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DIAGONAL CONTRACTIVE MAPS OVER METRIC SPACES

MIHAI TURINICI

ABSTRACT. A functional extension is given for the diagonal fixed point result in Cirić and Prešić [Acta Math. Comeniana, 76 (2007), 143-147] involving multistep type iterations.

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KEYWORDS: Metric space, diagonal fixed point, strong Prešić operator, Cirić-Prešić contraction, Matkowski admissible function.

1. INTRODUCTION

Let (X, d) be a metric space. Further, let $S \in \mathcal{F}(X)$ be a selfmap of X . [Here, for each couple A, B of nonempty sets, $\mathcal{F}(A, B)$ denotes the class of all functions from A to B ; when $A = B$, we write $\mathcal{F}(A)$ in place of $\mathcal{F}(A, A)$]. Denote $\text{Fix}(S) = \{x \in X; x = Sx\}$; each point of this set is referred to as *fixed* under S . Concerning the determination of such points, the basic result in this area is the 1922 one due to Banach [2]. Call the subset Y of X , *almost singleton* (in short: *asingleton*), provided $[y_1, y_2 \in Y \text{ implies } y_1 = y_2]$; and *singleton* if, in addition, Y is nonempty; note that in this case $Y = \{y\}$, for some $y \in X$. Then, let us say that S is *Banach $(d; \alpha)$ -contractive* (where $\alpha \geq 0$), if

$$d(Sx, Sy) \leq \alpha d(x, y), \text{ for all } x, y \in X.$$

Theorem 1.1. *Assume that S is Banach $(d; \alpha)$ -contractive, for some $\alpha \in [0, 1[$. In addition, let X be d -complete. Then,*

(11-a) *S is fix-singleton: $\text{Fix}(S) = \{z\}$, for some $z \in X$*

(11-b) *S is a strong Picard operator: $\lim_n S^n x = z, \forall x \in X$.*

This result (referred to as: *Banach fixed point theorem*) found some basic applications to the operator equations theory. As a consequence, many extensions for it were proposed. From the perspective of our present exposition, the *diagonal* ones are of interest; these consist in the initial selfmap S being viewed as the *diagonal part* of a mapping $T : X^k \rightarrow X$ (for some $k \geq 1$); i.e.,

$$Sx = T(x^k), x \in X;$$

where, for each $x \in X$, we denoted

$$x^k = \text{the element } (z_0, \dots, z_{k-1}) \in X^k, \text{ with } (z_i = x; 0 \leq i \leq k-1).$$

In this context, denote

$$\text{Fixd}(T) = \text{Fix}(S) (= \{z \in X; z = Sz\});$$

each element of this set will be called a *diagonal fixed point* of T .

The existence and uniqueness of such points is to be discussed in the context below. Given $U_0 := (u_0, \dots, u_{k-1}) \in X^k$, the sequence $(u_n; n \geq 0)$ in X given as

$$\text{(iter) } u_n = \text{the above one, } 0 \leq n \leq k-1; u_n = T(u_{n-k}, \dots, u_{n-1}), n \geq k$$

will be called the *iterative process* generated by U_0 ; and denoted (for simplicity) as $(u_n = T^n U_0; n \geq 0)$. The determination of these points is to be performed upon the directions below, comparable with the ones in Rus [18, Ch 2, Sect 2.2]:

pp-0) Let us say that T is *fixd-asingleton*, if $\text{Fixd}(T) = \text{Fix}(S)$ is an asingleton; and *fixd-singleton*, provided $\text{Fixd}(T) = \text{Fix}(S)$ is a singleton

pp-1) Call $U_0 = (u_0, \dots, u_{k-1}) \in X^k$, a *Prešić point* (modulo (d, T)), provided the associated iterative sequence $(u_n := T^n U_0; n \geq 0)$ is d -Cauchy. If each $U_0 = (u_0, \dots, u_{k-1}) \in X^k$ is a Prešić point (modulo (d, T)), then T will be referred to as a *Prešić operator* (modulo d)

pp-2) Call $U_0 = (u_0, \dots, u_{k-1}) \in X^k$, a *strong Prešić point* (modulo (d, T)), provided the associated iterative sequence $(u_n := T^n U_0; n \geq 0)$ is d -convergent and $z := \lim_n(u_n)$ is an element of $\text{Fixd}(T) = \text{Fix}(S)$. If each $U_0 = (u_0, \dots, u_{k-1}) \in X^k$ is a strong Prešić point (modulo (d, T)), then T will be referred to as a *strong Prešić operator* (modulo d).

Sufficient conditions for such properties involve metrical contractions. Given $\Gamma := (\gamma_0, \dots, \gamma_{k-1}) \in R_+^k$, let us say that T is *Prešić $(d; \Gamma)$ -contractive*, provided

$$\begin{aligned} \text{(P-contr)} \quad & d(T(x_0, \dots, x_{k-1}), T(x_1, \dots, x_k)) \leq \\ & \gamma_0 d(x_0, x_1) + \dots + \gamma_{k-1} d(x_{k-1}, x_k), \text{ for each } (x_0, \dots, x_k) \in X^{k+1}. \end{aligned}$$

The regularity condition imposed upon the vector Γ appearing here is

$$\begin{aligned} \text{(norm-sub)} \quad & \Gamma = (\gamma_0, \dots, \gamma_{k-1}) \text{ is } \textit{norm subunitary}: \\ & |\Gamma| := \gamma_0 + \dots + \gamma_{k-1} < 1. \end{aligned}$$

The following 1965 fixed point result obtained by Prešić [14] is our starting point.

