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Calculi, Countermodel Generation and Theorem Prover for Strong Logics of Counterfactual Reasoning*

Marianna Girlando[†] Björn Lellmann[‡] Nicola Olivetti[§]
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Abstract

We present hypersequent calculi for the strongest logics in Lewis' family of conditional systems, characterized by uniformity and total reflexivity. We first present a non-standard hypersequent calculus which allows a syntactic proof of cut elimination. We then introduce standard hypersequent calculi, in which sequents are enriched by additional structures to encode plausibility formulas and diamond formulas. Proof search using these calculi is terminating, and the completeness proof shows how a countermodel can be constructed from a branch of a failed proof search. We then describe **tuCLEVER**, a theorem prover which implements the standard hypersequent calculi. The prover provides a decision procedure for the logics, and it produces a countermodel in case of proof search failure. The prover **tuCLEVER** is inspired by the methodology of **lean** $T^A P$ and it is implemented in Prolog. Preliminary experimental results show that the performances of **tuCLEVER** are promising¹.

1 Introduction

Conditional logics are extensions of classical logic by means of a two-places modal operator, the *conditional operator*, denoted by $>$. They have a long history going back to the works of Stalnaker, Lewis, Nute, Chellas, Burgess

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¹This paper extends and revises a preliminary work presented in [10] and in [8].

and Pollock in the 60s-70s [30, 21, 22, 5, 4]. Conditional logics have since found an interest in several fields of knowledge representation, from reasoning about prototypical properties and non-monotonic reasoning [16] to modelling belief change. A successful attempt to relate conditional logics and belief update (as opposed to belief revision) was carried out by Grahne [12], who established a precise mapping between belief update operators and the logic $\forall\text{CU}$, one of the strongest systems of conditional logics. The relation is expressed by the so-called *Ramsey's Rule*:

$$(A \circ B) \rightarrow C \text{ holds if and only if } A \rightarrow (B > C) \text{ holds}$$

where the operator \circ is any *update* operator satisfying Katsuno and Mendelson's postulates [14], that are considered the "core" properties for any concrete and plausible operator of belief update. Ramsey's rule means that C is entailed by " A updated by B " if and only if the conditional $B > C$ is entailed by A . In this sense it can be said that the conditional $B > C$ expresses an hypothetical update of the information A .

One of the most important contributions to conditional logics is due to Lewis. In his seminal work [21], he proposed a formalization of conditional logics to capture hypothetical conditionals. His aim was to represent conditional sentences that cannot be captured by material implication and, in particular, *counterfactuals*, e.g., conditionals of the form "if A were the case, then B would be the case", where A is false. In [21] Lewis introduced a family of conditional logics semantically characterized by *sphere models*. Sphere models are possible-worlds models in which each world x is equipped with a *system of spheres*, $\text{SP}(x)$, which is a set of nested sets of worlds. Each set in $\text{SP}(x)$ is called a *sphere*, the intuition being that, according to x , worlds in inner spheres are more plausible than worlds belonging only to outer spheres.

Lewis takes as primitive the *comparative plausibility operator* \preceq , with a formula $A \preceq B$ meaning " A is at least as plausible as B ". A conditional formula $A > B$ can be defined in terms of the comparative plausibility operator by saying that "either A is impossible or $A \wedge \neg B$ is less plausible than $A \wedge B$ ", where the second disjunct can be simplified to " $A \wedge \neg B$ is less plausible than A ". Conversely, as we will show in Section 2, \preceq can be defined in terms of $>$.

Here we consider the logics of Lewis' family satisfying the following two semantic properties, which are natural to consider when modelling hypothetical reasoning and belief change:

- *Uniformity*: all worlds have the same set of accessible worlds, where the worlds accessible from a world x are those belonging to any sphere in $\text{SP}(x)$;
- *Total reflexivity*: every world x belongs to some sphere in $\text{SP}(x)$.

The basic logic satisfying these properties is VTU . We also consider some of its extensions, including the above mentioned $\forall\text{CU}$. All the logics considered in this paper contain modal logic S5 as a fragment: $\Box A$ can be defined as $\perp \preceq \neg A$ (or $\neg A > \perp$).

It is worth mentioning that conditional logics with uniformity and total reflexivity can be related with the logics of Comparative Concept Similarity studied in the context of ontologies [29]. These logics contain a connective \Leftarrow , which allows to express, e.g,

$$PicassoPainting \sqsubseteq BraquePainting \Leftarrow GiottoPainting$$

asserting that “Picasso’s paintings are more similar to Braque’s paintings than to Giotto’s ones”. The semantics is provided in terms of *distance space models*, defined as a set of worlds equipped with a distance function. It turns out that the basic logic of Comparative Concept Similarity coincides with Lewis’ logic $\mathbb{V}\mathbb{W}\mathbb{U}$, an extension of the basic system $\mathbb{V}\mathbb{T}\mathbb{U}$ with a property known as *weak centering*. Similarly, the logic defined by “minspace” distance models is equivalent to $\mathbb{V}\mathbb{C}\mathbb{U}$. Distance space models thus provide an alternative semantics, simple and natural, for conditional logics with uniformity.

In this paper we propose different systems of calculi for Lewis’ strongest logics, that are, $\mathbb{V}\mathbb{T}\mathbb{U}$ and its extensions, and a Prolog theorem prover, **tuCLEVER**, which automates root-first proof-search and countermodel construction for these logics.

In general, proof systems for a given logic may have several different features. In this paper, we aim to develop calculi having the the following properties:

- the calculi should be *internal*, where by *internal calculus* we mean a calculus where each configuration of a derivation can be translated into a formula of the corresponding logic. In contrast, *external calculi* make use of extra-logical elements, such as labels (terms representing worlds of a model) and relations on them;
- they should be *standard*, that is to say, each connective should be handled by a fixed finite set of rules, each having a finite and fixed set of premises;
- they should be *modular*, meaning that calculi for stronger logics are obtained by adding independent rules to a basic calculus;
- they should have good proof-theoretical properties, first of all allowing a syntactic proof of cut admissibility;
- they should provide a decision procedure for the respective logics (whenever the logics are decidable);
- they should be constructively complete with respect to the semantics: from *one* failed derivation, it should be possible to extract a countermodel of the input formula.

In general, it cannot be taken for granted that a *unique* type of calculi satisfies *all* these requirements; this motivates the investigation of alternative calculi, or alternative formulations of them.

In previous work [27, 9], we defined sequent calculi enjoying the above properties for several logics of the Lewis' family, including logics with the semantic property of *absoluteness*, which is stronger than uniformity, and states that all worlds have the same system of spheres. However, we have not treated systems with uniformity and, to the best of our knowledge, no internal calculi are known for these logics.

It is easy to see that sequents, even extended as in [9], are not sufficient to capture logics with uniformity: since modal logic **S5** can be embedded into $\forall\text{TTU}$, a sequent calculus for the latter would also yield a sequent calculus for **S5** and, as it is known, such a calculus does not exist².

In order to cope with logics with uniformity, we adopt the richer framework of *hypersequents* [2], where the basic objects are multisets of sequents, interpreted disjunctively. The additional power of hypersequents consists in the presence of rules which act on or combine several components (sequents) of an hypersequent at the same time.

As a starting point, we propose non-standard hypersequent calculi for $\forall\text{TTU}$ and its extensions. We prove that the calculi admit syntactic cut-elimination, a result which is not obvious, given the non-standard format of some rules. As a consequence, we obtain completeness of the calculi with respect the axiom systems of each logic (*syntactic completeness*). Moreover, the relation with modal logic **S5** can be appreciated at a proof-theoretical level: we show that by translating $\Box A$ as the external modality $\perp \preceq \neg A$ the calculi - restricted to such formulas - correspond to the well-known hypersequent calculus for **S5**. In other words, the hypersequent calculus for **S5** can be embedded in our calculus for $\forall\text{TTU}$ when the latter is restricted to formulas $\perp \preceq \neg A$.

We then introduce standard hypersequent calculi for all the logics. In order to define a standard calculus, the sequents composing an hypersequent are enriched by two types of structural connectives, called "blocks", to encode respectively disjunctions of plausibility formulas and \Diamond -formulas which, due to the condition of uniformity, need to be propagated. We also define hypersequent calculi for the logics with both absoluteness and reflexivity, which were missing in [9]³.

Our aim is to define calculi which provide both a decision procedure for the respective logics as well as countermodel extraction. Therefore, we introduce an *invertible* version of the standard hypersequent calculi, in which all principal formulas (or blocks) are kept on the premiss(es) of the rules, thus obtaining "kleened"⁴ version of the calculi. The invertible calculi allow to prove termination of root-first proof-search, whence they provide a decision procedure. Moreover, we give a direct proof of semantic completeness for the logics without absoluteness. The proof is constructive: from one failed saturated hypersequent, i.e., a leaf of a failed derivation, we define a countermodel of the formula/hypersequent at the root of the derivation. By the soundness of the

²We mean here a sequent calculus manipulating arbitrary formulas without pre-processing.

³Logics with absoluteness are included to show the modularity and the extent of the proposed framework, but given their simplified semantics they would be better handled by calculi with a tailored (simpler) structure.

⁴The name "kleened" is due to the introduction of rules copying the principal formulae into the premisses in Kleene's classic [15].

calculi, the semantic completeness provides a constructive proof of the finite model property of the logic: a satisfiable formula has a finite model.

The proposed calculi are not only meant for theoretical investigation, they also support automated reasoning. We present a Prolog implementation of the invertible standard hypersequent calculi. The program, called **tuCLEVER** (for *Total reflexivity and Uniformity Conditional LEwis logics theorem proVER*) is, to the best of our knowledge, the first and only existing prover for conditional logics with uniformity⁵. The conception of **tuCLEVER** is inspired by the methodology of **lean^{TAP}** [3]. The idea is that each axiom or rule of the sequent calculi is implemented by a single Prolog clause. No ad-hoc data structure is used. The resulting code is therefore simple and compact: the implementation of **tuCLEVER** for the basic system $\forall\text{TU}$ consists of only 3 predicates, 21 clauses and 118 lines of code.

The prover provides a decision procedure for the respective logics: since it implements the invertible version of the calculi, where the principal formula or block is kept in the premises of each rule, termination is obtained by simply avoiding *redundant* applications of the rules.

The prover implements root-first proof search, and it also provides countermodel construction in case of proof search failure. It implements the countermodel construction defined for the proof of semantic completeness, thus computing a countermodel of the input formula from the unprovable saturated sequent occurring in a leaf of the failed proof tree. Intuitively, an unprovable saturated hypersequent contains enough information to define a countermodel: each component of the hypersequent determines a world and the sequence of blocks within each components determines the system of spheres associated to it. The computation of worlds, sphere and propositional evaluations can be easily handled in **tuCLEVER** by inspecting the saturated hypersequent. Once more, to best of our knowledge, **tuCLEVER** is the only existing prover which computes countermodel for conditional logics.

Even if a set of benchmark formulae does not exist, the experimental results obtained so far show that the performances of **tuCLEVER** are promising. Being the unique theorem prover for conditional logics with Uniformity, **tuCLEVER** is not directly comparable with any other prover for conditional logics. Nonetheless, we show that on sets of formulae provable in other (weaker) conditional logics and on randomly generated formulas, the performances of **tuCLEVER** are surprisingly better than performances of other provers for conditional logics, notably **VINTE** [11] which covers weaker logics of the Lewis family. Further investigation might clarify whether this fact depends on the strength of the logic implemented by **tuCLEVER**, or on the features of the calculi, or on the implementation of **tuCLEVER**.

The program **tuCLEVER**, as well as all the Prolog source files, are available for free usage and download at <http://193.51.60.97:8000/tuclever/>.

This article is organized as follows. Section 2 introduces the axioms and

⁵The only possible exception is the theorem prover **CSLLean** [1] which implements a calculus for the logic of Comparative Concept Similarity over minspaces, which is equivalent to logic $\forall\text{CU}$.

the models of the logics under scope. In Section 3 we present the non-standard hypersequent calculi for the logic under investigation, and we prove the fundamental property of cut admissibility and syntactic completeness (Sec. 3.1); moreover we give a proof-theoretical account of the embedding of **S5** into the conditional logics studied in this work (Sec. 3.2). In Section 4 we present the standard hypersequent calculi, first in a simpler non-invertible version (Sec. 4.1) and then in their invertible version (Sec. 4.2), for which termination and semantic completeness are proved, the latter providing countermodel construction. In Section 5 we present the design of **tuCLEVER**, detailing the proof search procedure (Sec. 5.1) and the countermodel generation (Sec. 5.2) implemented by the prover. We conclude by analysing the performances of the prover (Sec. 5.3).

2 Lewis' Conditional Logics

We consider the *conditional logics* introduced by Lewis [21]. The set of *conditional formulae* is defined as follows, for $p \in \mathcal{V}$ is a propositional variable:

$$A ::= p \mid \perp \mid A \rightarrow A \mid A \preceq A$$

Intuitively, a formula $A \preceq B$ is interpreted as “ A is at least as plausible as B ”. The other propositional connectives and constants are standardly defined: $\neg A \equiv A \rightarrow \perp$, $A \wedge B \equiv \neg(A \rightarrow \neg B)$, $A \vee B \equiv \neg A \rightarrow B$, and $\top \equiv p \vee \neg p$. Moreover, Lewis' *conditional operator* $>$ is defined by

$$A > B \equiv (\perp \preceq A) \vee \neg((A \wedge \neg B) \preceq A)$$

where a formula $\perp \preceq C$ can be read as “ C is impossible”, and $\neg(C \preceq D)$ means “ D is less plausible than C ”. Conversely, as mentioned in the Introduction, the comparative plausibility operator \preceq can be defined in terms of the conditional operator $>$:

$$A \preceq B \equiv ((A \vee B) > \perp) \vee \neg((A \vee B) > \neg A)$$

The *outer modalities* \Box and \Diamond are defined by $\Box A \equiv (\perp \preceq \neg A)$ and $\Diamond A \equiv \neg(\perp \preceq A)$. The logics we consider are semantically defined as follows.

Definition 1 *A TU-model is a triple $\mathcal{M} = \langle W, \text{SP}, \llbracket \cdot \rrbracket \rangle$, consisting of a non-empty set W of elements, called worlds, a mapping $\text{SP} : W \rightarrow 2^{2^W}$, and a propositional valuation $\llbracket \cdot \rrbracket : \mathcal{V} \rightarrow 2^W$. Elements of $\text{SP}(x)$ are called spheres. We assume the following conditions:*

- For every $\alpha \in \text{SP}(w)$ we have $\alpha \neq \emptyset$ (non-emptiness)
- For every $\alpha, \beta \in \text{SP}(w)$ we have $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$ (sphere nesting)
- For all $w \in W$ we have $w \in \bigcup \text{SP}(w)$ (total reflexivity)
- For all $w, v \in W$ we have $\bigcup \text{SP}(w) = \bigcup \text{SP}(v)$ (uniformity)

The valuation $\llbracket \cdot \rrbracket$ is extended to all formulae, including the conditional, as follows:

$$\begin{aligned}
\llbracket \perp \rrbracket &= \emptyset \\
\llbracket A \rightarrow B \rrbracket &= (W - \llbracket A \rrbracket) \cup \llbracket B \rrbracket \\
\llbracket A \preceq B \rrbracket &= \{w \in W \mid \text{for all } \alpha \in \text{SP}(w), \text{ if } \llbracket B \rrbracket \cap \alpha \neq \emptyset, \text{ then } \llbracket A \rrbracket \cap \alpha \neq \emptyset\} \\
\llbracket A > B \rrbracket &= \{w \in W \mid \text{either } \bigcup \text{SP}(w) \cap \llbracket A \rrbracket = \emptyset \text{ or there exists} \\
&\quad \alpha \in \text{SP}(w) \text{ such that } \alpha \cap \llbracket A \rrbracket \neq \emptyset \text{ and } \alpha \cap \llbracket A \rrbracket \subseteq \llbracket B \rrbracket\}.
\end{aligned}$$

For a TU-model $\mathcal{M} = \langle W, \text{SP}, \llbracket \cdot \rrbracket \rangle$, we say that a formula A is satisfiable at a world w of \mathcal{M} , in symbols $\mathcal{M}, w \models A$, if $w \in \llbracket A \rrbracket$. Moreover, we say that A is satisfiable at \mathcal{M} , in symbols $\mathcal{M} \models A$, if for all $w \in W$, $\mathcal{M}, w \models A$. We say that A is valid in the class of TU-models if A is satisfiable at all TU-models.

The logic $\forall\text{TU}$ is defined as the set of formulae valid in all TU-models [21].

Definition 2 *Extensions of TU-models are defined by specifying the following additional conditions on the class of TU-models:*

- For all $\alpha \in \text{SP}(w)$ we have $w \in \alpha$ (weak centering)
- For all $w \in W$ we have $\{w\} \in \text{SP}(w)$ (centering)
- For all $w, v \in W$ we have $\text{SP}(w) = \text{SP}(v)$ (absoluteness)

These conditions identify five distinct classes of models, to which the notions of validity and satisfiability of formulae, defined above, can be immediately extended. These five classes of models in turn define five systems of logics, extensions of $\forall\text{TU}$. The logics are denoted by concatenating letters for the semantic properties characterizing each class of models: \mathbb{W} for weak centering, \mathbb{C} for centering, and \mathbb{A} for absoluteness.

$$\begin{array}{ll}
\forall\text{TU} & \forall\text{TA}: \quad \forall\text{TU} + \text{absoluteness} \\
\forall\text{WU}: \quad \forall\text{TU} + \text{weak centering} & \forall\text{WA}: \quad \forall\text{TA} + \text{weak centering} \\
\forall\text{CU}: \quad \forall\text{TU} + \text{centering} & \forall\text{CA}: \quad \forall\text{TA} + \text{centering}
\end{array}$$

Thus, $\forall\text{WU}$ is as defined as the set of formulas valid in all TU-models with weak centering, $\forall\text{CU}$ is defined to be the set of formulas valid in all TU-models with centering, and so on [21].

Some of the conditions defined above are incremental: centering implies weak centering, which in turn implies total reflexivity, and absoluteness implies uniformity. Moreover, the condition of uniformity states that for any two worlds $w, v \in W$, it holds that the union of the systems of spheres associated to w and v is the same. Without loss of generality, we can assume that for any $w \in W$, $\bigcup \text{SP}(w) = W$, meaning that if there exists a TU-model where $\bigcup \text{SP}(w) \subset W$ for any w , we can construct a (smaller) TU-model in which $\bigcup \text{SP}(w) = W$. As a consequence, the modal operators $\Box A \equiv (\perp \preceq \neg A)$ and $\Diamond A \equiv \neg(\perp \preceq \neg A)$ behave over the set $\bigcup \text{SP}(w)$ in the same way as the modal operators of S5 ⁶.

⁶ Notice that $\perp \preceq A$ means that A is impossible, since \perp is impossible and it is as plausible as A .

(CPR) $\frac{\vdash B \rightarrow A}{\vdash A \preceq B}$	(CPA) $(A \preceq A \vee B) \vee (B \preceq A \vee B)$
(TR) $(A \preceq B) \wedge (B \preceq C) \rightarrow (A \preceq C)$	(CO) $(A \preceq B) \vee (B \preceq A)$
(N) $\neg(\perp \preceq \top)$	(T) $(\perp \preceq \neg A) \rightarrow A$
(U1) $\neg(\perp \preceq A) \rightarrow (\perp \preceq (\perp \preceq A))$	(U2) $(\perp \preceq \neg A) \rightarrow (\perp \preceq \neg(\perp \preceq \neg A))$
(W) $A \rightarrow (A \preceq \top)$	(C) $(A \preceq \top) \rightarrow A$
(A1) $(A \preceq B) \rightarrow (\perp \preceq \neg(A \preceq B))$	(A2) $\neg(A \preceq B) \rightarrow (\perp \preceq (A \preceq B))$

$$\begin{aligned} \mathcal{A}_{\text{VTU}} &:= \{(\text{CPR}), (\text{CPA}), (\text{TR}), (\text{CO}), (\text{N}), (\text{T}), (\text{U1}), (\text{U2})\} \\ \mathcal{A}_{\text{VWU}} &:= \mathcal{A}_{\text{VTU}} \cup \{(W)\} & \mathcal{A}_{\text{VCU}} &:= \mathcal{A}_{\text{VTU}} \cup \{(W), (C)\} \\ \mathcal{A}_{\text{VTA}} &:= \mathcal{A}_{\text{VTU}} \cup \{(A1), (A2)\} & \mathcal{A}_{\text{VWA}} &:= \mathcal{A}_{\text{VTU}} \cup \{(W), (A1), (A2)\} \\ \mathcal{A}_{\text{VCA}} &:= \mathcal{A}_{\text{VTU}} \cup \{(W), (C), (A1), (A2)\} \end{aligned}$$

Figure 1: Lewis' logics and axioms.

For instance, a boxed formula is satisfied at any world of a TU-model: if $w \models \perp \preceq \neg A$, then for all $\alpha \in \text{SP}(w)$, all worlds in α satisfy A and, since $\bigcup \text{SP}(w) = W$, we have that all worlds in W satisfy A ⁷.

