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The twistor geometry of parabolic structures in rank two

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Abstract. Let X be a quasi-projective curve, compactified to (Y, D) with $X = Y - D$. We construct a Deligne–Hitchin twistor space out of moduli spaces of framed λ -connections of rank 2 over Y with logarithmic singularities and quasi-parabolic structure along D . To do this, one should divide by a Hecke-gauge groupoid. Tame harmonic bundles on X give preferred sections, and the relative tangent bundle along a preferred section has a mixed twistor structure with weights 0, 1, 2. The weight 2 piece corresponds to the deformations of the KMS structure including parabolic weights and the residues of the λ -connection.

Keywords. Parabolic structure; moduli space; twistor space; logarithmic connection; Higgs bundle.

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1. Introduction

In [17], Hitchin gave a hyperkähler structure on the moduli space of local systems over a smooth compact Riemann surface. By Penrose theory, this leads to a *twistor space*. Deligne gave an interpretation of the construction of the twistor space in terms of the moduli space of λ -connections. This viewpoint is amenable to generalization to the case of quasiprojective varieties. For rank 1 local systems on an open curve, a weight 2 property for the local monodromy transformations around the punctures came into view [42], and this was related to parabolic structures.

In the present paper, we would like to consider how to move to higher rank local systems on an open curve. We will look at the case of rank 2 bundles with logarithmic connection. The fundamental picture is an interplay between the notions of quasi-parabolic structure and parabolic structure, that were introduced by Seshadri [36] and developed further in [27] and the subsequent literature.

Before stating the results in §1.4, we will first have a look at the classical twistor space theory, the original case of a compact Riemann surface, then the rank 1 case over a quasiprojective curve.

This article is part of the “Special Issue in Memory of Professor C S Seshadri”.

1.1 Twistor space

Let $\mathbb{H} = \mathbb{R}\langle 1, I, J, K \rangle$ be the algebra of quaternions. The space of complex structures $\kappa \in \mathbb{H}$, $\kappa^2 = -1$ is identified with the two-sphere S^2 and provided with a complex structure making it into \mathbb{P}^1 . With coordinate on the main chart \mathbb{A}^1 denoted by λ , the complex structure I corresponds to $\lambda = 0$ and J corresponds to $\lambda = 1$. The antipodal involution $\kappa \mapsto -\kappa$ corresponds to $\sigma : \lambda \mapsto -\bar{\lambda}^{-1}$, it is a real structure on \mathbb{P}^1 with empty set of real points.

Suppose V is an \mathbb{H} -module, i.e., a quaternionic vector space. The product $V \times \mathbb{P}^1$ has a global complex structure inducing κ on $V \times \{\kappa\}$, making it into the total space of a vector bundle \mathcal{V}/\mathbb{P}^1 . We call the sections of \mathcal{V} of the form $\{v\} \times \mathbb{P}^1$ the *preferred sections*.

PROPOSITION 1

This twistor space construction is an equivalence between finite dimensional quaternionic vector spaces, and bundles \mathcal{V}/\mathbb{P}^1 of finite rank, provided with an involution covering σ , such that \mathcal{V} is semistable of slope 1 (i.e., $\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2d}$). The underlying \mathbb{R} -vector space is recovered as $V = \Gamma(\mathbb{P}^1, \mathcal{V})^\sigma$. For each $\kappa \in \mathbb{P}^1$, the projection $V \rightarrow \mathcal{V}_\kappa$ is an isomorphism of real vector spaces inducing the complex structure κ on V .

This linear picture extends to the nonlinear case [18]. If M is an integrable quaternionic manifold, its *twistor space* is

$$Tw(M) := M \times \mathbb{P}^1$$

with complex structure and antilinear involution σ obtained in the same way. The horizontal “preferred sections” $\{m\} \times \mathbb{P}^1$ are holomorphic, σ -invariant, and for any $\kappa \in \mathbb{P}^1$, the fiber $M \times \{\kappa\}$ is given the complex structure M_κ determined by the action of $\kappa \in \mathbb{H}$ on the tangent spaces of M . The “integrability” condition says that these are integrable complex structures, and the general Penrose theory yields an integrable complex structure on the total space $Tw(M)$.

The relative tangent bundle along a preferred section is the vector bundle given by Proposition 1, in particular, it is semistable of slope 1—we say it has *weight* 1. Locally around a preferred section, the σ -invariant sections are all preferred sections, and a neighborhood in the space of these sections maps isomorphically to a neighborhood in any one fiber M_κ , thanks to the weight 1 property. These however can fail for general σ -invariant sections not assumed to be near to preferred ones. A general study is given in [2, 3, 7].

1.2 Hitchin’s twistor space in the compact case

The moduli space M of local systems on a smooth compact Riemann surface X has a quaternionic and indeed hyperkähler structure [17], the latter meaning that there is a Riemannian metric Kähler for all the complex structures. We obtain the twistor space $Tw(M)$. In what follows, we look only at the smooth points of the moduli space without further mention.

For the complex structure $\kappa = I$ corresponding to $\lambda = 0$ in \mathbb{P}^1 , the complex moduli space M_0 is the moduli space of Hitchin pairs or “Higgs bundles” (E, φ) on X . We may call this the *Dolbeault moduli space* denoted M_{Dol} in view of the analogy with Dolbeault cohomology. For the complex structure $\kappa = J$ corresponding to $\lambda = 1$ in \mathbb{P}^1 , the complex

moduli space M_1 is the moduli space of vector bundles with integrable connection (E, ∇) on X . We call this the *de Rham moduli space* denoted M_{dR} in view of the analogy with de Rham cohomology.

Furthermore, for all the complex structures κ corresponding to $\lambda \neq 0, \infty$ the moduli spaces M_κ are naturally isomorphic, so they are all isomorphic to $M_{dR} = M_1$, which is in turn analytically isomorphic to the ‘‘Betti’’ moduli space of local systems or representations of the fundamental group.

Deligne, having discussed with Witten, gave a reinterpretation of $Tw(M)$ as follows: Each M_λ is the moduli space of *vector bundles with λ -connection* (E, ∇) . For $\lambda \neq 0$, the rescaling $\lambda^{-1}\nabla$ is just a connection, yielding the isomorphisms referred to above, whereas for $\lambda = 0$, a λ -connection is the same thing as a Higgs field φ .

We may make an algebraic geometry construction of the family of moduli spaces over $\mathbb{A}^1 \subset \mathbb{P}^1$, which for reasons of analogy with the Dolbeault and de Rham terminology, we call M_{Hod} for Hodge. This space together with its \mathbb{C}^* -action may be viewed as the ‘‘Hodge filtration’’ relating de Rham to Dolbeault [40].

The isomorphisms between different nonzero $\lambda \in \mathbb{A}^1 - \{0\} = \mathbb{G}_m$ fit together to give an analytic trivialization

$$(\mathbb{G}_m \times_{\mathbb{A}^1} M_{Hod})^{an} \cong (\mathbb{G}_m \times M_B)^{an},$$

where M_B is the ‘‘Betti’’ moduli space of representations of the fundamental group.

Then, the condition of existence of an antipodal involution covering σ motivated Deligne to define a *glueing* between $M_{Hod}(X)$ and $M_{Hod}(\bar{X})$, where \bar{X} denotes the complex conjugate variety, using the isomorphism

$$\pi_1(X) \cong \pi_1(\bar{X}) \quad \text{whence} \quad M_B(X) \cong M_B(\bar{X})$$

and applying the involution $\lambda \mapsto -\lambda^{-1}$ on \mathbb{G}_m . Glueing the two pieces together yields a space

$$M_{Hod}(X) \cup M_{Hod}(\bar{X}) =: M_{DH} \rightarrow \mathbb{P}^1$$

and one can define an antipodal involution using the fact that the moduli space M_{Hod} is a canonical algebraic geometry construction so it supports a complex conjugation operation.

PROPOSITION 2

The Deligne–Hitchin moduli space constructed by Deligne’s glueing is isomorphic to the twistor space for Hitchin’s quaternionic structure:

$$\begin{array}{ccc} M_{DH} & \cong & Tw(M) \\ & \searrow & \swarrow \\ & \mathbb{P}^1 & \end{array} .$$

The preferred sections of the twistor space correspond to sections of the fibration $M_{DH} \rightarrow \mathbb{P}^1$ that we also call ‘‘preferred sections’’. These are maps $\mathbb{P}^1 \rightarrow M_{DH}$ that are obtained whenever we have a *harmonic bundle*

$$(E, \partial, \bar{\partial}, \varphi, \varphi^\dagger, h)$$

corresponding to a solution of Hitchin’s equations. For $\lambda \in \mathbb{A}^1$, the point in the moduli space of holomorphic vector bundles with λ -connections is

$$(\mathcal{E}_\lambda := (E, \bar{\partial} + \lambda\varphi^\dagger), \nabla_\lambda := \lambda\partial + \varphi).$$

These preferred sections are compatible with the antipodal involution. The fact that this construction gives the twistor space of a quaternionic manifold comes from the following property.

PROPOSITION 3

Suppose $\rho : \mathbb{P}^1 \rightarrow M_{\text{DH}}$ is a preferred section defined as coming from a harmonic bundle in the above way. Let

$$T_\rho = \rho^*T(M_{\text{DH}}/\mathbb{P}^1)$$

be the pullback of the relative tangent bundle, or equivalently, the normal bundle of M_{DH} to the section. Then T_ρ is a semistable vector bundle of slope 1 over \mathbb{P}^1 .

An analogy with Hodge structures motivates us to call the property of being semistable of slope 1, a property of *weight 1*. Thus, the fact that we have a quaternionic structure on the moduli space comes from a weight 1 property for the tangent bundle to the preferred section.

Fundamentally, the calculation going into the proof uses the observation that the tangent space of the moduli space is an H^1 , calculated by some kind of harmonic forms. Then, the fact that they are 1-forms means that the transition functions needed to pass from the \mathbb{A}^1 neighborhood of $\lambda = 0$ to the \mathbb{A}^1 neighborhood of $\lambda = \infty$ involve λ^{-1} leading to the semistable of slope 1 property.