Theorem 1.2. *Suppose that T is Prešić $(d; \Gamma)$ -contractive, where $\Gamma = (\gamma_1, \dots, \gamma_k) \in R_+^k$ is norm subunitary. In addition, let X be d -complete. Then,*

(12-a) *T is fixd-singleton: $\text{Fixd}(T) = \text{Fix}(S) = \{z\}$, for some $z \in X$*

(12-b) *T is a strong Prešić operator (modulo d): for each (starting point) $U_0 := (u_0, \dots, u_{k-1}) \in X^k$, the iterative sequence $(u_n = T^n U_0; n \geq 0)$ fulfills $u_n \xrightarrow{d} z$.*

Concerning the imposed conditions, note that for each $x, y \in X$,

$$\begin{aligned} d(Sx, Sy) &= d(T(x^k), T(y^k)) \leq \\ & d(T(x^k), T(x^{k-1}, y)) + \dots + d(T(x, y^{k-1}), T(y^k)) \leq \\ & \gamma_0 d(x, y) + \dots + \gamma_{k-1} d(x, y) = |\Gamma| d(x, y). \end{aligned}$$

As a consequence of this,

$$S \text{ is Banach } (d; |\Gamma|)\text{-contractive, where } |\Gamma| < 1;$$

which, along with Banach fixed point theorem yields

$$\text{Fixd}(T) = \text{Fix}(S) = \text{singleton}.$$

In other words, the existence and uniqueness part of Theorem 1.2 are directly assured by the ambient conditions; so, its novelty consists in the underlying diagonal fixed point being approximated via Prešić type iterative processes.

In particular, when $k = 1$, Theorem 1.2 is just the Banach contraction principle. Hence, the question of extending it is not without interest. A basic contribution in this area is the 2007 one due to Ćirić and Prešić [4]. Let us say that T is *Ćirić-Prešić $(d; \beta)$ -contractive* (where $\beta \geq 0$), provided

$$\begin{aligned} \text{(CP-contr)} \quad & d(T(x_0, \dots, x_{k-1}), T(x_1, \dots, x_k)) \leq \\ & \beta \max\{d(x_0, x_1), \dots, d(x_{k-1}, x_k)\}, \text{ for each } (x_0, \dots, x_k) \in X^{k+1}. \end{aligned}$$

Theorem 1.3. *Assume that T is Cirić-Prešić $(d; \beta)$ -contractive, for a certain $\beta \in [0, 1[$. In addition, let X be d -complete. Then, the following conclusions hold:*

(13-a) T is fixed-asingleton, whenever

(S-snex) S is d -strictly-nonexpansive:

$$d(Sx, Sy) < d(x, y), \forall x, y \in X, x \neq y$$

(13-b) T is strong Prešić (modulo d): for each $U_0 := (u_0, \dots, u_{k-1}) \in X^k$, the iterative sequence $(u_n = T^n U_0; n \geq 0)$ fulfills $u_n \xrightarrow{d} z$ as $n \rightarrow \infty$, for some diagonal fixed point $z \in \text{Fixd}(T) = \text{Fix}(S)$.

Having these precise, it is our aim in the following to extend from a functional perspective this last class of results. Further aspects will be delineated elsewhere.

2. MAIN RESULT

Let (X, d) be a metric space, and $k \geq 1$ be a natural number. Further, let $T : X^k \rightarrow X$ be a mapping, $S : X \rightarrow X$ be its associated diagonal map, and $\text{Fixd}(T) = \text{Fix}(S)$ stand for the class of diagonal fixed points of T . The basic setting of our problem was already sketched. It is our aim in the following to give a functional version of the Cirić-Prešić fixed point result. Some conventions are in order. Given a function $\varphi \in \mathcal{F}(R_+)$, call it *regressive* provided

$$\varphi(0) = 0 \text{ and } \varphi(t) < t, \text{ for all } t \in R_+^0 :=]0, \infty[;$$

the class of such functions will be denoted as $\mathcal{F}(re)(R_+)$. Further, let us introduce the sequential properties over $\mathcal{F}(re)(R_+)$

(M-a) $\varphi \in \mathcal{F}(re)(R_+)$ is *Matkowski admissible*:

for each $(t_n; n \geq 0)$ in R_+^0 with $(t_{n+1} \leq \varphi(t_n), \forall n)$ we have $\lim_n t_n = 0$

(str-M-a) $\varphi \in \mathcal{F}(re)(R_+)$ is *strongly Matkowski admissible*:

for each $(t_n; n \geq 0)$ in R_+^0 with $(t_{n+1} \leq \varphi(t_n), \forall n)$ we have $\sum_n t_n < \infty$.

(These conventions are taken from the developments in Matkowski [9] and Turinici [23], respectively). In particular, when $\varphi \in \mathcal{F}(re)(R_+)$ is increasing, then

(M-1) φ is Matkowski admissible iff $\lim_n \varphi^n(t) = 0$, for each $t > 0$;

(M-2) φ is strongly Matkowski admissible iff $\sum_n \varphi^n(t) < \infty, \forall t > 0$;

here, as usual, φ^n means the n -th *iterate* of φ , for each $n \geq 0$.