Figure 1 presents the Hilbert-style axiom system from Lewis [21, Chp. 6] for all the above logics, with \vee and \wedge binding stronger than \preceq . Propositional axioms and rules are standardly defined.

To end this section, we recall some uses of conditionals, in particular within the logics considered in this paper.

All Lewis' logics, there including VTU and all its extensions, and with the exception of the systems VWA and VCA ⁸, are *non-monotonic*, meaning that adding formulas to the antecedent of a true conditional does not necessarily result in a true conditional. Let M = “Messi quits Barça”, G = “Griezmann quits Barça”, and W = “Barça wins the Champions League”. We have that the following formula is *not* valid:

$$(\neg M > W) \rightarrow ((\neg M \wedge G) > W)$$

However, all Lewis' logics satisfy a form of monotonicity, called *rational monotonicity*, expressed by the following formula, valid in all the logics⁹:

$$(S > T) \wedge \neg(S > \neg J) \rightarrow ((S \wedge J) > T)$$

⁷Moreover, using an easy semantic argument it can be shown that VWA collapses to S5 , and that VCA collapses to classical logic. A syntactic proof of both facts can be found in Proposition 18 of Section 3.2.

⁸Logics VWA and VCA collapse to classical systems: see previous footnote.

⁹The formula corresponds to axiom (CV) in axiomatization of conditional logics taking $>$ as primitive.

In the above formula, read S as “Simon is a student”, T as “Simon does not pay taxes” and J as “Simon has a summer job”.

Finally, one may wonder what is the role of the property of *uniformity*: according to this property, there is a unique set of accessible worlds, and this set is the *union* of the system of spheres of each world. As mentioned, because of this property, the inner modality $\Box A$, defined by $\perp \preceq \neg A$, or equally by $\neg A > \perp$, has the property of the \Box operator defined in S5. The property of uniformity may be used to enforce the preservation of possible, necessary or impossible formulas over right-nested conditionals, that can be viewed as subsequent updates (refer to the Introduction).

Consider the following example. In a very safe system, we have that an unauthorized access to classified files is (considered as) impossible. However, there might occur a crash of the system, with a subsequent recovery, and a password leak due to security failure. Let U stand for “an unauthorized access to classified files occurs”, R for “recovery from system crash occurs”, and P for “a password leak occurs”.

In all Lewis’ logics, even without Uniformity, we have that the following two formulas are valid:

$$\Box \neg U \rightarrow (R > \neg U) \quad \text{and} \quad \Box \neg U \rightarrow (P > \neg U)$$

That is, from the assumption $\Box \neg U$ we can conclude both $R > \neg U$ and $P > \neg U$. In contrast, the formulas

$$\Box \neg U \rightarrow (R > (P > \neg U)) \quad \text{and} \quad \Box \neg U \rightarrow (R > (P > \neg U))$$

are valid in $\mathbb{V}\text{CU}$, but they are not valid in logic without uniformity. Thus, without uniformity, from the assumption $\Box \neg U$, we cannot conclude neither $R > (P > \neg U)$, nor $P > (R > \neg U)$.

Intuitively, if we think of a conditional as representing an *update* operation, then by uniformity the set of worlds accessible from a given initial world is the same as the set of worlds accessible from the worlds determined by an update by R (or by P). Thus, the formula $\Box \neg U$, expressing safety of the system, is preserved in such worlds and in all the worlds determined by any subsequent update. On the other hand, without uniformity, we have that $\neg U$ holds in any world determined by a single update by R (or by P), but we cannot say anything about $\Box \neg U$, as the accessible worlds may be different. In this sense, we can say that the property of uniformity ensures that modal formulas are preserved along a sequence of updates.

3 Hypersequent Calculi for Lewis’ Logics

In this section we present some non-standard calculi for the logic $\mathbb{V}\text{TU}$ and its extensions. Our calculi are based on hypersequents, where as usual a *sequent* is a pair consisting of two multisets of formulae, written as $\Gamma \Rightarrow \Delta$.

Definition 3 *A hypersequent is a non-empty multiset of sequents, written $\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$, where $n \geq 1$ is the cardinality of the multiset.*

The conditional formula interpretation of a hypersequent is

$$\iota_{\preceq}(\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n) := \Box(\bigwedge \Gamma_1 \rightarrow \bigvee \Delta_1) \vee \dots \vee \Box(\bigwedge \Gamma_n \rightarrow \bigvee \Delta_n)$$

where \Box is the outer modality defined by $\Box A \equiv (\perp \preceq \neg A)$.

For \mathcal{L} any of the considered logics, the rules of the corresponding hypersequent calculus $\mathbf{H}_{\mathcal{L}}$ are given in Figure 2. We denote by $\mathbf{H}_{\mathcal{L}} \vdash \mathcal{G}$ derivability of hypersequent \mathcal{G} at $\mathbf{H}_{\mathcal{L}}$.

The rules of the calculi $\mathbf{H}_{\mathcal{L}}$ are based on an extension of the calculi from [19] to the hypersequent setting. We abbreviate multisets of formulae A_k, \dots, A_n to $[\vec{A}]_k^n$, and $C_k \preceq D_k, \dots, C_n \preceq D_n$ to $[C \preceq D]_k^n$ with the convention that $[\vec{A}]_k^n$ is empty if $k > n$. These calculi are *non-standard*, meaning that the rules have an unbounded number of premisses. Note also that they include the explicit contraction rules ICL, ICR . This is because for these calculi we are only interested in cut elimination. We will consider calculi in which contraction is admissible in Section 4.2 below. The crucial rule for uniformity is the rule trf_m . Intuitively, it unpacks a number of comparative plausibility formulae behaving like boxed formulae on the left hand side of a component in the conclusion into a different component in the rightmost premiss, most clearly seen in the case of $n = 1$. The leftmost set of premisses ensures that the comparative plausibility formulae indeed behave like boxed formulae. The rule T_m is the local version of trf_m , and essentially captures total reflexivity.

Theorem 4 For \mathcal{L} any of the considered logics, the calculus $\mathbf{H}_{\mathcal{L}}$ is sound for \mathcal{L} .

Proof. We need to show that whenever a hypersequent $\Rightarrow A$ is derivable in $\mathbf{H}_{\mathcal{L}}$, then the formula A is a theorem of \mathcal{L} . To do so we show below that all the rules preserve soundness with respect to ι_{\preceq} , i.e., whenever the formula translations of all premisses of a rule are theorems of \mathcal{L} , then so is the formula translation of its conclusion. This yields that whenever $\Rightarrow A$ is derivable, then $\Box(\top \rightarrow A)$ is a theorem of \mathcal{L} . Since $\Box B \rightarrow B$ is valid in all the logics, the formula $\top \rightarrow A$ is thus a theorem of \mathcal{L} , which is equivalent to A .

The fact that the rules preserve soundness with respect to the formula interpretation ι_{\preceq} is shown by constructing a countermodel for the formula interpretation of one of the premisses from a countermodel for the formula interpretation of the conclusion, using that the sphere system satisfies uniformity. For all the rules other than trf , abs_L , abs_R , spl this follows as in [19], using that $\Box A \rightarrow \Box \Box A$ is valid. For $\text{abs}_L, \text{abs}_R$ this follows straightforwardly from absoluteness, and for spl this follows from the fact that frames for \mathbf{VCA} are degenerate in the sense that $\text{SP}(w) = \{\{w\}\}$ for every world w .

For the rule trf , let $\mathfrak{M} = \langle W, \text{SP}, [\cdot] \rangle$ be a \mathbf{VTU} model, let $w \in W$, and suppose that

$$\mathfrak{M}, w \models \neg \iota_{\preceq}(\mathcal{G}) \wedge \diamond(\bigwedge \Sigma \wedge \bigwedge_{i=1}^m (C_i \preceq D_i) \wedge \neg \bigvee \Delta) \wedge \diamond(\bigwedge \Omega \wedge \neg \bigvee \Theta) . \quad (1)$$

Then in particular $\mathfrak{M}, w \models \neg(\iota_{\preceq}(\mathcal{G}) \vee \Box(\bigwedge \Sigma \rightarrow \Pi) \vee \Box(\bigwedge \Omega \rightarrow \Theta))$. Furthermore, suppose that for every $k \leq m$ we have

$$\mathfrak{M}, w \models \iota(\mathcal{G}) \vee \Box(\bigwedge \Sigma \rightarrow \bigvee \Pi) \vee \Box(\bigwedge \Omega \rightarrow \bigvee \Theta) \vee \Box(C_k \rightarrow \bigvee_{i=1}^{k-1} D_i) . \quad (2)$$

$$\begin{array}{c}
\frac{}{\mathcal{G} \mid \Gamma, p \Rightarrow p, \Delta} \text{init} \quad \frac{}{\mathcal{G} \mid \Gamma, \perp \Rightarrow \Delta} \perp_L \quad \frac{\mathcal{G} \mid \Gamma, A, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta} \text{ICL} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow A, A, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta} \text{ICR} \\
\frac{\mathcal{G} \mid \Gamma, B \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow \Delta} \rightarrow_L \quad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \rightarrow B, \Delta} \rightarrow_R \\
\frac{\{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid C_k \Rightarrow [\vec{D}]_1^{k-1}, [\vec{A}]_1^n : k \leq m\} \cup \{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid B_k \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n : k \leq n\}}{\mathcal{G} \mid \Sigma, [C \vec{\approx} D]_1^m \Rightarrow [A \vec{\approx} B]_1^n, \Pi} R_{m,n} \\
\frac{\{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow \Theta \mid C_k \Rightarrow [\vec{D}]_1^{k-1} : k \leq m\} \cup \{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow [\vec{D}]_1^m, \Theta\}}{\mathcal{G} \mid \Sigma, [C \vec{\approx} D]_1^m \Rightarrow \Pi \mid \Omega \Rightarrow \Theta} \text{trf}_m \\
\frac{\{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid C_k \Rightarrow [\vec{D}]_1^{k-1} : k \leq m\} \cup \{\mathcal{G} \mid \Sigma \Rightarrow [\vec{D}]_1^m, \Pi\}}{\mathcal{G} \mid \Sigma, [C \vec{\approx} D]_1^m \Rightarrow \Pi} T_m \\
\frac{\{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid C_k \Rightarrow [\vec{D}]_1^{k-1}, [\vec{A}]_1^n : k \leq m\} \cup \{\mathcal{G} \mid \Sigma \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, \Pi\}}{\mathcal{G} \mid \Sigma, [C \vec{\approx} D]_1^m \Rightarrow [A \vec{\approx} B]_1^n, \Pi} W_{m,n} \\
\frac{\mathcal{G} \mid \Sigma, C \Rightarrow \Pi \quad \mathcal{G} \mid \Sigma \Rightarrow D, \Pi}{\mathcal{G} \mid \Sigma, C \approx D \Rightarrow \Pi} R_C \quad \frac{\mathcal{G} \mid \Sigma \Rightarrow A, \Pi}{\mathcal{G} \mid \Sigma \Rightarrow A \approx B, \Pi} R_W \\
\frac{\mathcal{G} \mid \Sigma, \Gamma \Rightarrow \Pi, \Delta \mid \Omega \Rightarrow \Theta}{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega, \Gamma \Rightarrow \Theta, \Delta} \text{spl} \\
\frac{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega, C \approx D \Rightarrow \Theta}{\mathcal{G} \mid \Sigma, C \approx D \Rightarrow \Pi \mid \Omega \Rightarrow \Theta} \text{abs}_L \quad \frac{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega \Rightarrow A \approx B, \Theta}{\mathcal{G} \mid \Sigma \Rightarrow A \approx B, \Pi \mid \Omega \Rightarrow \Theta} \text{abs}_R
\end{array}$$

$$\begin{array}{l}
\mathbf{H}_{\text{VTU}} : \{\text{init}, \perp_L, \text{ICL}, \text{ICR}, \rightarrow_L, \rightarrow_R\} \cup \{R_{m,n} : m \geq 0, n \geq 1\} \cup \\
\quad \{\text{trf}_m : m \geq 1\} \cup \{T_m : m \geq 1\} \\
\mathbf{H}_{\text{VWU}} : \mathbf{H}_{\text{VTU}} \cup \{W_{m,n} : m + n \geq 1\} \quad \mathbf{H}_{\text{VCU}} : \mathbf{H}_{\text{VTU}} \cup \{R_C, R_W\} \\
\mathbf{H}_{\text{VTA}} : \mathbf{H}_{\text{VTU}} \cup \{\text{abs}_L, \text{abs}_R\} \quad \mathbf{H}_{\text{VWA}} : \mathbf{H}_{\text{VWU}} \cup \{\text{abs}_L, \text{abs}_R\} \\
\mathbf{H}_{\text{VCA}} : \mathbf{H}_{\text{VCU}} \cup \{\text{abs}_L, \text{abs}_R, \text{spl}\}
\end{array}$$

Figure 2: The hypersequent calculi for VTU and extensions.

Then from the case $k = 1$ of (2) we obtain $\mathfrak{M}, w \models \Box \neg C_1$. From this together with (1) and the fact that for every $v \in \bigcup \text{SP}(w)$ we have $\bigcup \text{SP}(v) = \bigcup \text{SP}(w)$ we then obtain $\mathfrak{M}, w \models \Box \neg D_1$. Similarly, using the case $k = 2$ of (2) we get $\mathfrak{M}, w \models \Box \neg D_2$ and continuing like this we get $\mathfrak{M}, w \models \Box \neg D_1 \wedge \dots \wedge \Box \neg D_m$. Together with (1) this gives $\mathfrak{M}, w \models \neg \iota(\mathcal{G}) \wedge \diamond(\bigwedge \Sigma \wedge \neg \bigvee \Pi) \wedge \diamond(\bigwedge \Omega \wedge \neg(D_1 \vee \dots \vee D_m \vee \bigvee \Theta))$ and hence we have a countermodel for the remaining premiss. \blacksquare

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{IW} \quad \frac{\mathcal{G}}{\mathcal{G} \mid \Rightarrow} \text{EW} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{mrg}$$

Figure 3: The structural rules of internal and external weakening and merge.

We further have derivability of generalised initial hypersequents:

Lemma 5 *For every formula A we have $\mathsf{H}_{\mathcal{L}} \vdash \mathcal{G} \mid \Gamma, A \Rightarrow A, \Delta$.*

Proof. By induction on the complexity of the formula A . For the propositional cases this is standard. If A is of the form $B \preceq C$, we have

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid B \Rightarrow B \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta \mid C \Rightarrow C, B}{\mathcal{G} \mid \Gamma, B \preceq C \Rightarrow B \preceq C, \Delta} R_{1,1}$$

where the premisses are derivable using the induction hypothesis. \blacksquare

As usual, a rule is *admissible* in $\mathsf{H}_{\mathcal{L}}$ if whenever the premisses are derivable in $\mathsf{H}_{\mathcal{L}}$, then so is its conclusion. By *depth* of a derivation we mean the length of its longest branch. A rule is *depth-preserving admissible* if the depth of the derivation of its conclusion is at most the maximal depth of the derivations of its premisses.

Lemma 6 *The rules IW, EW, mrg from Fig. 3 are depth-preserving admissible in $\mathsf{H}_{\mathcal{L}}$.*

Proof. By induction on the depth of the derivation in all cases. For mrg, if the last applied rule was trf_m , we might need to replace it with T_m . \blacksquare

Observe that from admissibility of mrg using the internal contraction rules we also immediately obtain admissibility of the external contraction rules, i.e., contraction on hypersequent components. We first show completeness of the systems with the *cut rule*:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \quad \mathcal{H} \mid A, \Sigma \Rightarrow \Pi}{\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{cut}$$

Cut-free completeness then will follow from cut elimination. In the following we write $\mathsf{H}_{\mathcal{L}}\text{cut}$ for the system $\mathsf{H}_{\mathcal{L}}$ with the cut rule.

Remark 7 *In order to improve readability, we use the standard rules for negation and conjunction when writing the derivations. The rules are trivially definable in the calculi, recalling that $\neg A \equiv A \rightarrow \perp$ and $A \wedge B \equiv \neg(A \rightarrow \neg B)$.*

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma, \neg A \Rightarrow \Delta} \neg_L \quad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \neg A, \Delta} \neg_R$$

$$\frac{\mathcal{G} \mid \Gamma, A, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \wedge B \Rightarrow \Delta} \wedge_L \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow A, \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \wedge B, \Delta} \wedge_R$$

can be permuted above logical rules is used to obtain a single occurrence of the cut formula in the left premiss of the cut, and the cut is permuted up in the right premiss. Again, contractions are swallowed up by a generalised induction hypothesis, and once the cut formula becomes principal in the last applied rule, its complexity is reduced (Lem. 11). For technical reasons we also include the rule mrg in the calculus when proving cut elimination. By Lem. 6 it is clear that all applications of this rule can then be eliminated in the cut-free system. In the following we write $\mathsf{H}_{\mathcal{L}}^*$ for the system $\mathsf{H}_{\mathcal{L}}$ with cut , mrg and with the rules $W_{m,n}^{\text{abs}}, R_C^{\text{abs}}, R_W^{\text{abs}}$ where applicable, and abbreviate $\underbrace{A, \dots, A}_{n\text{-times}}$ to A^n .

Definition 9 *The cut rank of a $\mathsf{H}_{\mathcal{L}}^*$ -derivation \mathcal{D} is the maximal complexity of a cut formula occurring in \mathcal{D} , written $\rho(\mathcal{D})$. A rule is cut-rank preserving admissible in $\mathsf{H}_{\mathcal{L}}^*$ if whenever its premiss(es) are derivable in $\mathsf{H}_{\mathcal{L}}^*$ with cut-rank n , then so is its conclusion.*

In order to be able to use the weakening rules in the cut elimination proof, we need that they are admissible in $\mathsf{H}_{\mathcal{L}}^*$ without increasing the cut-rank:

Lemma 10 *The rules EW, IW are depth- and cut-rank preserving admissible in $\mathsf{H}_{\mathcal{L}}^*$.*

Proof. Standard induction on the depth of the derivation. ■

Lemma 11 (Shift Right) *Suppose that for $k > 0$ and $n_1, \dots, n_k > 0$ there are $\mathsf{H}_{\mathcal{L}}^*$ -derivations \mathcal{D}_1 and \mathcal{D}_2 of $\mathcal{G} \mid \Omega \Rightarrow \Theta, A$ and $\mathcal{H} \mid A^{n_1}, \Xi_1 \Rightarrow \Upsilon_1 \mid \dots \mid A^{n_k}, \Xi_k \Rightarrow \Upsilon_k$ with $\rho(\mathcal{D}_1) < |A| > \rho(\mathcal{D}_2)$ and such that the displayed occurrence of A is principal in the last rule application in \mathcal{D}_1 . Then there is a $\mathsf{H}_{\mathcal{L}}^*$ -derivation \mathcal{D} of hypersequent $\mathcal{G} \mid \mathcal{H} \mid \Omega, \Xi_1 \Rightarrow \Theta, \Upsilon_1 \mid \dots \mid \Omega, \Xi_k \Rightarrow \Theta, \Upsilon_k$ and $\rho(\mathcal{D}) < |A|$.*

Proof. By induction on the depth of \mathcal{D}_2 . If none of the displayed occurrences of A is principal in the last rule in \mathcal{D}_2 , we apply the induction hypothesis on the premiss(es) of that rule, followed by the same rule (and possibly structural rules). If at least one of the displayed occurrences is principal in the last rule in \mathcal{D}_2 , we distinguish cases according to the last applied rule in \mathcal{D}_1 , with subcases according to the last rule in \mathcal{D}_2 . In order to not break the flow of the paper, here we only consider an exemplary case. The full proof is found in the appendix. Suppose the last rules in \mathcal{D}_1 and \mathcal{D}_2 are $R_{m,n+1}$ and trf_s respectively, that A is the formula $E \preceq F$ and that \mathcal{D}_1 ends in:

$$\frac{\begin{aligned} & \left\{ \mathcal{G} \mid \Omega \Rightarrow \Theta \mid C_j \Rightarrow [\vec{D}]_1^{j-1}, [\vec{A}]_1^n, E : 1 \leq j \leq m \right\} \\ & \cup \left\{ \mathcal{G} \mid \Omega \Rightarrow \Theta \mid B_j \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, E : 1 \leq j \leq n \right\} \\ & \cup \left\{ \mathcal{G} \mid \Omega \Rightarrow \Theta \mid F \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, E \right\} \end{aligned}}{\mathcal{G} \mid \Omega, [C \vec{\preceq} D]_1^m \Rightarrow [A \vec{\preceq} B]_1^n, E \preceq F, \Theta} R_{m,n+1}$$

First we apply the induction hypothesis to the conclusion of this and the premisses of trf_s to eliminate all the occurrences of $E \preceq F$ from the context.

