac{1}{\Delta x} (\mathcal{F}(u_{i-1}, \dots, u_{i+2}) - \mathcal{F}(u_{i-2}, \dots, u_{i+1})). \quad (2.6)$$

Let $t \in [0, T)$ being fixed. If we have for all $i \in \mathbb{Z}$,

$$\mathcal{F}(u_{i-1}, \dots, u_{i+2}) = f(u(x_{i+\frac{1}{2}}, t)) + \mathcal{O}_i(\Delta x^2) \quad (2.7)$$

where we have set for a given smooth enough function $u(x, t)$,

$$u_i(t) = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, t) dx, \quad (2.8)$$

and if there exists a constant $C(t)$ such that for all $i \in \mathbb{Z}$, $|\mathcal{O}_i(\Delta x^2)| \leq C(t)\Delta x^2$, then the scheme is second-order consistent in the weak sense.

Proof. Let w be a smooth solution on $\mathbb{R} \times [0, T)$ to (1.1) and consider the semi-discrete scheme defined by (2.6) associated with w . Let $t \in [0, T)$ and set for all $i \in \mathbb{Z}$, $\mathcal{F}_{i+\frac{1}{2}} = \mathcal{F}(w_{i-1}, \dots, w_{i+2})$

where $w_i(t) = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x, t) dx$. Let $\varphi \in C_c^\infty(\mathbb{R})$ then we have,

$$\begin{aligned} \int_{\mathbb{R}} \frac{dw_\Delta(t)}{dt} \varphi(x) dx &= \sum_{i \in \mathbb{Z}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{dw_i}{dt}(t) \varphi(x) dx = \sum_{i \in \mathbb{Z}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} -\frac{1}{\Delta x} (\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}) \varphi(x) dx \\ &= -\sum_{i \in \mathbb{Z}} (\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}) \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \varphi(x) dx = \sum_{i \in \mathbb{Z}} \mathcal{F}_{i+\frac{1}{2}} (\varphi_{i+1} - \varphi_i), \end{aligned}$$

where we have set $\varphi_i = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \varphi(x) dx$. Using the consistency of the numerical flux (2.7), we obtain

$$\int_{\mathbb{R}} \frac{dw_\Delta(t)}{dt} \varphi(x) dx = \sum_{i \in \mathbb{Z}} f(w(x_{i+\frac{1}{2}}, t)) (\varphi_{i+1} - \varphi_i) + \sum_{i \in \mathbb{Z}} \mathcal{O}_i(\Delta x^2) (\varphi_{i+1} - \varphi_i).$$

Since we have for all $i \in \mathbb{Z}$, $|\mathcal{O}_i(\Delta x^2)| \leq C(t)\Delta x^2$ where the constant $C(t)$ is uniform with respect to i , we obtain that the second term is uniformly a $\mathcal{O}(\Delta x^2)$ term because φ belongs to $C_c^\infty(\mathbb{R})$, namely:

$$\left| \sum_{i \in \mathbb{Z}} \mathcal{O}_i(\Delta x^2) (\varphi_{i+1} - \varphi_i) \right| \leq C(t)\Delta x^2 \sum_{i \in \mathbb{Z}} |\varphi_{i+1} - \varphi_i|.$$

To conclude the proof, it suffices to prove that $\sum_{i \in \mathbb{Z}} f(w(x_{i+\frac{1}{2}}, t)) (\varphi_{i+1} - \varphi_i) = \int_{\mathbb{R}} f(w(x, t)) \partial_x \varphi(x) dx + \mathcal{O}(\Delta x^2)$.

Decomposing the integral as a sum of integrals over each interval (x_i, x_{i+1}) we have

$$\int_{\mathbb{R}} f(w(x, t)) \partial_x \varphi(x) dx = \sum_{i \in \mathbb{Z}} \int_{x_i}^{x_{i+1}} f(w(x, t)) \partial_x \varphi(x) dx.$$

Using the smoothness of the functions f , w and φ and a Taylor-Lagrange expansion around $x_{i+\frac{1}{2}}$, we have the existence of a function $\xi : x \in [x_i, x_{i+1}) \mapsto \xi(x) \in (x, x_{i+\frac{1}{2}})$ such that

$$\begin{aligned} \int_{x_i}^{x_{i+1}} f(w(x, t)) \partial_x \varphi(x) dx &= f\left(w(x_{i+\frac{1}{2}}, t)\right) \partial_x \varphi(x_{i+\frac{1}{2}}) \Delta x \\ &+ \int_{x_i}^{x_{i+1}} \frac{(x - x_{i+\frac{1}{2}})^2}{2} \partial_{xx} (f(w(\cdot, t)) \partial_x \varphi(\cdot)) (\xi(x)) dx, \end{aligned}$$

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where the first-order term has vanished because $x_{i+\frac{1}{2}}$ is the mid point of the interval (x_i, x_{i+1}) and the function $x \mapsto (x - x_{i+\frac{1}{2}})$ is symmetric with respect to $x_{i+\frac{1}{2}}$. By summation, we thus have

$$\int_{\mathbb{R}} f(w(x, t)) \partial_x \varphi(x) dx = \sum_{i \in \mathbb{Z}} f\left(w(x_{i+\frac{1}{2}}, t)\right) \partial_x \varphi(x_{i+\frac{1}{2}}) \Delta x + \sum_{i \in \mathbb{Z}} \int_{x_i}^{x_{i+1}} \frac{(x - x_{i+\frac{1}{2}})^2}{2} \partial_{xx} (f(w(\cdot, t)) \partial_x \varphi(\cdot)) (\xi(x)) dx$$

Since φ is compactly supported the second sum is a $\mathcal{O}(\Delta x^2)$ term, so one has

$$\int_{\mathbb{R}} f(w(x, t)) \partial_x \varphi(x) dx = \sum_{i \in \mathbb{Z}} f\left(w(x_{i+\frac{1}{2}}, t)\right) \partial_x \varphi(x_{i+\frac{1}{2}}) \Delta x + \mathcal{O}(\Delta x^2).$$

Using again a Taylor-Lagrange expansion, we obtain that $\varphi_{i+1} - \varphi_i = \partial_x \varphi(x_{i+\frac{1}{2}}) \Delta x + \mathcal{O}(\Delta x^3)$ and by summation we eventually deduce

$$\int_{\mathbb{R}} f(w(x, t)) \partial_x \varphi(x) dx = \sum_{i \in \mathbb{Z}} f\left(w(x_{i+\frac{1}{2}}, t)\right) (\varphi_{i+1} - \varphi_i) + \mathcal{O}(\Delta x^2)$$

where the $\mathcal{O}(\Delta x^2)$ term is uniform with respect to $i \in \mathbb{Z}$ because φ is compactly supported. Using the fact that w is a smooth solution to (1.1) concludes the proof. \blacksquare

The numerical HLL flux function (2.2) is first-order consistent. Moreover, with (2.8), a standard Taylor expansion in a neighborhood of $x_{i+\frac{1}{2}}$ gives

$$\mathcal{F}_\lambda^{O1}(w_i^n, w_{i+1}^n) = f\left(w(x_{i+\frac{1}{2}}, t^n)\right) - \frac{\lambda \Delta x}{2} \partial_x w(x_{i+\frac{1}{2}}, t^n) + \mathcal{O}(\Delta x^2). \quad (2.9)$$

From the above Taylor expansion, we see that it is sufficient to propose a correction which is consistent with $+\frac{\lambda}{2} \partial_x w(x_{i+\frac{1}{2}}, t^n)$. We therefore consider second-order space accuracy numerical flux functions of the form

$$\mathcal{F}_\lambda^{O2}(w_{i-1}^n, \dots, w_{i+2}^n) = \mathcal{F}_\lambda^{O1}(w_i^n, w_{i+1}^n) + \frac{1}{2} (\alpha_i^n + \alpha_{i+1}^n), \quad (2.10)$$

where the corrective terms α_i^n (which will be defined here after) must satisfy the following consistency

$$\alpha_i^n = \frac{\lambda \Delta x}{2} \partial_x w(x_i, t^n) + \mathcal{O}(\Delta x^2), \quad (2.11)$$

for a smooth function w . Therefore the second-order numerical flux function \mathcal{F}_λ^{O2} will be composed of both an approximation of the term $-\frac{\lambda \Delta x}{2} \partial_x w$ which inherits from the HLL flux function \mathcal{F}_λ^{O1} and an approximation of the same term but with the opposite sign $+\frac{\lambda \Delta x}{2} \partial_x w$ which comes from the corrective term α_i . Of course, at the continuous level the sum of these two terms is equal to zero. However at the discrete level, since these two terms will not be discretized within the same stencil, they will not generally cancel. The difference will be asked to control the numerical dissipativity of the scheme and thus its stability. This is why our family of schemes are called dissipative. We thus propose the following definition of the second-order correction term α_i^n ,

$$\alpha_i^n = \frac{\lambda}{2} \Delta x \overline{\partial_x w_i^n} = \frac{\lambda}{2} [\Theta_i^n (w_{i+1}^n - w_i^n) + (I - \Theta_i^n) (w_i^n - w_{i-1}^n)], \quad (2.12)$$

where I stands for $d \times d$ identity matrix while Θ_i^n is a free $d \times d$ diagonal matrix parameter to be defined. The choice of the matrix parameter will be central to establish the global entropy inequality (1.4) in the general case. Namely, it will be asked to satisfy the inequality (4.4) which is the mathematical expression of the dissipativity relatively to an entropy η . Equipped with the correction terms α_i^n , we can establish the following.

Proposition 2.3 (Weak second-order consistency). *Let be given $u(x, t)$ a smooth function compactly supported in space and let u_i defined for all $i \in \mathbb{Z}$ by (2.8). Consider the numerical flux \mathcal{F}_λ^{O2} defined*

by (2.10),(2.12). Assume the sequence of matrices $(\Theta_i^n)_{i \in \mathbb{Z}}$ be uniformly bounded with respect to $i \in \mathbb{Z}$ and bounded as $\Delta x \rightarrow 0$. Then the scheme is second-order consistent in the weak-sense.

Proof. Invoking Lemma (2.2), it suffices to use a Taylor expansion of \mathcal{F}_λ^{O2} given by (2.10),(2.12) and the fact that u is compactly supported in space. \blacksquare

We have now to define the discretization in time. We may have restricted our attention to the standard forward Euler discretization in time, however as we shall see in the next section in the particular case of the linear transport equation, the global entropy inequality will be obtained under a parabolic type CFL condition. As a consequence, we consider an explicit second-order in time discretization based on the SSP Runge-Kutta methods introduced in [18, 19, 17], then the unlimited second-order HLL scheme reads

$$\begin{aligned} w_i^{n+\frac{1}{2}} &= w_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}_{i+\frac{1}{2}}^n - \mathcal{F}_{i-\frac{1}{2}}^n \right), \\ w_i^{n+1} &= \frac{w_i^n}{2} + \frac{1}{2} \left(w_i^{n+\frac{1}{2}} - \frac{\Delta t}{\Delta x} \left(\mathcal{F}_{i+\frac{1}{2}}^{n+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}^{n+\frac{1}{2}} \right) \right), \end{aligned} \quad (2.13)$$

with for all $m \in \{n, n + \frac{1}{2}\}$,

$$\begin{aligned} \mathcal{F}_{i+\frac{1}{2}}^m &= \mathcal{F}_\lambda^{O1}(w_i^m, w_{i+1}^m) + \frac{1}{2} (\alpha_{i+1}^m + \alpha_i^m), \\ \alpha_i^m &= \frac{\lambda}{2} \left(\Theta_i^m \delta_{i+\frac{1}{2}}^m + (I - \Theta_i^m) \delta_{i-\frac{1}{2}}^m \right), \\ \delta_{i+\frac{1}{2}}^m &= w_{i+1}^m - w_i^m, \end{aligned} \quad (2.14)$$

and \mathcal{F}_λ^{O1} is defined in (2.2).

