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Beyond Pairwise Comparisons in Social Choice: A Setwise Kemeny Aggregation Problem*

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Abstract

In this paper, we advocate the use of setwise contests for aggregating a set of input rankings into an output ranking. We propose a generalization of the Kemeny rule where one minimizes the number of k -wise disagreements instead of pairwise disagreements (one counts 1 disagreement each time the top choice in a subset of alternatives of cardinality at most k differs between an input ranking and the output ranking). After an algorithmic study of this k -wise Kemeny aggregation problem, we introduce a k -wise counterpart of the majority graph. This graph reveals useful to divide the aggregation problem into several sub-problems, which enables to speed up the exact computation of a consensus ranking. By introducing a k -wise counterpart of the Spearman distance, we also provide a 2-approximation algorithm for the k -wise Kemeny aggregation problem. We conclude with numerical tests.

Keywords: Computational social choice, Generalized Kemeny rule, Weighted majority graph, Computational complexity

1. Introduction

Rank aggregation aims at producing a single ranking from a collection of rankings of a fixed set of alternatives. In social choice theory (see, for example, the book by Moulin [2]), where the alternatives are candidates to an election and each ranking represents the preferences of a voter, aggregation rules are called *Social Welfare Functions* (SWFs). Apart

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6 from social choice, rank aggregation has proved useful in many applications, including
7 preference learning [3, 4], collaborative filtering [5], genetic map creation [6], similarity
8 search in database systems [7] and design of web search engines [8, 9]. In the following, we
9 use interchangeably the terms “input rankings” and “preferences”, “output ranking” and
10 “consensus ranking”, as well as “alternatives” and “candidates”.

11 The well-known Arrow’s impossibility theorem states that there exists no aggregation
12 rule satisfying a small set of desirable properties [10]. In the absense of an “ideal” rule, var-
13 ious aggregation rules have been proposed and studied. Following Fishburn’s classification
14 [11], we can distinguish between the SWFs for which the output ranking can be computed
15 from the *majority graph* alone, those for which the output ranking can be computed from
16 the *weighted majority graph* alone, and all other SWFs. The majority graph is obtained
17 from the input rankings by defining one vertex per alternative c and by adding an edge
18 from c to c' if c is preferred to c' in a strict majority of input rankings. In the weighted
19 majority graph, each edge is weighted by the majority margin. The many SWFs that rely
20 on these graphs alone take therefore only pairwise comparisons into account to determine
21 an output ranking. Note that Fishburn’s classification actually applies to social *choice*
22 functions, which prescribe a subset of winning alternatives from a collection of rankings,
23 but the extension to SWFs is straightforward. For a compendium of SWFs that rely solely
24 on (weighted) majority graphs, we refer the reader to the book chapter by Zwicker [12].

25 The importance of this class of SWFs can be explained by their connection with the
26 *Condorcet consistency* property, stating: if there is a Condorcet winner (i.e., an alternative
27 with outgoing edges to every other ones in the majority graph), then it should be ranked
28 first in the output ranking. Nevertheless, as shown by Baldiga and Green [13], the lack of
29 Condorcet consistency is not necessarily a bad thing, because this property may come into
30 contradiction with the objective of maximizing voters’ agreement with the output ranking.
31 The following example illustrates this point.

32 **Example 1** (Baldiga and Green [13]). *Consider an election with 100 voters and 3 can-*
33 *didates c_1, c_2, c_3 , where 49 voters have preferences $c_1 \succ c_2 \succ c_3$, 48 have preferences*
34 *$c_3 \succ c_2 \succ c_1$ and 3 have preferences $c_2 \succ c_3 \succ c_1$. Candidate c_2 is the Condorcet win-*
35 *ner, but is the top choice of only 3 voters. In contrast, candidate c_1 is in slight minority*
36 *against c_2 and c_3 , but c_1 is the top choice of 49 voters. This massive gain in agreement*
37 *may justify to put c_1 instead of c_2 in first position of the output ranking.*

38 Following Baldiga and Green [13], we propose to handle this tension between the pairwise
39 comparisons (leading to ranking c_2 first) and the plurality choice (leading to ranking c_1 first)
40 by using SWFs that take into account not only pairwise comparisons but *setwise* contests.
41 More precisely, given input rankings on a set C of candidates and $k \in \{2, \dots, |C|\}$, the
42 idea is to consider the plurality score of each candidate c for each subset $S \subseteq C$ such that

Table 1: Results of setwise contests in Example 1. The cell at the intersection of the row corresponding to a set S and the column corresponding to a candidate c displays the number of voters who rank c first in S .

set	c_1	c_2	c_3
$\{c_1, c_2\}$	49	51	–
$\{c_1, c_3\}$	49	–	51
$\{c_2, c_3\}$	–	52	48
$\{c_1, c_2, c_3\}$	49	3	48

43 $2 \leq |S| \leq k$, where the plurality score of c for S is the number of voters for which c is the top
 44 choice in S . The results of setwise contests for the preferences of Example 1 are given in
 45 Table 1 for $k=3$. Note that the three top rows obviously encode the same information as
 46 the weighted majority graph while the bottom row makes it possible to detect the tension
 47 between the pairwise comparisons and the plurality choice.