Hence we assume that the only occurrences of $E \preceq F$ in the conclusion of trf_s are principal and that \mathcal{D}_2 ends in:

$$\frac{\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1} : 1 \leq j \leq r \right\} \cup \\ \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid E \Rightarrow [\vec{H}]_1^r \right\} \cup \\ \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, F : r < j \leq s \right\} \cup \\ \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi, [\vec{H}]_1^s, F \right\} \end{array}}{\mathcal{H} \mid \Xi, [G \vec{\preceq} H]_1^r, E \preceq F, [G \vec{\preceq} H]_{r+1}^s \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi} \text{trf}_s$$

with $E \preceq F$ not occurring in $[G \vec{\preceq} H]_1^r$. Cuts on the formulae E and F then yield:

$$\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1} : 1 \leq j \leq r \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid C_j \Rightarrow [\vec{D}]_1^{j-1}, [\vec{A}]_1^n, [\vec{H}]_1^r : 1 \leq j \leq m \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid B_j \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r : 1 \leq j \leq n \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r : r < j \leq s \right\} \end{array}$$

Admissibility of internal weakening (Lem. 10) and an application of $R_{m+s, n+t}$ then gives:

$$\mathcal{G} \mid \mathcal{H} \mid \Omega, [G \vec{\preceq} H]_1^r, [C \vec{\preceq} D]_1^m, [G \vec{\preceq} H]_{r+1}^s \Rightarrow [A \vec{\preceq} B]_1^n, [I \vec{\preceq} J]_1^t, \Theta \mid \Xi \Rightarrow \Upsilon$$

Iterating this process to eliminate the remaining occurrences of $E \preceq F$ from $[G \vec{\preceq} H]_{r+1}^s$, followed by mrg and applications of contraction then yields the desired sequent. \blacksquare

Lemma 12 (Shift Left) *Suppose that for $k > 0$ and $n_1, \dots, n_k > 0$ there are $H_{\mathcal{L}}^*$ -derivations \mathcal{D}_1 and \mathcal{D}_2 of the hypersequents $\mathcal{G} \mid \Omega_1 \Rightarrow \Theta_1, A^{n_1} \mid \dots \mid \Omega_k \Rightarrow \Theta_k, A^{n_k}$ and $\mathcal{H} \mid A, \Xi \Rightarrow \Upsilon$ with $\rho(\mathcal{D}_1) < |A| > \rho(\mathcal{D}_2)$. Then there is a $H_{\mathcal{L}}^*$ -derivation \mathcal{D} of $\mathcal{G} \mid \mathcal{H} \mid \Omega_1, \Xi \Rightarrow \Theta_1, \Upsilon \mid \dots \mid \Omega_k, \Xi \Rightarrow \Theta_k, \Upsilon$ and $\rho(\mathcal{D}) < |A|$.*

Proof. By induction on the depth of \mathcal{D}_1 . If none of the displayed occurrences of A is principal in the last rule in \mathcal{D}_1 or the active formula of abs_R , we apply the induction hypothesis on the premiss(es) of the last rule in \mathcal{D}_1 followed by the same rule and possibly admissibility of weakening and contraction. If one of the occurrences of A is active in abs_R , we use admissibility of EW (Lem. 10) and abs_L on \mathcal{D}_2 to obtain $\mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid A \Rightarrow \cdot$. Then the induction hypothesis on this and the premiss of abs_R followed by mrg and IW yields the result. If an occurrence of A is principal in the last rule in \mathcal{D}_1 , we use the induction hypothesis to remove all the occurrences of A in the context of that rule. Then, in case this rule is $R_{m,n}, W_{m,n}, W_{m,n}^{\text{abs}}$, we apply contraction in the premisses and apply the same rule, so that only one occurrence of A is principal. Now Lem. 11 yields the result. \blacksquare

Theorem 13 (Cut Elimination) *Let $\mathcal{L} \in \{\text{VTU}, \text{VWU}, \text{VCU}, \text{VTA}, \text{VWA}, \text{VCA}\}$. If a hypersequent is derivable in $\text{H}_{\mathcal{L}}^*$, then it is derivable in $\text{H}_{\mathcal{L}}$.*

Proof. First we eliminate all applications of cut by induction on the tuples $\langle \rho(\mathcal{D}), \sharp(\mathcal{D}) \rangle$ under the lexicographic ordering, where $\sharp(\mathcal{D})$ is the number of applications of cut in \mathcal{D} with cut formula of complexity $\rho(\mathcal{D})$. Then applications of $W_{m,n}^{\text{abs}}, R_C^{\text{abs}}, R_W^{\text{abs}}$ are replaced with the $W_{m,n}, R_C, R_W$ and $\text{abs}_L, \text{abs}_R$, and mrg is eliminated using Lem. 6. It is straightforward to check that applications of $W_{m,n}^{\text{abs}}, R_C^{\text{abs}}, R_W^{\text{abs}}$ are only introduced in systems including the absoluteness rules. ■

As usual, the cut elimination result yields cut-free completeness:

Theorem 14 (Cut-free completeness) *If $A \in \mathcal{L}$, then $\text{H}_{\mathcal{L}} \vdash \Rightarrow A$.*

Proof. If $A \in \mathcal{L}$, then by Lem. 8 we have $\text{H}_{\mathcal{L}}\text{cut} \vdash \Rightarrow A$. Since $\text{H}_{\mathcal{L}}\text{cut}$ is a subsystem of $\text{H}_{\mathcal{L}}^*$ we thus have $\text{H}_{\mathcal{L}}^* \vdash \Rightarrow A$. Now Thm. 13 yields the result. ■

3.2 Connections to modal logic

The constructed hypersequent calculi provide purely syntactical proofs of results from Lewis [21] connecting the conditional logics to, e.g., modal logic S5. We write \mathcal{L}^{\square} for the *modal fragment* of a conditional logic \mathcal{L} , i.e., the fragment where comparative plausibility formulae are restricted to the shape $(\perp \preccurlyeq \neg A)$, and we write A^{\square} for the result of replacing every subformula $\perp \preccurlyeq \neg B$ of A with $\square B$. The proofs use the fact that the hypersequent calculus H_{S5} with the propositional rules of Fig. 2, the structural rules and the rules

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \square A, \Delta \mid \Rightarrow A}{\mathcal{G} \mid \Gamma \Rightarrow \square A, \Delta} \square_R \quad \frac{\mathcal{G} \mid \Gamma, \square A \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \square A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \square_L \quad \frac{\mathcal{G} \mid \Gamma, \square A, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \square A \Rightarrow \Delta} \top$$

is cut-free complete for S5 [28], see also [18]. The following derivations make use of the propositional rules from Remark 7.

Lemma 15 *If $A^{\square} \in \text{S5}$, then $A \in \mathcal{L}^{\square}$ for each of the logics \mathcal{L} considered here.*

Proof. By translating H_{S5} -derivations into $\text{H}_{\mathcal{L}}$ -derivations. E.g., \square_L is translated into:

$$\frac{\frac{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi \mid \neg \Rightarrow}{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg A, \perp \preccurlyeq \neg A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \perp_L \quad \frac{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg A \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg A \Rightarrow \Delta \mid \Sigma \Rightarrow \neg A, \Pi} \neg_L}{\frac{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg A, \perp \preccurlyeq \neg A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \perp \preccurlyeq \neg A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \text{trf}_1} \text{ICL}$$

The translations of \square_R, \top are similar, using $R_{0,1}$ and \top_1 respectively. ■

The backwards direction is similar, but translates into the calculus H_{S5} above with a form of Avron's *modal splitting rule* from [2]:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma, \square \Omega \Rightarrow \square \Theta, \Pi}{\mathcal{G} \mid \Gamma, \square \Omega \Rightarrow \square \Theta, \Delta \mid \Sigma \Rightarrow \Pi} \text{MS}$$

It is straightforward to check that the resulting calculus is sound for S5.

Lemma 16 *If $\mathcal{L} \neq \mathbb{VCA}$ and $A \in \mathcal{L}^\square$, then $A^\square \in \mathbb{S5}$.*

Proof. By translating derivations in $\mathbb{H}_{\mathcal{L}}$ into derivations in $\mathbb{H}_{\mathbb{S5}cut}$ and applying cut elimination. In particular, an application of the rule $R_{m,n}$

$$\frac{\begin{array}{l} \{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \perp \Rightarrow \neg D_1, \dots, \neg D_{j-1}, \perp^n : 1 \leq j \leq m\} \\ \cup \quad \{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \neg B_j \Rightarrow \neg D_1, \dots, \neg D_m, \perp^n : 1 \leq j \leq n\} \end{array}}{\mathcal{G} \mid \Gamma, \perp \preceq \neg D_1, \dots, \perp \preceq \neg D_m \Rightarrow \perp \preceq \neg B_1, \dots, \perp \preceq \neg B_n, \Delta} R_{m,n}$$

is translated into

$$\frac{\frac{\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \neg B_1 \Rightarrow \neg D_1, \dots, \neg D_m, \perp^n}{\mathcal{G} \mid \Gamma, \square D_1, \dots, \square D_m \Rightarrow \square B_1, \Delta \mid \neg B_1 \Rightarrow \neg D_1, \dots, \neg D_m, \perp^n} \text{IW}}{\mathcal{G} \mid \Gamma, \square D_1, \dots, \square D_m \Rightarrow \square B_1, \Delta \mid D_1, \dots, D_m \Rightarrow B_1} \text{prop}}{\frac{\mathcal{G} \mid \Gamma, \square D_1, \dots, \square D_m \Rightarrow \square B_1, \Delta}{\mathcal{G} \mid \Gamma, \square D_1, \dots, \square D_m \Rightarrow \square B_1, \Delta} \square_L, \square_R} \text{IW}$$

Here rule prop uses derivability of the inversions of the propositional rules using cut. Similarly, applications of \mathbb{T}_m and trf_m are translated using m applications of \square_L and \mathbb{T} respectively. Applications of $W_{m,n}$ and R_C are translated by \mathbb{T} , and R_W is replaced with weakening, using that whenever $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \perp$ is derivable in the system for $\mathbb{S5}$, then so is $\mathcal{G} \mid \Gamma \Rightarrow \Delta$. Finally, $\text{abs}_L, \text{abs}_R$ are replaced with the modalised splitting rule MS . ■

Theorem 17 ([21, Sec. 6.3]) *Let $\mathcal{L} \neq \mathbb{VCA}$. Then $A \in \mathcal{L}^\square$ iff $A^\square \in \mathbb{S5}$.*

The proof of the previous theorem is immediate from the preceding lemmas. It is then also straightforward to derive the known collapses of the counterfactual implication $>$ in \mathbb{VWA} and \mathbb{VCA} . Recall that $A > B \equiv (\perp \preceq A) \vee \neg((A \wedge \neg B) \preceq A)$.

Proposition 18 1. $\mathbb{H}_{\mathbb{VWA}} \vdash \Rightarrow (A > B) \leftrightarrow \square(A \rightarrow B)$

2. $\mathbb{H}_{\mathbb{VCA}} \vdash \Rightarrow A \leftrightarrow \square A$

3. $\mathbb{H}_{\mathbb{VCA}} \vdash \Rightarrow (A > B) \leftrightarrow (A \rightarrow B)$ and $\mathbb{H}_{\mathbb{VCA}} \vdash \Rightarrow (A \preceq B) \leftrightarrow (B \rightarrow A)$.

Proof. Statement 1 is proved by deriving formulas

$$\neg((A \wedge \neg B) \preceq A) \rightarrow \perp \preceq \neg(A \rightarrow B) \quad \perp \preceq A \rightarrow \perp \preceq \neg(A \rightarrow B)$$

using $R_{0,1}, \text{abs}_R, W_{0,1}$ and $R_{1,1}$, respectively. From these using \vee_L we derive the left-to-right direction of the implication. The other direction is derivable straightforwardly using $R_{2,1}$.

The derivation for statement 2 is

$$\frac{\frac{\frac{\perp \Rightarrow A}{\perp \preceq \neg A \Rightarrow A} \perp_L \quad \frac{\frac{\overline{A \Rightarrow A}}{\Rightarrow \neg A, A} \text{init}}{\Rightarrow \neg A, A} \neg_R}{\perp \preceq \neg A \Rightarrow A} R_C}{\Rightarrow (\perp \preceq \neg A) \rightarrow A} \rightarrow_R \quad \frac{\frac{\frac{\overline{\Rightarrow \mid A \Rightarrow A, \perp}}{\Rightarrow \mid A, \neg A \Rightarrow \perp} \text{init}}{\Rightarrow \mid \neg A \Rightarrow \perp} \neg_R}{A \Rightarrow \perp \preceq \neg A} \text{spl}}{A \Rightarrow \perp \preceq \neg A} R_{1,1}}{\Rightarrow A \rightarrow (\perp \preceq \neg A)} \rightarrow_R} \Rightarrow ((\perp \preceq \neg A) \rightarrow A) \wedge (A \rightarrow (\perp \preceq \neg A)) \wedge_R$$

For the first part of statement 3, since \mathbb{VCA} is an extension of \mathbb{VWA} from statement 1 we also obtain $\mathbb{H}_{\mathbb{VCA}} \vdash \Rightarrow A > B \leftrightarrow \Box(A \rightarrow B)$ (by replacing the application of $W_{0,1}$ with R_W). Hence using the first equivalence and cut elimination we obtain the equivalence of $A > B$ and $A \rightarrow B$.

The derivation of the last part of statement 3 is

$$\begin{array}{c}
\frac{}{B, A \Rightarrow A \mid \Rightarrow} \text{init} \\
\frac{}{B \Rightarrow A \mid A \Rightarrow} \text{spl} \\
\frac{}{A \preceq B, B \Rightarrow A} \\
\frac{}{\Rightarrow ((A \preceq B) \rightarrow (B \rightarrow A))} \rightarrow_R \\
\frac{}{\Rightarrow ((A \preceq B) \rightarrow (B \rightarrow A)) \wedge ((B \rightarrow A) \rightarrow (A \preceq B))} \wedge_R
\end{array}
\quad
\frac{}{A, B \Rightarrow A \mid \Rightarrow} \text{init} \quad \frac{}{B \Rightarrow A, B \mid \Rightarrow} \text{init} \rightarrow_L
\quad
\frac{}{B \rightarrow A, B \Rightarrow A \mid \Rightarrow} \\
\frac{}{B \rightarrow A \Rightarrow \mid B \Rightarrow A} \text{spl} \\
\frac{}{B \rightarrow A \rightarrow A \preceq B} R_{0,1} \\
\frac{}{\Rightarrow ((B \rightarrow A) \rightarrow (A \preceq B))} \rightarrow_R \\
\frac{}{\Rightarrow ((B \rightarrow A) \rightarrow (A \preceq B)) \wedge ((A \preceq B) \rightarrow (B \rightarrow A))} \wedge_R$$

■

4 Standard Calculi for Lewis' Logics

As mentioned in the Introduction, we call a calculus *standard* if it has a finite number of rules and each rule has a finite and fixed number of premisses. Our goal is to define standard hypersequent calculi for $\mathbb{V}\text{TLU}$ and its extension, and in this Section we shall do so, by introducing two families of proof systems, one non-invertible and one invertible.

The non-invertible calculi (Sec. 4.1) have the advantage of being easier to understand, and are thus introduced first. These calculi can be directly proved to be equivalent to the non-standard cut-free hypersequents presented in the previous Section.

The invertible calculi (Sec. 4.2) are defined by “kleeneing” the rules of the non-invertible calculi, i.e., by repeating in each premiss of a rule the principal formula and all the other formulas occurring in the conclusion of the rule (following the ideas of Kleene’s classic [15]). These calculi provide decision procedures for the corresponding logics and, for the logics without absoluteness, allow to extract a countermodel from a failed proof search branch. Thanks to these properties, the invertible calculi are the base of the prover **tuCLEVER**, described in Sec. 5.

4.1 Non-invertible calculi

To convert the non-standard calculi $\mathbb{H}_{\mathcal{L}}$ into standard calculi, we consider an extended notion of sequent, where the consequent contains additional structural connectives. These sequents extend those of [9, 27] with a connective $\langle \cdot \rangle$ interpreting possible formulae, i.e., \diamond -formulae.

Definition 19 A conditional block is a tuple $[\Sigma \triangleleft C]$ containing a multiset Σ of formulae and a single formula C . A transfer block is a multiset of formulae, written $\langle \Theta \rangle$. An extended sequent is a tuple $\Gamma \Rightarrow \Delta$ consisting of a multiset Γ of formulae and a multiset Δ containing formulae, conditional blocks, and

transfer blocks. An extended hypersequent is a multiset containing extended sequents, written $\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$

Recall that $\diamond A \equiv \neg(\perp \preceq A)$ and $\square A \equiv \perp \preceq \neg A$. The formula interpretation of an extended sequent is (all blocks shown explicitly):

$$\begin{aligned} \iota_e(\Gamma \Rightarrow \Delta, [\Sigma_1 \triangleleft C_1], \dots, [\Sigma_n \triangleleft C_n], \langle \Theta_1 \rangle, \dots, \langle \Theta_m \rangle) \\ := \bigwedge \Gamma \rightarrow \bigvee \Delta \vee \bigvee_{i=1}^n \bigvee_{B \in \Sigma_i} (B \preceq C_i) \vee \bigvee_{j=1}^m \diamond(\bigvee \Theta_j) \end{aligned}$$

The formula interpretation of an extended hypersequent is:

$$\iota_e(\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n) \quad := \quad \square \iota_e(\Gamma_1 \Rightarrow \Delta_1) \vee \dots \vee \square \iota_e(\Gamma_n \Rightarrow \Delta_n)$$

For \mathcal{L} any of the logics considered, we denote by $\text{SH}_{\mathcal{L}}$ the corresponding non invertible calculus, whose rules are detailed in Fig. 4. We denote by $\text{SH}_{\mathcal{L}} \vdash A$ derivability of a formula A at $\text{SH}_{\mathcal{L}}$.

The following lemma proves admissibility of the weakening rules, including their structural versions.

Lemma 20 (Admissibility of Weakening) *The rules IW, EW, CW, CIW, TW are depth-preserving admissible in $\text{SH}_{\mathcal{L}}^i$.*

$$\begin{array}{c} \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ IW} \quad \frac{\mathcal{G}}{\mathcal{G} \mid \Rightarrow} \text{ EW} \\ \\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft C]} \text{ CW} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft C]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, A \triangleleft C]} \text{ CIW} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle} \text{ TW} \end{array}$$

Proof. As usual by induction on the depth of the derivation we first obtain depth-preserving admissibility of IW, EW, CW and CIW. We then use depth-preserving admissibility of IW in the proof for TW for the case where the last applied rule is Con_S . \blacksquare

Theorem 21 (Soundness) *If $\text{SH}_{\mathcal{L}} \vdash \Rightarrow A$, then $A \in \mathcal{L}$.*

Proof. As for Thm. 4, by showing that the rules preserve validity under ι_e we obtain the first statement, i.e., whenever the formula translations of all premisses of a rule are theorems of \mathcal{L} , then so is the formula translation of its conclusion. Then using validity of $\square A \rightarrow A$ we obtain the second, i.e., whenever $\iota_e(\Rightarrow A)$ is a theorem of \mathcal{L} , then so is A . For the rules $\preceq_L, \preceq_R, \text{com}, \text{jump}, \text{W}, \text{C}$ the first statement is similar as in [9]. For rule T, if the interpretation of the conclusion is falsified in \mathfrak{M}, w , then there is a world $v \in \text{SP}(w)$ with $\mathfrak{M}, v \Vdash \bigwedge \Gamma \wedge (A \preceq B) \wedge \neg \bigvee \Delta \wedge \square \neg \bigvee \Theta$. If $\llbracket B \rrbracket = \emptyset$, then in particular $\mathfrak{M}, v \Vdash \square \neg (\bigvee \Theta \vee B)$, and the formula interpretation of the second premiss is falsified in \mathfrak{M}, v and therefore in \mathfrak{M}, w . Otherwise, from $\mathfrak{M}, v \Vdash A \preceq B$ we obtain a world $x \in \bigcup \text{SP}(v) = \bigcup \text{SP}(w)$ with $\mathfrak{M}, x \Vdash A$, and from $\mathfrak{M}, v \Vdash \square \neg \bigvee \Theta$ we also get that $\mathfrak{M}, x \Vdash \neg \bigvee \Theta$. Hence the formula interpretation of the first premiss is falsified at \mathfrak{M}, x , and therefore at \mathfrak{M}, w . The remaining cases are similar. \blacksquare

$$\begin{array}{c}
\frac{}{\mathcal{G} \mid \Gamma, \perp \Rightarrow \Delta} \perp_L \quad \frac{}{\mathcal{G} \mid \Gamma, p \Rightarrow \Delta, p} \text{init} \quad \frac{\mathcal{G} \mid \Gamma, B \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, A}{\mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow \Delta} \rightarrow_L \\
\frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta, B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \rightarrow B} \rightarrow_R \quad \frac{\mathcal{G} \mid \Gamma, A, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta} \text{ICL} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A, A}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A} \text{ICR} \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A], [\Sigma \triangleleft A]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]} \text{Cons}_S \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, A, A \triangleleft B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, A \triangleleft B]} \text{Con}_B \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle, \langle \Theta \rangle}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle} \text{Con}_T \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [A \triangleleft B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preceq B} \preceq_R \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [B, \Sigma \triangleleft C] \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]}{\mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \triangleleft C]} \preceq_L \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \triangleleft A] \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \triangleleft B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1 \triangleleft A], [\Sigma_2 \triangleleft B]} \text{com} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid A \Rightarrow \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]} \text{jump} \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid A \Rightarrow \Theta \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta, B \rangle}{\mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta, \langle \Theta \rangle} \text{T} \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \perp \rangle}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \text{intrf} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Theta, \Pi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle \mid \Sigma \Rightarrow \Pi} \text{jump}_U \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Theta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle} \text{jump}_T \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]} \text{W} \quad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta} \text{C} \quad \frac{\mathcal{G} \mid \Sigma, \Gamma \Rightarrow \Pi, \Delta \mid \Omega \Rightarrow \Theta}{\mathcal{G} \mid \Sigma \Rightarrow \Pi \mid \Omega, \Gamma \Rightarrow \Theta, \Delta} \text{spl} \\
\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma, A \preceq B \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \text{abs}_L \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow A \preceq B, \Pi}{\mathcal{G} \mid \Gamma \Rightarrow A \preceq B, \Delta \mid \Sigma \Rightarrow \Pi} \text{abs}_R
\end{array}$$

$$\begin{aligned}
\text{SH}_{\text{VTU}} &= \{ \perp_L, \text{init}, \rightarrow_L, \rightarrow_R, \text{ICL}, \text{ICR}, \text{Cons}_S, \text{Con}_B, \text{Con}_T \} \\
&\cup \{ \preceq_R, \preceq_L, \text{com}, \text{jump}, \text{T}, \text{intrf}, \text{jump}_U, \text{jump}_T \} \\
\text{SH}_{\text{VWU}} &= \text{SH}_{\text{VTU}} \cup \{ \text{W} \} \quad \text{SH}_{\text{VCU}} = \text{SH}_{\text{VWU}} \cup \{ \text{C} \} \quad \text{SH}_{\text{VTA}} = \text{SH}_{\text{VTU}} \cup \{ \text{abs}_L, \text{abs}_R \} \\
\text{SH}_{\text{VWA}} &= \text{SH}_{\text{VWU}} \cup \{ \text{abs}_L, \text{abs}_R \} \quad \text{SH}_{\text{VCA}} = \text{SH}_{\text{VCU}} \cup \{ \text{abs}_L, \text{abs}_R, \text{spl} \}
\end{aligned}$$

Figure 4: Non-invertible standard hypersequent calculi $\text{SH}_{\mathcal{L}}$ for VTU and extensions.