3. Quadratic stability for the scalar linear transport equation

The purpose of this section is to explicit (in a simple case) the dissipative property required by the second-order discretization in space, and to justify the need of a second-order in time discretization to obtain a stability property under a hyperbolic like CFL condition. We consider momentarily the scalar linear transport equation with velocity $a \neq 0$,

$$\begin{cases} \partial_t w + a \partial_x w = 0, & t > 0, \quad x \in \mathbb{R}, \\ w(x, t = 0) = w_0(x), \end{cases} \quad (3.1)$$

where the unknown $w(x, t) \in \mathbb{R}$. We also consider for the discussion the standard forward Euler time discretization associated with the semi-discrete scheme (2.10),(2.12). Then to approximate (3.1), we obtain the scheme:

$$w_i^{n+1} = w_i^n - \frac{\nu}{2} \left(\frac{a}{\lambda} \left(\delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n \right) - \delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n + \frac{1}{\lambda} (\alpha_{i+1}^n - \alpha_{i-1}^n) \right), \quad (3.2)$$

where $\nu = \lambda \Delta t / \Delta x$, α_i^n and $\delta_{i+\frac{1}{2}}^n$ are defined in (2.14). For the stability analysis, it will be convenient to consider the standard discrete Sobolev spaces defined by

$$l^2(\mathbb{Z}) = \left\{ v \in \mathbb{R}^{\mathbb{Z}} \mid \sum_{i \in \mathbb{Z}} |v_i|^2 \Delta x < +\infty \right\}, \quad (3.3)$$

$$h^2(\mathbb{Z}) = \left\{ v \in l^2(\mathbb{Z}) \mid \left(\frac{v_{i+1} - v_i}{\Delta x} \right)_{i \in \mathbb{Z}} \in l^2(\mathbb{Z}), \left(\frac{v_{i+1} - 2v_i + v_{i-1}}{\Delta x^2} \right)_{i \in \mathbb{Z}} \in l^2(\mathbb{Z}) \right\}. \quad (3.4)$$

We begin with the following easy preliminary lemma.

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Lemma 3.1 (Existence of dissipative corrections). *For any non zero sequence $(w_i^n)_{i \in \mathbb{Z}}$ in $h^2(\mathbb{Z})$, there exists $(\Theta_i^n)_{i \in \mathbb{Z}}$ such that the following inequality holds*

$$\sum_{i \in \mathbb{Z}} \left(\left(\delta_{i+\frac{1}{2}}^n - \delta_{i-\frac{1}{2}}^n \right)^2 - 2\Theta_i^n \left((\delta_{i+\frac{1}{2}}^n)^2 - (\delta_{i-\frac{1}{2}}^n)^2 \right) \right) > 0. \quad (3.5)$$

Proof. The case $\Theta_i^n = \Theta \equiv \text{const}$ for all $i \in \mathbb{Z}$ yields

$$\sum_{i \in \mathbb{Z}} \left(\left(\delta_{i+\frac{1}{2}}^n - \delta_{i-\frac{1}{2}}^n \right)^2 - 2\Theta \left((\delta_{i+\frac{1}{2}}^n)^2 - (\delta_{i-\frac{1}{2}}^n)^2 \right) \right) = \sum_{i \in \mathbb{Z}} \left(\delta_{i+\frac{1}{2}}^n - \delta_{i-\frac{1}{2}}^n \right)^2.$$

where the second sum vanishes because it is telescopic. Since the sequence is non zero and in $h^2(\mathbb{Z})$ the remaining sum is finite and positive. \blacksquare

The inequality (3.5) is the mathematical expression of the dissipative property relatively to the L^2 norm that we want to ensure for the second-order in space discretization. It takes a more general form when considering non linear hyperbolic systems with a general entropy η (see (4.4)). Several choices of the sequence $(\Theta_i^n)_{i \in \mathbb{Z}}$ are possibles. Another possibility is to choose the sequence $(\Theta_i^n)_{i \in \mathbb{Z}}$ in order to have $\sum_{i \in \mathbb{Z}} \Theta_i^n \left((\delta_{i+\frac{1}{2}}^n)^2 - (\delta_{i-\frac{1}{2}}^n)^2 \right) < 0$. For example the choice $\Theta_i^n = -\text{sign}((\delta_{i+\frac{1}{2}}^n)^2 - (\delta_{i-\frac{1}{2}}^n)^2)$ is also a possibility, some are very promising as we shall see in the numerical examples. We prove the following.

Proposition 3.2 (Quadratic stability with a parabolic CFL condition). *Let $(w_i^n)_{i \in \mathbb{Z}}$ be a non zero sequence in $h^2(\mathbb{Z})$ and $(\Theta_i^n)_{i \in \mathbb{Z}}$ be a sequence such that the inequality (3.5) holds. Let $\lambda > 0$ and set $\nu = \frac{\lambda \Delta t}{\Delta x}$ such that*

$$0 < \nu \leq \frac{\sum_{i \in \mathbb{Z}} \left(\left(\delta_{i+\frac{1}{2}}^n - \delta_{i-\frac{1}{2}}^n \right)^2 - 2\Theta_i^n \left((\delta_{i+\frac{1}{2}}^n)^2 - (\delta_{i-\frac{1}{2}}^n)^2 \right) \right)}{\sum_{i \in \mathbb{Z}} \left(\frac{a}{\lambda} \left(\delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n \right) - \delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n + \frac{1}{\lambda} (\alpha_{i+1}^n - \alpha_{i-1}^n) \right)^2}, \quad (3.6)$$

then the first-order in time scheme (3.2) is L^2 -stable, $\sum_{i \in \mathbb{Z}} |w_i^{n+1}|^2 \Delta x \leq \sum_{i \in \mathbb{Z}} |w_i^n|^2 \Delta x$.

Proof. Using the scheme (3.2), we develop the square

$$\begin{aligned} \sum_{i \in \mathbb{Z}} (w_i^{n+1})^2 \Delta x &= \sum_{i \in \mathbb{Z}} (w_i^n)^2 \Delta x \\ &- \nu \sum_{i \in \mathbb{Z}} w_i^n \left(\frac{a}{\lambda} \left(\delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n \right) - \delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n + \frac{1}{\lambda} (\alpha_{i+1}^n - \alpha_{i-1}^n) \right) \Delta x \\ &+ \frac{\nu^2}{4} \sum_{i \in \mathbb{Z}} \left(\frac{a}{\lambda} \left(\delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n \right) - \delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n + \frac{1}{\lambda} (\alpha_{i+1}^n - \alpha_{i-1}^n) \right)^2 \Delta x. \end{aligned}$$

Now, we develop the second term of the above equation and recall that $\delta_{i+\frac{1}{2}}^n = w_{i+1}^n - w_i^n$, so we have

$$\begin{aligned} & \sum_{i \in \mathbb{Z}} w_i^n \left(\frac{a}{\lambda} (\delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n) - \delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n + \frac{1}{\lambda} (\alpha_{i+1}^n - \alpha_{i-1}^n) \right) \\ &= \frac{a}{\lambda} \sum_{i \in \mathbb{Z}} w_i^n (w_{i+1}^n - w_{i-1}^n) - \sum_{i \in \mathbb{Z}} w_i^n \delta_{i+\frac{1}{2}}^n + \sum_{i \in \mathbb{Z}} w_i^n \delta_{i-\frac{1}{2}}^n \\ & \quad - \frac{1}{\lambda} \sum_{i \in \mathbb{Z}} \alpha_i^n (\delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n), \end{aligned}$$

where we have used a translation of indices for the third term. Using now the fact that $\alpha_i^n = \Theta_i^n \delta_{i+\frac{1}{2}}^n + (1 - \Theta_i^n) \delta_{i-\frac{1}{2}}^n$ and again a translation of indices, we have

$$\begin{aligned} & \sum_{i \in \mathbb{Z}} w_i^n \left(\frac{a}{\lambda} (\delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n) - \delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n + \frac{1}{\lambda} (\alpha_{i+1}^n - \alpha_{i-1}^n) \right) \\ &= \frac{1}{2} \sum_{i \in \mathbb{Z}} (\delta_{i+\frac{1}{2}}^n)^2 - \frac{1}{2} \sum_{i \in \mathbb{Z}} \Theta_i^n \left((\delta_{i+\frac{1}{2}}^n)^2 - (\delta_{i-\frac{1}{2}}^n)^2 \right) - \frac{1}{2} \sum_{i \in \mathbb{Z}} \delta_{i+\frac{1}{2}}^n \delta_{i-\frac{1}{2}}^n, \\ &= \frac{1}{4} \sum_{i \in \mathbb{Z}} (\delta_{i+\frac{1}{2}}^n - \delta_{i-\frac{1}{2}}^n)^2 - \frac{1}{2} \sum_{i \in \mathbb{Z}} \Theta_i^n \left((\delta_{i+\frac{1}{2}}^n)^2 - (\delta_{i-\frac{1}{2}}^n)^2 \right). \end{aligned}$$

As a consequence, we obtain

$$\begin{aligned} & \sum_{i \in \mathbb{Z}} (w_i^{n+1})^2 \Delta x = \sum_{i \in \mathbb{Z}} (w_i^n)^2 \Delta x \\ & \quad + \frac{\nu}{4} \left(- \sum_{i \in \mathbb{Z}} \left((\delta_{i+\frac{1}{2}}^n - \delta_{i-\frac{1}{2}}^n)^2 - 2\Theta_i^n \left((\delta_{i+\frac{1}{2}}^n)^2 - (\delta_{i-\frac{1}{2}}^n)^2 \right) \right) \Delta x \right. \\ & \quad \left. + \frac{\nu^2}{4} \sum_{i \in \mathbb{Z}} \left(\frac{a}{\lambda} (\delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n) - \delta_{i+\frac{1}{2}}^n + \delta_{i-\frac{1}{2}}^n + \frac{1}{\lambda} (\alpha_{i+1}^n - \alpha_{i-1}^n) \right)^2 \Delta x \right), \end{aligned}$$

Because of the inequality (3.5) the second sum after the equality is negative, therefore using the inequality (3.6) yields the result. \blacksquare

A consistency analysis of the CFL condition (3.6) for smooth solutions yields an order of magnitude which is such that $\frac{\Delta t}{\Delta x} \underset{\Delta x \rightarrow 0}{=} \mathcal{O}(\Delta x)$. The CFL condition is then of parabolic type. It is possible to recover a hyperbolic CFL magnitude $\frac{\Delta t}{\Delta x} \underset{\Delta x \rightarrow 0}{=} \mathcal{O}(1)$ using a second-order in time discretization. Indeed, in the case where the sequence $(\Theta_i^n)_{i \in \mathbb{Z}}$ is taken to be constant we have the following.