48 One can then define a new class of SWFs, those that rely on the results of setwise
 49 contests alone to determine an output ranking. The many works that have been carried
 50 out regarding voting rules based on the (weighted) majority graph can be revisited in this
 51 broader setting. This line of research has already been investigated by Lu and Boutilier
 52 [14] and Baldiga and Green [13]. However, note that both of these works consider a setting
 53 where candidates may become unavailable after voters express their preferences. We do
 54 not make this assumption. We indeed believe that this new class of SWFs makes sense in
 55 the standard setting where the set of candidates is known and deterministic, as it amounts
 56 to generate an output ranking by examining the choices that are made by the voters on
 57 subsets of candidates of various sizes (while usually only pairwise choices are considered).

58 A natural SWF in this class consists in determining an output ranking that minimizes
 59 the number of disagreements with the results of setwise contests for sets of cardinality at
 60 most k . This is a k -wise generalization of the Kemeny rule, obtained as a special case for
 61 $k=2$. We recall that the Kemeny rule consists in producing a ranking that minimizes the
 62 number of *pairwise* disagreements [15].

63 **Example 2.** *Let us come back to Example 1 and assume that we use the 3-wise Kemeny*
 64 *rule. Consider the output ranking $r = c_1 \succ c_2 \succ c_3$. For set $S = \{c_1, c_2\}$, the number of*
 65 *disagreements with the results of setwise contests is 51 because c_2 is the top choice in S for*
 66 *51 voters (see Table 1) while it is c_1 for r . Similarly, the number of disagreements induced*
 67 *by $\{c_1, c_3\}$, $\{c_2, c_3\}$ and $\{c_1, c_2, c_3\}$ are respectively 51, 48 and $3+48$. The total number*
 68 *of disagreements is thus $51+51+48+3+48=201$. This is actually the minimum number*
 69 *of disagreements that can be achieved for these input rankings, which makes r the k -wise*
 70 *Kemeny ranking.*

71 The purpose of this paper is to study the k -wise Kemeny aggregation problem. Section 2

72 formally defines the problem and reports on related work. Section 3 is devoted to some
73 axiomatic considerations of the corresponding voting rule, and to an algorithmic study
74 of the problem. More precisely, we show that the decision variant of the k -wise Kemeny
75 aggregation problem is NP-complete for any constant $k \geq 2$ and we provide an efficient
76 fixed-parameter algorithm for parameter m which relies on dynamic programming. We then
77 investigate a k -wise variant of the majority graph in Section 4. We prove that determining
78 this graph is easy for $k = 3$ but becomes NP-hard for $k > 3$, and we show how to use it
79 in a preprocessing step to speed up the computation of the output ranking. In Section 5,
80 we propose a 2-approximation algorithm for the k -wise Kemeny aggregation problem, by
81 introducing a k -wise variant of the Spearman distance. Numerical tests are presented
82 in Section 6 to assess both the efficiency of our exact methods and the accuracy of our
83 approximation algorithm.

84 2. Preliminaries

85 Adopting the terminology of social choice theory, we consider an election with a set
86 V of n voters and a set C of m candidates. Each voter v has a complete and transitive
87 preference order r_v over candidates (also called ranking). The collection of these rankings
88 defines a preference profile \mathcal{P} .

89 2.1. Notations and Definitions

90 Let us introduce some notations related to rankings. We denote by $\mathcal{R}(C)$ the set of $m!$
91 rankings over C . Given a ranking r and two candidates c and c' , we write $c \succ_r c'$ if c is in
92 a higher position than c' in r . Given a ranking r and a candidate c , $\mathbf{rk}(c, r)$ denotes the
93 rank of c in r . For instance, $\mathbf{rk}(c, r_v) = 1$ if c is the preferred candidate of voter v (the
94 candidate ranked highest in r_v). Given a ranking r and a set $S \subseteq C$, we define r_S as the
95 restriction of r to S and $\mathbf{top}_r(S)$ as the top choice (i.e., preferred candidate) in S according
96 to r . Similarly, given a preference profile \mathcal{P} and a set $S \subseteq C$, we define \mathcal{P}_S as the restriction
97 of \mathcal{P} to S . Lastly, we denote by $\mathbf{tail}_k(r)$ (resp. $\mathbf{head}_k(r)$) the subranking compounded of
98 the k least (resp. most) preferred candidates in r .

99 We are interested in SWFs which, given a preference profile \mathcal{P} , should return a consensus
100 ranking which yields a suitable compromise between the preferences in \mathcal{P} . One of the most
101 well-known SWFs is the *Kemeny rule*, which selects a ranking r with minimal Kendall tau
102 distance to \mathcal{P} . We recall:

103

Definition 1. *The Kendall tau distance between two rankings r and r' is defined by*

$$\delta_{\text{KT}}(r, r') = \sum_{(c, c') \in C^2} \mathbf{disagree}_{c, c'}(r, r')$$

104 where $\text{disagree}_{c,c'}(r, r') = 1$ if $c \succ_r c'$ and $c' \succ_{r'} c$, and 0 otherwise.