Theorem 22 (Completeness) *If $A \in \mathcal{L}$ then $\text{SH}_{\mathcal{L}} \vdash \Rightarrow A$.*

Proof. By converting a derivation in $\text{H}_{\mathcal{L}}$ into a derivation in $\text{SH}_{\mathcal{L}}$. Completeness then follows from completeness of $\text{H}_{\mathcal{L}}$ (Thm. 14). We simulate the derivations rule by rule, i.e., we show that all rules of $\text{H}_{\mathcal{L}}$ are derivable in $\text{SH}_{\mathcal{L}}$. Most of the rules are simulated as in [9], except for the rules trf_m, T_m . For T_m the derivation is given in Fig. 5, using admissibility of internal weakening (Lem. 20). The derivation of trf_m only replaces jump_T with jump_U . ■

$$\begin{array}{c}
\frac{\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid C_m \Rightarrow [\vec{D}]_1^{m-1}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid C_m \Rightarrow [\vec{D}]_1^{m-1}, \perp} \text{IW} \quad \frac{\frac{\mathcal{G} \mid \Gamma \Rightarrow [\vec{D}]_1^m, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow [\vec{D}]_1^m, \perp, \Delta} \text{IW} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow [\vec{D}]_1^m, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle [\vec{D}]_1^m, \perp \rangle} \text{jump}_T}{\mathcal{G} \mid \Gamma, C_m \preceq D_m \Rightarrow \Delta, \langle [\vec{D}]_1^{m-1}, \perp \rangle} \text{T} \\
\frac{\frac{\mathcal{G} \mid \Gamma, [C \overset{\vec{\cdot}}{\preceq} D]_2^m \Rightarrow \Delta \mid C_1 \Rightarrow \perp}{\mathcal{G} \mid \Gamma, [C \overset{\vec{\cdot}}{\preceq} D]_2^m \Rightarrow \Delta \mid C_1 \Rightarrow \perp} \text{IW} \quad \frac{\mathcal{G} \mid \Gamma, [C \overset{\vec{\cdot}}{\preceq} D]_2^m \Rightarrow \Delta, \langle D_1, \perp \rangle}{\mathcal{G} \mid \Gamma, [C \overset{\vec{\cdot}}{\preceq} D]_1^m \Rightarrow \Delta, \langle \perp \rangle} \text{T}}{\frac{\mathcal{G} \mid \Gamma, [C \overset{\vec{\cdot}}{\preceq} D]_1^m \Rightarrow \Delta, \langle \perp \rangle}{\mathcal{G} \mid \Gamma, [C \overset{\vec{\cdot}}{\preceq} D]_1^m \Rightarrow \Delta} \text{in}_{\text{trf}}} \text{T}
\end{array}$$

Figure 5: The derivation of T_m in SH_{VTU} .

4.2 Invertible calculi

The invertible, or “kleened”, versions of the calculi $\text{SH}_{\mathcal{L}}$ from Fig. 4 can be found in Fig. 6, and are denoted by $\text{SH}_{\mathcal{L}}^i$. These calculi are defined by adding to the premiss(es) of each rule the principal formula, and possibly the principal blocks, of the rule. The contraction rules, explicit in the calculi $\text{SH}_{\mathcal{L}}$, will be shown to be admissible in the invertible proof systems. The invertible calculi are better suited for proof search, and are implemented in **tuCLEVER**. Moreover, for the logics without absoluteness it is possible to give a direct proof of completeness, by constructing a countermodel from a failed proof search. This proof, called *semantic completeness*, does not depend on the cut elimination proof for the non-standard calculi in Sec. 3. As we will see, the theorem prover **tuCLEVER** implements both the decision procedure and the countermodel construction described in this section, respectively in Sec. 5.1 and in Sec. 5.2.

We start by showing equivalence of the invertible calculi with their non-invertible versions. This follows from admissibility of the structural rules.

Lemma 23 (Admissibility of Weakening) *The rules IW, EW, CW, CIW, TW are depth-preserving admissible in $\text{SH}_{\mathcal{L}}^i$.*

Proof. By induction on the depth of the derivation. We first show admissibility of IW, EW, CW, TW as usual. For CIW, in case the last applied rule was jump^i with the weakened formula in the active block, we first use the induction hypothesis on the premiss, then depth-preserving admissibility of IW, followed by jump^i . In case the last applied rule was \preceq_L^i or com^i , we might need to apply the induction hypothesis twice before applying the same rule. ■

We now show admissibility of contraction. The contraction rules can be found in Figure 4, and **mrg** is the following rule:

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{mrg}$$

$$\begin{array}{c}
\frac{\overline{\mathcal{G} \mid \Gamma, \perp \Rightarrow \Delta} \perp_L \quad \overline{\mathcal{G} \mid \Gamma, p \Rightarrow \Delta, p} \text{init}}{\mathcal{G} \mid \Gamma, A \rightarrow B, B \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow \Delta, A} \rightarrow_L^i \quad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta, A \rightarrow B, B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \rightarrow B} \rightarrow_R^i \\
\frac{\overline{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preceq B, [A \triangleleft B]} \preceq_R^i \quad \overline{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A] \mid A \Rightarrow \Sigma} \text{jump}^i}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preceq B} \preceq_R^i \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]} \text{jump}^i \\
\frac{\overline{\mathcal{G} \mid \Gamma, C \preceq D \Rightarrow \Delta, [D, \Sigma \triangleleft A]} \quad \overline{\mathcal{G} \mid \Gamma, C \preceq D \Rightarrow \Delta, [\Sigma \triangleleft A], [\Sigma \triangleleft C]} \preceq_L^i}{\mathcal{G} \mid \Gamma, C \preceq D \Rightarrow \Delta, [\Sigma \triangleleft A]} \preceq_L^i \\
\frac{\overline{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \triangleleft A], [\Sigma_2 \triangleleft B]} \quad \overline{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1 \triangleleft A], [\Sigma_1, \Sigma_2 \triangleleft B]} \text{com}^i}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma_1 \triangleleft A], [\Sigma_2 \triangleleft B]} \text{com}^i \\
\frac{\overline{\mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta, \langle \Theta \rangle \mid A \Rightarrow \Theta} \quad \overline{\mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta, \langle \Theta, B \rangle} \Gamma^i}{\mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta, \langle \Theta \rangle} \Gamma^i \\
\frac{\overline{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \perp \rangle} \text{in}_{\text{trf}}^i \quad \overline{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle \mid \Sigma \Rightarrow \Theta, \Pi} \text{jump}_U^i \quad \overline{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle, \Theta} \text{jump}_T^i}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \text{in}_{\text{trf}}^i \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle \mid \Sigma \Rightarrow \Theta, \Pi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle} \text{jump}_U^i \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle, \Theta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta \rangle} \text{jump}_T^i \\
\frac{\overline{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A], \Sigma} \text{W}^i \quad \overline{\mathcal{G} \mid \Gamma, C \preceq D, C \Rightarrow \Delta} \quad \overline{\mathcal{G} \mid \Gamma, C \preceq D \Rightarrow \Delta, \Delta} \text{C}^i}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]} \text{W}^i \quad \frac{\mathcal{G} \mid \Gamma, C \preceq D, C \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, C \preceq D \Rightarrow \Delta} \text{C}^i \\
\frac{\overline{\mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta \mid \Sigma, A \preceq B \Rightarrow \Pi} \text{abs}_L^i \quad \overline{\mathcal{G} \mid \Gamma \Rightarrow A \preceq B, \Delta \mid \Sigma \Rightarrow A \preceq B, \Pi} \text{abs}_R^i}{\mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \text{abs}_L^i \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow A \preceq B, \Delta \mid \Sigma \Rightarrow A \preceq B, \Pi}{\mathcal{G} \mid \Gamma \Rightarrow A \preceq B, \Delta \mid \Sigma \Rightarrow \Pi} \text{abs}_R^i
\end{array}$$

$$\begin{array}{l}
\text{SH}_{\text{VTU}}^i = \{ \perp_L, \text{init}, \rightarrow_L^i, \rightarrow_R^i \} \cup \{ \preceq_R^i, \preceq_L^i, \text{com}^i, \text{jump}^i, \Gamma^i, \text{in}_{\text{trf}}^i, \text{jump}_U^i, \text{jump}_T^i \} \\
\text{SH}_{\text{VWU}}^i = \text{SH}_{\text{VTU}}^i \cup \{ \text{W}^i \} \quad \text{SH}_{\text{VCU}}^i = \text{SH}_{\text{VWU}}^i \cup \{ \text{C}^i \} \quad \text{SH}_{\text{VTA}}^i = \text{SH}_{\text{VTU}}^i \cup \{ \text{abs}_L^i, \text{abs}_R^i \} \\
\text{SH}_{\text{VWA}}^i = \text{SH}_{\text{VWU}}^i \cup \{ \text{abs}_L^i, \text{abs}_R^i \} \quad \text{SH}_{\text{VCA}}^i = \text{SH}_{\text{VCU}}^i \cup \{ \text{abs}_L^i, \text{abs}_R^i \}
\end{array}$$

Figure 6: Invertible standard hypersequent calculi $\text{SH}_{\mathcal{L}}^i$ for VTU and extensions.

Lemma 24 (Admissibility of Contraction) *The rules ICL, ICR, Con_B , Con_S , mrg are depth-preserving admissible in $\text{SH}_{\mathcal{L}}^i$.*

Proof. Admissibility of ICL and ICR is shown by straightforward induction on the depth of the derivation. The proofs of admissibility of Con_B and Con_S are also by induction on the depth of the derivation, but also making use of admissibility of ICL and ICR. As an example, let us consider the case of Con_B when $\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma, A, A \triangleleft B]$ is obtained by an application of jump^i from $\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma, A, A \triangleleft B] \mid B \Rightarrow \Sigma, A, A$. By inductive hypothesis, we have a derivation of $\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma, A \triangleleft B] \mid B \Rightarrow \Sigma, A, A$. Admissibility of ICR yields $\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma, A \triangleleft B] \mid B \Rightarrow \Sigma, A$, to which we can apply jump^i to obtain a derivation of $\mathcal{G} \mid \Omega \Rightarrow \Theta, [\Sigma, A \triangleleft B]$. The other cases are similar. The proof of admissibility of Con_S also is by induction on the depth of the derivation, using admissibility of CIW if the last applied rule was \preceq_L^i or com^i . Finally, admissibility of mrg is also shown by induction on the depth of the derivation, making use of admissibility of Con_B if the last applied rule was com^i , and of admissibility of IW if the last applied rule was W^i . ■

From Lemmas 20, 23 and 24 it follows that:

Proposition 25 *The invertible and non-invertible calculi are equivalent.*

Proof. The rules of the non-invertible calculi are admissible in the invertible ones using admissibility of the various weakening rules (Lem. 23) to add the missing formulae / structures in the premisses and the corresponding invertible rule. Vice versa, the rules of the invertible calculi are admissible in the non-invertible ones using the corresponding non-invertible rule followed by applications of contraction on formulae or blocks. E.g., the rule T^i is simulated in the non-invertible calculus as follows:

$$\frac{\frac{\mathcal{G} \mid \Sigma, A \preceq B \Rightarrow \Pi, \langle \Theta \rangle \mid A \Rightarrow \Theta \quad \frac{\mathcal{G} \mid \Sigma, A \preceq B \Rightarrow \Pi, \langle \Theta \rangle, \langle \Theta \rangle}{\mathcal{G} \mid \Sigma, A \preceq B \Rightarrow \Pi, \langle \Theta \rangle, \langle \Theta \rangle} \text{TW}}{\mathcal{G} \mid \Sigma, A \preceq B \Rightarrow \Pi, \langle \Theta \rangle, \langle \Theta \rangle} \text{T}}{\frac{\mathcal{G} \mid \Sigma, A \preceq B \Rightarrow \Pi, \langle \Theta \rangle, \langle \Theta \rangle}{\mathcal{G} \mid \Sigma, A \preceq B \Rightarrow \Pi, \langle \Theta \rangle} \text{ICL}}{\mathcal{G} \mid \Sigma, A \preceq B \Rightarrow \Pi, \langle \Theta \rangle} \text{Con}_T$$

where the application of TW is admissible using Lem. 20. ■

In light of the completeness result proved for the non-invertible calculi, the previous result immediately implies completeness of the invertible systems. However, admissibility of the structural rules in the invertible calculi also means that in backwards proof search we do not need to consider redundant applications of rules, i.e., applications of rules where one of the premisses is equivalent to the conclusion modulo the structural rules. If for an extended hypersequent every possible rule application is redundant, then intuitively every possible rule has already been applied to it, and hence the extended hypersequent is saturated under the rules. To make this formally precise:

Definition 26 *An extended hypersequent \mathcal{G} is VTU-saturated if it is not an instance of \perp_L or init and it satisfies all of the following conditions:*

1. (\preceq_R) if $\Gamma \Rightarrow \Delta, A \preceq B \in \mathcal{G}$, then $[\Sigma, A \triangleleft B] \in \Delta$ for some Σ ;
2. (\preceq_L) if $\Gamma, C \preceq D \Rightarrow \Delta, [\Sigma \triangleleft A] \in \mathcal{G}$, then $D \in \Sigma$ or $[\Sigma \triangleleft C] \in \Delta$;
3. (com) if $\Gamma \Rightarrow \Delta, [\Sigma \triangleleft A], [\Pi \triangleleft B] \in \mathcal{G}$, then $\Sigma \subseteq \Pi$ or $\Pi \subseteq \Sigma$;
4. (jump) if $\Gamma \Rightarrow \Delta, [\Sigma \triangleleft A] \in \mathcal{G}$, then $A, \Theta \Rightarrow \Sigma, \Pi \in \mathcal{G}$ for some Θ, Π ;
5. (T) if $\Gamma, C \preceq D \Rightarrow \Delta, \langle \Theta \rangle \in \mathcal{G}$, then $D \in \Theta$ or $C, \Sigma \Rightarrow \Theta, \Pi \in \mathcal{G}$ for some Σ, Π ;
6. (in_{trf}) if $\Gamma \Rightarrow \Delta \in \mathcal{G}$, then $\langle \Theta \rangle \in \Delta$ for some Θ ;
7. ($\text{jump}_U, \text{jump}_T$) if $\Gamma \Rightarrow \Delta, \langle \Theta \rangle \in \mathcal{G}$ and $\Sigma \Rightarrow \Pi \in \mathcal{G}$, then $\Theta \subseteq \Pi$;
8. (\rightarrow_L) if $\Gamma, A \rightarrow B \Rightarrow \Delta \in \mathcal{G}$, then $B \in \Gamma$ or $A \in \Delta$;
9. (\rightarrow_R) if $\Gamma \Rightarrow A \rightarrow B, \Delta \in \mathcal{G}$, then $A \in \Gamma$ and $B \in \Delta$;

It is $\forall\text{WU}$ -saturated, resp. $\forall\text{CU}$ -saturated, if it also satisfies (W), resp. (C), below:

10. (W) if $\Gamma \Rightarrow \Delta, [\Sigma \triangleleft A] \in \mathcal{G}$, then $\Sigma \subseteq \Delta$;
11. (C) if $\Gamma, C \preceq D \Rightarrow \Delta \in \mathcal{G}$, then $C \in \Gamma$ or $D \in \Delta$;

We now proceed to prove the semantic completeness of the calculi $\text{SH}_{\mathcal{L}}^i$. We show that from a \mathcal{L} -saturated sequent of $\text{SH}_{\mathcal{L}}^i$ it is possible to define a countermodel of it in the respective logic \mathcal{L} .

Definition 27 Given a \mathcal{L} -saturated hypersequent

$$\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$$

We define the worlds and the valuation of the structure $\mathfrak{M}_{\mathcal{G}}$ as follows:

- $W := \{1, \dots, n\}$
- $\llbracket p \rrbracket := \{i \leq n : p \in \Gamma_i\}$

To define $\text{SP}(i)$, that is, the system of sphere associated to each world, for logic $\forall\text{TU}$ we proceed as follows. For each $i \leq n$, let

$$\Delta_i = \Delta'_i, [\Sigma_1 \triangleleft A_1], \dots, [\Sigma_k \triangleleft A_k]$$

where Δ'_i does not contain any conditional blocks. Δ'_i may contain a transfer block $\langle \Theta \rangle$, but this is not relevant for the definition. Define:

$$\text{Blocks}(\Delta_i) = \{[\Sigma_1 \triangleleft A_1], \dots, [\Sigma_k \triangleleft A_k]\}$$

Observe that due to saturation condition 3 we may assume without loss of generality that

$$\Sigma_1 \subseteq \Sigma_2 \subseteq \dots \subseteq \Sigma_k. \quad (3)$$

Moreover, by saturation condition 4 for every $j \leq k$ there is a component $\Gamma_{m_j} \Rightarrow \Delta_{m_j} \in \mathcal{G}$ with $A_j \in \Gamma_{m_j}$ and $\Sigma_j \subseteq \Delta_{m_j}$.

For each $[\Sigma_j \triangleleft A_j] \in \text{Blocks}(\Delta_i)$, define

$$\text{sp}([\Sigma_j \triangleleft A_j]) = \{m \in W \mid \Gamma_m \Rightarrow \Delta_m \in W \text{ and } A_j \in \Gamma_m \text{ and } \Sigma_j \subseteq \Delta_m\}$$

$$\text{sp}(\Sigma_j) = \bigcup \{\text{sp}([\Sigma_l \triangleleft A_l]) \mid [\Sigma_l \triangleleft A_l] \in \text{Blocks}(\Delta_i) \text{ and } \Sigma_l = \Sigma_j\}$$

Intuitively, $\text{sp}(\Sigma_j)$ defines a single sphere in the system of sphere associated to i . Observe that, trivially, if $\text{sp}([\Sigma_l \triangleleft A_l]), \text{sp}([\Sigma_j \triangleleft A_j]) \in \text{Blocks}(\Delta_i)$ and $\Sigma_l = \Sigma_j$, then $\text{sp}(\Sigma_l) = \text{sp}(\Sigma_j)$.

Finally, considering the ordering of blocks 3, define for each $\Gamma_i \Rightarrow \Delta_i$ the system of spheres associated to i :

$$\text{SP}(i) = \{\text{sp}(\Sigma_k), \text{sp}(\Sigma_k) \cup \text{sp}(\Sigma_{k-1}), \dots, \text{sp}(\Sigma_k) \cup \dots \cup \text{sp}(\Sigma_1), W\}$$

For $\forall\text{WU}$, the definition of $\text{SP}(i)$ is modified as follows:

$$\text{SP}(i)^w := \{\alpha \cup \{i\} \mid \alpha \in \text{SP}(i)\}.$$

For $\forall\text{CU}$, we take $\text{SP}(i)^c = \text{SP}(i)^w \cup \{i\}$.

Lemma 28 For a \mathcal{L} -saturated hypersequent \mathcal{G} the structure $\mathfrak{M}_{\mathcal{G}}$ is a \mathcal{L} -model.

Proof. Nesting of spheres is obvious from the fact that $\text{sp}(\Sigma_k) \subseteq \text{sp}(\Sigma_k) \cup \text{sp}(\Sigma_{k-1}) \subseteq \dots \subseteq \text{sp}(\Sigma_k) \cup \dots \cup \text{sp}(\Sigma_1) \subseteq W$. Reflexivity and uniformity follow from the fact that $W \in \text{SP}(i)$.