This weight 1 property is the nonabelian cohomology analogue of the statement in the usual Hodge theory that $H_{\text{dR}}^1(X)$ has a weight 1 Hodge structure, and similarly for a variety over \mathbb{F}_q that the étale cohomology $H_{\text{et}}^1(X_{\mathbb{F}_q}, \mathbb{Q}_\ell)$ has weight 1 in the sense that the eigenvalues of Frobenius have norm $q^{1/2}$.

1.3 A weight two property in the quasiprojective case

In usual cohomology, recall that for $X = \mathbb{P}^1 - \{0, \infty\}$, the mixed Hodge structure on $H^1(X)$ is one-dimensional of weight 2, and in the arithmetic setting $H_{\text{et}}^1(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(1)$ is a Tate twist having weight two. The weight two property is localized at the punctures, for example, from arithmetic geometry the inertia group has the form of a Tate twist, so it has weight -2 and the space of representations of the inertia group should be thought of as having weight 2.

Therefore, one may naturally conjecture that the weight 1 property for the twistor space would become a weight 2 property for the local monodromy transformations around punctures. Pridham [35] indeed does this for deformations of ℓ -adic representations. For the case of moduli spaces of local systems of rank 1 over an open curve, this was discussed in the paper [42].

Let us review some of the details under the simplifying assumption that $X = \mathbb{P}^1 - \{0, \infty\}$. The only data of a local system is then a single local monodromy at a puncture. A line bundle on X is trivial, and a logarithmic λ -connection has the form

$$\nabla(\lambda, a) = \lambda d + a \frac{dz}{z}.$$

There is an action of change of the trivialization making (λ, a) equivalent to $(\lambda, a + k\lambda)$ for any $k \in \mathbb{Z}$. The singularity of this action at $\lambda = 0$ is one of the difficulties of the open curve situation.

Let $\mathcal{G} \cong \mathbb{Z}$ be this “gauge group” acting. We note that the gauge transformations may also be viewed as Hecke transformations at the singular points. In this example, to maintain a trivial bundle, we do simultaneous Hecke transformations at both points, one up and one down.

The moduli space may be described as

$$M_{\text{Hod}} := \mathbb{A}^1 \times \mathbb{C}/\mathcal{G}$$

using \mathbb{A}^1 for the λ variable and \mathbb{C} for the coefficient a . Note that the group acts discretely over $\lambda \neq 0$ but the stabilizer group of the fiber over $\lambda = 0$ is the full $\mathcal{G} = \mathbb{Z}$. It is therefore not completely clear what kind of structure is best to accord to the quotient. We will discuss more on this aspect later.

The Riemann–Hilbert correspondence over $\lambda \neq 0$ sends the connection $\lambda^{-1}\nabla(\lambda, a)$ to the monodromy around the loop generating $\pi_1(X)$:

$$\mathbb{G}_m \times \mathbb{C}/\mathcal{G} \xrightarrow{\cong} \mathbb{G}_m \times \mathbb{C}^* \quad (\lambda, a) \mapsto (\lambda, \exp(2\pi ia/\lambda)).$$

We would like to use this identification to glue M_{Hod} to the other piece in the Deligne glueing. Let $\mu := -\lambda^{-1}$ denote the coordinate of the other chart $\mathbb{A}^1 \subset \mathbb{P}^1$. A point $(\mu, b) \in M_{\text{Hod}}(\bar{X})$ has monodromy transformation $\exp(2\pi ib/\mu)$ along the generating loop for $\pi_1(\bar{X})$.

The topological isomorphism $X^{\text{top}} \cong \bar{X}^{\text{top}}$ takes the generator to minus the generator, so the Deligne glueing should associate (λ, a) with (μ, b) (up to the \mathcal{G} action) when

$$\exp(2\pi ia/\lambda) = \exp(-2\pi ib/\mu).$$

Lifting over the action of the gauge group, this condition becomes

$$a/\lambda = -b/\mu = -b/(-\lambda^{-1}) = \lambda b, \quad \text{i.e., } a = \lambda^2 b.$$

This is the glueing condition for the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$ over \mathbb{P}^1 , yielding the weight 2 expression for the Deligne–Hitchin space:

$$M_{\text{DH}} = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(2))/\mathcal{G}.$$

There is also a natural antipodal involution, and the *preferred sections* are the sections that are compatible with σ . Recall from the compact case that we wanted to look at the space of σ -equivariant sections of $M_{\text{DH}}/\mathbb{P}^1$. Here let us lift over the action of the gauge group and look at the space of σ -equivariant sections of $\mathcal{O}_{\mathbb{P}^1}(2)$. Before asking for σ -equivariance, we have $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \cong \mathbb{C}^3$.

Lemma 4.

$$\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^\sigma \cong \mathbb{R}^3.$$

For any $\kappa \in \mathbb{P}^1$, the restriction morphism from these sections to the fiber $\mathbb{C}_\kappa = \mathcal{O}_{\mathbb{P}^1}(2)_\kappa$ is a surjection

$$\mathbb{R}^3 \rightarrow \mathbb{C}_\kappa.$$

There is a natural splitting as $\mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}_\kappa$ such that the generator of the gauge group has the form $(1, -\lambda(\kappa))$. \square

The extra real parameter, kernel of the restriction map, turns out to be the *parabolic weight parameter*. If $(E, \partial, \bar{\partial}, \varphi, \varphi^\dagger, h)$ is a tame harmonic bundle of rank 1 over X then it yields a σ -invariant section, and for $\lambda \in \mathbb{A}^1$, the corresponding point in $\mathbb{R} \times \mathbb{C}$ is (p, e) , where p is the parabolic weight and e the eigenvalue of the residue of the λ -connection. The parabolic weight expresses the growth rate of the harmonic metric h near a puncture. The fact that we have this extra real parameter may be seen as a manifestation of the fact that the monodromy transformations around punctures lie in a space whose Hodge weight is 2.

Recall that Mochizuki defines the notion of *KMS-spectrum* of a harmonic bundle on X at a puncture $y \in D$. This is the set of residual data consisting of a parabolic weight and an eigenvalue of the residue, for the asymptotic structure of the harmonic bundle near y . Each element of the KMS spectrum is a vector in the space \mathbb{R}^3 that we have seen above; such a point is interpreted as a pair (p, e) consisting of a parabolic weight and an eigenvalue, in a way that depends on λ .

Sabbah [37] and Mochizuki [29] gave formulas for the variation of parabolic weight p and eigenvalue e as a function of λ , generalizing my formulas [39] for $\lambda = 1$.

We will use the notations of [29] (Sabbah's notations are slightly different but equivalent). Starting with $(a, \alpha) \in \mathbb{R} \times \mathbb{C}$ at $\lambda = 0$, the parabolic weight of the parabolic structure at λ , giving the growth rate of holomorphic sections for the holomorphic structure $\partial + \lambda\varphi^\dagger$, is

$$p(\lambda, (a, \alpha)) = a + 2\operatorname{Re}(\lambda\bar{\alpha}). \quad (1)$$

The eigenvalue of the residue of the logarithmic λ -connection $\nabla_\lambda = \lambda\partial + \varphi$ is

$$e(\lambda, (a, \alpha)) = \alpha - a\lambda - \bar{\alpha}\lambda^2. \quad (2)$$

These formulas may be derived [42] as a consequence of the weight 2 twistor space interpretation, with the expression of $\mathbb{R} \times \mathbb{C}$, depending on λ , as corresponding to a unique \mathbb{R}^3 independent of λ .

1.4 The rank 2 case

We would like to extend this picture to higher rank local systems on an open curve. Let us look at some potential difficulties in light of the previous discussion. Many of these issues have been raised by Nitsure and others [16, 28, 29, 33, 34].

Although \mathcal{G} was a group acting in the above example, it is necessary, in general, to consider the action of a groupoid, that we will call the *Hecke-gauge groupoid*.

A first observation is that the action of this groupoid becomes singular over $\lambda = 0$, in the above example, the entire $\mathcal{G} = \mathbb{Z}$ stabilizes the full fiber over $\lambda = 0$. For this reason, we will tend to not really look at the quotient space, but to retain just the action groupoid instead. One possible solution here would be to invoke the notion of *diffeological space*.

A next observation is that we have side-stepped any discussion of stability. From the compact case, recall that the construction of M_{Dol} requires the notion of stability of a Higgs bundle, so this information is needed in the fiber over $\lambda = 0$ for the construction of M_{Hod} . However, in the quasiprojective case, defining stability (this is needed for all values of λ , see [21]) requires knowing the parabolic weights, but we are trying to recover the parabolic

weights from the twistor space construction itself as happened in rank 1. Without a notion of stability, we are going to be getting moduli spaces that are not of finite type but only locally of finite type, even over $\lambda \neq 0$ [15]. This should be accepted.

A third difficulty for making the construction is that, from the formulas (1), (2) for the variation of eigenvalue of the residue as a function of λ , a preferred section is always going to have some points where the eigenvalues are resonant, so we are not able to just impose a non-resonance condition on the residues. We do however impose some conditions on the fiber over 0 so as to improve somewhat the moduli problem. As a result, the discussion will work for most although not all “preferred sections”.

To get a mixed twistor property for the relative tangent space along a preferred section, we apply the theory of Sabbah [37] and Mochizuki [29]. Although we use only a small and early fraction of their theory, our application highlights some of the subtleties involved and might therefore serve as a gentle introduction.

Here is a summary of what we do, as stated in Theorems 6, 7 and 8. Those will then be proven in the subsequent sections.

Let Y be a compact Riemann surface and $D \subset Y$ a reduced divisor. Set $X := Y - D$, and choose a base-point $x \in X$. We look at bundles of rank 2.

A framed quasi-parabolic bundle with logarithmic λ -connection is

$$(\lambda, E, \nabla, F, \beta),$$

where $\lambda \in \mathbb{A}^1$, E is a rank 2 vector bundle on Y , ∇ is a logarithmic λ -connection, i.e.,

$$\nabla : E \rightarrow E \otimes \Omega_Y^1(\log D), \quad \nabla(ae) = a\nabla(e) + \lambda d(a)e,$$

$F = F_y \subset E_y$ is a one-dimensional subspace preserved by $\text{res}_y(\nabla)$, and $\beta : E_x \cong \mathbb{C}^2$ is a framing over the base-point.