Proposition 3.3 (Quadratic stability with hyperbolic CFL). *Consider $(\Theta_i^n)_{i \in \mathbb{Z}}$ and $(\Theta_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}}$ be constant sequences both equal to a constant $\Theta \in \mathbb{R}$ and $(w_i^n)_{i \in \mathbb{Z}} \in l^2(\mathbb{Z})$. Let $\lambda > 0$ and $\nu = \frac{\lambda \Delta t}{\Delta x}$ such that*

$$0 < \nu \max \left(1, \left(\frac{a}{\lambda} \right)^2, \left(1 - 2\Theta + \frac{a}{\lambda} \right)^2 \right) \leq 1, \quad (3.7)$$

then the second-order in time scheme (2.13) is L^2 -stable, $\sum_{i \in \mathbb{Z}} |w_i^{n+1}|^2 \Delta x \leq \sum_{i \in \mathbb{Z}} |w_i^n|^2 \Delta x$.

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Proof. We denote $\hat{w}_\Delta(\xi, t^n)$, the Fourier transform of $w_\Delta(x, t^n)$ given by (1.5). We have

$$\hat{w}_\Delta(\xi, t^n) = \frac{\sin(\pi\xi\Delta x)}{\pi\xi} \sum_{j \in \mathbb{Z}} w_j^n e^{-2\pi i \xi j \Delta x},$$

where $i^2 = -1$. Using the scheme (2.13), we deduce $\hat{w}_\Delta(\xi, t^{n+1}) = g(\xi\Delta x)\hat{w}_\Delta(\xi, t^n)$ where $g(\xi\Delta x)$ is the amplification factor at the frequency ξ associated with the scheme (2.13). Thanks to the Fourier isometry property, to prove the L^2 stability of the scheme (2.13) it is sufficient to prove $|g(\xi\Delta x)| \leq 1$ for all $\xi \in \mathbb{R}$. For the sake of conciseness in the notation, we set $\phi_\xi = \sin^2(\pi\xi\Delta x)$ and $h_\Theta = (1 - 2\Theta)\sin(\pi\xi\Delta x) + \frac{\alpha}{\lambda}$, therefore, it can be shown that the amplification factor $g(\xi\Delta x)$ verifies

$$\begin{aligned} |g(\xi\Delta x)|^2 &= (1 - 2\nu\phi_\xi (\phi_\xi - \nu\phi_\xi^3 + \nu(1 - \phi_\xi)h_\Theta^2))^2 + 4\phi_\xi(1 - \phi_\xi)(\nu h_\Theta(1 - 2\nu\phi_\xi^2))^2, \\ &= 1 - 4\nu\phi_\xi(\phi_\xi - \nu\phi_\xi^3 + \nu(1 - \phi_\xi)h_\Theta^2) + 4\nu^2\phi_\xi^2(\phi_\xi - \nu\phi_\xi^3 + \nu(1 - \phi_\xi)h_\Theta^2)^2 \\ &\quad + 4\phi_\xi(1 - \phi_\xi)(\nu h_\Theta(1 - 2\nu\phi_\xi^2))^2, \\ &= 1 - 4\nu\phi_\xi^2(1 - \nu\phi_\xi^2) + 4\nu^2\phi_\xi(1 - \phi_\xi)h_\Theta^2(-1 + (1 - 2\nu\phi_\xi^2)^2) \\ &\quad + 4\nu^2\phi_\xi^2(\phi_\xi - \nu\phi_\xi^3 + \nu(1 - \phi_\xi)h_\Theta^2)^2. \end{aligned}$$

We then notice that $-1 + (1 - 2\nu\phi_\xi^2)^2 = -4\nu\phi_\xi^2(1 - \nu\phi_\xi^2)$, so that

$$\begin{aligned} |g(\xi\Delta x)|^2 &= 1 - 4\nu\phi_\xi^2(1 - \nu\phi_\xi^2)(1 + 4\nu^2\phi_\xi(1 - \phi_\xi)h_\Theta^2) \\ &\quad + 4\nu^2\phi_\xi^2(\phi_\xi - \nu\phi_\xi^3 + \nu(1 - \phi_\xi)h_\Theta^2)^2. \end{aligned}$$

We remark that the restriction of $|g(\xi\Delta x)|$ in the case where $\phi_\xi = 1$, yields

$$|g(\xi\Delta x)|_{\phi_\xi=1}^2 = 1 - 4\nu(1 - \nu) + 4\nu^2(1 - \nu)^2 = (1 - 2\nu(1 - \nu))^2.$$

We see that if $\nu \geq 1$ then $|g(\xi\Delta x)|_{\phi_\xi=1}^2 \geq 1$, therefore it is necessary to consider $\nu \leq 1$. So we now consider $\nu \leq 1$. Since for all $\xi \in \mathbb{R}$, $0 \leq \phi_\xi \leq 1$ we have

$$0 \leq (1 - \phi_\xi) \leq (1 - \nu\phi_\xi^2) \quad \text{and} \quad (1 - \nu\phi_\xi^2)^2 \leq (1 - \nu\phi_\xi^2). \quad (3.8)$$

Consequently,

$$\begin{aligned} (\phi_\xi - \nu\phi_\xi^3 + \nu(1 - \phi_\xi)h_\Theta^2)^2 &= (\phi_\xi(1 - \nu\phi_\xi^2) + \nu(1 - \phi_\xi)h_\Theta^2)^2, \\ &= (1 - \nu\phi_\xi^2)^2\phi_\xi^2 + 2\nu\phi_\xi(1 - \nu\phi_\xi^2)(1 - \phi_\xi)h_\Theta^2 \\ &\quad + \nu^2(1 - \phi_\xi)^2h_\Theta^4, \\ &\leq (1 - \nu\phi_\xi^2)(\phi_\xi^2 + 2\nu\phi_\xi(1 - \phi_\xi)h_\Theta^2 + \nu^2(1 - \nu\phi_\xi^2)h_\Theta^4), \\ &\leq (1 - \nu\phi_\xi^2)(\phi_\xi^2 + 2\nu\phi_\xi(1 - \phi_\xi)h_\Theta^2 + \nu(1 - \nu\phi_\xi^2)h_\Theta^4), \end{aligned}$$

where we have used (3.8) and the fact that $\nu^2 \leq \nu$. We eventually obtain

$$\begin{aligned} |g(\xi\Delta x)|^2 &\leq 1 - 4\nu\phi_\xi^2(1 - \nu\phi_\xi^2)(1 + 2\nu^2\phi_\xi(1 - \phi_\xi)h_\Theta^2 - \nu\phi_\xi^2 - \nu^2(1 - \nu\phi_\xi^2)h_\Theta^4), \\ &\leq 1 - 2\nu^3\phi_\xi^2\phi_{2\xi}(1 - \nu\phi_\xi^2)h_\Theta^2 - 4\nu\phi_\xi^2(1 - \nu\phi_\xi^2)^2(1 - \nu^2h_\Theta^4), \end{aligned}$$

which is upper bounded by 1 because of the CFL condition (3.7) and the definition of h_Θ . \blacksquare

In this proof, the sequences $(\Theta_i^n)_{i \in \mathbb{Z}}$ and $(\Theta_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}}$ are taken to be constant which is convenient for the Fourier analysis. However, in this very particular case, the scheme does not enjoy the extra

dissipation in the inequality (3.5): the second sum vanishes. We believe that taking $(\Theta_i^n)_{i \in \mathbb{Z}}$ being non constant and ensuring (3.5) improves the stability.

4. Global entropy inequality in the general case

We establish the global entropy inequality (1.4) for the second-order in time and space scheme (2.13) in the general case of the system (1.1) with a given pair of entropy, entropy flux (η, G) defined in Ω . For the forthcoming developments, it is convenient to condense the scheme (2.13) in the form

$$\begin{aligned} w_i^{n+\frac{1}{2}} &= w_i^n + \frac{\Delta t}{\Delta x} \mathcal{R}_i^n, \\ w_i^{n+1} &= \frac{w_i^n}{2} + \frac{1}{2} \left(w_i^{n+\frac{1}{2}} + \frac{\Delta t}{\Delta x} \mathcal{R}_i^{n+\frac{1}{2}} \right), \end{aligned} \quad (4.1)$$

where

$$\mathcal{R}_i^m = -\frac{1}{2} (f(w_{i+1}^m) - f(w_{i-1}^m)) + \frac{\lambda}{2} (\delta_{i+\frac{1}{2}}^m - \delta_{i-\frac{1}{2}}^m) - \frac{1}{2} (\alpha_{i+1}^m - \alpha_{i-1}^m), \quad (4.2)$$

for all $m \in \{n, n + \frac{1}{2}\}$. Note that in the general case Ω is not necessarily equal to \mathbb{R}^d . Typically, for the Euler equations we have $\Omega = \{w = (\rho, \rho u, \rho E) \in \mathbb{R}^3 : \rho > 0, \rho E - \rho u^2/2 > 0\}$. Since our proof is based on Taylor expansion of the entropy of the updated state $\eta(w_i^{n+\frac{1}{2}})$, we have, at the very first, to make sure it belongs to Ω . For second-order accurate in space scheme, it is a challenging issue to establish it without using limitation techniques. We thus use a very simple topological argument which is also quite restrictive from the CFL point of view. It is summarized in the following lemma.

Lemma 4.1 (Scheme robustness). *Let Ω be a strict non empty convex open subset of \mathbb{R}^d . Let $(w_i^n)_{i \in \mathbb{Z}} \subset K^n$, where K^n is a compact set of Ω and assume the sequence of matrices $(\Theta_i^n)_{i \in \mathbb{Z}}$, $(\Theta_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}}$ be bounded. There exists two constants $c^n > 0$ and $c^{n+\frac{1}{2}} > 0$ such that if $0 < \frac{\Delta t}{\Delta x} \leq \min(c^n, c^{n+\frac{1}{2}})$ then the sequences $(w_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}}$, $(w_i^{n+1})_{i \in \mathbb{Z}}$ given by the scheme (4.1), or equivalently (2.13), are contained in Ω .*

Proof. We recall that the signed distance function to the boundary of Ω associated with the Euclidean norm $\|\cdot\|_2$ on \mathbb{R}^d is defined for all $w \in \mathbb{R}^d$ by:

$$\text{dist}(w, \partial\Omega) = \begin{cases} \inf_{y \in \partial\Omega} \|w - y\|_2, & \text{if } w \in \bar{\Omega}, \\ - \inf_{y \in \partial\Omega} \|w - y\|_2 & \text{if } w \in \mathbb{R}^d \setminus \bar{\Omega}, \end{cases}$$

It is a 1-Lipschitz continuous function on \mathbb{R}^d . Therefore using the scheme (4.1), we have for all $i \in \mathbb{Z}$,

$$\left| \text{dist}\left(w_i^{n+\frac{1}{2}}, \partial\Omega\right) - \text{dist}\left(w_i^n, \partial\Omega\right) \right| \leq \|w_i^{n+\frac{1}{2}} - w_i^n\|_2 = \frac{\Delta t}{\Delta x} \|\mathcal{R}_i^n\|_2. \quad (4.3)$$

We then obtain

$$\text{dist}\left(w_i^{n+\frac{1}{2}}, \partial\Omega\right) \geq \text{dist}\left(w_i^n, \partial\Omega\right) - \frac{\Delta t}{\Delta x} \|\mathcal{R}_i^n\|_2.$$

Since $(w_i^n)_{i \in \mathbb{Z}} \subset K^n$ and the signed distance function is continuous on the compact set K^n . One has $\text{dist}(w_i^n, \partial\Omega) \geq \min_{w \in K^n} \text{dist}(w, \partial\Omega) > 0$ where the minimum is positive because K^n is a compact set into an open set. Using the continuity of the physical flux f on the compact set K^n and the fact that the sequence $(\Theta_i^n)_{i \in \mathbb{Z}}$ is bounded, there exists a constant R^n such that for all $i \in \mathbb{Z}$, $\|\mathcal{R}_i^n\|_2 \leq R^n$. We thus

