



VERTEX TO VERTEX GEODESICS ON PLATONIC SOLIDS

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VERTEX TO VERTEX GEODESICS ON PLATONIC SOLIDS

SERGE TROUBETZKOY

ABSTRACT. We give a simple proof based on symmetries that there are no geodesics from a vertex to itself in the cube, tetrahedron, octahedron, and icosahedron.

A straight-line trajectory on the surface of a polyhedron is a straight line within a face that is uniquely extended over an edge so that the trajectory forms a straight line in the plane when the adjacent faces are unfolded to lie in the same plane. This is well-defined away from the vertices. Locally a straight-line trajectory is the shortest curve between points, thus it is a geodesic. By choosing a tangent vector at a vertex, one can consider the corresponding geodesic emanating from that vertex. Thus, while geodesics can start and end at a vertex they can not pass through a vertex. The study of geodesics on polyhedra was initiated quite some time ago in [7, 6]

We give a short simple proof of the following fact first proved in [4] and [5] and described in the expository article [2]. A proof close to ours, but dressed up in advanced terminology, is given in [3].

Theorem 1. *There are no geodesics connecting a vertex to itself on the cube, tetrahedron, octahedron, or icosahedron.*

Proof. The edges of all the polygons will be normalized to have length 1. We begin with the cube. Consider a geodesic segment γ which starts at a vertex and ends at a vertex. We will show that the two vertices can not coincide. For this we unfold the geodesic, in our unfoldings the squares will be parallel to the coordinate axes and the geodesic will start at a vertex of a square placed at the origin. The unfolding of γ is a line segment then starting at the origin and ending at vertex with coordinates $(p, q) \in \mathbb{N}^2$. The midpoint m of the geodesic segment has coordinates $(p/2, q/2)$. Since geodesics does not pass through vertices either p or q must be odd.

If both p and q are odd then m is the center of one of the squares of the unfolding (Figure 1 left), and thus the midpoint M of γ is located in the center of one of the faces of the cube. If p is even and q is odd; then the midpoints of the unfolding is located in the middle of a vertical edges of one of the squares of the unfolding (Figure 1 right), while if p is odd and q is even then it is located in the middle of one of the horizontal edges of the unfolding. In both of these last two cases

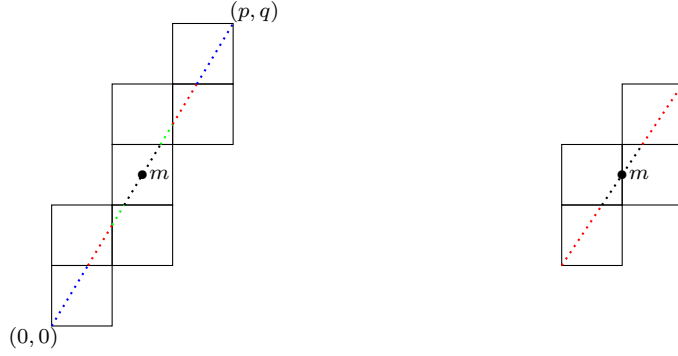


FIGURE 1. Center of symmetry of the unfolding.

the corresponding point M on the cube is in the middle of one of the edges of the cube.

In all the three cases the unlabeled unfolded figure is centrally symmetric about the point m . We will show that refolding this leads to a symmetry of the geodesic γ .

Suppose first that p and q are even let L be the line in \mathbb{R}^3 through M perpendicular to face of the cube containing m , so L passes through the center of the cube and the center of the opposite face. Consider the point m and follow the unfolded trajectory starting at m in both directions; we arrive at the first pair of centrally symmetric edges and we refold them, the resulting object is his yields a geodesic which is invariant under a rotation by 180° about the vertical line through m . We repeat this procedure each time we reach a pair or symmetric edges, in the end we obtain the geodesic γ on the cube, and since the symmetry is preserved at each step we conclude that γ is invariant under a rotation by 180° about L (Figure 2 top). We conclude that the two endpoints of γ are a rotation of each other and thus can not coincide.