Clearly, each strongly Matkowski admissible function is Matkowski admissible too; but the reciprocal is not in general true.

For practical reasons, it would be useful to determine sufficient conditions under which this last property holds. Call the function $h \in \mathcal{F}(R_+^0)$, *int-normal* provided

(i-n-1) $h(\cdot)$ is decreasing on R_+^0

(i-n-2) $H(t) := \int_0^t h(\xi) d\xi < \infty$, for each $t > 0$.

Note that, by the former condition (i-n-1),

$$\int_0^t h(\xi) d\xi := \lim_{s \rightarrow 0+} \int_s^t h(\xi) d\xi \text{ exists in } R_+ \cup \{\infty\}, \text{ for each } t > 0;$$

so, the latter condition (i-n-2) is meaningful. Moreover, by these definitions,

(H-1) $H(\cdot)$ is strictly increasing on R_+^0 ($t_1 < t_2 \implies H(t_1) < H(t_2)$)

(H-2) $H(\cdot)$ is continuous on R_+^0 and $H(0+) := \lim_{t \rightarrow 0+} H(t) = 0$.

Given $\varphi \in \mathcal{F}(re)(R_+)$, let us associate it the function $g \in \mathcal{F}(R_+^0)$, as

$$(g(t) = t/(t - \varphi(t)); t > 0); \text{ in short: } g = I/(I - \varphi).$$

Further, call $g \in \mathcal{F}(R_+^0)$, *int-subnormal* provided

$g(t) \leq h(t)$, $t \in R_+^0$, where $h \in \mathcal{F}(R_+^0)$ is int-normal.

The following int-subnormal type strongly Matkowski criterion is available.

Theorem 2.1. *Let the function $\varphi \in \mathcal{F}(re)(R_+)$ be such that the associated function $g = I/(I - \varphi)$ is int-subnormal.*

Then, φ is strongly Matkowski admissible.

Proof. By the imposed condition,

$$g(t) \leq h(t), t \in R_+^0, \text{ for some int-normal function } h \in \mathcal{F}(R_+^0).$$

Let the sequence $(t_n; n \geq 0)$ in R_+^0 be such that

$$t_{n+1} \leq \varphi(t_n), \text{ for all } n \geq 0; \text{ hence, } (t_n; n \geq 0) \text{ is strictly descending.}$$

Further, let $i \geq 0$ be arbitrary fixed. By the above choice,

$$t_i - \varphi(t_i) \leq t_i - t_{i+1}; \text{ whence, } 1 \leq (t_i - t_{i+1}) / (t_i - \varphi(t_i)).$$

Combining with the decreasing property of $h(\cdot)$ yields (by the definition of $H(\cdot)$)

$$t_i \leq (t_i - t_{i+1})g(t_i) \leq (t_i - t_{i+1})h(t_i) \leq H(t_i) - H(t_{i+1}).$$

Passing to limit in the relation between extremal members gives

$$\lim_i t_i = 0; \text{ whence, } \varphi \text{ is Matkowski admissible.}$$

On the other hand, from the same relation,

$$\sum_{i \leq n} t_i \leq H(t_0) - H(t_{n+1}), \text{ for each } n;$$

wherefrom, by a limit process,

$$\sum_n t_n \leq H(t_0) - H(0+) < \infty;$$

which tells us that the series $\sum_n t_n$ converges. \square

Returning to our initial setting, denote for each i, j with $i < j$

$$A(z_i, \dots, z_j) = \max\{d(z_i, z_{i+1}), \dots, d(z_{j-1}, z_j)\}, (z_i, \dots, z_j) \in X^{j-i+1}.$$

Given $\varphi \in \mathcal{F}(R_+)$, let us say that T is (d, φ) -contractive, provided

$$\begin{aligned} (d\text{-}\varphi) \quad & d(T(y_0, \dots, y_{k-1}), T(y_1, \dots, y_k)) \leq \varphi(A(y_0, \dots, y_{k-1}, y_k)), \\ & \text{for all } (y_0, \dots, y_{k-1}, y_k) \in X^{k+1}. \end{aligned}$$

Further, let us remember that S is called d -strictly-nonexpansive, provided

$$(s\text{-nexp}) \quad d(Sx, Sy) < d(x, y), \text{ whenever } x \neq y.$$

Our main result in this exposition is

Theorem 2.2. *Suppose that T is (d, φ) -contractive, for some increasing strongly Matkowski admissible $\varphi \in \mathcal{F}(re)(R_+)$. In addition, let X be d -complete. Then, the following conclusions hold:*

(22-a) *If (in addition) S is d -strictly-nonexpansive, then*

T is fixd-asingleton; i.e.: $\text{Fixd}(T) = \text{Fix}(S)$ is an asingleton

(22-b) *T is strong Prešić (modulo d): for each $U_0 := (u_0, \dots, u_{k-1}) \in X^k$, the iterative sequence $(u_n := T^n U_0; n \geq 0)$ fulfills $\lim_n u_n = z$, for some diagonal fixed point $z \in \text{Fixd}(T) = \text{Fix}(S)$.*

Proof. The first half of this statement is immediate, as results from

Step 0. Let $z_1, z_2 \in X$ be a couple of points in $\text{Fixd}(T) = \text{Fix}(S)$ (i.e.: $z_1 = Sz_1$, $z_2 = Sz_2$); and – contrary to the written conclusion – suppose that $z_1 \neq z_2$. By the d -strict-nonexpansive property of S ,

$$d(z_1, z_2) = d(Sz_1, Sz_2) < d(z_1, z_2); \text{ contradiction.}$$

Hence, necessarily, $z_1 = z_2$; and the claim follows.