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have for all $i \in \mathbb{Z}$,

$$\text{dist} \left(w_i^{n+\frac{1}{2}}, \partial\Omega \right) \geq \min_{w \in K^n} \text{dist} (w, \partial\Omega) - \frac{\Delta t}{\Delta x} R^n.$$

If $0 < \frac{\Delta t}{\Delta x} \leq \frac{\min_{w \in K^n} \text{dist}(w, \partial\Omega)}{2R^n}$ then for all $i \in \mathbb{Z}$, $\text{dist} \left(w_i^{n+\frac{1}{2}}, \partial\Omega \right) \geq \frac{1}{2} \min_{w \in K^n} \text{dist} (w, \partial\Omega) > 0$ which exactly means that $w_i^{n+\frac{1}{2}} \in \Omega$. We can refine the argument: using again the scheme (4.1) and the fact that $(w_i^n)_{i \in \mathbb{Z}}$ belongs to a compact set K^n , we deduce that the sequence $(w_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}}$ is also uniformly bounded. There exists a constant M^n such that for all $i \in \mathbb{Z}$, $\|w_i^{n+\frac{1}{2}}\|_2 \leq M^n$. Therefore if $0 < \frac{\Delta t}{\Delta x} \leq \frac{\min_{w \in K^n} \text{dist}(w, \partial\Omega)}{2R^n}$ the whole sequence $(w_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}}$ belongs to the subset of Ω ,

$$\left\{ w \in \mathbb{R}^d : \text{dist} (w, \partial\Omega) \geq \frac{1}{2} \min_{w \in K^n} \text{dist} (w, \partial\Omega), \|w\|_2 \leq M^n \right\}$$

which is compact since it is bounded and closed by continuity of the signed distance function. Repeting the above arguments therefore yields the same conclusion for $(w_i^{n+1})_{i \in \mathbb{Z}}$. \blacksquare

Our main result is the following.

Theorem 4.2 (Global entropy inequality). *Let Ω be a non empty convex open subset of \mathbb{R}^d . Consider $(\eta, G) \in C^2(\Omega, \mathbb{R}) \times C^1(\Omega, \mathbb{R})$ a pair of strictly convex entropy-entropy-flux which satisfies (1.2). Let $(w_i^n)_{i \in \mathbb{Z}}$ be a non zero sequence in $h^2(\mathbb{Z})$ and moreover such that $\sum_{i \in \mathbb{Z}} \eta(w_i^n) \Delta x$ is finite. Let $(\Theta_i^n)_{i \in \mathbb{Z}}$, $(\Theta_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}}$ two sequences of bounded matrices such that*

$$\mathcal{S}^m := \int_0^1 \sum_{i \in \mathbb{Z}} N_i^m(s) P_i^m(s) D_i^m \cdot D_i^m ds > 0, \quad \forall m \in \left\{ n, n + \frac{1}{2} \right\}, \quad (4.4)$$

where the block matrices $(D_i^m, N_i^m, P_i^m) \in \mathbb{R}^{2d} \times (\mathcal{M}_{2d}(\mathbb{R}))^2$ are respectively defined by

$$\begin{aligned} D_i^m &= \begin{pmatrix} \delta_{i-\frac{1}{2}}^m \\ \delta_{i+\frac{1}{2}}^m \end{pmatrix}, \\ N_i^m(s) &= \begin{pmatrix} \nabla^2 \eta \left(w_i^m - s \delta_{i-\frac{1}{2}}^m \right) & 0 \\ 0 & \nabla^2 \eta \left(w_i^m + s \delta_{i+\frac{1}{2}}^m \right) \end{pmatrix}, \\ P_i^m(s) &= \begin{pmatrix} (1-2s)I + \Theta_i^m & -\Theta_i^m \\ \Theta_i^m - I & 2(1-s)I - \Theta_i^m \end{pmatrix}, \end{aligned} \quad (4.5)$$

for all $s \in [0, 1]$. Also assume there exists a compact set $K^n \subset \Omega$ such that $(w_i^n)_{i \in \mathbb{Z}} \subset K^n$. Let the numerical diffusion λ be such that

$$\lambda > \max_{m \in \{n, n+\frac{1}{2}\}} \lambda^m, \quad (4.6)$$

where

$$\lambda^m = \frac{2 \max \left(0, \sum_{i \in \mathbb{Z}} \int_0^1 s \left(\int_0^1 N_i^m(us) du \right) F_i^m(s) D_i^m \cdot D_i^m ds \right)}{\sum_{i \in \mathbb{Z}} \int_0^1 N_i^m(s) P_i^m(s) D_i^m \cdot D_i^m ds} \geq 0, \quad (4.7)$$

$$F_i^m(s) = \begin{pmatrix} -\nabla f \left(w_i^m - s \delta_{i-\frac{1}{2}}^m \right) & 0 \\ 0 & \nabla f \left(w_i^m + s \delta_{i+\frac{1}{2}}^m \right) \end{pmatrix}, \quad \forall s \in [0, 1],$$

and $\frac{\Delta t}{\Delta x}$ be such that

$$0 < \frac{\Delta t}{\Delta x} \leq \frac{- \sum_{\substack{m \in \{n, n+\frac{1}{2}\}, \\ i \in \mathbb{Z}}} \nabla \eta(w_i^m) \cdot \mathcal{R}_i^m}{\int_0^1 (1-s) \sum_{\substack{m \in \{n, n+\frac{1}{2}\}, \\ i \in \mathbb{Z}}} \nabla^2 \eta \left(w_i^m + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^m \right) \mathcal{R}_i^m \cdot \mathcal{R}_i^m ds}. \quad (4.8)$$

If $\Omega = \mathbb{R}^d$, then one has the global entropy inequality,

$$\sum_{i \in \mathbb{Z}} \eta(w_i^{n+1}) \Delta x \leq \sum_{i \in \mathbb{Z}} \eta(w_i^n) \Delta x. \quad (4.9)$$

If $\Omega \neq \mathbb{R}^d$ then the global entropy inequality (4.9) still holds if moreover $0 < \frac{\Delta t}{\Delta x} \leq \min(c^n, c^{n+\frac{1}{2}})$ where c^n and $c^{n+\frac{1}{2}}$ are two positive non explicit constants given by the lemma 4.1.

We first establish the existence of the sequences $(\Theta_i^m)_{i \in \mathbb{Z}}$ verifying the condition (4.4).

Lemma 4.3 (Existence of dissipative corrections). *Let $(w_i^n)_{i \in \mathbb{Z}} \subset \Omega$, $(w_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}} \subset \Omega$ be non zero sequences in $h^2(\mathbb{Z})$ and let $\eta \in C^2(\Omega, \mathbb{R})$. If, for all $m \in \{n, n + \frac{1}{2}\}$, $(\Theta_i^m)_{i \in \mathbb{Z}}$ verifies*

$$\Theta_i^m = -\theta \operatorname{diag}_{1 \leq j \leq d} \left(\operatorname{sign} \left((\nabla \eta(w_{i+1}^m) - \nabla \eta(w_{i-1}^m))_j (\delta_{i+\frac{1}{2}}^m - \delta_{i-\frac{1}{2}}^m)_j \right) \right), \quad (4.10)$$

with

$$\theta > \frac{- \min \left(0, \int_0^1 \sum_{i \in \mathbb{Z}} Q_i^m(s) D_i^m \cdot D_i^m ds \right)}{\sum_{\substack{i \in \mathbb{Z} \\ j \in \{1, \dots, d\}}} |(\nabla \eta(w_{i+1}^m) - \nabla \eta(w_{i-1}^m))_j (\delta_{i+\frac{1}{2}}^m - \delta_{i-\frac{1}{2}}^m)_j|}, \quad (4.11)$$

$$Q_i^m(s) = \begin{pmatrix} (1-2s) \nabla^2 \eta(w_i^m - s \delta_{i-\frac{1}{2}}^m) & -\nabla^2 \eta(w_i^m + s \delta_{i+\frac{1}{2}}^m) \\ 0 & 2(1-s) \nabla^2 \eta(w_i^m + s \delta_{i+\frac{1}{2}}^m) \end{pmatrix},$$

for all $(s, m) \in [0, 1] \times \{n, n + \frac{1}{2}\}$, then the dissipative inequality (4.4) holds.

Proof. Since the matrix $(\Theta_i^m)_{i \in \mathbb{Z}}$ defined by (4.10) are symmetric (because they are diagonal) we have $\Theta_i^m a \cdot b = a \cdot \Theta_i^m b$, for all vectors $(a, b) \in (\mathbb{R}^d)^2$. As a consequence, from the definition of the matrices P_i^m , N_i^m given by (4.5) and the definition of \mathcal{S}^m given in (4.4), we have

$$\mathcal{S}^m = \int_0^1 \sum_{i \in \mathbb{Z}} Q_i^m(s) D_i^m \cdot D_i^m ds - \sum_{i \in \mathbb{Z}} \Theta_i^m (\nabla \eta(w_{i+1}^m) - \nabla \eta(w_{i-1}^m)) \cdot (\delta_{i+\frac{1}{2}}^m - \delta_{i-\frac{1}{2}}^m).$$

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Using the $(\Theta_i^m)_{i \in \mathbb{Z}}$ formula (4.10), we eventually obtain

$$\mathcal{S}^m = \int_0^1 \sum_{i \in \mathbb{Z}} Q_i^m(s) D_i^m \cdot D_i^m ds + \theta \sum_{\substack{i \in \mathbb{Z} \\ j \in \{1, \dots, d\}}} |((\nabla \eta(w_{i+1}^m) - (\nabla \eta(w_{i-1}^m)))_j (\delta_{i+\frac{1}{2}}^m - \delta_{i-\frac{1}{2}}^m)_j)|,$$

which is positive with θ verifying the inequality (4.11). ■

4.1. Reformulation of the global dissipation

We have the following.

Lemma 4.4 (Reformulation of the global entropy flux). *Let the sequences $(w_i^n)_{i \in \mathbb{Z}} \subset \Omega$, $(w_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}} \subset \Omega$ be in $h^2(\mathbb{Z})$ and let $(\eta, G) \in C^2(\Omega, \mathbb{R}) \times C^1(\Omega, \mathbb{R})$ a pair of strictly convex entropy-entropy-flux which satisfies (1.2). Then*

$$-\frac{1}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^m) \cdot (f(w_{i+1}^m) - f(w_{i-1}^m)) = \sum_{i \in \mathbb{Z}} \int_0^1 \frac{s}{2} \left(\int_0^1 N_i^m(us) du \right) F_i^m(s) D_i^m \cdot D_i^m ds, \quad \forall m \in \left\{ n, n + \frac{1}{2} \right\}, \quad (4.12)$$

where D_i^m , $N_i^m(s)$, $F_i^m(s)$ are defined in (4.5)-(4.7) respectively.