Hypothesis 5. Let $(\lambda, E, \nabla, F, \beta)$ be a framed quasi-parabolic logarithmic λ -connection. We make the following hypotheses:

1. The only quasi-parabolic endomorphisms of (E, ∇, F) are scalars.
2. If $\lambda = 0$, the spectral curve of $\varphi = \nabla$ (which is a Higgs field in this case) is irreducible of degree 2 over the base.
3. When speaking of a harmonic bundle, we assume furthermore that the two KMS spectrum elements (parabolic weight, residue eigenvalue) modulo \mathbb{Z} are distinct at each $y \in D$. This may be measured at $\lambda = 0$ in $(\mathbb{R}/\mathbb{Z}) \times \mathbb{C}$.

Note that (2) \Rightarrow (1) at $\lambda = 0$ since a Higgs bundle corresponds to a rank 1 torsion-free sheaf over the spectral curve and its only endomorphisms are scalars. We also note that (1) implies directly that the framed object is rigid, i.e., there are no nontrivial endomorphisms respecting the framing. We will see in Lemma 17 that (1) implies the moduli problem is unobstructed and hence smooth.

Theorem 6. *There exists a separated algebraic space, smooth and locally of finite type, parametrizing the framed quasi-parabolic logarithmic λ -connections $(\lambda, E, \nabla, F, \beta)$ satisfying Hypothesis 5,*

$$\tilde{M}_{\text{Hod}}(X) \xrightarrow{\lambda} \mathbb{A}^1.$$

Denote by $\tilde{M}_{\text{dR}}(X)$ the fiber over $1 \in \mathbb{A}^1$.

We would then like to define an equivalence relation on $\tilde{M}_{\text{dR}}(X)$ identifying different logarithmic connections that correspond to the same Betti local system data.

Let $\tilde{M}_B(X)$ denote the moduli space of tuples (L, F, β) , where L is a rank 2 local system on X , $F = \{F_y\}$ is a sub-local system of the restriction of L to a punctured disk around y , and $\beta : L_x \cong \mathbb{C}^2$ is a framing. Impose the analogue of Hypothesis 5(1), namely that the only endomorphisms preserving the subspaces are scalars. Let $\tilde{M}_B(X)$ be the moduli space for such framed quasi-parabolic local systems.

Forgetting the framing at x would provide a map to an open subset defined by 5(1) of the \mathcal{X} moduli space of Fock–Goncharov [14].

Define the *Betti gauge groupoid* \mathcal{G}_B acting on $\tilde{M}_B(X)$ to be the groupoid consisting of partially defined morphisms from $\tilde{M}_B(X)$ to itself, generated by the operation:

(P): defined on the open set where the eigenvalues of the local monodromy transformation are distinct, sending (L, F, β) to (L, F^\perp, β) , where F_y^\perp is the eigenspace of the local monodromy that is different from F_y .

This defines an étale groupoid, with quotient $M_B(X) = \tilde{M}_B(X)/\mathcal{G}_B$ a non-separated algebraic stack. This quotient stack behaves as in Kollár’s observation [23], since the operation is partially defined and étale.

Theorem 7. *There is an étale groupoid*

$$\mathcal{G}_{\text{dR}} \rightarrow \tilde{M}_{\text{dR}}(X) \times \tilde{M}_{\text{dR}}(X)$$

such that the Riemann–Hilbert correspondence gives an equivalence of analytic groupoids

$$(\tilde{M}_{\text{dR}}(X), \mathcal{G}_{\text{dR}}) \cong (\tilde{M}_B(X), \mathcal{G}_B).$$

This extends in a natural way to an étale groupoid on the Hodge moduli space

$$\mathcal{G}_{\text{Hod}} \rightarrow \tilde{M}_{\text{Hod}}(X) \times \tilde{M}_{\text{Hod}}(X).$$

The isomorphism of topological spaces between X^{top} and \tilde{X}^{top} gives an equivalence

$$(\tilde{M}_B(X), \mathcal{G}_B) \cong (\tilde{M}_B(\tilde{X}), \mathcal{G}_B).$$

Using the Riemann–Hilbert correspondence, we can then make a Deligne glueing. In terms of groupoids, this can be viewed as follows: let

$$\tilde{M}_{\text{DH}} := \tilde{M}_{\text{Hod}}(X) \sqcup \tilde{M}_{\text{Hod}}(\tilde{X})$$

with Hecke-gauge groupoid \mathcal{G}_{DH} combining the \mathcal{G}_{Hod} on both pieces, together with pieces identifying points that correspond to elements of the Betti moduli space that are identified under the previous equivalence:

$$M_{\text{DH}} = (\tilde{M}_{\text{DH}}, \mathcal{G}_{\text{DH}}).$$

The quotient of this groupoid would be some kind of non-separated analytic stack but with stabilizer groups that are not very well behaved. Although we do not identify the precise framework for such a quotient, we note that the tangent bundle of the “quotient” $M_{\text{DH}}(X)$ may be defined, since the groupoid is étale. This gives, in particular, the pullback of the tangent bundle by a section.

Theorem 8. *Suppose $(E_X, \partial, \bar{\partial}, \varphi, \varphi^\dagger, h)$ is a tame harmonic bundle on X , satisfying our Hypothesis 5, and choose a framing $\beta : E_x \cong \mathbb{C}^2$ taking h_x to the standard hermitian metric on \mathbb{C}^2 . Then we get in a natural way a section of the Deligne–Hitchin groupoid*

$$\rho : \mathbb{P}^1 \rightarrow (\tilde{M}_{\text{DH}}(X), \mathcal{G}_{\text{DH}}).$$

Let T_ρ be the pullback by ρ of the tangent bundle of $M_{\text{DH}}(X)$. It has a filtration

$$0 \subset W_0T_\rho \subset W_1T_\rho \subset W_2T_\rho = T_\rho,$$

where W_1T_ρ is the set of tangent vectors that preserve the eigenvalues of the residues, and W_0T_ρ is the tangent space of the change of framing. Then (T_ρ, W_\cdot) is a mixed twistor structure [41], meaning that W_k/W_{k-1} is a semistable bundle on \mathbb{P}^1 of slope k (for $k = 0, 1, 2$).

The weight 1 property of the graded piece corresponding to deformations fixing the eigenvalues of the residues, corresponds to the fact—well-known in the physics literature—that moduli spaces of flat bundles with fixed conjugacy classes have a Hitchin-type hyperkähler structure, see for example [19].

We will base our proof on the pure twistor \mathcal{D} -module theory of [29,37]. The more general and full theory of mixed twistor \mathcal{D} -modules [30] should allow for a more direct proof. Mochizuki has communicated the suggestion to consider the mixed twistor \mathcal{D} -module $\mathfrak{T}[*D][!x]$, where \mathfrak{T} is the extension associated to $\text{End}(E)$; one would still need to show a compatibility with the moduli space construction.

2. Logarithmic connections

Throughout this paper, Y is a smooth compact Riemann surface and $D = \{y_1, \dots, y_k\}$ is a nonempty reduced divisor. We set $X := Y - D$ and fix a base point $x \in X$.

A quasi-parabolic logarithmic λ -connection of rank 2 consists of $\lambda \in \mathbb{C}$, a vector bundle E of rank 2 over Y together with a logarithmic λ -connection operator

$$\nabla : E \rightarrow E \otimes_{\mathcal{O}_Y} \Omega_Y^1(\log D) \quad \nabla(ae) = a\nabla(e) + \lambda(da)e$$

and for each $y \in D$ a subspace $F_y \subset E_y$ of rank 1, preserved by the residue $\text{res}_y(\nabla)$.

A framing at the base point $x \in X$ is an isomorphism $\beta : E_x \cong \mathbb{C}^2$.

These all have versions relative to a base scheme S , where $\lambda : S \rightarrow \mathbb{A}^1$ and E becomes a bundle on $Y \times S$.

We usually consider Hypothesis 5 on $(\lambda, E, \nabla, F, \beta)$, implying, in particular, that the framed object is rigid.

Let $\mathcal{M}_{\text{Hod}}(Y, \log D, x)$ denote the moduli functor of quasi-parabolic logarithmic λ -connections of rank 2 on (Y, D) , framed at x and satisfying Hypothesis 5. This functor associates to a scheme S the set of relative data $(\lambda, E, \nabla, F, \beta)$ on $Y \times S$, up to isomorphism. It maps via λ to \mathbb{A}^1 .

Construction of a moduli space is by now a classical subject. Some references, including a few further directions are [1,4,5,8,9,15,16,21,22,24,25,31,32,43], but it would be impossible to mention all of the relevant articles here.

PROPOSITION 9

This moduli functor $\widetilde{\mathcal{M}}_{\text{Hod}}(Y, \log D, x)$ is represented by a separated algebraic space $\widetilde{M}_{\text{Hod}}(Y, \log D, x)$ that is locally of finite type over \mathbb{A}^1 .

Proof. We can cover the moduli functor by subsets \mathcal{U}_k consisting of points where the maximum degree of a subbundle of E is k . These are bounded. The usual theory allows us to represent each of these as a quotient of a quasiprojective scheme by the action of a group of the form $GL(N)$ for some large N . Since objects are rigid by Condition (1) of Hypothesis 5, the stabilizer groups are trivial. Luna's étale slice theorem implies that the quotient is an algebraic space. The union of these spaces of finite type is locally of finite type, although not of finite type [15, Lemma 4.13].

To show separatedness, we use the condition (2) of Hypothesis 5. Note that the moduli space is covered by finite type algebraic spaces and we may assume given two curves in one of those. That is to say, we assume given a pointed smooth curve $(S, 0)$ and denote by $S_\eta := S - \{0\}$, and we are given two maps $S \rightarrow \mathcal{U}$ that agree on S_η . We obtain two bundles E and E' on $Y \times S$ (together with all their data) that are isomorphic over $Y \times S_\eta$, and suppose either $\lambda(0) \neq 0$ or ∇_0 and ∇'_0 have irreducible spectral curves. We would like to show that the two maps agree on S , in other words, that the isomorphism extends to $Y \times S$.