It remains now to establish that T is strong Prešić (modulo d). There are several stages to be passed.

Step 1. Put, for simplicity ($\rho_n = d(u_n, u_{n+1})$, $n \geq 0$). Note that, by a previous convention, we have for each i, j with $i < j$,

$$A(u_i, \dots, u_j) = \max\{\rho_i, \dots, \rho_{j-1}\}.$$

For technical reasons, it would be useful to denote, for each $n \geq k$,

$$B_n = A(u_{n-k}, \dots, u_{n-1}, u_n); \text{ i.e.: } B_n = \max\{\rho_{n-k}, \dots, \rho_{n-1}\}.$$

Proposition 2.1. *Suppose that*

$$B_n = 0, \text{ for some } n \geq k.$$

Then

(21-1) *The subsequence ($y_i := u_{n-k+i}; i \geq 0$) is constant; i.e.,*

$$u_{n-k} = u_{n-k+1} = \dots = a, \text{ for some } a \in X.$$

(21-2) *In addition, we have*

$$a = T(a^k) = Sa;$$

hence, $a \in X$ is a diagonal fixed point of T .

Proof. (Proposition 2.1)

Let $n \geq k$ be such that $B_n = 0$. Then, by definition,

$$u_{n-k} = \dots = u_{n-1} = u_n = a, \text{ for some } a \in X.$$

Combining with the iterative procedure, we also get

$$a = u_n = T(u_{n-k}, \dots, u_{n-1}) = T(a^k);$$

so that, $a \in X$ is a diagonal fixed point of T . Finally, we have

$$\begin{aligned} u_{n+1} &= T(u_{n-k+1}, \dots, u_{n-1}, u_n) = T(a^k) = a, \\ u_{n+2} &= T(u_{n-k+2}, \dots, u_n, u_{n+1}) = T(a^k) = a, \dots; \end{aligned}$$

and, from this, ($u_{n-k+i} = a; i \geq 0$). □

As a consequence of these remarks, it follows that, whenever

$$B_n = 0, \text{ for some } n \geq k,$$

we are done; so, without loss, one may assume that

$$\text{(str-pos) } B_n > 0, \text{ for all } n \geq k.$$

The following auxiliary fact concentrates the "deep" part of our argument.

Proposition 2.2. *Under these conditions, we have*

(22-1) $\rho_n \leq \varphi(B_n) < B_n$, for each $n \geq k$

(22-2) $B_{n+1} \leq B_n$, $\forall n \geq k$; whence, ($B_{k+i}; i \geq 0$) is descending

(22-3) $\rho_{n+k} < B_{n+k} \leq \varphi(B_n) < B_n$, $\forall n \geq k$.

Proof. (Proposition 2.2)

i): Let $n \geq k$ be arbitrary fixed. By definition,

$$A(u_{n-k}, \dots, u_{n-1}, u_n) = B_n = \max\{\rho_{n-k}, \dots, \rho_{n-1}\}.$$

On the other hand, from our iterative construction,

$$\rho_n = d(u_n, u_{n+1}) = d(T(u_{n-k}, \dots, u_{n-1}), T(u_{n-k+1}, \dots, u_{n-1}, u_n));$$

and this, by the contractive property, gives

$$\rho_n \leq \varphi(A(u_{n-k}, \dots, u_{n-1}, u_n)) = \varphi(B_n);$$

proving the desired fact.

ii): From the representation above one has, for each $n \geq k$,

$$B_{n+1} = \max\{\rho_{n-k+1}, \dots, \rho_{n-1}, \rho_n\} \leq \max\{B_n, \rho_n\};$$

and this, along with $\rho_n < B_n$, yields the written relation.

iii): By the obtained facts one gets (via φ =increasing), for each $n \geq k$

$$\rho_{n+j} \leq \varphi(B_{n+j}) \leq \varphi(B_n), \forall j \in \{0, \dots, k-1\}.$$

This yields (by definition)

$$B_{n+k} = \max\{\rho_n, \dots, \rho_{n+k-1}\} \leq \varphi(B_n);$$

and proves our claim. \square

Having these established, we may now pass to the final part of our argument.

Step 2. From the evaluations above, one has

$$B_{(i+1)k} \leq \varphi^i(B_k), \text{ for all } i \geq 0.$$

As a consequence, we get (again for all ranks $i \geq 0$)

$$\rho_{ik} + \dots + \rho_{(i+1)k-1} \leq kB_{(i+1)k} \leq k\varphi^i(B_k);$$

wherefrom (by adding these inequalities)

$$\sum_n \rho_n \leq k \sum_i B_{(i+1)k} \leq k \sum_i \varphi^i(B_k) < \infty;$$

i.e.: the iterative sequence $(u_n; n \geq 0)$ is d -Cauchy. As X is d -complete,

$$u_n \xrightarrow{d} z \text{ as } n \rightarrow \infty, \text{ for some } z \in X.$$