Proof. Since $\eta \in C^2(\Omega; \mathbb{R})$, $G \in C^1(\Omega; \mathbb{R})$, f is assumed to be smooth and $(w_i^m)_{i \in \mathbb{Z}}$ belongs to Ω for $m = n, n + \frac{1}{2}$, a Taylor expansion gives,

$$\begin{aligned} G(w_{i+1}^m) &= G(w_i^m) + \int_0^1 \nabla G(w_i^m + s\delta_{i+\frac{1}{2}}^m) \cdot \delta_{i+\frac{1}{2}}^m ds, \\ \nabla \eta(w_i^m + s\delta_{i+\frac{1}{2}}^m) &= \nabla \eta(w_i^m) + s \int_0^1 \nabla^2 \eta(w_i^m + us\delta_{i+\frac{1}{2}}^m) \delta_{i+\frac{1}{2}}^m du, \\ f(w_{i+1}^m) &= f(w_i^m) + \int_0^1 \nabla f(w_i^m + s\delta_{i+\frac{1}{2}}^m) \delta_{i+\frac{1}{2}}^m ds, \end{aligned}$$

from which we deduce

$$\begin{aligned} \nabla \eta(w_i^m) \cdot (f(w_{i+1}^m) - f(w_i^m)) &= \int_0^1 \nabla \eta(w_i^m) \cdot \nabla f(w_i^m + s\delta_{i+\frac{1}{2}}^m) \delta_{i+\frac{1}{2}}^m ds, \\ &= \int_0^1 \nabla G(w_i^m + s\delta_{i+\frac{1}{2}}^m) \cdot \delta_{i+\frac{1}{2}}^m ds \\ &\quad - \int_0^1 \int_0^1 s \nabla^2 \eta(w_i^m + us\delta_{i+\frac{1}{2}}^m) \nabla f(w_i^m + s\delta_{i+\frac{1}{2}}^m) \delta_{i+\frac{1}{2}}^m \cdot \delta_{i+\frac{1}{2}}^m duds, \\ &= G(w_{i+1}^m) - G(w_i^m) \\ &\quad - \int_0^1 \int_0^1 s \nabla^2 \eta(w_i^m + us\delta_{i+\frac{1}{2}}^m) \nabla f(w_i^m + s\delta_{i+\frac{1}{2}}^m) \delta_{i+\frac{1}{2}}^m \cdot \delta_{i+\frac{1}{2}}^m duds. \end{aligned} \quad (4.13)$$

In the same way, we have

$$\begin{aligned} \nabla \eta(w_i^m) \cdot (f(w_i^m) - f(w_{i-1}^m)) &= G(w_i^m) - G(w_{i-1}^m) \\ &\quad + \int_0^1 \int_0^1 s \nabla^2 \eta(w_i^m - us\delta_{i-\frac{1}{2}}^m) \nabla f(w_i^m - s\delta_{i-\frac{1}{2}}^m) \delta_{i-\frac{1}{2}}^m \cdot \delta_{i-\frac{1}{2}}^m duds, \end{aligned} \quad (4.14)$$

Using the equations (4.13), (4.14) and the definitions (4.5), (4.7) we deduce the expected result. \blacksquare

Lemma 4.5 (Reformulation of the global dissipation). *Let the sequences $(w_i^m)_{i \in \mathbb{Z}} \subset \Omega$, $(w_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}} \subset \Omega$ be in $h^2(\mathbb{Z})$ and let $\eta \in C^2(\Omega, \mathbb{R})$. Then*

$$\begin{aligned} \frac{1}{2} \sum_{i \in \mathbb{Z}} \left(\lambda \nabla \eta(w_i^m) \cdot (\delta_{i+\frac{1}{2}}^m - \delta_{i-\frac{1}{2}}^m) - \nabla \eta(w_i^m) \cdot (\alpha_{i+1}^m - \alpha_{i-1}^m) \right) = \\ - \frac{\lambda}{4} \sum_{i \in \mathbb{Z}} \int_0^1 N_i^m(s) P_i^m(s) D_i^m \cdot D_i^m ds, \quad \forall m \in \left\{ n, n + \frac{1}{2} \right\}, \end{aligned} \quad (4.15)$$

where the block matrices D_i^m , N_i^m , P_i^m are defined in the equation (4.5).

Proof. Since for $m \in \{n, n + \frac{1}{2}\}$, $(w_i^m)_{i \in \mathbb{Z}}$ are included in Ω , using a Taylor expansion, we get

$$\begin{aligned} \eta(w_{i+1}^m) &= \eta(w_i^m) + \nabla \eta(w_i^m) \cdot \delta_{i+\frac{1}{2}}^m + \int_0^1 (1-s) \nabla^2 \eta \left(w_i^m + s \delta_{i+\frac{1}{2}}^m \right) \delta_{i+\frac{1}{2}}^m \cdot \delta_{i+\frac{1}{2}}^m ds, \\ \eta(w_{i-1}^m) &= \eta(w_i^m) - \nabla \eta(w_i^m) \cdot \delta_{i-\frac{1}{2}}^m + \int_0^1 (1-s) \nabla^2 \eta \left(w_i^m - s \delta_{i-\frac{1}{2}}^m \right) \delta_{i-\frac{1}{2}}^m \cdot \delta_{i-\frac{1}{2}}^m ds, \end{aligned} \quad (4.16)$$

which yields

$$\begin{aligned} \frac{\lambda}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^m) \cdot (\delta_{i+\frac{1}{2}}^m - \delta_{i-\frac{1}{2}}^m) &= \frac{\lambda}{2} \sum_{i \in \mathbb{Z}} (\eta(w_{i+1}^m) - 2\eta(w_i^m) + \eta(w_{i-1}^m)) \\ &- \frac{\lambda}{2} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^m + s \delta_{i+\frac{1}{2}}^m \right) \delta_{i+\frac{1}{2}}^m \cdot \delta_{i+\frac{1}{2}}^m ds \\ &- \frac{\lambda}{2} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} \nabla^2 \eta \left(w_i^m - s \delta_{i-\frac{1}{2}}^m \right) \delta_{i-\frac{1}{2}}^m \cdot \delta_{i-\frac{1}{2}}^m ds, \\ &= - \frac{\lambda}{2} \int_0^1 (1-s) \sum_{i \in \mathbb{Z}} N_i^m(s) D_i^m \cdot D_i^m ds. \end{aligned} \quad (4.17)$$

For the second-order correction term, using a translation of indices, we have

$$\begin{aligned} - \frac{1}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^m) \cdot (\alpha_{i+1}^m - \alpha_{i-1}^m) &= \frac{1}{2} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^m) - \nabla \eta(w_{i-1}^m)) \cdot \alpha_i^m, \\ &= \frac{\lambda}{4} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^m) - \nabla \eta(w_{i-1}^m)) \cdot \Theta_i^m \delta_{i+\frac{1}{2}}^m \\ &+ \frac{\lambda}{4} \sum_{i \in \mathbb{Z}} (\nabla \eta(w_{i+1}^m) - \nabla \eta(w_{i-1}^m)) \cdot (I - \Theta_i^m) \delta_{i-\frac{1}{2}}^m. \end{aligned}$$

But, using once again a Taylor expansion, we have

$$\begin{aligned} \nabla \eta(w_{i+1}^m) &= \nabla \eta(w_i^m) + \int_0^1 \nabla^2 \eta \left(w_i^m + s \delta_{i+\frac{1}{2}}^m \right) \delta_{i+\frac{1}{2}}^m ds, \\ \nabla \eta(w_{i-1}^m) &= \nabla \eta(w_i^m) - \int_0^1 \nabla^2 \eta \left(w_i^m - s \delta_{i-\frac{1}{2}}^m \right) \delta_{i-\frac{1}{2}}^m ds, \end{aligned}$$

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from which we infer

$$\begin{aligned}
 -\frac{1}{2} \sum_{i \in \mathbb{Z}} \nabla \eta(w_i^m) \cdot (\alpha_{i+1}^m - \alpha_{i-1}^m) &= \frac{\lambda}{4} \sum_{i \in \mathbb{Z}} \int_0^1 \nabla^2 \eta \left(w_i^m + s \delta_{i+\frac{1}{2}}^m \right) \Theta_i^m \delta_{i+\frac{1}{2}}^m \cdot \delta_{i+\frac{1}{2}}^m ds, \\
 + \frac{\lambda}{4} \sum_{i \in \mathbb{Z}} \int_0^1 \nabla^2 \eta \left(w_i^m + s \delta_{i+\frac{1}{2}}^m \right) (I - \Theta_i^m) \delta_{i-\frac{1}{2}}^m \cdot \delta_{i+\frac{1}{2}}^m ds \\
 + \frac{\lambda}{4} \sum_{i \in \mathbb{Z}} \int_0^1 \nabla^2 \eta \left(w_i^m - s \delta_{i-\frac{1}{2}}^m \right) \Theta_i^m \delta_{i+\frac{1}{2}}^m \cdot \delta_{i-\frac{1}{2}}^m ds \\
 + \frac{\lambda}{4} \sum_{i \in \mathbb{Z}} \int_0^1 \nabla^2 \eta \left(w_i^m - s \delta_{i-\frac{1}{2}}^m \right) (I - \Theta_i^m) \delta_{i-\frac{1}{2}}^m \cdot \delta_{i-\frac{1}{2}}^m ds.
 \end{aligned} \tag{4.18}$$

Rewriting (4.17) and (4.18) with the definitions (4.5) we deduce the result. \blacksquare

4.2. Proof of the main result

Let $(\eta, G) \in C^2(\Omega, \mathbb{R}) \times C^1(\Omega, \mathbb{R})$ a pair of strictly convex entropy, entropy-flux which satisfies (1.2). Consider the sequence $(w_i^n)_{i \in \mathbb{Z}}$ verifying the assumptions of the theorem 4.2. Consider the CFL condition $\frac{\Delta t}{\Delta x}$ given in the theorem 4.2. Therefore the sequences $(w_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}}$, $(w_i^{n+1})_{i \in \mathbb{Z}}$ are contained in Ω . Using the convexity of η , we then have the following

$$\begin{aligned}
 \eta(w_i^{n+\frac{1}{2}}) &= \eta \left(w_i^n + \frac{\Delta t}{\Delta x} \mathcal{R}_i^n \right), \\
 \eta(w_i^{n+1}) &\leq \frac{1}{2} \eta(w_i^n) + \frac{1}{2} \eta \left(w_i^{n+\frac{1}{2}} + \frac{\Delta t}{\Delta x} \mathcal{R}_i^{n+\frac{1}{2}} \right).
 \end{aligned}$$

Since $\eta \in C^2(\Omega; \mathbb{R})$, using a Taylor expansion, in the above equations, we deduce

$$\begin{aligned}
 \eta(w_i^{n+\frac{1}{2}}) &= \eta(w_i^n) + \frac{\Delta t}{\Delta x} \nabla \eta(w_i^n) \cdot \mathcal{R}_i^n \\
 &+ \left(\frac{\Delta t}{\Delta x} \right)^2 \int_0^1 (1-s) \nabla^2 \eta \left(w_i^n + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^n \right) \mathcal{R}_i^n \cdot \mathcal{R}_i^n ds, \\
 \eta(w_i^{n+1}) &\leq \frac{1}{2} \eta(w_i^n) + \frac{1}{2} \eta(w_i^{n+\frac{1}{2}}) + \frac{\Delta t}{2\Delta x} \nabla \eta(w_i^{n+\frac{1}{2}}) \cdot \mathcal{R}_i^{n+\frac{1}{2}} \\
 &+ \frac{1}{2} \left(\frac{\Delta t}{\Delta x} \right)^2 \int_0^1 (1-s) \nabla^2 \eta \left(w_i^{n+\frac{1}{2}} + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^{n+\frac{1}{2}} \right) \mathcal{R}_i^{n+\frac{1}{2}} \cdot \mathcal{R}_i^{n+\frac{1}{2}} ds.
 \end{aligned}$$