The conditions of *weak centering* for $\forall\text{WU}$ and of *centering* for $\forall\text{CU}$ are equally obvious. \blacksquare

Lemma 29 Let $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ be a \mathcal{L} -saturated hypersequent and let $\mathfrak{M}_{\mathcal{G}}$ be defined as above, for a world i associated to component $\Gamma_i \Rightarrow \Delta_i$. Then:

- a) given a formula A , if $A \in \Gamma_i$ then $\mathfrak{M}_{\mathcal{G}}, i \Vdash A$
- b) given a formula A , if $A \in \Delta_i$ then $\mathfrak{M}_{\mathcal{G}}, i \nVdash A$
- c) given a block $[\Sigma \triangleleft C]$, if $[\Sigma \triangleleft C] \in \Delta_i$, then $\mathfrak{M}_{\mathcal{G}}, i \nVdash \bigvee_{B \in \Sigma} (B \preccurlyeq C)$
- d) given a formula B , if $\langle \Theta, B \rangle \in \Delta_i$ for some Θ , then $\mathfrak{M}_{\mathcal{G}}, i \nVdash \diamond B$

Proof. We first consider the case of $\forall\text{TU}$. We then show how to extend the proof to the cases of *weak centering* for $\forall\text{WU}$ and of *centering* for $\forall\text{CU}$. We prove statements a) and b) by mutual induction on the complexity of A . The base case (atomic A) and the propositional cases are straightforward, hence we consider $A = E \preccurlyeq F$. Let $i \in W$ be associated to $\Gamma_i \Rightarrow \Delta_i$ with $\Delta_i = \Delta_i^*, [\Sigma_1 \triangleleft D_1], \dots, [\Sigma_k \triangleleft D_k], \langle \Theta \rangle$, where Δ_i^* contains no conditional block or transfer block and $\Sigma_1 \subseteq \Sigma_2 \subseteq \dots \subseteq \Sigma_k$.

- Suppose $E \preccurlyeq F \in \Gamma_i$. We have to show that for any $\alpha \in \text{SP}(i)$,

- i) either for all worlds $x \in \alpha$, $x \nVdash F$,
- ii) or there exists a worlds $y \in \alpha$ such that $y \Vdash E$.

In case $\alpha \neq W$ we have $\alpha = \text{sp}(\Sigma_k) \cup \dots \cup \text{sp}(\Sigma_j)$, for $j \leq k$. By definition, $[\Sigma_j \triangleleft D_j] \in \text{Blocks}(\Delta_i)$ and, saturation condition (\preccurlyeq_L), that is, 2 of Def. 26, either $F \in \Sigma_j$ or $[\Sigma_j \triangleleft E] \in \text{Blocks}(\Delta_i)$.

Suppose $F \in \Sigma_j$, and let $m \in \alpha$. Then $m \in \text{sp}(\Sigma_\ell)$, for some $j \leq \ell \leq k$ and, by definition, $m \in \text{sp}([\Sigma_\ell \triangleleft D_\ell])$, for some formula D_ℓ . Again by definition, there is a component $\Gamma_m \Rightarrow \Delta_m \in \mathcal{G}$, with $D_\ell \in \Gamma_m$ and $\Sigma_\ell \subseteq \Delta_m$. Since $\Sigma_j \subseteq \Sigma_\ell$ and $F \in \Sigma_j$, then $F \in \Sigma_\ell$. Moreover, by inductive hypothesis we have that $m \nVdash F$ and, since this reasoning holds for an arbitrary element, we conclude that case i) holds.

Now suppose $[\Sigma_j \triangleleft E] \in \text{Blocks}(\Delta_i)$, and take $m \in \text{sp}([\Sigma_j \triangleleft E]) \subseteq \text{sp}(\Sigma_j) \subseteq \alpha$. Then there is a component $\Gamma_m \Rightarrow \Delta_m \in \mathcal{G}$ such that $E \in \Gamma_m$ (and $\Sigma_j \subseteq \Delta_m$). By inductive hypothesis, $m \Vdash E$, and we conclude that ii) holds.

In case $\alpha = W$, by saturation condition (T) either $F \in \langle \Theta \rangle$, or $E, \Lambda \Rightarrow \Pi, \Theta \in \mathcal{G}$ for some Λ, Π . In the latter case for the world j associated

to the component $E, \Lambda \Rightarrow \Pi, \Theta$ by induction hypothesis on E we get $\mathfrak{M}_{\mathcal{G}, j} \Vdash E$, whence *ii*) holds. In the former case we have $F \in \langle \Theta \rangle$. Any $k \in W$ (including $k = i$) is associated to a component $\Gamma_k \Rightarrow \Delta_k$, but by saturation condition ($\text{jump}_T, \text{jump}_U$) we have $\Theta \subseteq \Delta_k$, whence $F \in \Delta_k$; by induction hypothesis on F we have $\mathfrak{M}_{\mathcal{G}, k} \nVdash F$, showing that *i*) holds.

- Suppose $E \preceq F \in \Delta_i$. We have to show that there exists an $\alpha \in \text{SP}(i)$ such that

iii) there exist a world $x \in \alpha$ such that $x \Vdash F$, and

iv) for all worlds $y \in \alpha$, $y \nVdash E$.

By the saturation condition 1 (\preceq_R), there is $[\Sigma_j \triangleleft F] \in \text{Blocks}(\Delta_i)$ with $E \in \Sigma_j$. Let $m \in \text{sp}([\Sigma_j \triangleleft F]) \subseteq \text{sp}(\Sigma_j)$. By definition, there is some component $\Gamma_m \Rightarrow \Delta_m \in \mathcal{G}$ such that $F \in \Gamma_m$ and $\Sigma_j \subseteq \Delta_m$. Let us consider the sphere $\alpha = \text{sp}(\Sigma_k) \cup \dots \cup \text{sp}(\Sigma_j) \in \text{SP}(i)$. By inductive hypothesis we have that $m \Vdash F$ and, since $m \in \alpha$, condition *iii*) holds. For any ℓ such that $j \leq \ell \leq k$, let any $h \in \text{sp}(\Sigma_\ell)$. By definition, to h is associated a component $\Gamma_h \Rightarrow \Delta_h \in \mathcal{G}$, such that $\Sigma_\ell \subseteq \Delta_h$. Since $E \in \Sigma_j$, and $\Sigma_j \subseteq \Sigma_\ell$, we have that $E \in \Delta_h$, whence by inductive hypothesis, $h \nVdash E$. Therefore, condition *iv*) is satisfied.

The proof of *c*) uses *b*), recalling that a block is a disjunction of \preceq -formulas. The proof of *d*) uses *b*) with an argument similar to the one in the proof of *a*) for the case of $\alpha = W$, with $B \in \langle \Theta \rangle$.

We consider now the case of VWU . According to Def. 29, the system of sphere is defined as:

$$\text{SP}(i)^w := \{\alpha \cup \{i\} \mid \alpha \in \text{SP}(i)\}.$$

It suffices to show that the proof of *a*) and *b*) still work in this case. As before have to show that for any $\alpha \in \text{SP}(i)$,

i) either for all worlds $x \in \alpha$, $x \nVdash F$,

ii) or there exists a worlds $y \in \alpha$ such that $y \Vdash E$.

We only have to check the case $\alpha \neq W$, with $\alpha = \text{sp}(\Sigma_k) \cup \dots \cup \text{sp}(\Sigma_j) \cup \{i\}$, for $j \leq k$. As before, $[\Sigma_j \triangleleft D_j] \in \text{Blocks}(\Delta_i)$ and, by saturation condition (\preceq_L), either $F \in \Sigma_j$ or $[\Sigma_j \triangleleft E] \in \text{Blocks}(\Delta_i)$. We only have to worry about the case $F \in \Sigma_j$, as the case $[\Sigma_j \triangleleft E] \in \text{Blocks}(\Delta_i)$ is exactly the same as before. Let $F \in \Sigma_j$, we want to show as in the previous proof that *i*) holds, that is: for all $x \in \alpha$, $x \nVdash F$. We only have to check the claim in the additional case $i \in \alpha$. To this purpose by the saturation condition (W) we have that $\Sigma_j \subseteq \Delta_i$, whence $F \in \Delta_i$. Thus by induction hypothesis we obtain that $\mathfrak{M}_{\mathcal{G}, i} \nVdash F$ and we are done.

We now prove *b*), that is, $E \preceq F \in \Delta_i$. By saturation, there is $[\Sigma_j \triangleleft F] \in \text{Blocks}(\Delta_i)$ with $E \in \Sigma_j$. But also by the saturation condition (W), we have that $\Sigma_j \subseteq \Delta_i$, whence $E \in \Delta_i$. By the induction hypothesis, we get that $\mathfrak{M}_{\mathcal{G}, i} \nVdash E$. Thus by taking $\alpha = \text{sp}(\Sigma_k) \cup \dots \cup \text{sp}(\Sigma_j) \cup \{i\} \in \text{SP}(i)$, and by

reasoning as before, we obtain that for all $y \in \alpha$, $y \not\vdash E$ (claim *iv*) and that there is $x \in \alpha$ such that $x \vdash F$ (claim *iii*).

In the case of \mathbb{VCU} the countermodel is defined as follows:

$$\text{SP}(i)^c = \text{SP}(i)^w \cup \{i\}.$$

We only have to check the additional case of *a*) whenever $\alpha = \{i\} \in \text{SP}(i)^c$. If $E \preceq F \in \Gamma_i$, then by the saturation condition (C) either $E \in \Gamma_i$ or $F \in \Delta_i$. By induction hypothesis we obtain that either $\mathfrak{M}_{\mathcal{G}, i} \vdash E$, or $\mathfrak{M}_{\mathcal{G}, i} \not\vdash F$, whence we trivially get that either claim *ii*) or claim *i*) hold for $\alpha = \{i\}$. ■

From Lemma 29 we obtain:

Lemma 30 *For $\mathcal{L} \in \{\mathbb{VTU}, \mathbb{VWU}, \mathbb{VCU}\}$ let $\mathcal{G} = \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ be a \mathcal{L} -saturated hypersequent and let $\mathfrak{M}_{\mathcal{G}}$ be defined as above, then*

- *for any $i \in W$ associated to sequent $\Gamma_i \Rightarrow \Delta_i$ we have $\mathfrak{M}_{\mathcal{G}, i} \not\vdash \iota_e(\Gamma_i \Rightarrow \Delta_i)$*
- *for any $i \in W$ we have $\mathfrak{M}_{\mathcal{G}, i} \not\vdash \iota_e(\mathcal{G})$*

To use these results in a decision procedure, we consider a strategy of *local loop checking*: rules are not applied if there is a premiss from which the conclusion is derivable using structural rules, that is, a rule is not applied to an hypersequent if the saturation condition for that rule is satisfied by the hypersequent. Since all the structural rules are admissible in $\text{SH}_{\mathcal{L}}^i$, application of the loop check does not jeopardise completeness.

Proposition 31 *Backwards proof search with local loop checking terminates and every leaf of the resulting derivation is an axiom or a saturated hypersequent.*

Proof. By Lemmas 23 and 24, the structural rules are depth-preserving admissible. Hence the local loop checking condition does not influence completeness. Termination is seen by checking that every rule satisfies the subformula property, and hence every rule application which satisfies the local loop checking condition adds a new formula, block or component to the hypersequent. Since the number of such new formulae, blocks and components is bounded in the number of subformulae of the input hypersequent, proof search with local loop checking terminates. Furthermore, every leaf is either an axiom or a saturated sequent, since otherwise another rule could be applied. ■

Theorem 32 (Completeness) *For $\mathcal{L} \in \{\mathbb{VTU}, \mathbb{VWU}, \mathbb{VCU}\}$, it holds that if $\iota_e(\mathcal{G}) \in \mathcal{L}$, then $\text{SH}_{\mathcal{L}} \vdash \mathcal{G}$.*

Proof. By Prop. 31 backwards proof search with root \mathcal{G} terminates and every leaf of it is an axiom or a saturated sequent. By invertibility of the rules each sequent \mathcal{G}' occurring as a leaf is valid. But then \mathcal{G}' must be an axiom, since otherwise, by Lem. 30 we can build a countermodel $\mathfrak{M}_{\mathcal{G}'}$ falsifying $\iota_e(\mathcal{G}')$ and hence by monotonicity also $\iota_e(\mathcal{G})$. ■

$$\begin{array}{c}
\frac{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b], \langle \perp \rangle, \perp}{a \rightarrow \neg b \Rightarrow a, b, b, \langle \perp \rangle, \perp \mid \neg a \rightarrow b, a \Rightarrow b, \langle \perp \rangle, \perp} \text{fail} \\
\frac{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b], \langle \perp \rangle, \perp}{a \rightarrow \neg b \Rightarrow a, b, b, \langle \perp \rangle, \perp \mid \neg a \rightarrow b, a \Rightarrow b} (*) \\
\frac{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b], \langle \perp \rangle, \perp}{a \rightarrow \neg b \Rightarrow a, b, b \mid \neg a \rightarrow b, a \Rightarrow b} (*) \\
\frac{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b], \langle \perp \rangle, \perp}{\mid a \rightarrow \neg b \Rightarrow a, b, b \mid \neg a \rightarrow b \Rightarrow b, \neg a} \neg_R^i \\
\frac{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b], \langle \perp \rangle, \perp}{\mid a \rightarrow \neg b \Rightarrow a, b, b \mid \neg a \rightarrow b \Rightarrow b} \rightarrow_L^i \\
\frac{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b], \langle \perp \rangle, \perp}{a \rightarrow \neg b, \neg b \Rightarrow a, b \mid \neg a \rightarrow b \Rightarrow b} \neg_L^i \\
\frac{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b], \langle \perp \rangle, \perp}{a \rightarrow \neg b \Rightarrow a, b \mid \neg a \rightarrow b \Rightarrow b} \rightarrow_L^i \\
\frac{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b], \langle \perp \rangle, \perp \mid a \rightarrow \neg b \Rightarrow a, b}{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b], \langle \perp \rangle, \perp} \text{jump}^i \\
\frac{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b], \langle \perp \rangle, \perp}{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b]} (*) \\
\frac{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b]}{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b]} \text{com}^i \\
\frac{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a \triangleleft a \rightarrow \neg b]}{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b)} \preccurlyeq_R^i \\
\frac{\Rightarrow a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b)}{\neg(a \preccurlyeq (a \rightarrow \neg b)) \Rightarrow (b \preccurlyeq (\neg a \rightarrow b))} \neg_L^i \\
\frac{\Rightarrow \neg(a \preccurlyeq (a \rightarrow \neg b)) \Rightarrow (b \preccurlyeq (\neg a \rightarrow b))}{\Rightarrow \neg(a \preccurlyeq (a \rightarrow \neg b)) \rightarrow (b \preccurlyeq (\neg a \rightarrow b))} \rightarrow_R^i \\
\sim * \sim \\
1 : a \preccurlyeq (a \rightarrow \neg b), b \preccurlyeq (\neg a \rightarrow b), [a, b \triangleleft a \rightarrow \neg b], [b \triangleleft \neg a \rightarrow b], \langle \perp \rangle, \perp \\
2 : a \rightarrow \neg b \Rightarrow a, b, b, \langle \perp \rangle, \perp \\
3 : \neg a \rightarrow b, a \Rightarrow b, \langle \perp \rangle, \perp \\
W_{\mathcal{H}} = \{1, 2, 3\} \\
\text{SP}_{\mathcal{H}}(1) = \{\{2\}, \{2, 3\}, W\} \quad \text{SP}_{\mathcal{H}}(2) = \text{SP}_{\mathcal{H}}(3) = W \\
\llbracket a \rrbracket_{\mathcal{H}} = \{3\} \quad \llbracket b \rrbracket_{\mathcal{H}} = \emptyset
\end{array}$$

Figure 7: *Above*: The derivation branch used to construct a countermodel for the formula $\neg(a \preccurlyeq (a \rightarrow \neg b)) \rightarrow (b \preccurlyeq (\neg a \rightarrow b))$. The grey formulas are those added to achieve invertibility, and are in grey only to increase readability. For reasons of space, some propositional formula are not repeated upwards in the branch. (*) stands for an application of in_{trf} followed by jump_T^i . *Below*: The countermodel for the formula extracted from the upper sequent of the derivation branch.

Example 33 We show how to construct a countermodel for the formula $\neg(a \preccurlyeq (a \rightarrow \neg b)) \rightarrow (b \preccurlyeq (\neg a \rightarrow b))$ from a branch of a failed SH_{VTTU} proof search. The failed derivation branch is shown in Fig. 4.2. The Figure only shows one branch, and thus one premiss of the two-premisses rules com^i , \rightarrow_L^i applied in

the derivation¹⁰.

Let us call \mathcal{H} the upper hypersequent of the derivation branch. A countermodel $\mathfrak{M}_{\mathcal{H}} = \langle W_{\mathcal{H}}, \mathbf{SP}_{\mathcal{H}}, \llbracket \cdot \rrbracket_{\mathcal{H}} \rangle$ for the formula is built from \mathcal{H} according to the strategy described in Def. 27. First observe that \mathcal{H} is saturated: it is not an instance of the initial rules, and no other rule of $\mathbf{SH}_{\mathbf{VTTU}}$ can be applied to it without breaking one of the saturation conditions listed in Def. 26. The components of the model are described in the lower part of the Figure. The worlds of $\mathfrak{M}_{\mathcal{H}}$ are taken to be the three components of \mathcal{H} . The systems of spheres associated to each world are defined through the blocks occurring in the corresponding hypersequent component, and the valuation is defined by looking at the atomic formulas in the antecedent of the component. It is immediate to verify that $1 \not\models a \preceq (a \rightarrow \neg b)$ and that $1 \not\models b \preceq (\neg a \rightarrow b)$; thus, $\mathfrak{M}_{\mathcal{H}}$ is a countermodel for the considered formula.

We now analyse the complexity of backwards proof search with a local loop checking. Prop. 31 ensures termination of proof search which, at a first glance, could be thought as providing a 2-EXPTIME decision procedure: each hypersequent might contain an exponential number of components, and each component might have an exponential size. However, a more accurate termination argument shows that a lower CO-NEXPTIME complexity bound for the decision procedure can be obtained. This bound is yet not optimal, as it is known that the logics of this section are EXPTIME-complete [6]. The remaining of this Section exposes the refined termination argument.

If for a given input sequent $\Rightarrow G$ there is no \mathcal{L} -saturated extended hypersequent containing only subformulae of G and containing a component $\Gamma \Rightarrow \Delta, G$, then every leaf encountered in backwards proof search for $\Rightarrow G$ must be an axiom, and hence backwards proof search produces a derivation of $\Rightarrow G$. But if there is such a saturated extended hypersequent, then by Lem. 30 it gives rise to a countermodel for G . Thus to check validity of $\Rightarrow G$, it is enough to verify that there is no such saturated extended hypersequent. We can do this by nondeterministically choosing a possible candidate for such an extended hypersequent and then verifying that it is saturated. In order to bound the size of the candidate hypersequent we make use of the following facts:

1. The conditional blocks occurring in a single component of the saturated hypersequent are linearly ordered by set inclusion, i.e., If the component contains the conditional blocks $[\Sigma_1 \triangleleft A_1], \dots, [\Sigma_n \triangleleft A_n]$, then w.l.o.g. we have $\Sigma_1 \subseteq \Sigma_2 \subseteq \dots \subseteq \Sigma_n$ (see also Def. 27).
2. W.l.o.g. every component of the saturated hypersequent contains only a single transfer block.

The first fact follows essentially immediately from the saturation condition (com). The second fact follows from the following lemmas.

Lemma 34 *If $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta_1 \rangle, \langle \Theta_2 \rangle$ is \mathcal{L} -saturated, then so is $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta_1, \Theta_2 \rangle$.*

¹⁰The invertible rules for negation applied in the derivation can be immediately defined from the rules in Fig. 6 and $\neg A \equiv A \rightarrow \perp$. The rules can be found in Fig. 5.1.