Step 3. We now claim that the obtained limit z is a diagonal fixed point for T ;

i.e.: $z = T(z^k)(= Sz)$. To verify this, it will suffice proving that

$$u_n \xrightarrow{d} Sz(= T(z^k)), \text{ as } n \rightarrow \infty.$$

In fact, by the triangle inequality and contractive condition, we have for each $n \geq k$

$$\begin{aligned} d(u_n, Sz) &= d(T(u_{n-k}, \dots, u_{n-1}), T(z^k)) \leq \\ &d(T(u_{n-k}, \dots, u_{n-2}, u_{n-1}), T(u_{n-k+1}, \dots, u_{n-1}, z)) + \\ &d(T(u_{n-k+1}, \dots, u_{n-2}, u_{n-1}, z), T(u_{n-k+2}, \dots, u_{n-1}, z^2)) + \dots + \\ &d(T(u_{n-1}, z^{k-1}), T(z^k)) \leq \\ &\varphi(\max\{\rho_{n-k}, \dots, \rho_{n-2}, d(u_{n-1}, z)\}) + \\ &\varphi(\max\{\rho_{n-k+1}, \dots, \rho_{n-2}, d(u_{n-1}, z)\}) + \dots + \varphi(d(u_{n-1}, z)); \end{aligned}$$

and this (via φ =increasing) yields

$$d(u_n, Sz) \leq k\varphi(\max\{\rho_{n-k}, \dots, \rho_{n-2}, d(u_{n-1}, z)\}), \forall n \geq k.$$

Passing to limit as $n \rightarrow \infty$, we get

$$\lim_n d(u_n, Sz) = 0; \text{ so that (by uniqueness), } z = Sz(= T(z^k));$$

and this gives us the desired fact. The proof is thereby complete. \square

In particular, when φ is linear; i.e.,

$$\varphi(t) = \beta t, \text{ } t \in R_+ \text{ (for some } \beta \in [0, 1]),$$

the obtained result is just the one in Cirić and Prešić [4]. On the other hand, when φ is continuous, the corresponding version of our main result includes a related 1981 statement in Rus [17], refined in Turinici [24, Paper 1-4]. Some coincidence point extensions of these facts are possible, under the lines in George and Khan [6], Murthy [10], Păcurar [12, 13], Pathak et al [11] and Yeşilkaya [25]. On the other hand, multivalued versions of these facts may be obtained under the lines in Shahzad and Shukla [19], Rajagopalan [15], and Latif et al [8]. For direct applications of

these facts to convergence questions involving real sequences, we refer to Chen [3], Khan et al [7], and the references therein.

3. TASKOVIĆ APPROACH

In the following, we show that the 2007 Cirić-Prešić result [4] (subsumed to Theorem 1.3) is “almost” deductible from a 1976 statement in Tasković [21].

Given $k \geq 1$, let $f : R_+^k \rightarrow R_+$ be a function; and $\Gamma := (\gamma_0, \dots, \gamma_{k-1})$ be an element in R_+^k . We say that f is *semi admissible*, provided

(s-adm-1) f is *increasing*:

$$u_i \leq v_i, i \in \{1, \dots, k\} \text{ imply } f(u_1, \dots, u_k) \leq f(v_1, \dots, v_k)$$

(s-adm-2) f is *semi homogeneous*:

$$f(\lambda x_1, \dots, \lambda x_k) \leq \lambda f(x_1, \dots, x_k), \forall (x_1, \dots, x_k) \in R_+^k, \forall \lambda \geq 0.$$

In addition, let us introduce the concept

(tas) (f, Γ) is *Tasković*: the associated map $g : R_+ \rightarrow R_+$

defined as $(g(t) = f(\gamma_0 t, \dots, \gamma_{k-1} t^k); t \in R_+)$ is continuous at $t = 1$.

Finally, let us introduce the conditions

(tele-sub) (f, Γ) is *telescopic subunitary*

$$(\alpha := f(\gamma_0, 0, \dots, 0) + \dots + f(0, \dots, 0, \gamma_{k-1}) < 1)$$

(sub) (f, Γ) is *subunitary* ($\beta := f(\gamma_0, \dots, \gamma_{k-1}) < 1$).

Having these precise, let us say that $T : X^k \rightarrow X$ is *Tasković* ($d; f, \Gamma$)-*contractive* (where $f \in \mathcal{F}(R_+^k, R_+)$, $\Gamma := (\gamma_0, \dots, \gamma_{k-1}) \in R_+^k$), provided

(T-contr) $d(T(x_0, \dots, x_{k-1}), T(x_1, \dots, x_k)) \leq$

$$f(\gamma_0 d(x_0, x_1), \dots, \gamma_{k-1} d(x_{k-1}, x_k)), \text{ for each } (x_0, \dots, x_k) \in X^{k+1}.$$

The following 1976 fixed point statement in Tasković [21] is our starting point.