Stated differently, δ_{KT} measures the distance between two rankings by the number of pairwise disagreements between them. The distance $\delta_{\text{KT}}(r, \mathcal{P})$ between a ranking r and a preference profile \mathcal{P} is then obtained by summation:

$$\delta_{\text{KT}}(r, \mathcal{P}) = \sum_{r' \in \mathcal{P}} \delta_{\text{KT}}(r, r').$$

105

106 However, the Kendall tau distance only takes into account pairwise comparisons, which
 107 may entail counterintuitive results as illustrated by Example 1. To address this issue, the
 108 Kendall tau distance can be generalized to take into consideration disagreements on sets of
 109 cardinality greater than two. Given a set $S \subseteq C$ and $t \leq m$, we denote by $\Delta^t(S)$ the set
 110 of subsets of S of cardinality lower than or equal to t , i.e., $\Delta^t(S) = \{S' \subseteq S \text{ s.t. } |S'| \leq t\}$.
 111 When S is not specified, it is assumed to be C , i.e., $\Delta^t = \Delta^t(C)$.

Definition 2. Let $k \geq 2$ be an integer. The k -wise Kendall tau distance δ_{KT}^k between r and r' is defined by:

$$\delta_{\text{KT}}^k(r, r') = \sum_{S \in \Delta^k} \text{disagree}_S(r, r')$$

112 where $\text{disagree}_S(r, r') = 1$ if $\text{top}_r(S) \neq \text{top}_{r'}(S)$, and 0 otherwise.

In other words, δ_{KT}^k measures the distance between two rankings by the number of top-choice disagreements on sets of cardinality lower than or equal to k . It is not hard to see that $\delta_{\text{KT}}^k(r, r')$ can also be computed by using the following formula:

$$\begin{aligned} \delta_{\text{KT}}^k(r, r') &= \sum_{(c,c') \in C^2} \text{disagree}_{c,c'}(r, r') |\Delta^{k-2}(\text{Below}_c(r) \cap \text{Below}_{c'}(r'))| \\ &= \sum_{(c,c') \in C^2} \text{disagree}_{c,c'}(r, r') \sum_{i=0}^{k-2} \binom{|\text{Below}_c(r) \cap \text{Below}_{c'}(r')|}{i} \end{aligned} \quad (1)$$

113 where $\text{Below}_c(r) = \{x \in C \text{ s.t. } c \succ_r x\}$ is the set of candidates that are ranked below c
 114 in r . Formula 1 amounts to counting, for any pair $\{c, c'\}$ of candidates such that $c \succ_r c'$
 115 and $c' \succ_{r'} c$, the number of sets in Δ^k on which there is a disagreement because the
 116 top choice is c for r while it is c' for r' . Such sets are of the form $S \cup \{c, c'\}$, where
 117 $S \in \Delta^{k-2}(\text{Below}_c(r) \cap \text{Below}_{c'}(r'))$, otherwise c and c' would not be the top choices. Hence
 118 the formula.

119 Several observations can be made regarding δ_{KT}^k . Firstly, the following result states that
 120 δ_{KT}^k has all the properties of a distance:

121 **Proposition 1.** *The function δ_{KT}^k has all the properties of a distance: non-negativity, iden-*
 122 *tity of indiscernibles, symmetry and triangle inequality.*

123 *Proof.* Non-negativity and symmetry are obvious from the definition of δ_{KT}^k . It also verifies
 124 identity of indiscernibles: if $\delta_{\text{KT}}^k(r, r') = 0$, then rankings r and r' must in particular agree
 125 on each pairwise comparison, hence $\delta_{\text{KT}}(r, r') = 0$ and $r = r'$ because δ_{KT} verifies identity of
 126 indiscernibles. Lastly, triangular inequality comes from the fact that given three rankings
 127 r_1, r_2 and r_3 and a set S , $\text{disagree}_S(r_1, r_3) \leq \text{disagree}_S(r_1, r_2) + \text{disagree}_S(r_2, r_3)$. \square

128 Secondly, as mentioned in the introduction, we have $\delta_{\text{KT}}^2 = \delta_{\text{KT}}$. Thirdly and maybe most
 129 importantly, $\delta_{\text{KT}}^k(r, r')$ can be computed in polynomial time in the number m of candidates:

130 **Proposition 2.** *Given two rankings r and r' , $\delta_{\text{KT}}^k(r, r')$ can be computed in $O(m^3)$ by using*
 131 *Formula 1.*

132 *Proof.* We prove the $O(m^3)$ complexity of the method. First note that the computation of
 133 all binomial coefficients $\binom{p}{i}$ for $i \in \{0, \dots, k-2\}$ and $p \in \{i, \dots, m-2\}$ can be performed
 134 in $O(mk)$ thanks to Pascal's formula $\binom{p}{i} + \binom{p}{i+1} = \binom{p+1}{i+1}$. Then the computation of the
 135 sums $\sum_{i=0}^{k-2} \binom{p}{i}$ for $p \in \{0, \dots, m-2\}$ can also be computed in $O(mk)$. For each pair
 136 $\{c, c'\}$ of candidates such that $\text{top}_r(\{c, c'\}) = c$ and $\text{top}_{r'}(\{c, c'\}) = c'$, the computation
 137 of $|\text{Below}_c(r) \cap \text{Below}_{c'}(r')|$ can be performed in $O(m)$. As there are at most $O(m^2)$ such
 138 pairs, the overall complexity of the method is $O(m^3 + mk) = O(m^3)$. \square