In the case $\lambda(0) \neq 0$, reason complex-analytically. The framed representation space of the fundamental group of $X = Y - D$ is separated, so the isomorphism extends to an isomorphism of flat holomorphic bundles on $X \times S$, hence to an isomorphism of bundles on $Y \times S$, by Hartogs' theorem. Again by Hartogs' theorem, the connection operator also extends, and the framing extends since $x \in X$. Our isomorphism preserves the subbundles $F_{\{y\} \times S}$ away from the origin of S , it follows that these subbundles are also preserved at $0 \in S$. Although the construction was analytic, the resulting isomorphism of bundles is algebraic since Y is proper.

We need to treat the case $\lambda(0) = 0$ assuming that the spectral varieties of $\nabla(0)$ and $\nabla'(0)$ are irreducible. Let t denote the coordinate on S . Let $g : E_\eta \cong E'_\eta$ be the isomorphism of bundles with respective connections. There is a power of t so that $t^a g$ extends to a morphism

$$t^a g : E \rightarrow E',$$

that is, nonzero for $t = 0$. But this morphism preserves the connections, i.e., the Higgs fields at $t = 0$. Since the spectral varieties are irreducible, this implies that the morphism is an isomorphism over $t = 0$ too. Now the condition that g preserves the framing implies that $a = 0$, so g extends to an isomorphism as required. As before, it preserves the sub-bundles $F_{\{y\} \times S}$. \square

This proposition is the construction of moduli spaces for Theorem 6, see Corollary 18 for smoothness.

We note that the rescaling of a λ -connection to $\lambda^{-1}\nabla$ provides an isomorphism on the open set $\mathbb{G}_m \subset \mathbb{A}^1$, where $\lambda \neq 0$ and

$$\widetilde{M}_{\lambda \neq 0}(Y, \log D, x) \xrightarrow{\cong} \mathbb{G}_m \times \widetilde{M}_{\lambda=1}(Y, \log D, x).$$

This is equivariant for the \mathbb{G}_m -action by rescaling on the left-hand side and the trivial action on the right-hand side. Here and below, we write $\tilde{M}_{\lambda \neq 0}$, etc. for the fibers of \tilde{M}_{Hod} over various values of λ .

2.1 Riemann–Hilbert morphism

For each $y \in D$, choose a point $\eta_y \in X$ near y . Given a local system L on X , the *nearby fiber* to $y \in D$ is the fiber L_{η_y} , and it has a local monodromy operator induced by the loop based at η_y going once around y .

Let $\tilde{M}_B(X, D, x)$ denote the moduli space of local systems L on X provided with a subspace F_y of the nearby fiber at each $y \in D$, invariant under the local monodromy, such that Hypothesis 5 is satisfied, together with a framing $\beta : L_x \cong \mathbb{C}^2$.

Consider the open subset of points called *nonresonant*,

$$\tilde{M}_{\lambda \neq 0}^{\text{nr}}(Y, \log D, x) \subset \tilde{M}_{\lambda \neq 0}(Y, \log D, x)$$

defined to be the set of $(\lambda, E, \nabla, F, \beta)$ such that, if a_y denotes the eigenvalue of $\text{res}_y(\nabla)$ on F_y , and b_y the eigenvalue on E_y/F_y , then

$$b_y - a_y \notin \lambda \cdot \mathbb{Z}_{<0}$$

for any $y \in D$.

Let U_y be a small disk around y . If $(\lambda, E, \nabla, F, \beta)$ satisfies the non-resonance condition, then letting $L := E^{\lambda^{-1}\nabla}$ denote the local system of flat sections, there is a unique sub-connection of rank 1,

$$F_{U_y} \subset E|_{U_y}$$

whose fiber over y is F_y . We denote by $F_{L,y}$ the fiber of F_{U_y} at the nearby point η_y . Hypothesis 5 implies that (L, F) has only scalar endomorphisms (the condition for inclusion in \tilde{M}_B). This serves to define the *Riemann–Hilbert morphism*. Beyond Deligne [11], some other references include [9, 10, 12, 21, 33, 34], although it would again be impossible to give a complete list.

PROPOSITION 10

The Riemann–Hilbert correspondence is a morphism of complex analytic spaces

$$\tilde{M}_{\lambda \neq 0}^{\text{nr}}(Y, \log D, x) \xrightarrow{RH} \mathbb{G}_m \times \tilde{M}_B(X, D, x)$$

sending $(\lambda, E, \nabla, F, \beta)$ to (λ, L, F_L, β) , where $L := E^{\lambda^{-1}\nabla}$ and β is the same framing on L as on E , and $F_{L,y}$ is defined as in the previous paragraph.

2.2 Morphisms and equivalences of groupoids

If Z is an algebraic or analytic space, a *groupoid* acting on Z is an algebraic (resp. analytic) space \mathcal{G} with maps

$$(s, t) : \mathcal{G} \rightarrow Z \times Z, \quad m : \mathcal{G} \times_Z \mathcal{G} \rightarrow \mathcal{G}, \quad e : Z \rightarrow \mathcal{G}, \quad i : \mathcal{G} \rightarrow \mathcal{G}$$

making (Z, \mathcal{G}) into a groupoid in the category of algebraic (resp. analytic) spaces. If Z' is another space, then a map $Z' \rightarrow (Z, \mathcal{G})$ consists of an étale open covering (resp. usual open covering) $\{U_i\}$ of Z' , morphisms $z_i : U_i \rightarrow Z$, and morphisms $g_{ij} : U_{ij} \rightarrow \mathcal{G}$ compatible with the previous morphisms via the source and target maps, and satisfying a cocycle condition. We may similarly define the notion of a map of groupoids $(Z', \mathcal{G}') \rightarrow (Z, \mathcal{G})$: a quick way is to view it as a functor, defined after possibly replacing Z' by an open covering. The previous definition may be seen in this manner.

A *natural isomorphism* between two maps is a natural transformation of functors after refinement, given in the concrete notation by a common refinement $\{U_k''\}$ of the two coverings plus a collection of maps $U_k'' \rightarrow \mathcal{G}$ satisfying the natural intertwining condition between the source and target map data. The set of maps as objects related by natural isomorphisms forms a groupoid $\text{Hom}((Z', \mathcal{G}'), (Z, \mathcal{G}))$ in the algebraic sense. In general, elements here can have automorphisms, but in the case where (s, t) is a monomorphism (as shall be the case for us over $\lambda \neq 0$), the groupoid of maps is equivalent to a set.

A map is an *equivalence* if there is a quasi-inverse, that is to say, a map going in the opposite direction such that the two compositions are equivalent by natural isomorphisms to the identities. These standard definitions (going back to Ehresmann and Satake) serve as a replacement for taking some kind of quotient stack of the groupoid, allowing us to sidestep the issue of how precisely to view the quotient stacks.

One says that the groupoid is *étale* if the source and target maps $\mathcal{G} \rightarrow Z$ are étale, and in this case that the groupoid is *smooth* if Z is smooth. We can then define the tangent bundle, and maps have differentials in the usual way.

2.3 Hecke-gauge groupoid on residues

For $y \in D$, we define the *residual space at y* to be $R_y := \mathbb{A}^1 \times \mathbb{C}^2$ with coordinates noted as (λ, a_y, b_y) . Of course, this is just \mathbb{C}^3 , the notation is meant to distinguish the λ -direction from the two residual ones. If $(\lambda, E, \nabla, F, \beta) \in \tilde{M}_{\text{Hod}}(Y, \log D, x)$, then we get the residue

$$\text{Res}_y(\lambda, E, \nabla, F, \beta) = (\lambda, a_y, b_y) \in R_y$$

where a_y is the eigenvalue of $\text{res}_y(\nabla)$ on F_y , and b_y is the eigenvalue on E_y/F_y . Set

$$R := R_{y_1} \times_{\mathbb{A}^1} \times \cdots \times_{\mathbb{A}^1} R_{y_k}$$

and we obtain the residue vector $\text{Res}(\lambda, E, \nabla, F, \beta) = (\lambda, a, b) \in R$.

We define a residual Hecke-gauge groupoid acting on R_y , in a way intended to be compatible with the Hecke-gauge groupoid that we will define on \tilde{M}_{Hod} below. Consider the operations (the third one being only partially defined)

$$\mathbf{h}_y, \mathbf{h}_y^{-1}, \mathbf{p}_y : R_y \rightarrow R_y$$

given by

$$\mathbf{h}_y(\lambda, a, b) = (\lambda, b - \lambda, a), \quad \mathbf{h}_y^{-1}(\lambda, a, b) = (\lambda, b, a + \lambda),$$

and

$$\mathbf{p}_y(\lambda, a, b) = (\lambda, b, a) \quad \text{defined when } a \neq b.$$

This latter operation means that the graph of \mathbf{p}_y is the open subset of R complement of the diagonal.

Let $\mathcal{G}_{R,y}$ be the groupoid generated by these operations subject to the relations that \mathbf{h}_y^{-1} is inverse to \mathbf{h}_y , $\mathbf{p}_y^2 = 1$, and \mathbf{p}_y commutes with \mathbf{h}_y^2 .

Here is a more explicit description. Note that $\mathbf{h}_y \mathbf{p}_y(\lambda, a, b) = (\lambda, a - \lambda, b)$ defined when $a \neq b$. We have $(\mathbf{h}_y \mathbf{p}_y)^k(\lambda, a, b) = (\lambda, a - k\lambda, b)$ defined on the open subset, where

$$b \notin \{a, a - \lambda, \dots, a - (k - 1)\lambda\}.$$

For $\epsilon = 0, 1, k \in \mathbb{N}$ and $m \in \mathbb{Z}$, let

$$\mathbf{g}_y(\epsilon, k, m) := \mathbf{p}_y^\epsilon (\mathbf{h}_y \mathbf{p}_y)^k \mathbf{h}_y^m,$$

and we can similarly write down the open subsets of definition of these operations depending on ϵ, k, m . The graph is a locally closed subset $\text{Graph}(\mathbf{g}_y(\epsilon, k, m)) \subset R_y \times R_y$, isomorphic to the open subset of definition of $\mathbf{g}_y(\epsilon, k, m)$ in R_y .