Proof. Since the only difference between the two hypersequents is in the transfer blocks, all saturation conditions except for (T) , (in_{trf}) , $(\text{jump}_U, \text{jump}_T)$ carry over immediately to $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta_1, \Theta_2 \rangle$. Of these, (in_{trf}) is trivial. For $(\text{jump}_U, \text{jump}_T)$, if $\Sigma \Rightarrow \Pi \in \mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta_1, \Theta_2 \rangle$, then by saturation of $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta_1 \rangle, \langle \Theta_2 \rangle$ we have $\Theta_1 \subseteq \Pi$ and $\Theta_2 \subset \Pi$ and hence $\Theta_1, \Theta_2 \subset \Pi$. For (T) , if $C \preceq D \in \Gamma$, then by saturation of $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta_1 \rangle, \langle \Theta_2 \rangle$ we have $D \in \Theta_1 \subset \Theta_1, \Theta_2$ or there are Σ, Π with $C, \Sigma \Rightarrow \Theta_1, \Pi$ occurring in the hypersequent. But by the saturation condition $(\text{jump}_U, \text{jump}_T)$ for this component we then have $\Theta_2 \subset \Theta_1, \Pi$, and hence there is a component $C, \Sigma \Rightarrow \Theta_1, \Theta_2, \Pi$ (possibly modulo contractions). ■

Lemma 35 *If $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta_1 \rangle, \langle \Theta_2 \rangle$ is derivable, then so is $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \langle \Theta_1, \Theta_2 \rangle$.*

Proof. By induction on the depth of the derivation, using admissibility of weakening for the rules jump_T^i and jump_U^i . ■

Thus if n is the number of subformulae of the input $\Rightarrow G$, the size of a single component in such a duplication-free and size-bound saturated hypersequent is bounded by the following: Each side of the component contains at most n many formulae; the number of blocks $[\Sigma \triangleleft A]$ is bounded by n (the number of possible Σ) times n (the number of possible A), each block with size at most $n + 1$; the single transfer block contains at most n many formulae. Thus the size of a component is polynomial in n . Since the number of conditional blocks is bounded by n^2 and there is only one transfer block, there are only exponentially many possible components. Thus we can non-deterministically choose such a hypersequent and check that it is saturated in exponential time, resulting in a CO-NEXPTIME complexity bound.

Of course this bound is not optimal, as it is known that the logics of this section are EXPTIME-complete [6].

5 The Theorem Prover tuCLEVER

In this section we present a Prolog implementation of the invertible standard hypersequent calculi presented in Sec. 4.2. The program, called tuCLEVER for *Total reflexivity and Uniformity Conditional LEwis logics theorem proVER*, is inspired by the “lean” methodology of *lean^{TAP}*, and contributes to the recent literature of theorem proving for conditional logics [24, 23, 25, 26, 11].

We first describe the prover and the proof search procedure it implements (Sec. 5.1), then we show how tuCLEVER computes a countermodel from failed proof search (Sec. 5.2). We discuss the performances of the prover in Sec. 5.3. The system tuCLEVER has also a graphical user interface implemented in the form of a responsive Web Application. Some pictures of tuCLEVER are shown in the Appendix. The prover tuCLEVER, as well as all the Prolog source files, can be downloaded for free or used in a web interface, both available at <http://193.51.60.97:8000/tuclever/>.

5.1 The proof search procedure

The prover is composed of a set of Prolog clauses, each implementing a sequent rule or an initial rule of the calculi presented in Fig. 6. Moreover, **tuCLEVER** contains a clause for \top and a clause for each left and right rule for the propositional connectives defined in our language, i.e., \neg , \wedge and \vee , along with a left and right rule for the conditional operator, $>$. These sequent rules are reported in Fig. 5.1. The proof search is provided for free by the mere depth-first search mechanism of Prolog, without the need to add *ad hoc* strategies. Termination of the proof search procedure is ensured by the fact that **tuCLEVER** implements the local loop check strategy described in Proposition 31.

$$\begin{array}{c}
\frac{}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \top} \top_R \qquad \frac{\mathcal{G} \mid \Gamma, \neg A \Rightarrow \Delta, A}{\mathcal{G} \mid \Gamma, \neg A \Rightarrow \Delta} \neg_L^i \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow \Delta, \neg A}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \neg A} \neg_R^i \\
\frac{\mathcal{G} \mid \Gamma, A \wedge B, A, B \Rightarrow \Delta, A}{\mathcal{G} \mid \Gamma, A \wedge B \Rightarrow \Delta} \wedge_L^i \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \wedge B, A \quad \mathcal{G} \mid \Gamma \Rightarrow \Delta, A \wedge B, B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \wedge B} \wedge_R^i \\
\frac{\mathcal{G} \mid \Gamma, A \vee B, A \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, A \vee B, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \vee B \Rightarrow \Delta} \vee_L^i \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \vee B, A, B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \vee B} \vee_R^i \\
\frac{\mathcal{G} \mid \perp \leq A, A > B, \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid A > B, \Gamma \Rightarrow \Delta, [A \wedge \neg B \triangleleft A]}{\mathcal{G} \mid A > B, \Gamma \Rightarrow \Delta} >_L^i \\
\frac{\mathcal{G} \mid (A \wedge \neg B) \leq A, \Gamma \Rightarrow \Delta, A > B, [\perp \triangleleft A]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A > B} >_R^i
\end{array}$$

Figure 8: Additional invertible rules implemented by **tuCLEVER**.

An *extended hypersequent* is represented in **tuCLEVER** as a Prolog list of extended sequents. In turn, an *extended sequent* is represented by means of a pair (that is, a Prolog list with two elements) of Prolog lists:

$$[\text{Gamma}, \text{Delta}]$$

where **Gamma** and **Delta** represent the left-hand and the right-hand side of the extended sequent, respectively. An extended sequent might contain conditional blocks and transfer blocks. A *conditional block* $[\Sigma \triangleleft C]$ is encoded by a pair $[\text{Sigma}, \text{C}]$, where **Sigma** is a list of formulas. A *transfer block* $\langle \Theta \rangle$ is implemented by a term **transfer Theta**, where **Theta** is a Prolog list. Symbols \top and \perp are represented by constants **true** and **false**, respectively, whereas connectives \neg , \wedge , \vee , \rightarrow , \leq , and $>$ are represented by `-`, **and**, **or**, `->`, `<`, and `=>`. Propositional variables are represented by Prolog atoms. For example, the sequent

$$A, \neg B \vee C \Rightarrow A \wedge C, D, A \rightarrow B, \langle \perp \rangle, [A \leq C, B \triangleleft A \vee C]$$

is represented by the list:

$$[[a, -b \text{ or } c], [a \text{ and } c, d, a \rightarrow b, \text{transfer}[\text{false}], [[a < c, b], a \text{ or } c]]].$$

The hypersequent calculi are implemented for each logic by the predicate

```
prove(Hypersequent,ProofTree).
```

This predicate succeeds if and only if the extended hypersequent represented by the Prolog list `Hypersequent` is derivable. When it succeeds, the output term `ProofTree` matches with a representation of the derivation found by the prover. For instance, in order to prove the formula $(A \preceq A \vee B) \vee (B \preceq A \vee B)$ in $\mathbb{V}\text{TU}$, one queries `tuCLEVER` with the goal:

```
prove([],[(a < a or b) or (b < a or b)],ProofTree).
```

To construct a proof search tree for an extended hypersequent, `tuCLEVER` proceeds as follows. If the hypersequent is an instance of either \perp_L or \top_R or `init`, the goal will succeed immediately by using one of the following clauses for the axioms:

```
1 prove(Hypersequent,tree(...)) :-
2   member([Gamma,Delta],Hypersequent),member(false,Gamma),!.
3 prove(Hypersequent,tree(...)) :-
4   member([Gamma,Delta],Hypersequent),member(true,Delta),!.
5 prove(Hypersequent,tree(...)) :- member([Gamma,Delta],Hypersequent),
6   member(X,Gamma),member(X,Delta),atom(X),!.
```

If the hypersequent is not an instance of the initial rules, then the first applicable rule will be chosen. For instance, if a component $\Gamma \Rightarrow \Delta$ of the hypersequent has a formula $A \preceq B$ occurring in the Δ , then the clause implementing the \preceq_R^i rule will be chosen, and `tuCLEVER` will be recursively invoked on the unique premise of the rule, which introduces a conditional block $[A \triangleleft B]$. `tuCLEVER` proceeds in a similar way for the other rules. The ordering of the clauses is such that the application of the branching rules is postponed as much as possible.

By means of example, we show the clauses implementing the rules \preceq_R^i , jump_U^i and \top^i , which are used in the calculus for $\mathbb{V}\text{TU}$ as well as in all the calculi for the extensions of the logic.

The clause implementing \preceq_R^i is the following:

```
1 prove(Hypersequent,tree(condR,Hypersequent,[Gamma,Delta],no,SubTree1,no)) :-
2   select([Gamma,Delta],Hypersequent,Remainder),
3   member(A < B,Delta),
4   \+findBlock(Delta,[[A],B]),!,
5   prove([[Gamma,[[A],B]|Delta]]|Remainder],SubTree1).
```

In line 4, the auxiliary predicate `findBlock` is invoked in order to implement the local loop checking of Proposition 31. For a conditional formula $A \preceq B$, the predicate checks whether the block $[A \triangleleft B]$, represented by the Prolog pair $[[A],B]$, already occurs in Δ . If this is the case, the predicate is evaluated as `false` due to the the negation as failure `\+`, and application of the rule to the

conditional formula $A \preceq B$ is blocked. Since the rule is invertible, Prolog cut `!` is used in line 4 to block possible backtracking.

The following clause implements the rule jump_U^i :

```

1  prove(Hypersequent,tree(jumpU,Hypersequent,[Gamma,Delta],[Gamma2,Delta2],
2      SubTree1,no)) :-
3      select([Gamma,Delta],Hypersequent,Remainder),
4      member(transfer Theta, Delta),
5      select([Gamma2,Delta2],Remainder,Remainder2),
6      \+subset(Theta,Delta2),
7      append(Delta2,Theta,NewDelta2),
8      !,
9      prove([[Gamma,Delta],[Gamma2,NewDelta2]|Remainder2],SubTree1).

```

In line 4, the predicate `member` checks whether a conditional block $\langle \Theta \rangle$, represented by the Prolog term `transfer Theta`, occurs in the list `Delta` representing the consequent of a component `[Gamma,Delta]`. If this is the case, and if the formulas in `transfer Theta` do not already occur in the consequent side `Delta2` of a second component `[Gamma2,Delta2]` (line 6), then the predicate `prove` is recursively invoked on the only premise of the rule (line 9), where the premiss of the rule is constructed by adding the formulas in Θ to the consequent `Delta2` of the second component (line 7).

The Prolog clause implementing the rule T^i is as follows:

```

1  prove(Hypersequent,tree(t,Hypersequent,[Gamma,Delta],no,SubTree1,SubTree2)) :-
2      select([Gamma,Delta],Hypersequent,Remainder),
3      member(A < B,Gamma),
4      select(transfer Theta,Delta,Delta2),
5      \+member(B,Theta),
6      \+findSequent(Hypersequent,[[A],Theta]),
7      !,
8      prove([[Gamma,Delta],[A],Theta]|Remainder],SubTree1),
9      prove([[Gamma,[transfer [B|Theta]|Delta2]]|Remainder],SubTree2).

```

The rule has two premisses, which are implemented by invoking `prove` twice, on two different hypersequents representing the premisses of the rule (line 8 and 9). In line 8, an extended sequent $A \Rightarrow \Theta$ is added as a new component to the extended hypersequent, which thus represents the left-hand side premiss of T^i . In line 9, the formula B is added to the conditional block $\langle \Theta \rangle$, represented by `transfer Theta`, occurring in the consequent of the sequent `[Gamma,Delta]` under consideration. The hypersequent in the argument of `prove` at line 9 represents the right-hand side premiss of T^i . Lines 5 and 6 implement the local loop checking strategy, by avoiding redundant applications of the rule in case either B already belongs to Θ or an extended sequent $A \Rightarrow \Theta$ already occurs in the extended hypersequent.

Implementations of the calculi for extensions of VTU are built on the implementation of VTU, by adding clauses for the rules for the extensions. By

means of example, we show the clauses implementing the rule W^i , belonging to the implementations of the systems involving axiom (W), characterising the logics with the condition of *weak centering*, and rules abs_L^i and abs_R^i , which characterize systems with *absoluteness*.

```

1 prove(Hypersequent,tree(w,Hypersequent,[Gamma,Delta],no,SubTree1,no)) :-
2   select([Gamma,Delta],Hypersequent,Remainder),
3   member([Sigma,_],Delta),
4   \+subset(Sigma,Delta),
5   append(Delta,Sigma,NewDelta),!,
6   prove([[Gamma,NewDelta]|Remainder],SubTree1).

```

In line 4 the predicate `\+subset(Sigma,Delta)` succeeds only if the Prolog list `Sigma` is *not* contained in the list `Delta`, in order to avoid multiple applications of the rule to the same `Sigma`, thus implementing the local loop checking of Proposition 31.

```

1 prove(Hypersequent,tree(absL,Hypersequent,[Gamma,Delta],[Gamma2,Delta2],
2   SubTree1,no)) :-
3   select([Gamma,Delta],Hypersequent,Remainder),
4   member(A < B,Gamma),
5   select([Gamma2,Delta2],Remainder,Remainder2),
6   \+member(A < B,Gamma2),
7   !,
8   prove([[Gamma,Delta],[A < B|Gamma2],Delta2]!Remainder2],SubTree1).

```

```

1 prove(Hypersequent,tree(absR,Hypersequent,[Gamma,Delta],[Gamma,Delta],
2   SubTree1,no)) :-
3   select([Gamma,Delta],Hypersequent,Remainder),
4   member(A < B,Delta),
5   select([Gamma2,Delta2],Remainder,Remainder2),
6   \+member(A < B,Delta2),
7   !,
8   prove([[Gamma,Delta],[Gamma2,[A < B|Delta2]]|Remainder2],SubTree1).

```

These rules are applied to a sequent $\Gamma \Rightarrow \Delta$ with a formula $A \preceq B$ belonging (member predicate in lines 4) to either Γ (abs_L^i) or Δ (abs_R^i), by selecting another sequent $\Gamma_2 \Rightarrow \Delta_2$ in the hypersequent under consideration (line 5 of both clauses). Again, line 6 of both clauses implements the local loop checking: the rules can be applied only if $A \preceq B$ does not already belong to Γ_2 for abs_L^i and to Δ_2 for abs_R^i . In lines 8 the `prove` predicate is recursively invoked on the unique premise of the rules, by adding $A \preceq B$ to Γ_2 in abs_L^i and to Δ_2 in abs_R^i .

As already mentioned, the code of `tuCLEVER` is simple and compact: the implementation of the basic system $\mathbb{V}\text{TU}$ consists of only 3 predicates, 21 clauses and 118 lines of code¹¹.

¹¹The countermodel construction for systems $\mathbb{V}\text{TU}$, $\mathbb{V}\text{WU}$, and $\mathbb{V}\text{CU}$ described in the next section requires only further 79 lines of code.

5.2 Countermodel generation

For the systems VTU, VWU, and VCU, in case the construction of the proof search described in the previous section fails, the program `tuCLEVER` builds a countermodel for the extended hypersequent (and thus: for the formula) occurring at the root of the proof search tree. The countermodel is constructed from a saturated hypersequent computed by the proof search procedure, implementing the construction of Lemmas 28 and 29.

The very last clause of the `prove` predicate, which is invoked when all the other clauses implementing the rules of the invertible calculi are no longer applicable, makes use of the non standard predicate `assert` in order to explicitly add to the Prolog working memory the current hypersequent:

```
1 prove(Hypersequent,_) :-
2     assert(satHyper(Hypersequent)),
3     fail.
```

This predicate adds the extended hypersequent `satHyper(Hypersequent)` to the set of `tuCLEVER`'s clauses, in order to save it across the backtracking mechanism before forcing the backtracking itself in line 3 by means of `fail`: this is needed in order to allow `tuCLEVER` to restart the computation with the construction of the countermodel. The extended hypersequent represented by `satHyper(Hypersequent)` is *saturated*: since it is stored with the very last clause of the `prove` predicate, it holds that it is not an instance of the initial rules and that there are no other rules applicable to it.

A top level predicate, devoted to wrap the whole countermodel construction procedure and interact with the graphical interface, will then invoke a predicate `buildCounterModel`, whose aim is to extract a model falsifying the initial formula from the saturated extended hypersequent `satHyper(Hypersequent)`.

```
1 prove_wrapper(Formula,...) :-
2     writeln("Not valid. Generating counter-model ... "),
3     retract(satHyper(SaturatedHypersequent)),
4     buildCounterModel(Formula,SaturatedHypersequent,
5         Worlds,SpheresSystems,Valutations),
6     write("Saturated Hypersequent => "),
7     writeln(SaturatedHypersequent),
8     write("W => "),
9     writeln(Worlds),
10    write("SP => "),
11    writeln(SpheresSystems),
12    write("V => "),
13    writeln(Valutations).
```

The predicate `buildCounterModel` has three output parameters, namely `Worlds`, `SpheresSystems`, and `Valutations`. They match Prolog representations of the components of the model after the execution of suitable auxiliary

predicates extracting such components from `satHyper(Hypersequent)`.

```

1 buildCounterModel(Formula,SaturatedHypersequent,
2   Worlds,SpheresSystems,Valutations) :-
3   length(SaturatedHypersequent,N),
4   generateWorlds(1,N,Worlds),
5   generateSpheresSystems(1,N,SaturatedHypersequent,Worlds,SpheresSystems),
6   getPropVars(Formula,Vars),
7   generateValutations(N,Vars,SaturatedHypersequent,Valutations).

```

More in detail, given a saturated extended hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$, the countermodel is computed as follows::

- a) the predicate `generateWorlds` computes the set of worlds of the model as a Prolog list $W = [1, 2, \dots, n]$;
- b) the predicate `generateSpheresSystems` builds the set of spheres by means of the following Prolog predicate, recursively invoked for each world:

```

1 generateSphereForAWorld(W,SaturatedHypersequent,
2   [Gamma,Delta],LastSphere,[NewSphere|T]) :-
3   select([Sigma,A],Delta,Remainder),
4   \+checkForInclusiveBlock(Remainder,Sigma),
5   findall(B,member([Sigma,B],Remainder),ListAllBs),
6   generateSphereForABlock(SaturatedHypersequent,Sigma,
7     LastSphere,[A|ListAllBs],Temp),
8   generateCommonSphereForCousinsBlocks(Temp,NewSphere),
9   generateSphereForAWorld(W,SaturatedHypersequent,
10    [Gamma,Remainder],NewSphere,T).

```

For a component $\Gamma \Rightarrow \Delta$ in the saturated hypersequent and a conditional block $[\Sigma \triangleleft A]$ belonging to Δ , `tuCLEVER` implements the construction described in Definition 27 in the lines 4-8. The auxiliary predicates `generateSphereForABlock` and `generateCommonSphereForCousinsBlocks` build a sphere for $[\Sigma \triangleleft A]$ by also checking the presence of other conditional blocks of the form $[\Sigma \triangleleft B]$ by means of the Prolog built-in predicate `findall`. The returned sphere corresponds to the list of spheres built for each conditional block by means of the following predicate:

```

1 generateSphereForABlock(_,_,_,[],[]):-!.
2 generateSphereForABlock(SaturatedHypersequent,Sigma,LastSphere,
3                         [A|ListAllBs],[SphereA|OtherBlocks]) :-
4     nth1(Index,SaturatedHypersequent,[Gamma2,Delta2]),
5     member(A,Gamma2),
6     subset(Sigma,Delta2),
7     !,
8     append(LastSphere,[Index],SphereA),
9     generateSphereForABlock(SaturatedHypersequent,Sigma,LastSphere,
10                            ListAllBs,OtherBlocks).

```

where, for a block $[\Sigma \triangleleft A]$, the predicate looks for another sequent $\Gamma_2 \Rightarrow \Delta_2$ such that $A \in \Gamma_2$ and $\Sigma \subset \Delta_2$ (lines 4–6), according to the construction of Definition 27;

- c) the predicate `getPropVars` extracts atomic formulas from `Formula` by a recursive definition on its structure and by exploiting the Prolog predicate `atom/1`;
- d) the predicate `generateValuations` builds a Prolog list `Valuations` having the form $[[p_1, L_1], [p_2, L_2], \dots, [p_k, L_k]]$, for each propositional variable p_i , where L_i is a Prolog list of the worlds of the model where p_i is true.

5.3 Performance of tuCLEVER

The performances of tuCLEVER are promising. We have tested it by running SWI Prolog 7.6.4 on an Acer Aspire E5-575G, 2.7 GHz Intel Core i7 7500U, 16GB RAM, Ubuntu 19.04 amd64 machine. In absence of theorem provers specifically tailored for Lewis' logics with uniformity and total reflexivity, we have compared the performances of tuCLEVER with those of VINTE [11]. VINTE implements standard and internal calculi for systems of Lewis' logics weaker than those implemented by tuCLEVER, and the comparison was made by considering on formulas provable in both systems. We have performed two kinds of experiments. On the one hand, we have tested tuCLEVER with randomly generated formulas, therefore including not provable ones. On the other hand, we have tested the two provers over a set of valid formulas.

We have still to test the performances of the countermodel construction implemented by tuCLEVER. We plan to do in future work, by defining a set of unprovable benchmark formulas on which to evaluate the size of the generated countermodel with respect to the size of the formula.

5.3.1 Tests over randomly generated formulas

We have tested tuCLEVER over randomly generated formulas, fixing two different time limits, namely 1 second and 10 seconds, and varying the depth of a formula (i.e. the maximum level of nesting of connectives) as well as the number of different propositional variables. We have considered the system $\mathbb{V}\mathbb{T}\mathbb{U}$

as well as all the extensions, obtaining the percentages of timeouts in Figures 9 and 10.