The distance δ_{KT}^k induces a new SWF, the k -wise Kemeny rule, which, given a profile \mathcal{P} ,
 returns a ranking r with minimal distance δ_{KT}^k to \mathcal{P} , where:

$$\delta_{\text{KT}}^k(r, \mathcal{P}) = \sum_{r' \in \mathcal{P}} \delta_{\text{KT}}^k(r, r').$$

Note that this coincides with the rule we used in the introduction, by commutativity of
 addition:

$$\sum_{r' \in \mathcal{P}} \sum_{S \in \Delta^k} \text{disagree}_S(r, r') = \sum_{S \in \Delta^k} \sum_{r' \in \mathcal{P}} \text{disagree}_S(r, r').$$

139 Determining a consensus ranking for this rule induces an optimization problem that we
 140 term the k -wise Kemeny Aggregation Problem (k -KAP for short).

k -WISE KEMENY AGGREGATION PROBLEM (k -KAP)

141 *INSTANCE:* A profile \mathcal{P} with n voters and m candidates.

142 *SOLUTION:* A ranking r of the m candidates minimizing $\delta_{\text{KT}}^k(r, \mathcal{P})$.

143 When discussing the complexity of the problem, we may also refer to the decision version
 144 of the problem, k -KAP-DEC:

k -KAP-DEC

145 *INSTANCE:* A profile \mathcal{P} with n voters and m candidates and a threshold $\tau \in \mathbb{N}$.

146 *QUESTION:* Does there exist a ranking r of the m candidates such that $\delta_{\text{KT}}^k(r, \mathcal{P}) \leq \tau$?

147 *2.2. Related Work*

148 Several other variants of the Kemeny rule have been proposed in the literature, either
 149 to obtain generalizations able to deal with partial or weak orders [9, 16], to penalize more
 150 some pairwise disagreements than others [17], or to account for candidates that may become
 151 unavailable after voters express their preferences [13, 14].

152 Despite its popularity, the Kemeny rule has received several criticisms. One of them is
 153 that the Kendall tau distance counts equally the disagreements on every pair of candidates.
 154 This property is undesirable in many settings. For instance, with a web search engine, a
 155 disagreement on a pair of web pages with high positions in the considered rankings should
 156 have a higher cost than a disagreement on pairs of web pages with lower ones. This drawback
 157 motivated the introduction of weighted Kendall tau distances by Kumar and Vassilvitskii
 158 [17].

159 *2.2.1. Comparison between the k -wise and the weighted Kendall tau distances*

160 As mentioned above, Kumar and Vassilvitskii proposed that disagreements on highly
 161 ranked candidates be more costly than disagreements on lowly ranked ones. To achieve this,
 162 they defined a position-weighted version of Kendall tau, denoted by K_w , where an inversion
 163 of the two candidates at positions i and $i - 1$ has a cost w_i . For convenience, a cost $w_1 = 1$
 164 is also defined. Given the costs w_i , one can then measure the average swap-cost of moving
 165 a candidate from position i to j by computing the ratio $\frac{p_i - p_j}{i - j}$ where $p_i = \sum_{j=1}^i w_j$. This
 166 observation motivated the following definition for a position-weighted version of Kendall
 167 tau [17]:

Definition 3. Let $w = (w_1, \dots, w_m)$ be a vector of m weights $w_i > 0$. The position-weighted
 Kendall tau distance K_w between r and r' is defined by:

$$K_w(r, r') = \sum_{(c, c') \in C^2} \text{disagree}_{c, c'}(r, r') \bar{p}(r, r', c) \bar{p}(r, r', c')$$

168 where $\bar{p}(r, r', c) = \frac{p_{\text{rk}(c, r)} - p_{\text{rk}(c, r')}}{\text{rk}(c, r) - \text{rk}(c, r')}$ and $\bar{p}(r, r', c) = 1$ if $\text{rk}(c, r) = \text{rk}(c, r')$.

169 Put another way, if two rankings r and r' disagree on a pairwise comparison between
170 two candidates c and c' , then the cost of this disagreement is weighted by the product of the
171 average swap-cost of moving c from its position in r to its position in r' with the average
172 swap-cost of moving c' from its position in r to its position in r' . Note that if $w_i = 1$ for
173 all i in $\{1, \dots, m\}$, one obtains the usual Kendall tau distance.

174 Both the position-weighted Kendall tau distance and the k -wise Kendall tau distance
175 can be used in order to penalize more strongly disagreements on candidates with high
176 ranks (i.e., candidates that appear near the top of the ranking). For the k -wise Kendall tau
177 distance, this property results from the fact that for a pair $\{c, c'\}$ such that $\text{top}_r(\{c, c'\}) \neq$
178 $\text{top}_{r'}(\{c, c'\})$ the number of resulting subsets S for which $\text{top}_r(S) \neq \text{top}_{r'}(S)$ is all the larger
179 as c and c' are ranked high in r and r' . Note however that the position-weighted Kendall tau
180 distance requires to specify the $m - 1$ parameters w_2, \dots, w_m , the tuning of which does not
181 seem to be obvious. In comparison, the k -wise Kendall tau distance only requires to choose
182 the value of k , from which the cost of each disagreement on a pair $\{c, c'\}$ of candidates is
183 naturally entailed: it corresponds to the number of subsets of C of size less than or equal to
184 k for which the top choice in r is c while the top choice in r' is c' (see Section 2, Equation 1
185 for the formal expression of swap-costs according to k).