Lemma 11. This groupoid $\mathcal{G}_{R,y}$ is étale, and has the following description as a disjoint sum of locally closed subvarieties of $R_y \times R_y$:

$$\mathcal{G}_{R,y} = \coprod_{\epsilon, k, m} \text{Graph}(\mathbf{g}_y(\epsilon, k, m)).$$

The map $\mathcal{G}_{R,y} \rightarrow R_y \times_{\mathbb{A}^1} R_y$ is a monomorphism over $\lambda \neq 0$.

Now define \mathcal{G}_R to be the product groupoid acting on R , where the operations at different points commute. It is an étale groupoid with an analogous structure statement, and again $\mathcal{G}_R \rightarrow R \times_{\mathbb{A}^1} R$ is a monomorphism. Notice that, although the image is a closed subset, \mathcal{G}_R is not isomorphic to its image.

The quotient R/\mathcal{G}_R would be a non-separated space somewhat along the lines of the “bug-eye” spaces introduced by Kollár [23].

We may similarly define the Betti residual space $R_{B,y} = \mathbb{C}^* \times \mathbb{C}^*$. Then $R_B := \prod_{y \in D} R_{B,y}$, and we let \mathcal{G}_{R_B} be the groupoid generated by the partially defined operations $\mathbf{p}_y(\alpha_y, \beta_y) := (\beta_y, \alpha_y)$ defined when $\alpha_y \neq \beta_y$, subject to the relations $\mathbf{p}_y^2 = 1$ and they commute for different values of y .

Lemma 12. We have an analytic equivalence of groupoids

$$(R, \mathcal{G}_R)_{\lambda \neq 0}^{\text{an}} \xrightarrow{\cong} (\mathbb{G}_m \times (R_B, \mathcal{G}_{R_B}))^{\text{an}}$$

given by

$$(\lambda, a, b) \mapsto (\lambda, e^{2\pi ia}, e^{2\pi ib}).$$

2.4 Hecke-gauge groupoid

Define, at each point $y \in D$, an operation H_y , its inverse H_y^{-1} and a partially defined operation P_y on $\tilde{M}_{\text{Hod}}(Y, \log D, x)$. The first H_y is the well-known Hecke operation, or elementary transformation, as has been considered in [21] and more recently by [13, 20, 26]. Given $(\lambda, E, \nabla, F, \beta)$, we set

$$E' := \ker(E \rightarrow E_y/F_y)$$

and let F'_y be the image of $E(-y)_y$ in E'_y . At the other points, $y_i \neq y$ and keep the same $F'_{y_i} := F_{y_i}$; note that $E'|_X = E|_X$. The condition that F is preserved by the residue of ∇ implies that $\nabla|_X$ extends to a λ -connection ∇' on E' , again with residue preserving F' . We can take $\beta' := \beta$ since $x \in X$.

Let $T_y(E) := E \otimes \mathcal{O}_Y(y)$ with the induced subspaces, and put $H_y^{-1} := T_y H_y = H_y T_y$. This is inverse to H_y .

The operation P_y is going to be only partially defined. Let

$$\tilde{M}_{\text{Hod}}(X, \log D, x)^{(y)} \subset \tilde{M}_{\text{Hod}}(X, \log D, x)$$

denote the open subset consisting of points where the eigenvalues of $\text{res}_y(\nabla)$ are distinct. We define

$$P_y : \tilde{M}_{\text{Hod}}(Y, \log D, x)^{(y)} \rightarrow \tilde{M}_{\text{Hod}}(Y, \log D, x)$$

to be the operation that replaces the subspace F_y by its complementary eigenspace of $\text{res}_y(\nabla)$, keeping the same F at the other points of D .

Define the *Hecke-gauge groupoid* \mathcal{G}_{Hod} to be the groupoid of operations on $\tilde{M}_{\text{Hod}}(Y, \log D, x)$ generated by the operations H_y, H_y^{-1} and P_y subject to the relations that the first two are inverses, that $P_y^2 = 1$, that P_y commutes with H_y^2 , and that the operations for different values of y commute.

PROPOSITION 13

This defines an étale groupoid with the structural map

$$\mathcal{G}_{\text{Hod}} \rightarrow \tilde{M}_{\text{Hod}}(Y, \log D, x) \times_{\mathbb{A}^1} \tilde{M}_{\text{Hod}}(Y, \log D, x)$$

that is a monomorphism over $\lambda \neq 0$. The residue gives a morphism of groupoids

$$(\tilde{M}_{\text{Hod}}(Y, \log D, x), \mathcal{G}_{\text{Hod}}) \xrightarrow{\text{Res}} (R, \mathcal{G}_R).$$

Proof. Hypothesis 5 is preserved by our operations. As before, one may consider the composed operations

$$G_y(\epsilon, k, m) := P_y^\epsilon (H_y P_y)^k H_y^m$$

for $\epsilon = 0, 1, k \in \mathbb{N}$ and $m \in \mathbb{Z}$. The domain of definition of $G_y(\epsilon, k, m)$ is the pullback, under the residue at y , of the domain of definition of $\mathbf{g}_y(\epsilon, k, m)$. Using the given relations, any element of \mathcal{G}_{Hod} may be expressed uniquely as a product over $y \in D$ of $G_y(\epsilon, k, m)$, and this gives the expression

$$\mathcal{G}_{\text{Hod}} = \prod_{y \in D} \left[\coprod_{\epsilon, k, m} \text{Graph} (G_y(\epsilon, k, m)) \right].$$

We see that \mathcal{G}_{Hod} is an étale groupoid. The operations H_y, H_y^{-1}, P_y on $\tilde{M}_{\text{Hod}}(X, \log D, x)$ are compatible with the operations $\mathbf{h}_y, \mathbf{h}_y^{-1}, \mathbf{p}_y$ on residues, so the residue map descends to a map of groupoids.

The (source, target) structural map is a monomorphism over $\lambda \neq 0$, since the different pieces of the decomposition map to the corresponding pieces of \mathcal{G}_R and these are disjoint. \square

The Betti gauge groupoid is defined in a corresponding way. Let $\tilde{M}_B(X, D, x)^{(y)}$ denote the subspace where the eigenvalues of local monodromy operator at $y \in D$ are distinct, and let $P_{B,y} : \tilde{M}_B(X, D, x)^{(y)} \rightarrow \tilde{M}_B(X, D, x)^{(y)}$ denote the operation of replacing the eigenspace F_y by its complementary eigenspace. Let \mathcal{G}_B be the groupoid acting on $\tilde{M}_B(X, D, x)$, generated by the partially defined operations $P_{B,y}$ with relations $P_{B,y}^2 = 1$ and that for distinct values of y the operations commute.

This is again an étale groupoid, and the local monodromy maps give a morphism of groupoids

$$(\tilde{M}_B(X, D, x), \mathcal{G}_B) \rightarrow (R_B, \mathcal{G}_{R_B}).$$

In this case, the groupoid has an algebraic space quotient

$$\tilde{M}_B(X, D, x) \rightarrow M_B(X, D, x)/\mathcal{G}_B.$$

Theorem 14. *The Riemann–Hilbert correspondence gives an equivalence of analytic groupoids*

$$(\tilde{M}_{\lambda \neq 0}(Y, \log D, x), \mathcal{G}) \xrightarrow{\cong} \mathbb{G}_m \times (\tilde{M}_B(X, D, x), \mathcal{G}_B)$$

compatible with the residue morphisms.

Proof. It suffices to consider the fiber over $\lambda = 1$. Using the operations of \mathcal{G}_{Hod} and tensoring with a rank 1 connection, we may move any small neighborhood in $\tilde{M}_{\lambda \neq 0}(Y, \log D, x)$ into a small neighborhood where the residues (a_y, b_y) lie in $U_y \times V_y$ for $U_y, V_y \subset \mathbb{C}$ small neighborhoods such that the map $w \mapsto \exp(2\pi i w)$ is injective on $U_x \cup V_y$. These map isomorphically to the corresponding neighborhoods in $\tilde{M}_B(X, D, x)$ by [11]. The action of \mathcal{G}_B corresponds to that of \mathcal{G}_{Hod} whenever U_y and V_y overlap. \square

This completes the proof of Theorem 7.

2.5 The Deligne–Hitchin twistor space and preferred sections

Let

$$\tilde{M}_{\text{DH}} := \tilde{M}_{\text{Hod}}(X, \log D, x) \sqcup \tilde{M}_{\text{Hod}}(\bar{X}, \log \bar{D}, \bar{x}).$$

On this, we have a groupoid \mathcal{G}_{DH} defined as the disjoint union of \mathcal{G}_{Hod} for X with the same for \bar{X} , together with the pieces defining the identification obtained by Theorem 14 from

$$(\tilde{M}_B^{\text{an}}(X, D, x), \mathcal{G}_{B,X}) \cong (\tilde{M}_B^{\text{an}}(\bar{X}, \bar{D}, \bar{x}), \mathcal{G}_{B,\bar{X}}).$$

We obtain a complex analytic space with étale groupoid

$$(M_{\text{DH}}, \mathcal{G}_{\text{DH}}) \rightarrow \mathbb{P}^1.$$

Theorem 15. *Suppose $(E_X, \partial, \bar{\partial}, \varphi, \varphi^\dagger, h)$ is a tame harmonic bundle of rank 2 on X satisfying the conditions of Hypothesis 5. Fix a framing $\beta : E_X \cong \mathbb{C}^2$. Then we obtain a canonical corresponding preferred section to the groupoid*

$$\rho : \mathbb{P}^1 \rightarrow (\tilde{M}_{\text{DH}}, \mathcal{G}_{\text{DH}})$$

yielding the standard construction of family of Higgs bundles and λ -connections over X .