<i>Depth / var</i>	1 s	10 s	<i>Depth / var</i>	1 s	10 s
5/3	0%	0%	5/3	0%	0%
6/3	2%	0%	6/3	1%	0%
7/3	4%	2%	7/3	3%	2%
8/3	7%	5%	8/3	7%	4%
5/5	0%	0%	5/5	0%	0%
6/5	2%	1%	6/5	2%	1%
7/5	6%	4%	7/5	6%	4%
8/5	10%	7%	8/5	10%	6%

Figure 9: Percentage of timeouts in $\text{SH}_{\forall\text{TU}}^i$ (left) and $\text{SH}_{\forall\text{WU}}^i$ (right).

<i>Depth / var</i>	1 s	10 s	<i>Depth / var</i>	1 s	10 s
5/3	0%	0%	5/3	6%	3%
6/3	2%	1%	6/3	12%	9%
7/3	5%	3%	7/3	21%	17%
8/3	8%	5%	8/3	25%	22%
5/5	0%	0%	5/5	8%	7%
6/5	4%	2%	6/5	20%	16%
7/5	7%	5%	7/5	27%	20%
8/5	11%	9%	8/5	31%	28%

Figure 10: Percentage of timeouts in $\text{SH}_{\forall\text{CU}}^i$ (left) and $\text{SH}_{\forall\text{TA}}^i$ (right).

In all cases, the quite low percentages of timeouts suggest that the performance of **tuCLEVER** are encouraging.

We have also compared the performance of **tuCLEVER** with those of the theorem prover **CSLLean** [1], which, as already mentioned, implements a labelled calculus for the logic of Comparative Concept Similarity over minspaces which is equivalent to logic $\forall\text{CU}$. We have repeated the above tests over randomly generated formulas and, in all cases, the percentage of timeouts for **CSLLean** is over the 20% with a time limit fixed in 10 seconds.

5.3.2 Tests over valid formulas

First of all, we have tested both **tuCLEVER** and **VINTE** over 76 valid formulas in Lewis' conditional logic \forall [21]. Logic \forall is the weakest system of Lewis' logics: its formulas can be tested using **VINTE**, and formulas valid in it will also be valid in the stronger system $\forall\text{TU}$, where the tests are performed using **tuCLEVER**. The formulas used for this first comparison are obtained by translating valid formulas of the basic modal logic **K** [13] provided by Heuerding in conditional formulas: $\Box A$ is replaced by $\top > A$ ¹², whereas $\Diamond A$ is replaced by $\neg(\top > \neg A)$. We have observed the results in Figure 11 concerning the number of

¹²It is worth noticing that this translation introduces an exponential blowup.

timeouts, witnessing a significant increase of performances with respect to those of VINTE.

<i>Theorem prover</i>	1 s	60 s	180 s
VINTE	49	34	31
tuCLEVER	8	3	3

Figure 11: Percentage of timeouts for tuCLEVER and VINTE over valid formulas.

This result could be explained by the fact that, even if tuCLEVER manipulates “heavier” hypersequents, all rules implemented by tuCLEVER are invertible, avoiding backtracking points that are present in VINTE.

We have then compared the performance of tuCLEVER and VINTE over valid formulas obtained as instances of three different schemas, by fixing a time limit of 60 seconds, and by letting a parameter n vary, starting from $n = 1$. The first schema is as follows:

$$(A_1 \preceq A_2) \vee (A_2 \preceq A_3) \vee \cdots \vee (A_n \preceq A_1),$$

We have observed that tuCLEVER is able to answer also with $n = 25$, whereas VINTE is able to answer only until $n = 9$. Similarly, we have compared the performance of the provers on:

$$(A_1 \preceq A_2) \wedge (A_2 \preceq A_3) \wedge \cdots \wedge (A_{n-1} \preceq A_n) \rightarrow (A_1 \preceq A_n)$$

obtaining that tuCLEVER is able to answer also with $n = 15$, whereas VINTE is able to answer only until $n = 5$. The prover VINTE has, however, better performances than those of tuCLEVER over formulas following the following schema:

$$(A_1 \preceq (A_1 \vee A_2 \vee \cdots \vee A_n)) \vee (A_2 \preceq (A_1 \vee A_2 \vee \cdots \vee A_n)) \vee \dots \\ \dots \vee (A_n \preceq (A_1 \vee A_2 \vee \cdots \vee A_n))$$

where tuCLEVER is able to answer with $n = 4$, whereas VINTE is able to answer also for $n = 15$.

Finally, we have compared the performances of the provers using a set of valid formulas generated by translating the rules $R_{n,m}$ of the sequent calculus for \forall according to the translation from rules to axioms described in [20]. In this case, we have fixed a time limit of 120 seconds, obtaining the results in Figure 12. Each column is labelled by the values adopted for n/m . Higher values, missing in the table, lead to a timeout in both tuCLEVER and VINTE.

<i>Theorem prover</i>	1/1	1/2	1/3	1/4	2/1	2/2	3/1	4/1
VINTE	✓	✓	✓	✓	✓	✓	✓	✓
tuCLEVER	✓	✓	✓	✓	✓	✗	✗	✗

Figure 12: Answers within 120 seconds for translations of the rules $R_{n,m}$.

Here we can observe that VINTE offers better results than tuCLEVER, which is not able to answer when either $m > 1$ or $n > 2$ within the fixed time limit. This happens because the hypersequents generated by these formulas have many components, whereas in VINTE only one component is kept. Thus, tuCLEVER keeps more information its hypersequents than VINTE does in its sequents and, if this is an advantage in most of the cases, as backtrack is not needed, on some formulas it might be a disadvantage, as tuCLEVER has to manipulate more formulas than VINTE does.

6 Conclusions and Future Issues

We have introduced two kinds of calculi for the strongest logics in Lewis' family, that is, logics with uniformity and total reflexivity. We have first introduced *non-standard calculi*, for which we have proved cut elimination, and *standard calculi*, in both a non-invertible and an invertible version. The non-invertible version is in direct correspondence with the non-standard calculi, while the invertible calculi provide a decision procedure for all the logics considered, and a countermodel construction for the logics without uniformity.

Limiting the discussion to Lewis' family of conditional logics, non-standard calculi were defined in Lellmann and Pattinson [19] and in Lellman [17]. Standard calculi for the logics were defined in De Swart [31], in Giordano, Gliozzi, Olivetti, and Schwind [7], in Olivetti and Pozzato [27] and in Girlando, Lellmann, Olivetti and Pozzato [9]. The latter two references present proof theoretical strategies similar to the ones proposed here. However, of all the standard calculi proposed, only [7] treats logics with uniformity, and in [7] the authors define a tableau calculus, using modalities indexed with worlds. Thus, their approach is not comparable with the calculi presented in our paper. To the best of our knowledge, the calculi presented in this paper are the first internal and standard sequent calculi for Lewis' conditional logics.

We have then introduced tuCLEVER, a theorem prover implementing the proposed standard and invertible hypersequent calculi. When the formula to be tested is valid in the selected system, tuCLEVER generates and displays a derivation of it in the proposed calculi; otherwise, for the systems $\mathbb{V}\mathbb{T}\mathbb{U}$, $\mathbb{V}\mathbb{W}\mathbb{U}$, and $\mathbb{V}\mathbb{C}\mathbb{U}$, tuCLEVER computes a countermodel. We have compared the performance of tuCLEVER with those of VINTE, a theorem prover for the weaker Lewis' logics, as well as with those of CSLLean, a theorem prover for the logic of Comparative Concept Similarity over minspaces which is equivalent to the system $\mathbb{V}\mathbb{C}\mathbb{U}$: in all cases, the performances of tuCLEVER are promising, with the only exception of a set of valid formulas generated by translating the rules $R_{n,m}$ of the sequent calculus for \mathbb{V} according to the translation from rules to axioms in [20], for which the prover VINTE has better results.

In future work, we plan to evaluate the performances of the countermodel generation procedure. Moreover, it would be interesting to have a sequent calculus, and a prover, specifically tailored for Lewis' logics with the condition of *absoluteness*.

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APPENDIX

Proof of the Shift Right Lemma 11

For convenience we repeat the statement of the lemma:

Lemma 11 (Shift Right) *Suppose that for $k > 0$ and $n_1, \dots, n_k > 0$ there are $H_{\mathcal{L}}^*$ -derivations \mathcal{D}_1 and \mathcal{D}_2 of $\mathcal{G} \mid \Omega \Rightarrow \Theta, A$ and $\mathcal{H} \mid A^{n_1}, \Xi_1 \Rightarrow \Upsilon_1 \mid \dots \mid A^{n_k}, \Xi_k \Rightarrow \Upsilon_k$ with $\rho(\mathcal{D}_1) < |A| > \rho(\mathcal{D}_2)$ and such that the displayed occurrence of A is principal in the last rule application in \mathcal{D}_1 . Then there is a $H_{\mathcal{L}}^*$ -derivation \mathcal{D} with endhypersequent $\mathcal{G} \mid \mathcal{H} \mid \Omega, \Xi_1 \Rightarrow \Theta, \Upsilon_1 \mid \dots \mid \Omega, \Xi_k \Rightarrow \Theta, \Upsilon_k$ and $\rho(\mathcal{D}) < |A|$.*

Proof. As mentioned in the main text, the proof is by induction on the depth of the derivation \mathcal{D}_2 . In case none of the displayed occurrences of A are principal in the last rule application in \mathcal{D}_2 , we apply the induction hypothesis on the premiss(es) of that rule, followed by the same rule. If that rule was **mr_g** we might need to apply contractions to eliminate duplicates afterwards. In case at least one of the displayed occurrences is principal in the last rule application in \mathcal{D}_2 , we distinguish cases according to the last applied rule in the derivation \mathcal{D}_1 . Since the displayed occurrence of A is principal in the last applied rule in \mathcal{D}_1 , that rule must be one of $\rightarrow_R, R_{m,n}, W_{m,n}, W_{m,n}^{\text{abs}}, R_W, R_W^{\text{abs}}$.

(I): The case of \rightarrow_R is dealt with as usual: Assuming that A is the formula $B \rightarrow C$, and assuming w.l.o.g. that the principal occurrence of $B \rightarrow C$ is in the first displayed component, we first apply the induction hypothesis to the premisses of the last applied rule \rightarrow_L in \mathcal{D}_2 to obtain

$$\mathcal{G} \mid \mathcal{H} \mid \Omega, \Xi_1, C \Rightarrow \Theta, \Upsilon_1 \mid \Omega, \Xi_2 \Rightarrow \Theta, \Upsilon_2 \mid \dots \mid \Omega, \Xi_k \Rightarrow \Theta, \Upsilon_k$$

and

$$\mathcal{G} \mid \mathcal{H} \mid \Omega, \Xi_1 \Rightarrow B, \Theta, \Upsilon_1 \mid \Omega, \Xi_2 \Rightarrow \Theta, \Upsilon_2 \mid \dots \mid \Omega, \Xi_k \Rightarrow \Theta, \Upsilon_k$$

Now applying cut to these sequents and the premiss

$$\mathcal{G} \mid \Omega, B \Rightarrow C, \Theta$$

of the application of \rightarrow_R in \mathcal{D}_1 followed by applications of **mr_g** and contraction yields

$$\mathcal{G} \mid \mathcal{H} \mid \Omega, \Xi_1 \Rightarrow \Theta, \Upsilon_1 \mid \dots \mid \Omega, \Xi_k \Rightarrow \Theta, \Upsilon_k$$

and we are done.

(II): Suppose the last applied rule in \mathcal{D}_1 is one of the rules **R_{m,n}**. In the following we assume that A is the formula $E \preceq F$ and that the derivation \mathcal{D}_1 ends in:

$$\frac{\begin{array}{l} \left\{ \mathcal{G} \mid \Omega \Rightarrow \Theta \mid C_j \Rightarrow [\vec{D}]_1^{j-1}, [\vec{A}]_1^n, E : 1 \leq j \leq m \right\} \\ \cup \left\{ \mathcal{G} \mid \Omega \Rightarrow \Theta \mid B_j \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, E : 1 \leq j \leq n \right\} \\ \cup \left\{ \mathcal{G} \mid \Omega \Rightarrow \Theta \mid F \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, E \right\} \end{array}}{\mathcal{G} \mid \Omega, [C \vec{\preceq} D]_1^m \Rightarrow [A \vec{\preceq} B]_1^n, E \preceq F, \Theta} R_{m,n+1} \quad (4)$$

For the sake of a uniform presentation, in the cases where the absoluteness rules are present we assume that both Ω and Θ are empty. This can always be achieved by applying admissibility of external weakening (Lem. 10) to the premisses of the application of $R_{m,n+1}$ and moving the principal formulae to this new component. This of course means that after applying the transformation below, we need to apply the rules $\mathbf{abs}_L, \mathbf{abs}_R$ several times to move the formulae $[C \overset{\vec{\cdot}}{\preceq} D]_1^m, [A \overset{\vec{\cdot}}{\preceq} B]_1^n$ to the correct component.

In a first step, then, we apply the induction hypothesis to the conclusion of $R_{m,n+1}$ and the premiss(es) of the last applied rule in \mathcal{D}_2 to eliminate all the occurrences of $E \preceq F$ from the context. Following this we apply the same rule to the so obtained premisses. Hence we may assume that the only occurrences of $E \preceq F$ in the conclusion of the last applied rule in \mathcal{D}_2 are principal formulae. We then distinguish cases according to the last applied rule in \mathcal{D}_2 .

(II.1): Suppose that the last applied rule in \mathcal{D}_2 is $R_{s+1,t}$. Then this derivation ends in

$$\frac{\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, [\vec{I}]_1^t : 1 \leq j \leq r \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid E \Rightarrow [\vec{H}]_1^r, [\vec{I}]_1^t \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, F, [\vec{I}]_1^t : r < j \leq s \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid J_j \Rightarrow [\vec{H}]_1^s, F, [\vec{I}]_1^t : 1 \leq j \leq t \right\} \end{array}}{\mathcal{H} \mid \Xi, [G \overset{\vec{\cdot}}{\preceq} H]_1^r, E \preceq F, [G \overset{\vec{\cdot}}{\preceq} H]_{r+1}^s \Rightarrow [I \overset{\vec{\cdot}}{\preceq} J]_1^t, \Upsilon} R_{s+1,t}$$

Here we assume that the displayed principal occurrence of $E \preceq F$ is the first one, i.e., that none of the formulae in $[G \overset{\vec{\cdot}}{\preceq} H]_1^r$ is $E \preceq F$. However some of the formulae $[G \overset{\vec{\cdot}}{\preceq} H]_{r+1}^s$ might be $E \preceq F$. Now applying cut on formulae of complexity smaller than $|E \preceq F|$ to the premisses of this rule application and the premisses of (4) yields

$$\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, [\vec{I}]_1^i : 1 \leq j \leq r \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid C_j \Rightarrow [\vec{D}]_1^{j-1}, [\vec{A}]_1^n, [\vec{H}]_1^r, [\vec{I}]_1^t : 1 \leq j \leq m \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid B_j \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r, [\vec{I}]_1^t : 1 \leq j \leq n \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r, [\vec{I}]_1^t, [\vec{I}]_1^t : r < j \leq s \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid J_j \Rightarrow [\vec{H}]_1^s, [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r, [\vec{I}]_1^t, [\vec{I}]_1^t : 1 \leq j \leq t \right\} \end{array}$$

Using admissibility of internal weakening (Lem. 10) followed by an application of $R_{m+s,n+t}$ we then obtain

$$\mathcal{G} \mid \mathcal{H} \mid \Omega, [G \overset{\vec{\cdot}}{\preceq} H]_1^r, [C \overset{\vec{\cdot}}{\preceq} D]_1^m, [G \overset{\vec{\cdot}}{\preceq} H]_{r+1}^s \Rightarrow [A \overset{\vec{\cdot}}{\preceq} B]_1^n, [I \overset{\vec{\cdot}}{\preceq} J]_1^t, \Theta \mid \Xi \Rightarrow \Upsilon$$

Iterating this process to eliminate the remaining occurrences of $E \preceq F$ from $[G \overset{\vec{\cdot}}{\preceq} H]_{r+1}^s$, followed by mrg and applications of contraction then yields the desired sequent.

(II.2): Suppose that the last applied rule in \mathcal{D}_2 is trf_s . Then this derivation ends in

$$\frac{\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1} : 1 \leq j \leq r \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid E \Rightarrow [\vec{H}]_1^r \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, F : r < j \leq s \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi, [\vec{H}]_1^s, F \right\} \end{array}}{\mathcal{H} \mid \Xi, [G \vec{\approx} H]_1^r, E \preccurlyeq F, [G \vec{\approx} H]_{r+1}^s \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi} \text{trf}_s$$

Again we assume that the displayed occurrence of $E \preccurlyeq F$ is the first one, so that it doesn't occur in $[G \vec{\approx} H]_1^r$. As above using cuts of complexity smaller than $|E \preccurlyeq F|$ on the premisses we obtain

$$\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1} : 1 \leq j \leq r \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid C_j \Rightarrow [\vec{D}]_1^{j-1}, [\vec{A}]_1^n, [\vec{H}]_1^r : 1 \leq j \leq m \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid B_j \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r : 1 \leq j \leq n \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r : r < j \leq s \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi, [\vec{H}]_1^s, [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r \right\} \end{array}$$

Discarding the last premiss of these and applying Weakening gives the set of premisses for an application of the rule $R_{m+s,n}$ with conclusion

$$\mathcal{G} \mid \mathcal{H} \mid \Omega, [G \vec{\approx} H]_1^r, [C \vec{\approx} D]_1^m, [G \vec{\approx} H]_{r+1}^s \Rightarrow [A \vec{\approx} B]_1^n, \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi$$

Iterating this process to eliminate all relevant occurrences of $E \preccurlyeq F$ from $[G \vec{\approx} H]_{r+1}^s$ and applying mrg and contractions to eliminate duplicates in the context yields the desired conclusion.

(II.3): If the last applied rule in \mathcal{D}_2 is \mathbb{T}_s the transformation is the same as for trf_s .

(II.4): In case the last rule in \mathcal{D}_2 is $W_{s+1,t}$, this derivation ends in

$$\frac{\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, [\vec{I}]_1^t : 1 \leq j \leq r \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid E \Rightarrow [\vec{H}]_1^r, [\vec{I}]_1^t \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, F, [\vec{I}]_1^t : r < j \leq s \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi, [\vec{H}]_1^s, F, [\vec{I}]_1^t \right\} \end{array}}{\mathcal{H} \mid \Xi, [G \vec{\approx} H]_1^r, E \preccurlyeq F, [G \vec{\approx} H]_{r+1}^s \Rightarrow [I \vec{\approx} J]_1^t, \Upsilon \mid \Sigma \Rightarrow \Pi} W_{s+1,t}$$

Again we assume that the displayed occurrence of $E \preceq F$ is the first one, so that it doesn't occur in $[G \vec{\preceq} H]_1^r$. As above using cuts of complexity smaller than $|E \preceq F|$ on the premisses we obtain:

$$\begin{aligned} & \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, [\vec{I}]_1^t : 1 \leq j \leq r \right\} \\ \cup & \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid C_j \Rightarrow [\vec{D}]_1^{j-1}, [\vec{A}]_1^n, [\vec{H}]_1^r, [\vec{I}]_1^t : 1 \leq j \leq m \right\} \\ \cup & \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid B_j \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r, [\vec{I}]_1^t : 1 \leq j \leq n \right\} \\ \cup & \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, [\vec{I}]_1^t, [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r, [\vec{I}]_1^t : r < j \leq s \right\} \\ \cup & \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi, [\vec{H}]_1^s, [\vec{I}]_1^t, [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r, [\vec{I}]_1^t \right\} \end{aligned}$$

Again, discarding superfluous premisses and eliminating duplicates using contractions gives all the premisses of an application of $W_{m+s, n+t}$ with conclusion

$$\mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma, [G \vec{\preceq} H]_1^r, [C \vec{\preceq} D]_1^m, [G \vec{\preceq} H]_{r+1}^s \Rightarrow [A \vec{\preceq} B]_1^n, [I \vec{\preceq} J]_1^t, \Pi$$

and iterating the process produces the desired conclusion.

(II.5): If the last rule in \mathcal{D}_2 is $W_{s+1, t}^{\text{abs}}$ the transformation is as in the previous case.

(II.6): If the last rule in \mathcal{D}_2 is R_C , that derivation ends in

$$\frac{\mathcal{H} \mid \Xi, E \Rightarrow \Upsilon \quad \mathcal{H} \mid \Xi \Rightarrow F, \Upsilon}{\mathcal{H} \mid \Xi, E \preceq F \Rightarrow \Upsilon} R_C$$

Applying cuts of complexity smaller than $|E \preceq F|$ to the premisses of this rule application and the premisses of (4) yields

$$\begin{aligned} & \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi, C_j \Rightarrow [\vec{D}]_1^{j-1}, [\vec{A}]_1^n, \Upsilon : 1 \leq j \leq m \right\} \\ \cup & \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi, B_j \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, \Upsilon : 1 \leq j \leq n \right\} \\ \cup & \left\{ \mathcal{G} \mid \mathcal{H} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Omega \Rightarrow \Theta \mid \Xi, \Xi \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, \Upsilon, \Upsilon \right\} \end{aligned}$$

Now applying mrg and contractions to eliminate duplicates, followed by an application or R_C yields the hypersequents

$$\begin{aligned} & \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid C_m \preceq D_m, C_j \Rightarrow [\vec{D}]_1^{j-1}, [\vec{A}]_1^n, \Upsilon : 1 \leq j \leq m-1 \right\} \\ \cup & \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi, C_m \preceq D_m \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, \Upsilon \right\} \end{aligned}$$

Continuing like this with successive applications of R_C produces the hypersequent

$$\mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi, [C \vec{\preceq} D]_1^m \Rightarrow [\vec{A}]_1^n, \Upsilon$$

Finally, n applications of R_W to this yield the desired result.