186 Let us illustrate with the following example that the k -wise Kendall tau distance is also
187 well suited to penalize more the disagreements involving alternatives at the top of the input
188 rankings.

189 **Example 3.** Consider rankings r_1, r_2, r_3 defined by $c_1 \succ_{r_1} c_2 \succ_{r_1} c_3$, $c_1 \succ_{r_2} c_3 \succ_{r_2} c_2$, and
190 $c_2 \succ_{r_3} c_1 \succ_{r_3} c_3$. We have $\delta_{\text{KT}}(r_1, r_2) = \delta_{\text{KT}}(r_1, r_3) = 1$ while $\delta_{\text{KT}}^3(r_1, r_2) = 1 < 2 = \delta_{\text{KT}}^3(r_1, r_3)$
191 because r_1 and r_3 disagree on both subsets $\{c_1, c_2\}$ and $\{c_1, c_2, c_3\}$. Put another way,
192 $\delta_{\text{KT}}^3(r_1, r_3) > \delta_{\text{KT}}^3(r_1, r_2)$ because r_1 and r_3 disagree on their top-ranked alternatives whereas
193 r_1 and r_2 disagree on the alternatives ranked in the last places.

194 2.3. Comparison with aggregation models where candidates may become unavailable

195 The two works closest to ours are related to another extension of the Kemeny rule.
196 This extension considers a setting in which, besides the fact that voters have preferences
197 over a set C , the election will in fact occur on a subset $S \subseteq C$ drawn according to a
198 probability distribution [13, 14]. The optimization problem considered is then to find a
199 consensus ranking r which minimizes, in expectation, the number of voters' disagreements
200 with the chosen candidate in S (a voter v disagrees if $\text{top}_{r_v}(S) \neq \text{top}_r(S)$). The differences
201 between the work of Baldiga and Green [13] and the one of Lu and Boutilier [14] are then
202 twofold. Firstly, while Baldiga and Green mostly focused on the axiomatic properties of
203 this aggregation procedure, the work of Lu and Boutilier has more of an algorithmic flavor.
204 Secondly, while Baldiga and Green mostly study a setting in which the probability $\mathbb{P}(S)$ of
205 S is only dependent on its cardinality (i.e., $\mathbb{P}(S)$ is only a function of $|S|$), Lu and Boutilier

206 study a setting that can be viewed as a special case of the former, where each candidate is
 207 absent of S independently of the others with a probability p (i.e., $\mathbb{P}(S) = p^{|C \setminus S|}(1 - p)^{|S|}$).
 208 The Kemeny aggregation problem can be formulated in both settings, either by defining
 209 $\mathbb{P}(S) = 0$ for $|S| \geq 3$, or by defining a probability p that is “sufficiently high” w.r.t. the size
 210 of the instance [14]. Lu and Boutilier conjectured that the determination of a consensus
 211 ranking is NP-hard in their setting, designed an exact method based on mathematical
 212 programming, two approximation greedy algorithms and a polynomial-time approximation
 213 scheme.

214 Our model can be seen as a special case of the model of Baldiga and Green where the
 215 set S is drawn uniformly at random within the set of subsets of C of cardinality smaller
 216 than or equal to a given constant $k \geq 2$. While it cannot be casted in the specific setting
 217 studied by Lu and Boutilier, our model is closely related and may be used to obtain new
 218 insights on their work.

219 3. Aggregation with the k -wise Kemeny Rule

220 In this section, we investigate the axiomatic properties of the k -wise Kemeny rule, and
 221 then we turn to the algorithmic study of k -KAP.

222 3.1. Axiomatic Properties of the k -wise Kemeny Rule

223 Several properties of the k -wise Kemeny rule have already been studied by Baldiga and
 224 Green [13], because their setting includes the k -wise Kemeny rule as a special case. Among
 225 other things, they showed that the rule is not *Condorcet consistent*. That is to say, a
 226 Condorcet winner may not be ranked first in any consensus ranking even when one exists,
 227 as illustrated by Example 2. The example indeed shows that the k -wise Kemeny rule is not
 228 Condorcet consistent for $k = 3$, as the Condorcet winner is not ranked first in the unique
 229 consensus ranking for the 3-wise Kemeny rule. Note that any $k > 3$ would yield the same
 230 consensus ranking, as there are only $m = 3$ candidates in the profile, hence the result holds
 231 for any $k \geq 3$. The example can be generalized to show that the result holds even if $m \geq k$:

232 **Proposition 3.** *The k -wise Kemeny rule is not Condorcet consistent for any $k \geq 3$, even*
 233 *if $m \geq k$.*

234 *Proof.* We provide an example similar to Example 2. Consider an election with k candidates
 235 c_1, c_2, \dots, c_k and 100 voters, with:

- 236 – 49 voters having preferences r_1 defined by $c_1 \succ c_2 \succ c_3 \succ c_4 \succ \dots \succ c_k$;
- 237 – 49 voters having preferences r_2 defined by $c_3 \succ c_2 \succ c_1 \succ c_4 \succ \dots \succ c_k$;
- 238 – 2 voters having preferences r_3 defined by $c_2 \succ c_1 \succ c_3 \succ c_4 \succ \dots \succ c_k$.