Theorem 3.1. *Assume that T is Tasković ($d; f, \Gamma$)-contractive, where f is a semi admissible function in $\mathcal{F}(R_+^k, R_+)$ and $\Gamma = (\gamma_0, \dots, \gamma_{k-1})$ is a vector in R_+^k . In addition, let X be d -complete. Then, the following conclusions hold:*

(31-a) *If (f, Γ) is telescopic subunitary, then T is fixd-asingleton, in the sense:*

$$\text{Fixd}(T) = \text{Fix}(S) \text{ is an asingleton}$$

(31-b) *If (f, Γ) is Tasković and subunitary, then T is strong Prešić (modulo d): for each $U_0 := (u_0, \dots, u_{k-1}) \in X^k$, the iterative sequence $(u_n = T^n U_0; n \geq 0)$ fulfills $u_n \xrightarrow{d} z$, for some diagonal fixed point $z \in \text{Fixd}(T) = \text{Fix}(S)$.*

Before passing to the verification of this result, some technical remarks are in order. Given $f \in \mathcal{F}(R_+^k, R_+)$, call it *homogeneous*, provided

$$f(\lambda x_1, \dots, \lambda x_k) = \lambda f(x_1, \dots, x_k), \forall (x_1, \dots, x_k) \in R_+^k, \forall \lambda \geq 0.$$

Clearly, any such function is semi homogeneous. But, the reciprocal inclusion is also true; as it results from

Proposition 3.1. *For each function $f \in \mathcal{F}(R_+^k, R_+)$, we have*

$$\text{semi homogeneous} \implies \text{homogeneous; whence,}$$

$$\text{semi homogeneous} \iff \text{homogeneous.}$$

Proof. Suppose that $f : R_+^k \rightarrow R_+$ is semi homogeneous; i.e.:

$$f(\lambda x_1, \dots, \lambda x_k) \leq \lambda f(x_1, \dots, x_k), \forall (x_1, \dots, x_k) \in R_+^k, \forall \lambda \geq 0.$$

Putting $\lambda = 0$ in this relation gives

$$0 \leq f(0, \dots, 0) \leq 0; \text{ whence, } f(0, \dots, 0) = 0;$$

i.e.: the homogeneous property is fulfilled in case of $\lambda = 0$. It remains then to verify the underlying property in case of $\lambda > 0$. Denote for simplicity $\mu := 1/\lambda$; and let $(x_1, \dots, x_k) \in R_+^k$ be arbitrary fixed. By the semi homogeneous property, we have

$$\begin{aligned} f(x_1, \dots, x_k) &= f(\mu\lambda x_1, \dots, \mu\lambda x_k) \leq \mu f(\lambda x_1, \dots, \lambda x_k); \\ \text{or, equivalently (by our notation): } \lambda f(x_1, \dots, x_k) &\leq f(\lambda x_1, \dots, \lambda x_k). \end{aligned}$$

This, along with the semi homogeneous inequality, gives

$$f(\lambda x_1, \dots, \lambda x_k) = \lambda f(x_1, \dots, x_k), \forall (x_1, \dots, x_k) \in R_+^k, \forall \lambda > 0;$$

and proves our claim. \square

As a consequence of this, the semi-homogeneous property imposed to f is, ultimately, a homogeneous property of the same. On the other hand, the Tasković property (involving the associated function $g(\cdot)$) may be removed, as we will see. Summing up, the following simplified variant of Theorem 3.1 is to be considered. Given $k \geq 1$, let $f : R_+^k \rightarrow R_+$ be a function; we call it *admissible*, provided

$$\begin{aligned} \text{(adm-1) } f &\text{ is increasing:} \\ u_i \leq v_i, i \in \{0, \dots, k-1\} &\text{ imply } f(u_0, \dots, u_{k-1}) \leq f(v_0, \dots, v_{k-1}) \\ \text{(adm-2) } f &\text{ is homogeneous:} \\ f(\lambda x_0, \dots, \lambda x_{k-1}) &= \lambda f(x_0, \dots, x_{k-1}), \forall (x_0, \dots, x_{k-1}) \in R_+^k, \forall \lambda \geq 0. \end{aligned}$$

Theorem 3.2. *Assume that T is Tasković $(d; f, \Gamma)$ -contractive, where f is an admissible function in $\mathcal{F}(R_+^k, R_+)$ and $\Gamma = (\gamma_0, \dots, \gamma_{k-1})$ is a vector in R_+^k . In addition, let X be d -complete. Then, the following conclusions hold:*

(32-a) *If (f, Γ) is telescopic subunitary, then T is fixd-asingleton, in the sense:*

$$\text{Fixd}(T) = \text{Fix}(S) \text{ is an asingleton}$$

(32-b) *If (f, Γ) is subunitary, then T is strong Prešić (modulo d): for each $U_0 := (u_0, \dots, u_{k-1}) \in X^k$, the iterative sequence $(u_n = T^n U_0; n \geq 0)$ fulfills $u_n \xrightarrow{d} z$, for some diagonal fixed point $z \in \text{Fixd}(T) = \text{Fix}(S)$.*

Concerning the relationships between these results and the Cirić-Prešić one, the following answer is available.

Proposition 3.2. *Under the above conventions,*

(32-1) *Theorem 1.3 (first half) \implies Theorem 3.1 (first half) \implies Theorem 3.2 (first half)*

(32-2) *The inclusions Theorem 3.1 (first half) \implies Theorem 1.3 (first half) and Theorem 3.2 (first half) \implies Theorem 1.3 (first half) are not in general true*

(32-3) *Theorem 3.1 (second half) \implies Theorem 1.3 (second half) \implies Theorem 3.2 (second half) \implies Theorem 3.1 (second half); hence, all these statements are equivalent to each other.*

Proof. There are three parts to be passed.