It implies

$$\begin{aligned}
 \sum_{i \in \mathbb{Z}} \eta(w_i^{n+1}) \Delta x &\leq \sum_{i \in \mathbb{Z}} \eta(w_i^n) \Delta x + \frac{1}{2} \frac{\Delta t}{\Delta x} \sum_{\substack{m \in \{n, n+\frac{1}{2}\}, \\ i \in \mathbb{Z}}} \nabla \eta(w_i^m) \cdot \mathcal{R}_i^m \Delta x \\
 &+ \frac{1}{2} \left(\frac{\Delta t}{\Delta x} \right)^2 \int_0^1 (1-s) \sum_{\substack{m \in \{n, n+\frac{1}{2}\}, \\ i \in \mathbb{Z}}} \nabla^2 \eta \left(w_i^m + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^m \right) \mathcal{R}_i^m \cdot \mathcal{R}_i^m ds \Delta x.
 \end{aligned}$$

To establish the global entropy inequality (4.9), it is sufficient to prove the following inequality

$$\sum_{\substack{m \in \{n, n + \frac{1}{2}\}, \\ i \in \mathbb{Z}}} \nabla \eta(w_i^m) \cdot \mathcal{R}_i^m + \frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{\substack{m \in \{n, n + \frac{1}{2}\}, \\ i \in \mathbb{Z}}} \nabla^2 \eta \left(w_i^m + s \frac{\Delta t}{\Delta x} \mathcal{R}_i^m \right) \mathcal{R}_i^m \cdot \mathcal{R}_i^m \, ds \leq 0. \quad (4.19)$$

Thanks to the lemmas 4.4 and 4.5, we deduce

$$\sum_{i \in \mathbb{Z}} \nabla \eta(w_i^m) \cdot \mathcal{R}_i^m = \sum_{i \in \mathbb{Z}} \int_0^1 \left(\frac{s}{2} \left(\int_0^1 N_i^m(us) \, du \right) F_i^m(s) - \frac{\lambda}{4} N_i^m(s) P_i^m(s) \right) D_i^m \cdot D_i^m \, ds, \quad (4.20)$$

for all $m \in \{n, n + \frac{1}{2}\}$. But, as the sequences of matrices $(\Theta_i^n)_{i \in \mathbb{Z}}$ and $(\Theta_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}}$ are selected in order to satisfy the inequality (4.4), and λ is such that the inequality (4.6) is verified, we have

$$\sum_{\substack{m \in \{n, n + \frac{1}{2}\}, \\ i \in \mathbb{Z}}} \nabla \eta(w_i^m) \cdot \mathcal{R}_i^m < 0.$$

We may remark at this point that the sequences $(w_i^n)_{i \in \mathbb{Z}}$, $(w_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}}$ are in $h^2(\mathbb{Z})$, they are in particular bounded. The Hessian of the entropy $\nabla^2 \eta$ is positive (because η is strictly convex) and continuous on a compact set that depends on the two sequences $(w_i^n)_{i \in \mathbb{Z}}$, $(w_i^{n+\frac{1}{2}})_{i \in \mathbb{Z}}$. So for λ^n defined in (4.7) the numerator is finite because $(w_i^n)_{i \in \mathbb{Z}}$ is in $h^2(\mathbb{Z})$. The denominator is non zero because $(w_i^n)_{i \in \mathbb{Z}}$ is a non zero sequence, so λ^n is finite. We can show that it is also the case for $\lambda^{n+\frac{1}{2}}$. Therefore $\max\{\lambda^n, \lambda^{n+\frac{1}{2}}\}$ is also finite. The ratio $\frac{\Delta t}{\Delta x}$ verifies the inequality (4.8). In this inequality, the right hand side is positive because the numerator is positive by construction of the sequences $(\Theta_i^m)_{i \in \mathbb{Z}}$ and the denominator is finite because the sequences $(w_i^m)_{i \in \mathbb{Z}}$ are bounded and in $h^2(\mathbb{Z})$ so the sum is convergent. We thus deduce that (4.19) is satisfied. It ends the proof of the theorem 4.2.

To conclude this section, we mention that a formal consistency analysis (which is laborious) for a smooth compactly supported solution of the CFL condition (4.8) yields formally $\frac{\Delta t}{\Delta x} \underset{\Delta x \rightarrow 0}{=} \mathcal{O}(1)$. It is nevertheless a non standard hyperbolic CFL condition and probably far from being optimal.

5. Numerical results

We give several numerical examples that illustrate the accuracy and the stability of the schemes (2.13). Let us now explain how the parameters of the scheme are chosen. Being given a strictly convex entropy η , at each iteration $m = n, n + \frac{1}{2}$, we design the matrix parameter $(\Theta_i^m)_{i \in \mathbb{Z}}$ that satisfies the inequality (4.4). Then, on the contrary to the theorem 4.2, we fix the values of the numerical viscosity coefficient λ and the time step Δt according to the standard hyperbolic CFL condition, $\lambda = \max_i |\mu(w_i^m)|$ and $\frac{\lambda \Delta t}{\Delta x} \leq \frac{1}{2}$. We verify systematically at the numerical level that these choices ensure the decrease of the mass of the entropy.

Equipped with this numerical parameters, we perform numerical simulations considering mainly the scalar Burgers equation and the Euler equations. For all these cases, we propose three different choices of the matrix parameter $\Theta_i^m \in \{\Theta_{a,i}^m, \Theta_{b,i}^m, \Theta_{c,i}^m, \Theta_{d,i}^m\}$ which are given here after. We systematically measure the error in L^1, L^2 and L^∞ norms between the numerical solutions and the exact solution. Plots of the obtained numerical solutions and the total entropy are also given.

5.1. Burgers equation

The Burgers equation consists in taking $\Omega = \mathbb{R}$ and the flux function is given by $f(w) = w^2/2$ for all $w \in \Omega$. We consider the entropy function $\eta(w) = w^2/2$. We shall present several tests with the

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following parameters:

$$\begin{aligned}
 \Theta_{a,i}^m &= -\theta_a \operatorname{sign} \left((\delta_{i+\frac{1}{2}}^m)^2 - (\delta_{i-\frac{1}{2}}^m)^2 \right), \\
 \Theta_{b,i}^m &= -\theta_b \tanh \left((\delta_{i+\frac{1}{2}}^m)^2 - (\delta_{i-\frac{1}{2}}^m)^2 \right), \\
 \Theta_{c,i}^m &= \frac{\left((\delta_{i-\frac{1}{2}}^m)^2 - (\delta_{i+\frac{1}{2}}^m)^2 \right) \left((\delta_{i-\frac{1}{2}}^m)^2 + (\delta_{i+\frac{1}{2}}^m)^2 \right)}{\left((\delta_{i-\frac{1}{2}}^m)^2 + (\delta_{i+\frac{1}{2}}^m)^2 \right)^2 + \varepsilon}, \\
 \Theta_{d,i}^m &= \frac{1}{2},
 \end{aligned} \tag{5.1}$$

for all $m \in \{n, n + \frac{1}{2}\}$, where θ_a satisfies (4.11), θ_b is taken equal to $\theta_a + 20$ and we fix $\varepsilon = 10^{-12}$. We verify that these choices of the matrix parameter $(\Theta_i^m)_{i \in \mathbb{Z}}$ always satisfy the inequality (4.4).

5.1.1. Smooth solution

We take a smooth initial data $w_0(x) = 0.25 + 0.5 \sin(\pi x)$ over a periodic domain $[-1, 1)$. With a final time small enough, here given by $t = 0.3$, the exact solution remains smooth so that the order of accuracy can be evaluated.

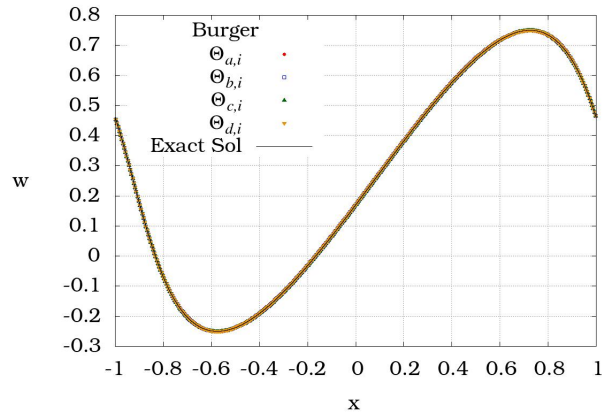


FIGURE 1. Second-order accurate approximation of the smooth Burgers with a mesh made of 400 cells.

Second-order scheme errors $\Theta_i^m = \Theta_{a,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	5.7E-04	-	6.8E-04	-	1.8E-03	-
200	1.4E-04	2.0	1.7E-04	2.0	4.5E-04	2.0
400	3.5E-05	2.0	4.1E-05	2.0	1.1E-04	2.0
800	8.8E-06	2.0	1.0E-05	2.0	2.8E-05	2.0
1600	2.2E-06	2.0	2.6E-06	2.0	6.9E-06	2.0
Second-order scheme errors $\Theta_i^m = \Theta_{b,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	5.7E-04	-	6.8E-04	-	1.8E-03	-
200	1.4E-04	2.0	1.7E-04	2.0	4.5E-04	2.0
400	3.5E-05	2.0	4.1E-05	2.0	1.1E-04	2.0
800	8.8E-06	2.0	1.0E-05	2.0	2.8E-05	2.0
1600	2.2E-06	2.0	2.6E-06	2.0	6.9E-06	2.0
Second-order scheme errors $\Theta_i^m = \Theta_{c,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	1.4E-03	-	1.6E-03	-	4.1E-03	-
200	2.4E-04	2.5	2.8E-04	2.6	7.6E-04	2.4
400	3.9E-05	2.7	4.3E-05	2.7	1.1E-04	2.8
800	8.7E-06	2.1	1.0E-05	2.1	2.7E-05	2.0
1600	2.2E-06	2.0	2.6E-06	2.0	6.8E-06	2.0
Second-order scheme errors $\Theta_i^m = \Theta_{d,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	4.4E-04	-	4.8E-04	-	1.1E-03	-
200	1.1E-04	2.0	1.1E-04	2.1	2.7E-04	2.1
400	2.6E-05	2.0	2.8E-05	2.0	6.6E-05	2.0
800	6.5E-06	2.0	6.8E-06	2.0	1.6E-05	2.0
1600	1.6E-06	2.0	1.7E-06	2.0	4.0E-06	2.0

TABLE 1. Errors and order evaluations for the second-order accurate scheme with the smooth Burgers solution for Θ_i^m defined in (5.1).

The numerical solutions are displayed in Fig. 1. We notice the very good behavior of the approximations. This remark is completed by Table 1 where the order of accuracy are presented for different strong norms. We observe asymptotically the second-order accuracy.

5.1.2. Discontinuous solution

We take a discontinuous initial data over the periodic domain $[-1,1)$ defined by $w_0(x) = \begin{cases} 1 & \text{if } -0.25 \leq x \leq 0.25, \\ 0 & \text{otherwise.} \end{cases}$ The exact solution is made of both rarefaction and shock waves. With a final time $t = 0.3$, the waves do not interact. The numerical simulations are presented in Fig. 2. With the specific choice $\Theta_{c,i}$, we observe the exceptional behavior of the approximate solution where no spurious oscillations appear. The Table 2 gives the measured orders of accuracy.