239 Note that c_2 is a Condorcet winner and that the ranking r_1 has a k -wise Kemeny score

240 worth $0 + 49 \times (2^{k-2} + 2^{k-3} + 2^{k-3}) + 2 \times 2^{k-2} = 100 \times 2^{k-2}$ while ranking r_3 has a score
 241 worth $49 \times 2^{k-2} + 49 \times (2^{k-2} + 2^{k-3}) + 0 = 122.5 \times 2^{k-2}$. It is easy to convince oneself that
 242 r_3 is the best possible ranking if c_2 is ranked first. Indeed, candidates c_1 , c_2 and c_3 are
 243 ranked higher than other candidates in all three rankings r_1 , r_2 and r_3 and therefore should
 244 be placed at the top in a consensus ranking (see Lemma 1). Additionally, c_1 and c_3 have
 245 symmetric positions in rankings r_1 and r_2 , and c_1 is preferred to c_3 in r_3 , thus c_1 should be
 246 placed before c_3 in a consensus ranking. The result follows. \square

247 The authors also show that the k -wise Kemeny rule is neutral, i.e., all candidates are
 248 treated equally, and that for $k \geq 3$ it is different from any positional method or any method
 249 that uses only the pairwise majority margins (among which is the standard Kemeny rule).
 250 We provide here some additional properties satisfied by the k -wise Kemeny rule:

- 251 • *Monotonicity*: up-ranking cannot harm a winner; down-ranking cannot enable a loser
 252 to win. Let us state this axiom more formally. Let $\mathcal{R}_{\mathcal{P}}^*$ denote the set of consensus
 253 rankings for preference profile \mathcal{P} . Then for any candidate $c \in C$ and profiles \mathcal{P}
 254 and \mathcal{P}' such that \mathcal{P}' can be obtained from \mathcal{P} by decreasing the position of c in
 255 some ranking in \mathcal{P} (all other things being equal): $c \in \{\text{top}_r(C) : r \in \mathcal{R}_{\mathcal{P}'}^*\}$ implies
 256 $c \in \{\text{top}_r(C) : r \in \mathcal{R}_{\mathcal{P}}^*\}$ and $c \notin \{\text{top}_r(C) : r \in \mathcal{R}_{\mathcal{P}}^*\}$ implies $c \notin \{\text{top}_r(C) : r \in \mathcal{R}_{\mathcal{P}'}^*\}$.
- 257 • *Unanimity*: if all voters rank c before c' , then c is ranked before c' in any consensus
 258 ranking.
- 259 • *Reinforcement*: let $\mathcal{R}_{\mathcal{P}}^*$ and $\mathcal{R}_{\mathcal{P}'}^*$ denote the sets of consensus rankings for preference
 260 profiles \mathcal{P} and \mathcal{P}' respectively. If $\mathcal{R}_{\mathcal{P}}^* \cap \mathcal{R}_{\mathcal{P}'}^* \neq \emptyset$ and \mathcal{P}'' is the profile obtained by
 261 concatenating \mathcal{P} and \mathcal{P}' , then $\mathcal{R}_{\mathcal{P}''}^* = \mathcal{R}_{\mathcal{P}}^* \cap \mathcal{R}_{\mathcal{P}'}^*$.

262 Examples of voting rules for which the monotonicity property does not hold are *plurality with*
 263 *run-off* (a two-round election system where, if some candidate is top ranked by a majority
 264 of the voters, it wins in round 1; otherwise, round 2 consists of the majority rule applied to
 265 the two candidates with highest plurality score in round 1) and *single transferable vote* (at
 266 each stage, the candidate with lowest plurality score is dropped from all votes and each vote
 267 for which this candidate was top ranked is transferred to the next remaining candidate in
 268 the ranking; at the first stage for which some candidate c sits atop a majority of the votes,
 269 c is declared the winner). For more details, the reader may refer to the book chapter by
 270 Zwicker [12]. The unanimity property is obviously desirable for any reasonable voting rule.
 271 Finally, the reinforcement property has been introduced by Young [18], originally calling it
 272 *consistency*, for the axiomatization of Borda's rule viewed as a *social choice function*, i.e.,
 273 returning a (set of) winning candidate(s). Reinforcement states that a candidate elected
 274 by two disjoint electorates should remain a winning candidate if one merges the voters,

275 and that a candidate elected by only one electorate is in some sense not as “good” as a
 276 candidate elected by both electorates. The adaptation of the property to *social welfare*
 277 *functions*, i.e., functions returning a (set of) consensus ranking(s), has been proposed by
 278 Young and Levenglick [19] for an axiomatic characterization of Kemeny’s rule: they show
 279 that it is the unique Condorcet consistent social welfare function that satisfy neutrality and
 280 reinforcement, where the neutrality property states that all candidates are treated equally
 281 (in the sense that permuting the names of the candidates in the input rankings result in
 282 the same permutation in the consensus rankings).