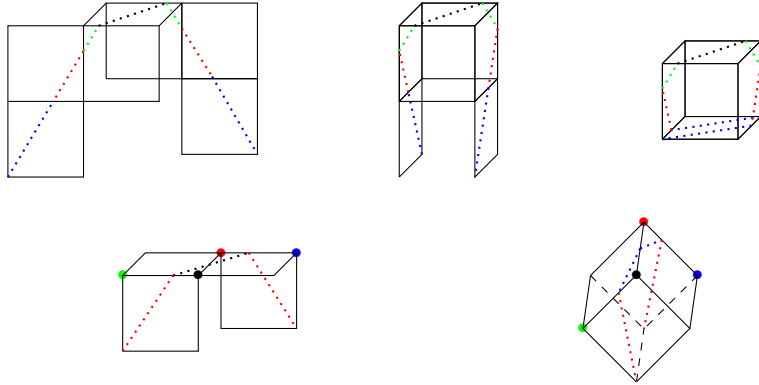


FIGURE 2. Rotational symmetry is preserved by the re-folding process.

In the other two cases the corresponding line is the line L which passes through M , the center of the cube, and the midpoint of the opposite edge of the cube. To understand these cases consider the vertical line passing through m . Repeating the same procedure as above yields a trajectory which has been refolded everywhere except along the edge e (Figure 2 bottom left). Finally we fold the edge e in such a way that the vertical symmetry axis becomes the bisector of the angle, and thus becomes the line L (Figure 2 bottom right). Again the two endpoints of γ are a rotation of each other and thus can not coincide.

Now we adapt this argument to the other three polygons. We start with a geodesic segment γ which starts and ends at a vertex and unfold it to a straight line segment starting and ending at vertices of the equilateral triangle tiling of the plane whose sides are parallel to the unit vectors $v_1 := (1, 0)$, $v_2 := (\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $v_3 := (-\frac{\sqrt{3}}{2}, \frac{1}{2})$. We consider v_1 and v_2 as a basis of the plane, and suppose that the unfolded trajectory goes from the origin to a point (p, q) . As above the midpoint m has coordinates $(p/2, q/2)$. By the definition of a geodesic either p or q must be odd, i.e., of the form $(k, l + \frac{1}{2})$, $(k + \frac{1}{2}, l)$ or $(k + \frac{1}{2}, l + \frac{1}{2})$. In the first case m is in the middle of an edge in the direction v_2 , in the second case m is in the middle of an edge in the direction v_1 , while in the last case m is in the middle of an edge in the direction v_3 and the unfolding is centrally symmetric around the point m (Figure 3).

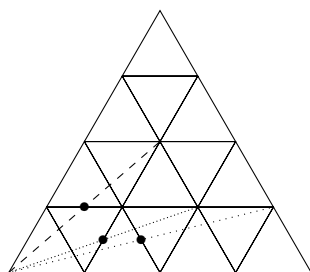


FIGURE 3. The three possible cases for the triangular lattice.

Refolding again leads to an axis L of rotational symmetry of the geodesic γ . In each of the three case the line of symmetry connects the midpoint of an edge to the center of the polyhedron and then to the midpoint of another edge. The rotation by 180° about L does not fix any vertices, thus the endpoints of γ are distinct. \square

There is one more platonic solid, the dodecahedron. It turns out that there are geodesics from a vertex to itself on the dodecahedron [1, 2], of course our proof can not work in this case since pentagons do not tile the plane.

The symmetries of platonic solids have been extremely well studied, our proof yields a classification of the possible pairs of vertices which

can be connected by a geodesic. The proof in [4] and [5] have already given this.

On the tetrahedron there is a simpler proof since the net of the tetrahedron is an equilateral triangle, so it tiles the plane. Thus each vertex of the triangular lattice corresponds to a unique vertex of the polygon once we fix the correspondance at the origin. This is not the case for the three other platonic solids we treat since their nets do not tile the plane.

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