Proof. Suppose $y \in D$. Let $u = (a, \alpha)$ and $u' = (a', \alpha')$ be the two parabolic weight and residual eigenvalue pairs at $\lambda = 0$ for the point y . Hypothesis 5(3) says that these KMS spectrum elements are distinct modulo \mathbb{Z} , i.e., $u - u' \notin \mathbb{Z} \times \{0\}$. Recall the formulas from [29], given in the Introduction as (1) and (2) giving the parabolic weight $\mathfrak{p}(\lambda)$ and eigenvalue $\epsilon(\lambda)$ corresponding to u , and $\mathfrak{p}'(\lambda)$ and $\epsilon'(\lambda)$ corresponding to u' .

The main problem, related to what Mochizuki [29] calls “difficulty (b)” and to Sabbah’s picture [37, page 70], is that the parabolic weights might become equal, for some values of λ , and similarly, the eigenvalues could become resonant. Luckily, these two things do not happen simultaneously. Indeed the modifications given by (1) and (2) are always bijective. Thus, the pairs $(\mathfrak{p}(\lambda), \epsilon(\lambda))$ and $(\mathfrak{p}'(\lambda), \epsilon'(\lambda))$ are distinct modulo the action of \mathbb{Z} for each λ .

This allows us to use the groupoid \mathcal{G}_{Hod} to move around enough to define the quasi-parabolic structures in a holomorphically varying way.

Look at a small neighborhood $\lambda_0 \in U \subset \mathbb{P}^1$. We may assume one of the two cases: either

- (a) $\mathfrak{p}(\lambda) \neq \mathfrak{p}'(\lambda)$ in \mathbb{R}/\mathbb{Z} for all $\lambda \in U$, or
- (b) $\mathfrak{p}(\lambda_0) = \mathfrak{p}'(\lambda_0) + k$, $k \in \mathbb{Z}$, but $\epsilon(\lambda) \neq \epsilon'(\lambda) - k\lambda$ for all $\lambda \in U$.

We can assume (possibly by reducing the size of U) that there is some $b \in \mathbb{R}/\mathbb{Z}$ that is distinct from $\mathfrak{p}(\lambda)$ and $\mathfrak{p}'(\lambda)$ for all $\lambda \in U$. This leads to a family of bundles with logarithmic λ -connections $(E(\lambda), \nabla(\lambda))$ for $\lambda \in U$.

For $y \in D$, in case (a), the parabolic structure on $E(\lambda)$ has distinct parabolic weights for all λ , so we get a rank 1 subbundle of E_y (these are a bundle with subbundle in terms of the parameter λ). In case (b) at y , the two eigenvalues of the residue on $E(\lambda)$ are $\epsilon(\lambda)$ and $\epsilon'(\lambda) - k\lambda$; since they are distinct, we can choose in a uniform way one of the two eigenspaces of the residue of $\nabla(\lambda)$ over $\lambda \in U$.

Either way, we obtain a rank 1 subbundle of E_y required to define a quasi-parabolic structure.

We note that at each point λ , the associated quasi-parabolic logarithmic λ -connection satisfies the conditions of Hypothesis 5. Indeed for $\lambda = 0$, the irreducibility of the spectral curve is an assumption on our harmonic bundle. In turn, this implies that the harmonic bundle is indecomposable, so the associated canonical parabolic objects are stable for any λ . In our construction, the quasi-parabolic structure may be different so an argument is needed when $\lambda \neq 0$. There exists a set of parabolic weights for the quasi-parabolic structure that makes it stable. In case (a), we keep the given parabolic weights, whereas in case (b) we choose parabolic weights for the subbundle that are very close to $\mathfrak{p}(\lambda) \approx \mathfrak{p}'(\lambda) + k$. Stability of the canonical parabolic object implies stability of this parabolic object. Now, the only quasi-parabolic endomorphisms would be endomorphisms of the stable parabolic object so they are scalars, giving part (1) of Hypothesis 5 in the case $\lambda \neq 0$.

Now that we know Hypothesis 5 holds, this gives the data required to define a section $U \rightarrow \tilde{M}_{\text{Hod}}(X, \log D, x)$ if $U \subset \mathbb{P}^1 - \{\infty\}$, or similarly $U \rightarrow \tilde{M}_{\text{Hod}}(\tilde{X}, \log \tilde{D}, \tilde{x})$ if $U \subset \mathbb{P}^1 - \{0\}$.

A different choice of b and/or a different choice of one of the eigenvalues, leads to a section that differs by a section $U \rightarrow \mathcal{G}_{\text{Hod}}$. Therefore, on intersections of open sets U these glue together to give a well-defined section to the target modulo the groupoid, as stated. \square

This theorem gives the first part of Theorem 8.

3. Tangent spaces and cohomology

3.1 Deformations of quasi-parabolic logarithmic connections

The deformation theory is well-known, see for example [24] and subsequent literature for the Higgs case, or [21] for parabolic logarithmic connections.

Suppose (E, ∇, F, β) is a framed quasi-parabolic bundle with logarithmic λ -connection on (Y, D) . We would like to write down the complex governing its deformations. Let $\text{End}(E) = E^* \otimes E$ denote the endomorphism bundle. For $y \in Y$, we have a map $\text{End}(E) \rightarrow \text{Hom}(F_y, E_y/F_y)$. Combining these together, define the sheaf of *quasi-parabolic endomorphisms* to be the kernel in the exact sequence

$$0 \rightarrow \text{End}_Q(E) \rightarrow \text{End}(E) \rightarrow \bigoplus_{y \in D} \text{Hom}(F_y, E_y/F_y) \rightarrow 0,$$

where the sheaf on the right is a direct sum of skyscraper sheaves located at $y \in D$. Let $\mathcal{R}_y := \text{End}(F_y) \oplus \text{End}(E_y/F_y)$ denote the space of residue eigenvalues at y . We have a map $\text{End}_Q(E) \rightarrow \mathcal{R}_y$ and we define the sheaf of *strongly quasiparabolic endomorphisms* to be the kernel in the exact sequence

$$0 \rightarrow \text{End}_{SQ}(E) \rightarrow \text{End}_Q(E) \rightarrow \bigoplus_{y \in D} \mathcal{R}_y \rightarrow 0.$$

Furthermore, recall that $x \in X$ is our base point; let $\text{End}_{Q,x}(E)$ be the kernel of $\text{End}_Q(E) \rightarrow \text{End}(E_x)$.

Consider the Zariski tangent space $T := T_{(E, \nabla, F, \beta)} \tilde{M}_\lambda(Y, D, x)$. Let $W_0T \subset T$ be the tangent space to the changes of framing, and let W_1T be the subspace of deformations that preserve the eigenvalues of the residue of ∇ on F_y and E_y/F_y . Set $W_{-1}T := 0$ and $W_2T := T$.

Lemma 16. *The operator ∇ induces a logarithmic λ -connection on $\text{End}(E)$ and this restricts to an operator $\text{End}_Q(E) \xrightarrow{d_\nabla} \text{End}_{SQ}(E) \otimes \Omega_Y^1(\log D)$. We get a complex, and also, complexes from any compositions in the sequence*

$$\begin{array}{ccc} \text{End}_{Q,x}(E) & \longrightarrow & \text{End}_Q(E) \\ & & \downarrow \\ & & \text{End}_{SQ}(E) \otimes \Omega_Y^1(\log D) \longrightarrow \text{End}_Q(E) \otimes \Omega_Y^1(\log D). \end{array}$$

The Zariski tangent space $T = T_{(E, \nabla, F, \beta)} \tilde{M}_\lambda(Y, D, x)$ to the moduli space at (E, ∇, F, β) is the first hypercohomology

$$T = \mathbb{H}^1(\text{End}_{\mathcal{O}, x}(E) \xrightarrow{d_\nabla} \text{End}_{\mathcal{O}}(E) \otimes \Omega_Y^1(\log D)).$$

The subquotients $W_k T / W_n T$ are those induced by the various complexes obtained from the above sequence, for example,

$$W_1 T / W_0 T = \mathbb{H}^1(\text{End}_{\mathcal{O}}(E) \xrightarrow{d_\nabla} \text{End}_{S\mathcal{O}}(E) \otimes \Omega_Y^1(\log D)).$$

Lemma 17. Suppose that E has no strictly quasi-parabolic endomorphisms, i.e., there are no ∇ -invariant sections of $\text{End}_{S\mathcal{O}}(E)$. Then

$$\mathbb{H}^2(\text{End}_{\mathcal{O}}(E) \xrightarrow{d_\nabla} \text{End}_{\mathcal{O}}(E) \otimes \Omega_Y^1(\log D)) = 0$$

and the moduli functor is smooth at $(\lambda, E, \nabla, F, \beta)$. Suppose the only quasi-parabolic endomorphisms are scalars. Then the sequences

$$0 \rightarrow \mathbb{C} \rightarrow \text{End}(E_x) \rightarrow W_0 T \rightarrow 0$$

and

$$0 \rightarrow W_2 T / W_1 T \rightarrow \bigoplus_{y \in D} \mathcal{R}_y \rightarrow \mathbb{C} \rightarrow 0$$

are exact.

Proof. We may apply Serre duality to the complex

$$[\text{End}_{\mathcal{O}}(E) \xrightarrow{d_\nabla} \text{End}_{\mathcal{O}}(E) \otimes \Omega_Y^1(\log D)].$$

Some care must be taken with the fact that the differential is a first-order operator; the Serre dual becomes

$$\begin{aligned} & [\text{End}_{\mathcal{O}}(E)^*(-D) \xrightarrow{d_\nabla} \text{End}_{\mathcal{O}}(E)^* \otimes \Omega_Y^1] \\ & \cong [\text{End}_{S\mathcal{O}}(E) \xrightarrow{d_\nabla} \text{End}_{S\mathcal{O}}(E) \otimes \Omega_Y^1(\log D)]. \end{aligned}$$

Thus, the \mathbb{H}^2 in the first part of the lemma is dual to the space of strictly quasi-parabolic endomorphisms, yielding the first statement. Furthermore, \mathbb{H}^2 is the obstruction space for the deformation theory, so if it vanishes, the moduli functor is smooth. For the last part, we use some exact sequences together with the hypothesis that

$$\mathbb{H}^0(\text{End}_{\mathcal{O}}(E) \xrightarrow{d_\nabla} \text{End}_{S\mathcal{O}}(E) \otimes \Omega_Y^1(\log D)) = \mathbb{C}.$$

□

COROLLARY 18

The moduli space $\tilde{M}_{\text{Hod}}(Y, \log D, x)$ is smooth and hence also the disjoint union $\tilde{M}_{\text{DH}}(Y, \log D, x)$ is smooth.