283 While the fact that the k -wise Kemeny rule satisfies neutrality and reinforcement is
 284 quite obvious from its definition, the two following results state that the monotonicity and
 285 unanimity conditions also hold.

286 **Proposition 4.** *The k -wise Kemeny rule satisfies monotonicity.*

287 *Proof.* Let r be a ranking and v be a voter such that c and c' are consecutive in r_v and
 288 $c' \succ_{r_v} c$. Let us denote by $r_v^{c \leftrightarrow c'}$ the ranking obtained from r_v by switching the positions
 289 of c and c' . Then, the sets S for which $\text{top}_{r_v}(S) \neq \text{top}_{r_v^{c \leftrightarrow c'}}(S)$ are of the form $\{c, c'\} \cup S'$
 290 where $S' \subseteq \text{Below}_c(r_v)$. Furthermore, if such a set S contains a candidate c'' such that
 291 $c'' \succ_r \text{top}_r(\{c, c'\})$, then we will both have $\text{top}_r(S) \neq \text{top}_{r_v}(S)$ and $\text{top}_r(S) \neq \text{top}_{r_v^{c \leftrightarrow c'}}(S)$.
 292 Hence, the sets which account for the difference between $\delta_{\text{KT}}^k(r, r_v)$ and $\delta_{\text{KT}}^k(r, r_v^{c \leftrightarrow c'})$ are of
 293 the form $\{c, c'\} \cup S'$ where $S' \subseteq \text{Below}_c(r_v) \cap \text{Below}_{\text{top}_r(\{c, c'\})}(r)$.

294 More precisely, using Equation 1, we obtain that $\delta_{\text{KT}}^k(r, r_v^{c \leftrightarrow c'})$ is equal to:

295 $\bullet \delta_{\text{KT}}^k(r, r_v) - \sum_{i=0}^{k-2} \binom{|\text{Below}_{c'}(r_v) \cap \text{Below}_c(r)|}{i}$ if $c \succ_r c'$;
 296 $\bullet \delta_{\text{KT}}^k(r, r_v) + \sum_{i=0}^{k-2} \binom{|\text{Below}_c(r_v) \cap \text{Below}_{c'}(r)|}{i}$ if $c' \succ_r c$.

297 Hence, $\delta_{\text{KT}}^k(r, r_v)$ will decrease if r ranks c before c' and the decrease is maximal when c is
 298 ranked first in r (because it maximizes $|\text{Below}_{c'}(r_v) \cap \text{Below}_c(r)|$). Repeating this argument
 299 shows that no winner is harmed by up-ranking. Similarly, $\delta_{\text{KT}}^k(r, r_v)$ will increase if r ranks
 300 c' before c and the increase is maximal when c' is ranked first in r (because it maximizes
 301 $|\text{Below}_c(r_v) \cap \text{Below}_{c'}(r)|$). Repeating this argument shows that no loser can win by down-
 302 ranking. \square

303 **Proposition 5.** *The k -wise Kemeny rule satisfies unanimity.*

304 *Proof.* Let $c, c' \in C$ be two candidates and \mathcal{P} be a preference profile such that for all ranking
 305 r'' in \mathcal{P} , $c \succ_{r''} c'$. Let r be a ranking such that $c' \succ_r c$, and r' be the ranking obtained from
 306 r by exchanging the positions of c' and c . Moreover, let K denote the set of candidates
 307 between c and c' in r . Let us assume for the sake of contradiction that r minimizes $\delta_{\text{KT}}^k(\cdot, \mathcal{P})$.

308 We will prove that for any $r'' \in \mathcal{P}$, $\delta_{\text{KT}}^k(r', r'') < \delta_{\text{KT}}^k(r, r'')$. Let $r'' \in \mathcal{P}$ and $S \subseteq C$ such
309 that $\text{top}_{r'}(S) \neq \text{top}_{r''}(S)$ and $\text{top}_r(S) = \text{top}_{r''}(S)$. Then S must contain either c or c' , and
310 does not contain any element ranked higher than c' in r because otherwise we would have
311 $\text{top}_r(S) = \text{top}_{r'}(S)$. This implies that either $\text{top}_{r'}(S) = c$ (if $c \in S$) or $\text{top}_r(S) = c'$ (if
312 $c' \in S$). These situations are exclusive: there cannot be both c and c' in S as we cannot
313 have $\text{top}_{r''}(S) = c'$ if $c \in S$. To sum up, there are two possibilities:

- 314 1. $\text{top}_{r'}(S) = c$ and $\text{top}_{r''}(S) = \text{top}_r(S) = c''$ is in K . This implies that c'' is the second
315 choice of r' in S and that $c'' \succ_{r''} c'$ as $c'' \succ_{r''} c$. In this case, necessarily, $c' \notin S$ and
316 we consider $S' = (S \setminus \{c\}) \cup \{c'\}$.
- 317 2. $\text{top}_{r'}(S) \in K$ and $\text{top}_{r''}(S) = \text{top}_r(S) = c'$. In this case, necessarily, $c \notin S$ and we
318 consider $S' = S \cup \{c\}$.