Part 1. Assume that f is semi admissible (or, equivalently: admissible) and (f, Γ) is telescopic subunitary:

$$\alpha := f(\gamma_0, 0, \dots, 0) + \dots + f(0, \dots, 0, \gamma_{k-1}) < 1.$$

We claim that (the diagonal map) S is $(d; \alpha)$ -contractive. In fact, let $x, y \in X$ be arbitrary fixed with $x \neq y$. From the Tasković contractive condition (and f -homogeneous), one derives

$$\begin{aligned}
(\text{S-con}) \quad & d(Sx, Sy) = d(T(x^k), T(y^k)) \leq \\
& d(T(x^k), T(x^{k-1}, y)) + \dots + d(T(x, y^{k-1}), T(y^k)) \leq \\
& f(\gamma_0 d(x, y), 0, \dots, 0) + \dots + f(0, \dots, 0, \gamma_{k-1} d(x, y)) = \alpha d(x, y);
\end{aligned}$$

and this, via $\alpha < 1$, establishes the first inclusion. Further, the second inclusion is trivial, in view of (admissible \iff semi admissible).

Part 2. Suppose by absurd that Theorem 3.1 (first half) \implies Theorem 1.3 (first half). This, by the above, means a deduction of (S-con) from (S-snex); which is clearly impossible. The case of Theorem 3.2 (first half) \implies Theorem 1.3 (first half) is handled in the same way.

Part 3. Suppose that T is Cirić-Prešić $(d; \beta)$ -contractive. Taking (f, Γ) as

$$\begin{aligned}
f(t_0, \dots, t_{k-1}) &= \max\{t_0, \dots, t_{k-1}\}, (t_0, \dots, t_{k-1}) \in R_+^k; \\
\Gamma &= (\gamma_0, \dots, \gamma_{k-1}): \gamma_i = \beta, 0 \leq i \leq k-1
\end{aligned}$$

one derives that f is semi admissible (or, equivalently: admissible) and (f, Γ) is Tasković, subunitary; and this tells us that the first inclusion holds.

Further, suppose that f is admissible and (f, Γ) is subunitary. From the imposed properties, we have (under our notations)

$$f(\gamma_0 t_0, \dots, \gamma_{k-1} t_{k-1}) \leq \beta \max\{t_0, \dots, t_{k-1}\}, (t_0, \dots, t_{k-1}) \in R_+^k.$$

The mapping T is therefore Cirić-Prešić $(d; \beta)$ -contractive; and this tells us that the second inclusion holds.

Finally, the third inclusion is evident, by definition. The proof is complete. \square

Summing up, Theorem 1.3 is the most general result in the area; but, its second half is equivalent with the second part of Theorem 3.1 and Theorem 3.2. In other words, passing to these functional extensions of Theorem 1.3 does not bring any degree of generality upon it. Further aspects may be found in Tasković [22].

4. ABBAS-ILIĆ-NAZIR STATEMENT

Let us now return to our initial setting. So, take a metric space (X, d) ; as well as a natural number $k \geq 1$. Further, let $T : X^k \rightarrow X$ be a mapping; and $S : X \rightarrow X$ be its associated diagonal operator. Remember that, our main diagonal fixed point result is essentially based upon strongly Matkowski admissible functions. Precisely, denote for each i, j with $i < j$

$$A(z_i, \dots, z_j) = \max\{d(z_i, z_{i+1}), \dots, d(z_{j-1}, z_j)\}, (z_i, \dots, z_j) \in X^{j-i+1}.$$

Given $\varphi \in \mathcal{F}(R_+)$, we say that T is (d, φ) -contractive, provided

$$\begin{aligned}
d(T(y_0, \dots, y_{k-1}), T(y_1, \dots, y_k)) &\leq \varphi(A(y_0, \dots, y_{k-1}, y_k)), \\
\text{for all } (y_0, \dots, y_{k-1}, y_k) &\in X^{k+1}.
\end{aligned}$$

The basic conditions imposed upon φ are

- (in-re) φ is increasing and regressive ($\varphi(0) = 0$ and $\varphi(t) < t, \forall t > 0$)
- (str-M) φ is strongly Matkowski admissible ($\sum_n \varphi^n(t) < \infty, \forall t > 0$).

A natural question to be posed is that of conclusion in our main result remaining valid if the strong admissible property above is replaced by its weaker counterpart:

- (M) φ is Matkowski admissible ($\lim_n \varphi^n(t) = 0, \forall t > 0$).

Technically speaking, such extensions (to the Matkowski admissible case) are not in general possible; because, the contractive condition imposed upon T connects two *consecutive* members of our iterative process. This must be true as well for the

fixed point statement in Abbas et al [1] to be stated below. Let us say that the function $\varphi \in \mathcal{F}(R_+)$ is *normal*, provided

- (nor-1) $\varphi(0) = 0, t \geq \varphi(t) > 0, \forall t \in R_+^0$
- (nor-2) φ is lower semicontinuous (in short: lsc) on R_+ .

Given such an object, define the function $\psi \in \mathcal{F}(R_+)$ as

$$\psi(t) = t - \varphi(t), t \in R_+ \text{ (in short: } \psi = I - \varphi\text{);}$$

clearly, ψ is regressive and upper semicontinuous (usc) on R_+ .

Theorem 4.1. *Suppose that there exists a normal function $\varphi \in \mathcal{F}(R_+)$, such that (under the above convention)*

$$d(T(x_1, \dots, x_k), T(x_2, \dots, x_{k+1})) \leq \psi(\max\{d(x_i, x_{i+1}); 1 \leq i \leq k\}),$$

for all $(x_1, \dots, x_{k+1}) \in X^{k+1}$.