Proof. Recall that \tilde{M}_{Hod} was defined as the moduli space of objects satisfying Hypothesis 5. Condition (1) of the hypothesis implies, since by assumption $D \neq \emptyset$, that there are no strictly quasi-parabolic endomorphisms, so Lemma 17 applies. \square

This corollary completes the proof of Theorem 6.

Suppose $(E_X, \partial, \bar{\partial}, \varphi, \varphi^\dagger, h)$ is a tame harmonic bundle on X , satisfying Hypothesis 5, leading to a preferred section $\rho : \mathbb{P}^1 \rightarrow (\tilde{M}_{\text{DH}}, \mathcal{G}_{\text{DH}})$ by Theorem 15. By Corollary 18, \tilde{M}_{DH} is smooth. We obtain the pullback by ρ of the relative tangent bundle $TM_{\tilde{M}_{\text{DH}}}/\mathbb{P}^1$, that is furnished with glueing data to obtain a bundle T_ρ over \mathbb{P}^1 .

Let W_0T_ρ denote the subbundle of relative tangent vectors corresponding to change of framing β . Let W_1T_ρ be the subbundle of relative tangent vectors whose projection to the tangent of the space of residues (R, \mathcal{G}_R) is trivial, and let $W_2T_\rho := T_\rho$ and $W_{-1}T_\rho := 0$. These definitions are compatible with the ones in Lemma 16.

The goal of this section is to prove that with this weight filtration, T_ρ becomes a mixed twistor structure, in other words, $W_kT_\rho/W_{k-1}T_\rho$ is a semistable bundle of slope k on \mathbb{P}^1 . The idea is to apply [29,37].

3.2 Identification with a pure twistor cohomology

We look in the neighborhood of a point $y \in D$.

Let $u = (a, \alpha)$ and $u' = (a', \alpha')$ be the two KMS spectrum elements, in coordinates at $\lambda = 0$. Hypothesis 5(3) says that $u - u' \notin \mathbb{Z} \times \{0\}$. Let $\mathfrak{p}(\lambda)$ and $\epsilon(\lambda)$ be the parabolic weight and eigenvalue at λ corresponding to u , and $\mathfrak{p}'(\lambda)$ and $\epsilon'(\lambda)$ corresponding to u' .

The endomorphism bundle decomposes as a direct sum

$$\text{End}(E) = \mathcal{O} \oplus S,$$

where $S := \text{End}^0(E)$ is the trace-free endomorphism bundle. This latter has rank 3, and the KMS spectrum elements are $(u - u'), 0, (u' - u)$.

The case of rank 1 systems was dealt with in [42] so we may focus here on the deformations parametrized by the trace-free part S .

Choose a point $\lambda_0 \neq 0$, and set

$$p := \mathfrak{p}(\lambda_0), \quad p' := \mathfrak{p}'(\lambda_0), \quad e := \epsilon(\lambda_0), \quad e' := \epsilon'(\lambda_0).$$

There are two basic cases, depending on whether $p - p'$ is an integer.

Case 1. Suppose $p - p' \in \mathbb{Z}$. The parabolic weights of S are integers. Let

$$\epsilon''(\lambda) := \epsilon'(\lambda) - \lambda(p - p')$$

be the eigenvalue corresponding to the transition of $\epsilon'(\lambda)$ from parabolic weight p' to parabolic weight $p'' := p$, and let $e'' := \epsilon''(\lambda_0)$.

We obtain the locally free sheaf in the parabolic structure $E := E_{p+\epsilon}$ for some small ϵ , defined in a small neighborhood $U^{(\lambda_0)}$ of λ_0 in the λ -line. Put

$$G_E := E_{p+\epsilon}/E_{p-\epsilon}.$$

It is a rank two bundle over $\{y\} \times U^{(\lambda_0)}$. We recall from [29] that it has a decomposition according to the eigenvalues of the residue. Those eigenvalues are, as functions of λ ,

$$\epsilon(\lambda), \quad \epsilon''(\lambda).$$

The hypothesis of distinct KMS spectrum elements implies that $e \neq e''$, so we may identify the sections $\epsilon(\lambda)$ and $\epsilon''(\lambda)$ by the notations e and e'' . The eigenspaces of $\text{res}_y(\nabla)$ acting on G_E may therefore be denoted by $G_{E,e}$ and $G_{E,e''}$ and

$$G_E = G_{E,e} \oplus G_{E,e''}.$$

This decomposition is valid over the neighborhood $U^{(\lambda_0)}$, so $\text{res}_y(\nabla)$ acts by $\epsilon(\lambda)$ on $G_{E,e}$ and by $\epsilon''(\lambda)$ on $G_{E,e''}$.

Choose one of the two subspaces, say for example, $G_{E,e}$ to be F_y in the quasi-parabolic structure over our neighborhood $U^{(\lambda_0)}$.

For the trace-free endomorphism bundle, we obtain a locally free sheaf from the parabolic structure $S = S_\epsilon$ for some small ϵ , defined in the neighborhood $U^{(\lambda_0)}$ possibly reducing its size. Put

$$G := S_\epsilon/S_{-\epsilon}.$$

It is a rank three bundle over $\{y\} \times U^{(\lambda_0)}$. It has a decomposition according to the eigenvalues of the residue. We recall that there are three different eigenvalues here:

$$\epsilon(\lambda) - \epsilon''(\lambda), \quad 0, \quad \epsilon''(\lambda) - \epsilon(\lambda).$$

Denote the corresponding subspaces by $G_{(e-e'')}$, G_0 and $G_{(e''-e)}$, so

$$G = G_{(e-e'')} \oplus G_0 \oplus G_{(e''-e)}.$$

We have locally free subsheaves

$$Q' := \text{End}_{S_Q}^0(E) \subset Q := \text{End}_Q^0(E) \subset S = \text{End}^0(E)$$

of trace-free endomorphisms that strictly preserve (resp. preserve) the quasi-parabolic structure. Along $\{y\} \times U^{(\lambda_0)}$, the endomorphisms not preserving the parabolic structure are the ones that send $G_{E,e}$ to $G_{E,e''}$, that is to say they have eigenvalue $(e'' - e)$. The ones in Q that act trivially on the graded pieces are Q' , and these are the ones mapping to zero in G_0 . Thus, we have exact sequences

$$0 \rightarrow Q \rightarrow S_\epsilon \rightarrow G_{(e''-e)} \rightarrow 0 \tag{3}$$

and

$$0 \rightarrow Q' \rightarrow S_\epsilon \rightarrow G_0 \oplus G_{(e''-e)} \rightarrow 0. \tag{4}$$

In order to define the bundle over $U^{(\lambda_0)} \subset \mathbb{A}^1$ that corresponds to the Sabbah–Mochizuki pure twistor structure on cohomology, following the discussion in the Appendix of [29], we should consider the germs of holomorphic bundles of sections of the parabolic extension that are locally L^2 with respect to the Poincaré metric. Here these are germs around the point λ_0 in the λ -line.

We will denote these by \mathcal{L}^2 , and we have the complex with two bundles

$$\mathcal{L}^2(S(*D)) \rightarrow \mathcal{L}^2(S \otimes \Omega^1(*D)).$$

In the present setting, $\mathcal{L}^2(S) = \mathcal{L}^2(S(*D))$ is the sheaf of sections of S_ϵ whose projection into G lies in the piece G_0 ; in other words, we have an exact sequence

$$0 \rightarrow \mathcal{L}^2(S) \rightarrow S_\epsilon \rightarrow G_{(e-e'')} \oplus G_{(e''-e)} \rightarrow 0.$$

We recall that this is due to the fact that a section projecting into one of the other pieces will go, on a half-disk centered at λ_0 , into a piece where the parabolic weight is > 0 , so it would have a growth rate of $|z|^a$ with $a > 0$ for values of λ in that half-disk of $U^{(\lambda_0)}$. By definition, here we are looking at germs around λ_0 .

A similar description works for holomorphic one-forms, taking coefficients with logarithmic poles. However, in that case we should introduce the W_{-2} term of the weight filtration on the G_0 piece. Here, all the KMS spectrum eigenspaces have rank 1 so the weight filtration is trivial, thus $W_{-2} = 0$ in the G_0 piece. This means we have an exact sequence

$$0 \rightarrow \mathcal{L}^2(S \otimes \Omega_Y^1(*D)) \rightarrow S_\epsilon \otimes \Omega_Y^1(\log D) \rightarrow G_{(e-e'')} \oplus G_{(e''-e)} \oplus G_0 \rightarrow 0$$

hence

$$\mathcal{L}^2(S \otimes \Omega_Y^1(*D)) = S_{-\epsilon} \otimes \Omega_Y^1(\log D) = S_\epsilon \otimes \Omega_Y^1.$$

A word about notation: these objects are all really over $U^{(\lambda_0)}$ but we do not write, for e.g., $D \times U^{(\lambda_0)} \subset Y \times U^{(\lambda_0)}$ etc., and also the space of residues $\Omega^1(*D)_Y$ is a trivial bundle (over $U^{(\lambda_0)}$). A trivialization is chosen and used in the expressions, for example, the one-prior exact sequence.