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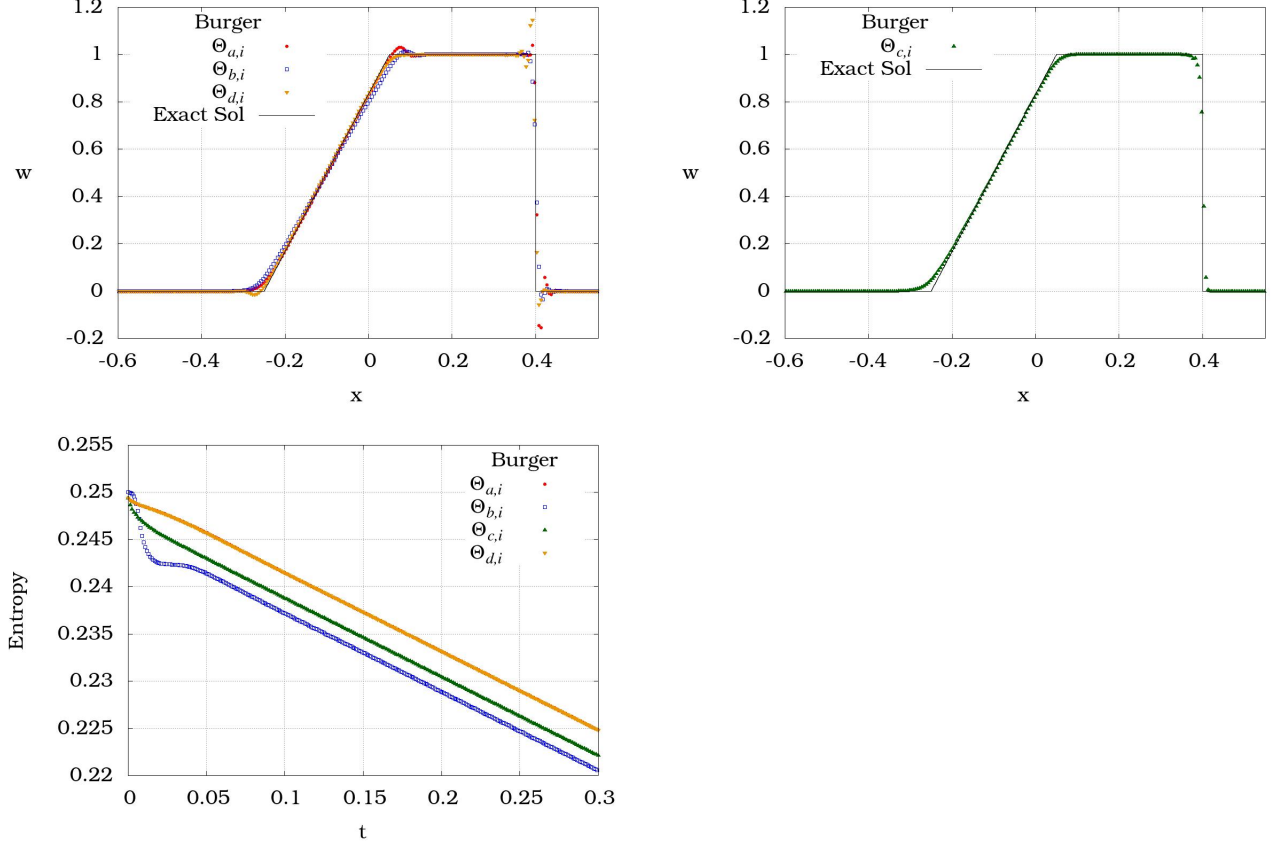


FIGURE 2. Second-order accurate approximation of the Burgers solution made of rarefaction and shock waves with a mesh made of 400 cells.

Second-order scheme errors $\Theta_i^m = \Theta_{a,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	3.4E-02	-	6.4E-02	-	3.3E-01	-
200	1.7E-02	1.0	4.3E-02	0.6	3.3E-01	0.0
400	8.5E-03	1.0	3.0E-02	0.5	3.2E-01	0.0
800	4.3E-03	1.0	2.1E-02	0.5	3.2E-01	0.0
1600	2.1E-03	1.0	1.5E-02	0.5	3.2E-01	0.0
Second-order scheme errors $\Theta_i^m = \Theta_{b,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	5.4E-02	-	9.0E-02	-	3.8E-01	-
200	2.7E-02	1.0	5.9E-02	0.6	3.8E-01	0.0
400	1.4E-02	1.0	4.0E-02	0.6	3.8E-01	0.0
800	7.0E-03	1.0	2.7E-02	0.6	3.8E-01	0.0
1600	3.5E-03	1.0	1.8E-02	0.5	3.8E-01	0.0

Second-order scheme errors $\Theta_i^m = \Theta_{c,\varepsilon,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-02	-	7.1E-02	-	3.6E-01	-
200	1.8E-02	0.9	4.9E-02	0.5	3.6E-01	0.0
400	9.2E-03	1.0	3.4E-02	0.5	3.6E-01	0.0
800	4.6E-03	1.0	2.3E-02	0.5	3.6E-01	0.0
1600	2.3E-03	1.0	1.6E-02	0.5	3.6E-01	0.0
Second-order scheme errors $\Theta_i^m = \Theta_{d,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	3.1E-02	-	5.8E-02	-	2.8E-01	-
200	1.4E-02	1.1	4.0E-02	0.6	2.8E-01	0.0
400	7.1E-03	1.0	2.8E-02	0.5	2.8E-01	0.0
800	3.5E-03	1.0	1.9E-02	0.5	2.8E-01	0.0
1600	1.7E-03	1.0	1.4E-02	0.5	2.8E-01	0.0

TABLE 2. Errors and order evaluations for the second-order accurate schemes with the Burgers solution made of rarefaction and shock waves, for Θ_i^m defined in (5.1).

5.2. Euler system

We consider the Euler system where $\Omega = \{(\rho, \rho u, \rho E) \in \mathbb{R}^3 : \rho > 0, \rho E - \rho u^2/2 > 0\}$, for a perfect diatomic gas where the unknown vector is $w = (\rho, \rho u, \rho E)^T$ and the flux function is $f(w) = (\rho u, \rho u^2 + p, \rho E u + p u)^T$, with $p = (\gamma - 1)(\rho E - \frac{\rho u^2}{2})$. We fix $\gamma = 1.4$ and we consider the entropy $\eta(w) = -\rho \ln\left(\frac{p}{\rho^\gamma}\right)$. For the Euler problem, we use the following matrix parameter

$$\begin{aligned}
\Theta_{a,i}^n &= -\theta_a \operatorname{diag}_{1 \leq j \leq d} \left(\operatorname{sign} \left((\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n))_j (\delta_{i+\frac{1}{2}}^m - \delta_{i-\frac{1}{2}}^m)_j \right) \right), \\
\Theta_{b,i}^n &= -\theta_b \operatorname{diag}_{1 \leq j \leq d} \left(\tanh \left((\nabla \eta(w_{i+1}^n) - \nabla \eta(w_{i-1}^n))_j (\delta_{i+\frac{1}{2}}^m - \delta_{i-\frac{1}{2}}^m)_j \right) \right), \\
\Theta_{c,\varepsilon,i}^m &= \operatorname{diag}_{1 \leq j \leq d} \left(\frac{\left((\delta_{i-\frac{1}{2}}^m)_j^2 - (\delta_{i+\frac{1}{2}}^m)_j^2 \right) \left((\delta_{i-\frac{1}{2}}^m)_j^2 + (\delta_{i+\frac{1}{2}}^m)_j^2 \right)}{\left((\delta_{i-\frac{1}{2}}^m)_j^2 + (\delta_{i+\frac{1}{2}}^m)_j^2 \right)^2 + \varepsilon} \right), \tag{5.2}
\end{aligned}$$

for all $m \in \{n, n + \frac{1}{2}\}$, where θ_a is defined in (4.11), θ_b is taken equal to $\theta_a + 50$ and ε is equal to 10^{-12} . We perform two numerical simulations : one with a smooth solution and one with a shock tube solution.

5.2.1. Smooth solution

The initial data is given as follows over the periodic domain $[-1, 1)$: $\rho_0(x) = 1 + 0.5 \sin^2(\pi x)$, $u_0(x) = 0.5$, $p_0(x) = 1$. For such an initial data the Euler equations reduces to a linear transport problem and the solution remains smooth for all time $t > 0$.

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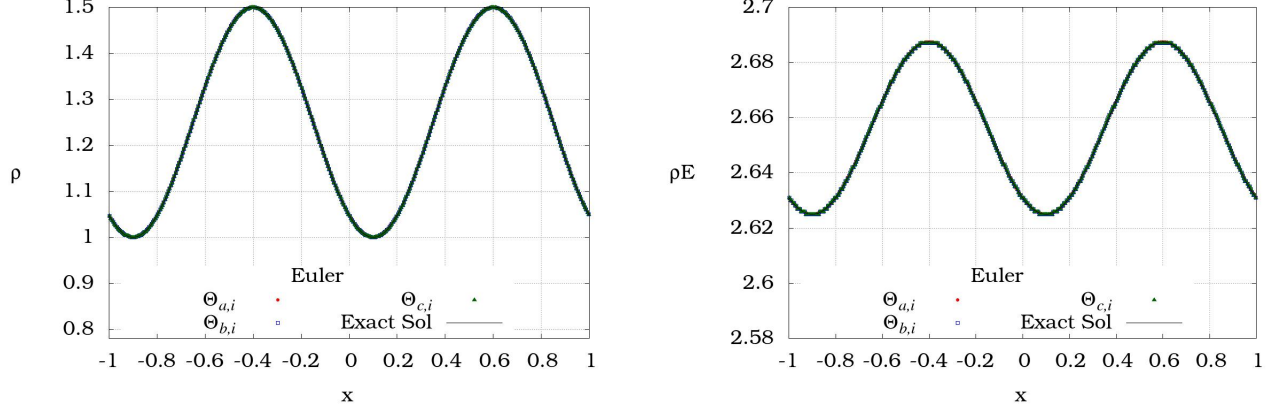


FIGURE 3. Second-order accurate approximation of the smooth Euler solution and entropy with a mesh made of 400 cells.

Second-order scheme errors $\Theta_i^m = \Theta_{a,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-03	-	1.9E-03	-	1.7E-03	-
200	8.7E-04	2.0	4.7E-04	2.0	4.2E-04	2.0
400	2.2E-04	2.0	1.2E-04	2.0	1.0E-04	2.0
800	5.4E-05	2.0	2.9E-05	2.0	2.6E-05	2.0
1600	1.4E-05	2.0	7.4E-06	2.0	6.5E-06	2.0
Second-order scheme errors $\Theta_i^m = \Theta_{b,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	3.5E-03	-	1.9E-03	-	1.7E-03	-
200	8.7E-04	2.0	4.7E-04	2.0	4.2E-04	2.0
400	2.2E-04	2.0	1.2E-04	2.0	1.0E-04	2.0
800	5.4E-05	2.0	2.9E-05	2.0	2.6E-05	2.0
1600	1.4E-05	2.0	7.4E-06	2.0	6.5E-06	2.0
Second-order scheme errors $\Theta_i^m = \Theta_{c,\varepsilon,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	1.2E-02	-	7.6E-03	-	1.1E-02	-
200	2.4E-03	2.3	1.6E-03	2.2	2.9E-03	1.9
400	3.4E-04	2.8	2.0E-04	3.0	3.7E-04	3.0
800	6.0E-05	2.5	3.1E-05	2.7	2.6E-05	3.8
1600	1.4E-05	2.1	7.4E-06	2.1	6.5E-06	2.0

TABLE 3. Errors and order evaluations for the second-order accurate schemes with the continuous Euler solution and for Θ_i^m described in (5.2).