(II.7): If R_C^{abs} is the last rule in \mathcal{D}_2 , the transformation is similar to the previous case.

(III): Suppose that the last rule in \mathcal{D}_1 is the rule $W_{m,n+1}$. Again, we distinguish cases according to the last applied rule in \mathcal{D}_2 and assume that A is the formula $E \preceq F$. The derivation \mathcal{D}_1 then ends in

$$\frac{\begin{array}{l} \left\{ \mathcal{G} \mid \Omega \Rightarrow \Theta \mid C_j \Rightarrow [\vec{D}]_1^{j-1}, [\vec{A}]_1^n, E : 1 \leq j \leq m \right\} \\ \cup \left\{ \mathcal{G} \mid \Omega \Rightarrow \Theta \mid B_j \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, E : 1 \leq j \leq n \right\} \\ \cup \left\{ \mathcal{G} \mid \Omega \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, E, \Theta \right\} \end{array}}{\mathcal{G} \mid \Omega, [C \vec{\preceq} D]_1^m \Rightarrow [A \vec{\preceq} B]_1^n, E \preceq F, \Theta} W_{m,n+1} \quad (5)$$

Again, in the first step we apply the induction hypothesis to the conclusion of this rule and the premisses of the last applied rule in \mathcal{D}_2 to delete every occurrence of $E \preceq F$ from the context. Then we distinguish cases according to the last applied rule in \mathcal{D}_2 . Since in this case the system must be $H_{\forall\forall\cup}$ or $H_{\forall\forall\Delta}$, the only possibilities then are $R_{m,n}$, trf_m , \top_M , $W_{m,n}$ or $W_{m,n}^{\text{abs}}$.

(III.1): If the last applied rule in \mathcal{D}_2 was $R_{s+1,t}$, then this derivation ends in

$$\frac{\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, [\vec{I}]_1^t : 1 \leq j \leq r \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid E \Rightarrow [\vec{H}]_1^r, [\vec{I}]_1^t \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, F, [\vec{I}]_1^t : r < j \leq s \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid J_j \Rightarrow [\vec{H}]_1^s, F, [\vec{I}]_1^t : 1 \leq j \leq t \right\} \end{array}}{\mathcal{H} \mid \Xi, [G \vec{\preceq} H]_1^r, E \preceq F, [G \vec{\preceq} H]_{r+1}^s \Rightarrow [I \vec{\preceq} J]_1^t, \Upsilon} R_{s+1,t}$$

where we again assume that $E \preceq F$ does not occur in $[G \vec{\preceq} H]_1^r$. Applying cuts with smaller complexity to the premisses of (5) and this rule yields

$$\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, [\vec{I}]_1^t : 1 \leq j \leq r \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid C_j \Rightarrow [\vec{D}]_1^{j-1}, [\vec{A}]_1^n, [\vec{H}]_1^r, [\vec{I}]_1^t : 1 \leq j \leq m \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid B_j \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r, [\vec{I}]_1^t : 1 \leq j \leq n \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r, [\vec{I}]_1^t, \Theta \mid \Xi \Rightarrow \Upsilon \right\} \end{array}$$

Again, using weakening (Lem. 10) followed by an application of $W_{m+r,n+t}$ we then obtain

$$\mathcal{G} \mid \mathcal{H} \mid \Omega, [G \vec{\preceq} H]_1^r, [C \vec{\preceq} D]_1^m \Rightarrow [A \vec{\preceq} B]_1^n, [I \vec{\preceq} J]_1^t, \Theta \mid \Xi \Rightarrow \Upsilon$$

Now adding the missing formulae from $[G \vec{\preceq} H]_{r+1}^s$ using weakening, followed by mrg yields the desired sequent.

(III.2): Suppose that the last applied rule in \mathcal{D}_2 is trf_{s+1} . Then this derivation ends in

$$\frac{\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1} : 1 \leq j \leq r \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid E \Rightarrow [\vec{H}]_1^r \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, F : r < j \leq s \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi, [\vec{H}]_1^s, F \right\} \end{array}}{\mathcal{H} \mid \Xi, [G \vec{\approx} H]_1^r, E \approx F, [G \vec{\approx} H]_{r+1}^s \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi} \text{trf}_{s+1}$$

Again we assume that the displayed occurrence of $E \approx F$ is the first one, so that it doesn't occur in $[G \vec{\approx} H]_1^r$. As above using cuts of complexity smaller than $|E \approx F|$ on the premisses we obtain

$$\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid G_j \Rightarrow [\vec{H}]_1^{j-1} : 1 \leq j \leq r \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid C_j \Rightarrow [\vec{D}]_1^{j-1}, [\vec{A}]_1^n, [\vec{H}]_1^r : 1 \leq j \leq m \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \mid B_j \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r : 1 \leq j \leq n \right\} \\ \cup \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, [\vec{H}]_1^r, \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi \right\} \end{array}$$

Weakening followed by an application of $W_{m+r,n}$ then gives

$$\mathcal{G} \mid \mathcal{H} \mid \Xi, [G \vec{\approx} H]_1^r, [C \vec{\approx} D]_1^m \Rightarrow [A \vec{\approx} B]_1^n, \Theta \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi$$

and, again, weakening and mrg yield the desired result.

(III.3): If the last applied rule in \mathcal{D}_2 is T_{s+1} , the transformation is as in the last case.

(III.4): For the last rule being $W_{s+1,t}$, the transformation is as for $R_{s+1,t}$.

(III.5): If the last rule is $W_{s+1,t}^{\text{abs}}$, the transformation is as for $R_{s+1,t}$.

(III.6): The case where the last applied rule in the derivation \mathcal{D}_2 is R_C does not occur.

(III.7): The case where the last applied rule in the derivation \mathcal{D}_2 is R_C^{abs} does not occur either.

(IV): Suppose that the last applied rule in \mathcal{D}_1 was $W_{m,n}^{\text{abs}}$, and that the formula A is $E \approx F$. Then the derivation \mathcal{D}_1 ends in

$$\frac{\begin{array}{l} \left\{ \mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Omega \Rightarrow \Theta \mid C_j \Rightarrow [\vec{D}]_1^{j-1}, [\vec{A}]_1^n, E : 1 \leq j \leq m \right\} \\ \cup \left\{ \mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Omega \Rightarrow \Theta \mid B_j \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, E : 1 \leq j \leq n \right\} \\ \cup \left\{ \mathcal{G} \mid \Gamma \Rightarrow [\vec{D}]_1^m, [\vec{A}]_1^n, E, \Delta \mid \Omega \Rightarrow \Theta \right\} \end{array}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Omega, [C \vec{\approx} D]_1^m \Rightarrow [A \vec{\approx} B]_1^n, E \approx F, \Theta} W_{m,n+1}^{\text{abs}} \quad (6)$$

As above, after applying the induction hypothesis to the conclusion of this rule and the premisses of the last applied rule in \mathcal{D}_2 to eliminate occurrences of

$E \preceq F$ from the context, we distinguish cases according to the last applied rule in \mathcal{D}_2 . The transformations are analogous to those in the previous case (III).

(V): Suppose that the last applied rule in \mathcal{D}_1 was R_W . Then that derivation ends in

$$\frac{\mathcal{G} \mid \Omega \Rightarrow E, \Theta}{\mathcal{G} \mid \Omega \Rightarrow E \preceq F, \Theta} R_W \quad (7)$$

As above, we first eliminate all occurrences of $E \preceq F$ from the context of the last applied rule in \mathcal{D}_2 by applying the induction hypothesis on the premiss(es) of that rule and the conclusion of R_W , then distinguish cases according to the last rule in \mathcal{D}_2 . Since the last applied rule in \mathcal{D}_1 was R_W , the system must be H_{VCU} or H_{VCA} , and the last applied rule in \mathcal{D}_2 hence is one of $R_{m,n}$, trf_m , T_m , R_C , R_C^{abs} .

(V.1): If the last applied rule in \mathcal{D}_2 was $R_{s+1,t}$, then that derivation ends in

$$\frac{\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, [\vec{I}]_1^t : 1 \leq j \leq r \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid E \Rightarrow [\vec{H}]_1^r, [\vec{I}]_1^t \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, F, [\vec{I}]_1^t : r < j \leq s \right\} \\ \cup \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid J_j \Rightarrow [\vec{H}]_1^s, F, [\vec{I}]_1^t : 1 \leq j \leq t \right\} \end{array}}{\mathcal{H} \mid \Xi, [G \vec{\preceq} H]_1^r, E \preceq F, [G \vec{\preceq} H]_{r+1}^s \Rightarrow [I \vec{\preceq} J]_1^t, \Upsilon} R_{s+1,t}$$

Applying cuts of smaller complexity to the premisses of this rule and the premiss of (7) yields the sequents

$$\begin{array}{l} \left\{ \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid G_j \Rightarrow [\vec{H}]_1^{j-1}, [\vec{I}]_1^t : 1 \leq j \leq r \right\} \\ \left\{ \mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow [\vec{H}]_1^r, [\vec{I}]_1^t, \Theta \mid \Xi \Rightarrow \Upsilon \right\} \end{array}$$

Now weakening to assimilate the contexts, followed by an application of R_C yields

$$\mathcal{G} \mid \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Omega, G_r \preceq H_r \Rightarrow [\vec{H}]_1^{r-1}, [\vec{I}]_1^t, \Theta$$

Continuing like this we obtain

$$\mathcal{G} \mid \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Omega, [G \vec{\preceq} H]_1^r \Rightarrow [\vec{I}]_1^t, \Theta$$

and t applications of R_W followed by weakening and mrg yield the desired result.

(V.2): In case the last applied rule in \mathcal{D}_2 is trf_{s+1} , the transformation is as for $R_{s+1,t}$.

(V.3): If the last applied rule in \mathcal{D}_2 is T_{s+1} , again the transformation is as for $R_{s+1,t}$.

(V.4): If the last applied rule in \mathcal{D}_2 is R_C , then the derivation ends in

$$\frac{\mathcal{H} \mid \Xi, E \Rightarrow \Upsilon \quad \mathcal{H} \mid \Xi \Rightarrow F, \Upsilon}{\mathcal{H} \mid \Xi, E \preceq F \Rightarrow \Upsilon} R_C$$

and a cut on E yields the desired

$$\mathcal{G} \mid \mathcal{H} \mid \Omega, \Xi \Rightarrow \Theta, \Upsilon .$$

(V.5): In case the last applied rule in \mathcal{D}_2 is R_C^{abs} , that derivation ends in:

$$\frac{\mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma, E \Rightarrow \Pi \quad \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow F, \Pi}{\mathcal{H} \mid \Xi, E \preceq F \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi} R_C^{\text{abs}}$$

In this case a cut on the left premiss of this rule application and the premiss of (7) with cut formula E yields the hypersequent

$$\mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Sigma, \Xi \Rightarrow \Pi, \Upsilon$$

and an application of the split rule **spl** yields the desired

$$\mathcal{G} \mid \mathcal{H} \mid \Omega, \Xi \Rightarrow \Theta, \Upsilon \mid \Sigma \Rightarrow \Pi$$

Note that this case shows why we need to add the rule **spl** to the rule set for \mathbb{VCA} .

(VI): Suppose that the last applied rule in \mathcal{D}_1 was R_W^{abs} . Then the derivation ends in

$$\frac{\mathcal{G} \mid \Omega \Rightarrow \Theta \mid \Gamma \Rightarrow E, \Delta}{\mathcal{G} \mid \Omega \Rightarrow E \preceq F, \Theta \mid \Gamma \Rightarrow \Delta} R_W^{\text{abs}} \quad (8)$$

Again we eliminate all context occurrences of $E \preceq F$ from the premisses of the last rule application in \mathcal{D}_2 by applying the induction hypothesis, and then distinguish cases according to the last applied rule in \mathcal{D}_2 . The possibilities in this case are $R_{m,n}$, trf_m , T_m , R_C , R_C^{abs} .

(VI.1): If the last applied rule in \mathcal{D}_2 was $R_{s+1,t}$, the transformation is analogous to the previous case (V.1).

(VI.2): In case the last applied rule in \mathcal{D}_2 is trf_{s+1} , the transformation is as in (V.2).

(VI.3): For the last applied rule in \mathcal{D}_2 being T_{s+1} , the transformation is as in (V.3).

(VI.4): If the last applied rule in \mathcal{D}_2 is R_C , then that derivation ends in

$$\frac{\mathcal{H} \mid \Xi, E \Rightarrow \Upsilon \quad \mathcal{H} \mid \Xi \Rightarrow F, \Upsilon}{\mathcal{H} \mid \Xi, E \preceq F \Rightarrow \Upsilon} R_C$$

A cut on the left premiss of this and the premiss of (8) yields

$$\mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Gamma, \Xi \Rightarrow \Delta, \Upsilon$$

and an application of the split rule **spl** yields the desired result.

(VI.5): In case the last applied rule in \mathcal{D}_2 is R_C^{abs} , that derivation ends in

$$\frac{\mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma, E \Rightarrow \Pi \quad \mathcal{H} \mid \Xi \Rightarrow \Upsilon \mid \Sigma \Rightarrow F, \Pi}{\mathcal{H} \mid \Xi, E \preceq F \Rightarrow \Upsilon \mid \Sigma \Rightarrow \Pi} R_C^{\text{abs}}$$

A cut on the left premiss of this and the premiss of (8) yields the premiss of the following derivation:

$$\frac{\frac{\mathcal{G} \mid \mathcal{H} \mid \Omega \Rightarrow \Theta \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi \mid \Xi \Rightarrow \Upsilon}{\mathcal{G} \mid \mathcal{H} \mid \Omega, \Xi \Rightarrow \Theta, \Upsilon \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{mrg}}{\frac{\mathcal{G} \mid \mathcal{H} \mid \Omega, \Xi \Rightarrow \Theta, \Upsilon \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi \mid \Rightarrow}{\mathcal{G} \mid \mathcal{H} \mid \Omega, \Xi \Rightarrow \Theta, \Upsilon \mid \Gamma \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi} \text{EW}} \text{spl}$$

The conclusion of this is the desired hypersequent. ■

The Graphical Interface of tuCLEVER

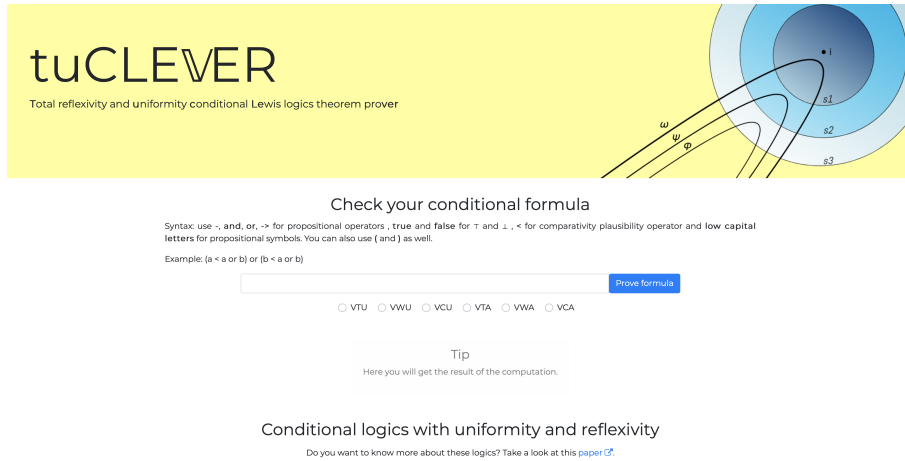


Figure 13: Home page of tuCLEVER. When the users want to check whether a formula F is valid, then (i) they select the conditional logic to use, (ii) they type F in the form and (iii) click the button in order to execute the proof search procedure.

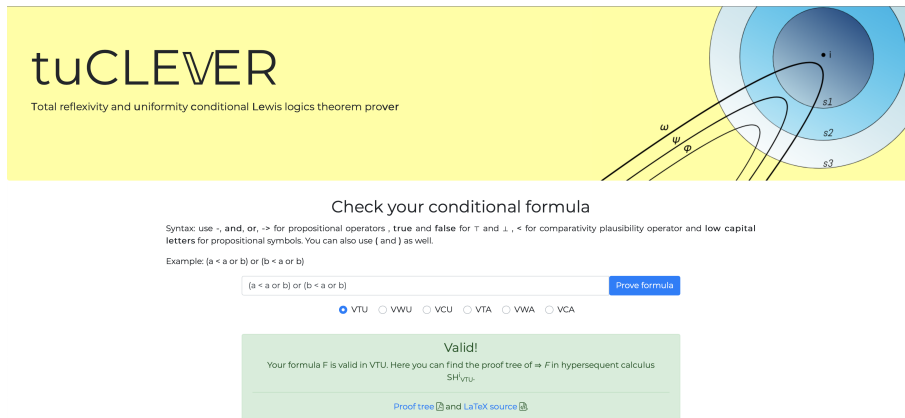


Figure 14: When the formula is valid, tuCLEVER computes both a pdf containing the derivation found by the prover and its \LaTeX source file.

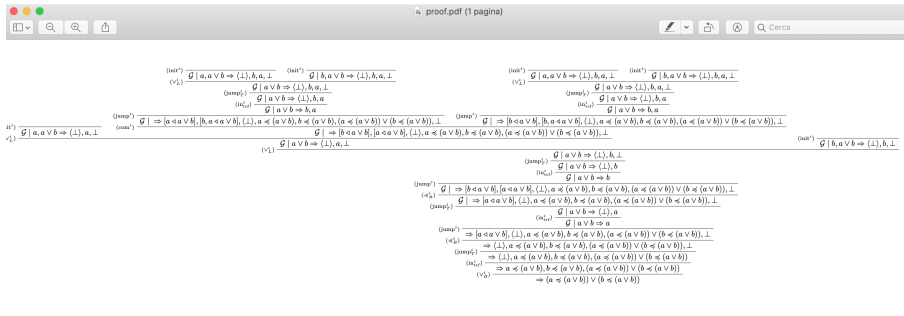


Figure 15: When the submitted formula is valid, then the user can have a look at the derivation built by tuCLEVER, stored in a pdf file. As an alternative, the user can download the \LaTeX source file of the derivation.

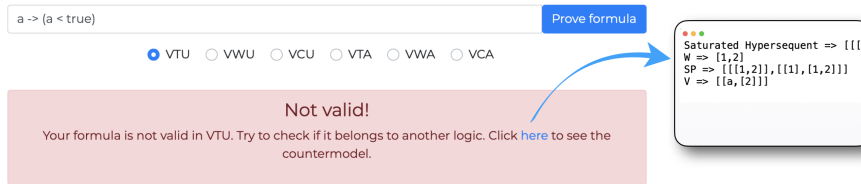


Figure 16: For the systems VTU, VWU, and VCU, when the submitted formula is not valid, tuCLEVER generates a countermodel stored in a txt file. We are currently working on improving the interface in order to let tuCLEVER show the generated countermodel in a graphical window.

$$\begin{array}{c}
 \frac{(\perp_1)}{\mathcal{G} \mid \perp \Rightarrow \perp} \quad \frac{(\text{jump}_T^+)}{\perp \not\leq T \Rightarrow (T, \perp), \neg(\perp \not\leq T), \perp} \\
 \frac{(\text{jump}_T^-)}{\perp \not\leq T \Rightarrow (\perp), \neg(\perp \not\leq T), \perp} \\
 \frac{(\text{init}_T)}{\perp \not\leq T \Rightarrow (\perp), \neg(\perp \not\leq T)} \\
 \frac{(\neg_k)}{\Rightarrow \neg(\perp \not\leq T)}
 \end{array}$$

Detailed proof trees in \LaTeX

If a formula F is valid in a conditional logic L , then tuCLEVER is able to generate a text file containing the proof tree of $\Rightarrow F$ in hypersequent calculus SH_L .

You need [bussproofs.sty](#) to compile the resulted .tex file.

tuCLEVER is entirely implemented in Prolog

Here you can find the source codes for all hypersequent calculi.

You need [SWI-Prolog](#) to run them.

Calculus	Source code
SH _{VTU}	source
SH _{VWU}	source
SH _{VCU}	source
SH _{VTA}	source
SH _{VWA}	source
SH _{VCA}	source

Experiments

We have done several kind of experiments. We have tested tuCLEVER

- over 76 K-valid formulas provided by Heuerding
- over VTU/VWU/VCU/VTA-valid formulas
- over valid formulas obtained by the translations of the rules R_{SH} of the sequent calculus for V
- over valid formulas in the style of Lewis' axioms

The performances of tuCLEVER are promising and consistent with theoretical expectations: in fact, hypersequent calculi SH_L provide non-optimal decision procedures for VTU and its extensions. In this respect, it is known that deciding

Type of test	Source code
experiment 1	source
experiment 2	source
experiment 3	source
experiment 4	source

Figure 17: All Prolog source files, including those for testing the performance of tuCLEVER, are available on the web page.