Using the description of [29], the bundle of cohomology of the pure twistor \mathscr{D} -module has, as germ around λ_0 , the first hypercohomology of the complex $\mathcal{L}^2(S) \rightarrow \mathcal{L}^2(S \otimes \Omega_Y^1(*D))$ or isomorphically

$$[\mathcal{L}^2(S) \rightarrow S_{-\epsilon} \otimes \Omega_Y^1(\log D)].$$

Let us compare this with the complex that governs the deformations of the quasi-parabolic logarithmic bundle (E, ∇, F) . Recall from the previous discussion that this complex is

$$[Q \rightarrow Q' \otimes \Omega^1(\log D)]$$

and that we have exact sequences (3) and (4). Comparing with the exact sequences for the L^2 sheaves, we get

$$0 \rightarrow \mathcal{L}^2(S) \rightarrow Q \rightarrow G_{(e-e'')} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{L}^2(S \otimes \Omega^1(*D)) \rightarrow Q' \otimes \Omega^1(\log D) \rightarrow G_{(e-e'')} \rightarrow 0.$$

Therefore our two complexes fit into a diagram

$$\begin{array}{ccc} \mathcal{L}^2(S) & \longrightarrow & \mathcal{L}^2(S \otimes \Omega^1(*D)) \\ \downarrow & & \downarrow \\ Q & \longrightarrow & Q' \otimes \Omega^1(\log D) \\ \downarrow & & \downarrow \\ G_{(e-e'')} & \xrightarrow{\cong} & G_{(e-e'')} \end{array} .$$

COROLLARY 19

In Case 1, there is a natural quasi-isomorphism of complexes

$$[\mathcal{L}^2(S) \rightarrow \mathcal{L}^2(S \otimes \Omega^1(*D))] \xrightarrow{q.i.} [Q \rightarrow Q' \otimes \Omega^1(\log D)].$$

Case 2. Suppose $p - p' \notin \mathbb{Z}$. Then the parabolic weights of S are not integers. This holds at λ_0 so we may choose our small neighborhood $U^{(\lambda_0)}$ so that it holds on the neighborhood, and furthermore the neighborhood is made small enough so that the various inequalities we will state below also hold on $U^{(\lambda_0)}$.

Choose a representative (up to the Hecke-gauge group action and tensoring with a rank 1 system) such that the parabolic weights $b, b - q$ of E , depending on λ , that are in the interval $(-1, 0)$ differ by q with $0 < q < 1$.

We use $E := E_{b(\lambda_0)+\epsilon}$ as the bundle, and this is locally free. The parabolic weight subspace

$$F_y := E_{b+\epsilon-q}/E_{b+\epsilon-1} \hookrightarrow E_{b+\epsilon}/E_{b+\epsilon-1} = E|_{\{y\} \times U^{(\lambda_0)}}$$

will be used to define our quasi-parabolic structure over the neighborhood $U^{(\lambda_0)}$.

The bundle of trace-free endomorphisms $S = \text{End}^0(E)$ has a three-step parabolic structure whose parabolic weights in the interval $(-1, 0]$ are

$$0, -q, q - 1.$$

Although the weights $q - 1$ and $-q$ might cross over, they remain bounded away from 0 and -1 over the neighborhood so the sheaves S_ϵ and $S_{-\epsilon}$ are well defined.

The sub-bundle of endomorphisms that preserve the quasi-parabolic structure is $Q = \text{End}_Q^0(E) = S_\epsilon$, whereas the sub-bundle of those that also act trivially on the graded pieces is $Q' = \text{End}_{S_Q}^0(E) = S_{-\epsilon}$. This is uniform over $\lambda \in U^{(\lambda_0)}$.

The locally L^2 sections of S (which are the same as those of $S(*D)$) are

$$\mathcal{L}^2(S) = S_\epsilon$$

since the KMS spectrum element has constant eigenvalue equal to 0 here and the eigenspace has rank 1, so it is equal to its W_0 piece. The W_{-2} piece is zero in the associated-graded, so

$$\mathcal{L}^2(S \otimes \Omega^1(*D)) = W_{-2}S_\epsilon \otimes \Omega^1(D) = S_{-\epsilon} \otimes \Omega^1(D).$$

In this case, we conclude that the L^2 complex and the deformation theory complex are actually equal.

COROLLARY 20

In Case 2, the complex of locally L^2 forms is equal to the deformation complex for the quasi-parabolic structure

$$[\mathcal{L}^2(S) \rightarrow \mathcal{L}^2(S \otimes \Omega^1(*D))] = [Q \rightarrow Q' \otimes \Omega^1(\log D)].$$

3.3 Globalization

Now go back to the global case and put these two corollaries together. Let $\mathbf{H}^1(\text{End}^0(\mathfrak{E}))$ denote the Sabbah–Mochizuki [29,37] pure twistor structure for the cohomology of the

pure twistor \mathcal{D} -module corresponding to the trace-free endomorphisms of our harmonic bundle. For the central or scalar part, let $H_{\text{DH}}^i(X)$ denote the twistor structure of weight i associated to the pure Hodge structure on the cohomology of X .

The relative tangent bundle of our Deligne–Hitchin groupoid is pulled back along the preferred section to give $\rho^*T(M_{\text{DH}}/\mathbb{P}^1)$. We defined a three-step filtration

$$0 = W_{-1} \subset W_0 \subset W_1 \subset W_2 = \rho^*T(M_{\text{DH}}/\mathbb{P}^1),$$

where W_0 is the deformation of the framing, W_1/W_0 is the deformation of the quasi-parabolic logarithmic λ -connection conserving the eigenvalues of the residue, and W_2/W_1 is the deformation of the residual data. The filtrations coming from the two M_{Hod} pieces glue over \mathbb{G}_m , because they coincide with the similarly defined filtrations on the tangent to M_B pulled back along the preferred section.

Theorem 21. *We have a natural isomorphism*

$$Gr_1^W \rho^*T(M_{\text{DH}}/\mathbb{P}^1) \cong \mathbf{H}^1(\text{End}^0(\mathcal{E})) \oplus H_{\text{DH}}^1(X).$$

In particular, Gr_1^W is a pure twistor structure of weight 1. For $i = 0, 2$, we also have that Gr_i^W is a pure twistor structure of weight i ; in other words, our tangent space with the given weight filtration is a mixed twistor structure.

Proof. In the previous subsections, we have shown that there is a natural quasi-isomorphism between the complex calculating deformations of the quasi-parabolic logarithmic λ -connection, and the complex of locally L^2 forms shown by [29] to calculate the twistor \mathbf{H}^1 . This was done for germs in the neighborhood of any $\lambda_0 \in \mathbb{A}^1$.

It needs to be checked that this natural isomorphism is compatible with the glueing to the other chart of the twistor \mathbb{P}^1 . In the Sabbah–Mochizuki theory, this glueing is done using the sesquilinear pairing in Sabbah’s definition of \mathcal{R} -triple [37], whereas for the tangent of deformation theory, it comes from comparison with the Betti moduli spaces.

To make the comparison with [37], recall that an \mathcal{R} -triple has two \mathcal{R}_X -modules that are related by a sesquilinear pairing. The first module is the minimal extension of the harmonic bundle (in this case, $S = \text{End}^0(E)$) from $\mathbb{A}^1 \times X$ to $\mathbb{A}^1 \times Y$, and the second one is the same for \bar{X} , but complex-conjugated back to being an object over $\mathbb{A}^1 \times Y$. The sesquilinear pairing, defined over the unit circle $|\lambda| = 1$, takes values in distributions. The precise structure is complicated near points of the divisor D since the modules involve meromorphic sections, so this brings into play the division of distributions [38] and Mellin transform. Over points of X , the pairing is just the same identification between local systems on X and \bar{X} that we are using.

The higher direct image in [37] is calculated using the Dolbeault resolution, then the pairing on the cohomology (i.e. higher direct image to a point) involves wedging forms, pairing the \mathcal{R}_X -module coefficients, and integrating it [37, §1.6.d].

We can avoid having to look too closely at the behavior near singularities. This is because we need to understand the Betti glueing vs. the sesquilinear pairing, for classes in H^1 . It suffices to verify that the identifications are the same for general values of λ on the unit circle. We may therefore assume that the two residue eigenvalues at any $y \in D$ do not differ by integer multiples of λ . Our cohomology space in question then has the property that it is the image of the map $H_c^1(X, S^{\nabla\lambda}) \rightarrow H^1(X, S^{\nabla\lambda})$ from compactly supported cohomology to cohomology over X . We may therefore represent cohomology classes by forms that are

compactly supported on X , and pair them with other forms that are compactly supported on X in order to check the identifications.

In this setting, the pairing formula from [37, §1.6.d] is just the usual cup-product of cohomology classes via the identification between de Rham cohomology of a λ -connection and Betti cohomology. That identification is the one that occurs for the tangent spaces of our moduli spaces under the identification between tangent spaces, hence deformation spaces and cohomology spaces. This gives the compatibility.

From the general theory of [29, 37], we get that $\mathbf{H}^1(\text{End}^0(\mathcal{E}))$ is a pure twistor structure of weight 1 meaning that as a bundle it is a direct sum of copies of $\mathcal{O}_{\mathbb{P}^1}(1)$. We therefore obtain that required weight property for $Gr_1^W \rho^* T(M_{\text{DH}}/\mathbb{P}^1)$.

For the Gr_0^W piece, it is easy to see that the deformations of change of framing give a trivial bundle over \mathbb{P}^1 since the framing does not depend on anything. The space of deformations of change of framing, globalized over the twistor line, is $\text{End}(\mathcal{O}_{\mathbb{P}^1}^2)/H_{\text{DH}}^0(X)$ taking the quotient by the subspace $H_{\text{DH}}^0(X) = \mathcal{O}_{\mathbb{P}^1}$ of scalar endomorphisms of the bundle. This has a weight 0 twistor structure.

For the Gr_2^W piece, we refer to the discussion of [42] for the weight two property of the space of deformations of the residual data. Here again there is a modification by $H_{\text{DH}}^2(X)$, namely, we have an exact sequence

$$0 \rightarrow Gr_2^W \rightarrow \bigoplus_{y \in D} \mathcal{R}_{y, \text{DH}} \rightarrow H_{\text{DH}}^2(X) \rightarrow 0$$

corresponding to the condition that the sums of all the residues should vanish. It is a condition happening on the determinant bundle. The morphism on the right is a morphism of weight 2 twistor structures so the kernel Gr_2^W is a weight 2 twistor structure.

This concludes the proof that the full relative tangent space, along the preferred section, has a mixed twistor structure with weights 0, 1, 2. \square

This in turn, completes the proof of Theorem 8.

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