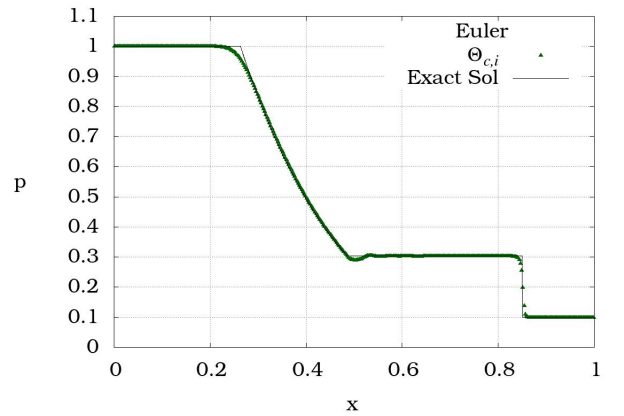
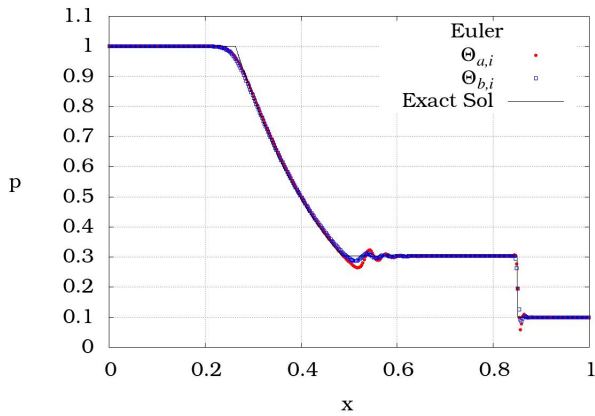
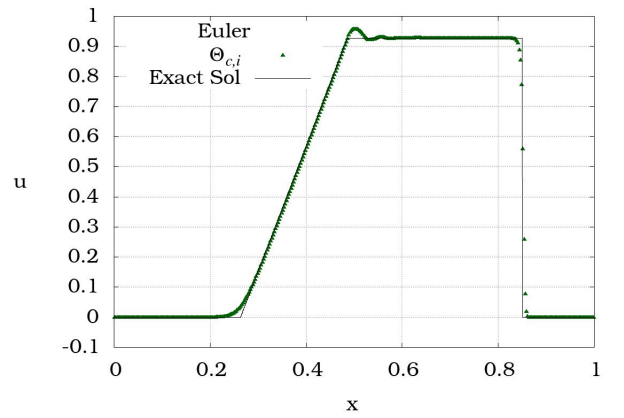
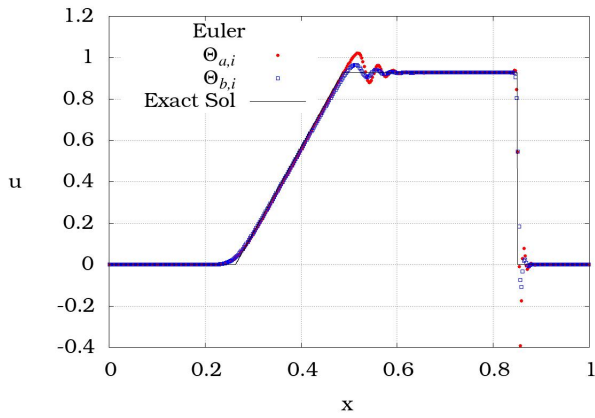
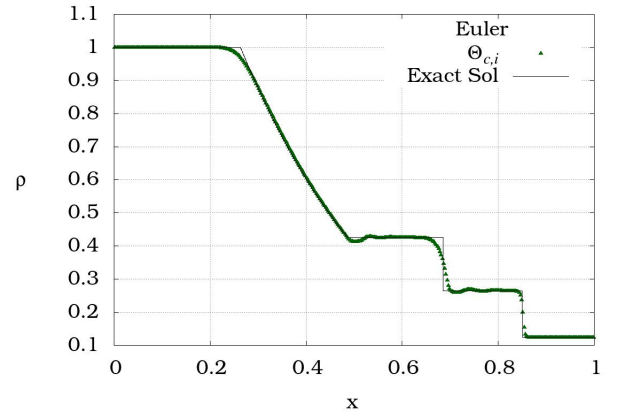
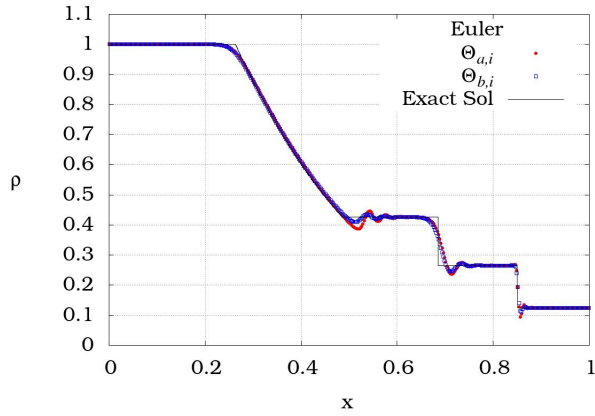
The numerical solutions are displayed in Fig. 3. The second-order accuracy is observed in Table 3.

5.2.2. Shock tube solution

We consider the initial data given by

$$\rho_0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 0.125 & \text{otherwise,} \end{cases} \quad u_0(x) = 0, \quad p_0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 0.1 & \text{otherwise,} \end{cases}$$

over the domain $[0, 1]$. The final time is 0.2. To respect the periodic conditions on the boundaries, we work on the domain $[-1, 1]$ and we symmetrize the shock tube problem on $[-1, 0]$.



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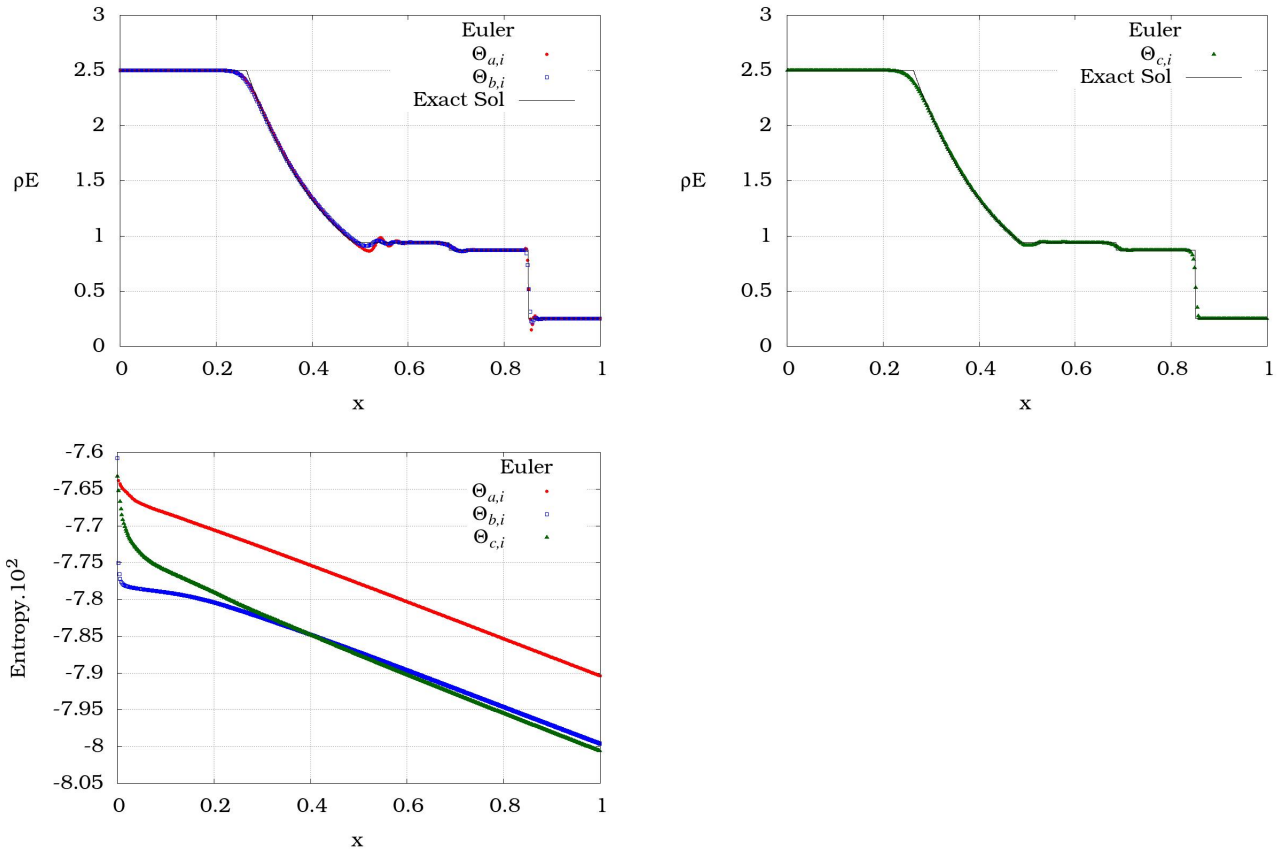


FIGURE 4. Second-order accurate approximation of the shock tube Euler solution and entropy with a mesh made of 400 cells.

Second-order scheme errors $\Theta_i^m = \Theta_{a,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	7.2E-02	-	7.1E-02	-	2.7E-01	-
200	4.0E-02	0.8	4.5E-02	0.7	2.4E-01	0.1
400	2.2E-02	0.9	2.9E-02	0.6	2.4E-01	0.0
800	1.2E-02	0.9	1.9E-02	0.6	1.9E-01	0.4
1600	6.4E-03	0.9	1.3E-02	0.6	1.7E-01	0.1
Second-order scheme errors $\Theta_i^m = \Theta_{b,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	5.9E-02	-	6.2E-02	-	1.9E-01	-
200	3.3E-02	0.8	3.9E-02	0.7	1.9E-01	0.0
400	1.8E-02	0.9	2.5E-02	0.6	1.9E-01	0.0
800	9.4E-03	0.9	1.5E-02	0.7	1.3E-01	0.6
1600	5.2E-03	0.9	1.1E-02	0.5	1.6E-01	0.4
Second-order scheme errors $\Theta_i^m = \Theta_{c,\varepsilon,i}^m$						
cells	L^1	order	L^2	order	L^∞	order
100	6.0E-02	-	6.3E-02	-	2.5E-01	-
200	3.2E-02	0.9	4.0E-02	0.7	2.3E-01	0.1
400	1.7E-02	0.9	2.6E-02	0.6	2.4E-01	0.0
800	8.7E-03	1.0	1.6E-02	0.7	1.7E-01	0.5
1600	4.5E-03	0.9	1.1E-02	0.5	2.2E-01	0.4

TABLE 5. Errors and order evaluations for the second-order accurate schemes with the shock tube Euler solution and for Θ_i^m described in (5.2).

The numerical solutions are displayed Fig. 4. We remark only little spurious oscillations. In Table 5, we give the measured orders of accuracy.

6. Conclusion

We have presented a family of unlimited and formally second-order accurate finite volume schemes. They are designed in such a way that a dissipative property is required for the second-order discretization in space. We have given a proof of a global entropy inequality under a CFL which is not standard and probably far from being optimal. The proof requires the strict convexity of the entropy. At a numerical level, we observe the very good behavior of the schemes both for smooth and discontinuous solutions. In particular, there is one choice of the second-order discretization in space which seems to preserve the maximum principle without losing the order of accuracy. The analysis of this particular second-order discretization in space will be the purpose of a future work.

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