In addition, let X be d -complete. Then, the following conclusions hold:

(41-a) For each $(x_1, \dots, x_k) \in X^k$, the iterative sequence (x_n) introduced as

$$x_{n+k} = T(x_n, \dots, x_{n+k-1}), n \geq 1$$

converges to some $u \in X$ with $u = T(u, \dots, u)$

(41-b) Moreover, if

$$d(T(x, \dots, x), T(y, \dots, y)) \leq \psi(d(x, y)), \text{ for each } x, y \in X \text{ with } x \neq y,$$

then $u \in X$ is uniquely determined with such a property.

Proof. According to authors, the argument consists in the following.

Part 1. Using the contractive condition, we conclude that

$$(1;\text{decr}) \text{ the sequence } (d(x_{n+k-1}, x_{n+k}); n \geq 0) \text{ is monotone decreasing}$$

$$\text{and } \lim_n d(x_{n+k-1}, x_{n+k}) = 0.$$

Part 2. Further, (x_n) is a Cauchy sequence. For, given any $n, m \in N$ with $n \leq m$, we have (using the contractive condition)

$$(2;A) \quad d(x_{k+n}, x_{k+m}) =$$

$$d(T(x_1, \dots, x_{k+n-1}), T(x_2, \dots, x_{k+m-1})) \leq$$

$$d(T(x_1, \dots, x_{k+n-1}), T(x_2, \dots, x_{k+n})) +$$

$$d(T(x_2, \dots, x_{k+n}), T(x_3, \dots, x_{k+n+1})) + \dots +$$

$$d(T(x_2, \dots, x_{k+n}), T(x_3, \dots, x_{k+m-1})) \leq$$

$$\psi(\max\{d(x_{i+n}, x_{i+n+1}); 1 \leq i \leq k-1\}) +$$

$$\psi(\max\{d(x_{i+n}, x_{i+n+1}); 1 \leq i \leq k\}) + \dots +$$

$$\psi(\max\{d(x_{i+n}, x_{i+n+1}); 1 \leq i \leq k+m-1\}).$$

Taking the upper limit as $n, m \rightarrow \infty$, implies that

$$(2;B) \quad \lim_{n \rightarrow \infty} d(x_{k+n}, x_{k+m}) = 0.$$

Hence, (x_n) is a Cauchy sequence.

Part 3. Since (X, d) is complete, there exists $u \in X$ such that

$$\lim_{n, m \rightarrow \infty} d(x_n, x_m) = \lim_{n \rightarrow \infty} d(x_n, u).$$

Using the contractive condition gives $u = T(u, \dots, u)$.

Part 4. Assume that there exists $v \neq u$ such that $v = T(v, \dots, v)$. As $d(u, v) > 0$, we must have $\varphi(d(u, v)) > 0$. Combining with the imposed extra condition yields

$$d(u, v) = d(T(u, \dots, u), T(v, \dots, v)) \leq \psi(d(u, v)) < d(u, v);$$

a contradiction. Hence, $u = v$; and the last conclusion of statement follows. \square

Unfortunately, the proposed reasoning cannot be accepted. The basic motivation for this concerns the evaluation of iterates x_{k+n} and x_{k+m} appearing in relation (2;A). Precisely, the authors' evaluation of the former iterate

$$x_{k+n} = T(x_1, \dots, x_{k+n-1})$$

is not correct; because it involves $k + n - 1$ variables in the right hand side; and this, along with

$$k + n - 1 > k, \text{ whenever } n \geq 2$$

is in contradiction with the construction of our iterative sequence

$$x_{n+k} = T(x_n, \dots, x_{n+k-1}), n \geq 2;$$

which involves, at any stage, k variables in the right hand side. Likewise, the authors' evaluation of the latter iterate

$$x_{k+m} = T(x_2, \dots, x_{k+m-1})$$

is again incorrect; because it involves $k + m - 2$ variables in the right hand side; and this, along with

$$k + m - 2 > k, \text{ whenever } m \geq 3$$

is in contradiction with the construction of our iterative sequence

$$x_{m+k} = T(x_m, \dots, x_{m+k-1}), m \geq 3;$$

which involves, at any stage, k variables in the right hand side.

A direct way of correcting this argument is that of using our main result, with ψ in place of φ . Precisely, the following variant of Theorem 4.1 is available.

Theorem 4.2. *Let the normal function $\varphi \in \mathcal{F}(re)(R_+)$ be such that*

$$\psi := I - \varphi \text{ is increasing strongly Matkowski admissible and } T \text{ is } (d, \psi)\text{-contractive.}$$

In addition, let X be d -complete. Then, the following conclusions hold:

(42-a) *If (in addition) S is d -strictly-nonexpansive, then*

T is fixd-asingleton; i.e.: $\text{Fixd}(T) = \text{Fix}(S)$ is an asingleton

(42-b) *T is strong Prešić (modulo d): for each $U_0 := (u_0, \dots, u_{k-1}) \in X^k$, the iterative sequence $(u_n := T^n U_0; n \geq 0)$ fulfills $u_n \xrightarrow{d} z$, for some diagonal fixed point $z \in \text{Fixd}(T) = \text{Fix}(S)$.*

The verification is immediate; so, we do not give details. Further aspects may be found in Rao et al [16], and Gholidahaneh et al [5]; see also Shukla et al [20].

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