319 In both cases, we obtain a set S' such that $\text{top}_{r'}(S') = \text{top}_{r''}(S')$ and $\text{top}_r(S') \neq \text{top}_{r''}(S')$.
320 Note that any set S will induce a different S' and that $\{c, c'\}$ is not one of these sets S' .
321 As we also have $\text{top}_{r'}(\{c, c'\}) = \text{top}_{r''}(\{c, c'\})$ and $\text{top}_r(\{c, c'\}) \neq \text{top}_{r''}(\{c, c'\})$, this proves
322 that for any $r'' \in \mathcal{P}$, $\delta_{\text{KT}}^k(r', r'') < \delta_{\text{KT}}^k(r, r'')$ and hence the claim. \square

323 Besides, the k -wise Kemeny rule does not satisfy *Independence of irrelevant alternatives*,
324 i.e., the relative positions of two candidates in a consensus ranking can depend on the
325 presence of other candidates. Let us illustrate this point with the following example.

326 **Example 4.** *Considering the preference profile from Example 1, the only consensus ranking*
327 *for δ_{KT}^3 is $c_1 \succ c_2 \succ c_3$. Yet, without c_3 the only consensus ranking would be $c_2 \succ c_1$.*

328 Lastly, note that there exists a noise model such that the k -wise Kemeny rule can be
329 interpreted as a maximum likelihood estimator [20]. In this view of voting, one assumes
330 that there exists a “correct” ranking r , and each vote corresponds to a noisy perception
331 of this correct ranking. Consider the conditional probability measure \mathbb{P} on $\mathcal{R}(C)$ defined
332 by $\mathbb{P}(r'|r) \propto e^{-\delta_{\text{KT}}^k(r, r')}$. It is easy to convince oneself that the k -wise Kemeny rule returns
333 a ranking r^* that maximizes $\mathbb{P}(\mathcal{P} | r^*) = \prod_{r' \in \mathcal{P}} \mathbb{P}(r' | r^*)$ and is thus a maximum likelihood
334 estimate of r .

3.2. Computational Complexity of k -KAP

335 We now turn to the algorithmic study of k -KAP. After providing a hardness result, we
336 will design an efficient Fixed Parameter Tractable (FPT) algorithm for parameter m .

337 While k -KAP is obviously NP-hard for $k = 2$ as it then corresponds to determining a
338 consensus ranking w.r.t. the Kemeny rule, we strengthen this result by showing that k -
339 KAP-DEC is also NP-complete for *any* constant value $k \geq 3$. To prove this result, we first
340 need two lemmas.
341

342 **Lemma 1.** *If the candidates in a set $S \subseteq C$ are ranked in the $|S|$ last positions by all*
 343 *voters and in the same order, then for any $k \geq 2$, any consensus ranking w.r.t. the k -wise*
 344 *Kemeny rule has the same property.*

345 *Proof.* This is a simple consequence of the unanimity property that is satisfied by the k -wise
 346 Kemeny rule. □

Lemma 2. *For any $p \in \mathbb{N}^*$ and $\varepsilon \in (0, \frac{1}{2p})$ we have the following inequality:*

$$(1 + \varepsilon)^p < 1 + 2p\varepsilon$$

Proof. We prove the claim by induction. It is obvious for $p = 1$. Consider the claim true
 for $p = k$, then for $\varepsilon \in (0, \frac{1}{2(k+1)})$

$$\begin{aligned} (1 + \varepsilon)^{k+1} &= (1 + \varepsilon)(1 + \varepsilon)^k < (1 + \varepsilon)(1 + 2k\varepsilon) \\ &= 1 + 2k\varepsilon + \varepsilon + 2k\varepsilon^2 < 1 + 2(k + 1)\varepsilon \end{aligned}$$

347 where the first inequality uses the induction hypothesis and the second inequality uses the
 348 fact that $2k\varepsilon^2 < \varepsilon$ because $2k\varepsilon < 1$ for $\varepsilon \in (0, \frac{1}{2(k+1)})$. □

349 We can now prove the hardness result, by using a reduction from 2-KAP-DEC.

350 **Theorem 1.** *For any constant $k \geq 3$, k -KAP-DEC is NP-complete, even if the number of*
 351 *voters equals 4 or if the average range of candidates is less than or equal to 2 (where the*
 352 *range of a candidate c is defined by $\max_{r \in \mathcal{P}} \mathbf{rk}(c, r) - \min_{r \in \mathcal{P}} \mathbf{rk}(c, r) + 1$ and the average*
 353 *is taken over all candidates).*

354 *Proof.* Membership in NP follows from Proposition 2. We obtain our hardness result via a
 355 reduction from the standard Kemeny aggregation problem (2-KAP-DEC), which is known
 356 to be NP-complete [21]. Consider a preference profile \mathcal{P} with $n \geq 1$ voters and $m \geq k \geq 3$
 357 candidates and an integer τ . We wish to determine if there exists a ranking r of the m
 358 candidates such that $\delta_{\text{KT}}(r, \mathcal{P}) \leq \tau$. Stated otherwise, we wish to determine if the Kemeny
 359 score of a consensus ranking is lower than or equal to τ . Note that we can assume $m \geq k$
 360 as k is a constant and 2-KAP is fixed parameter tractable w.r.t. m [22]. We add to the
 361 problem $\lambda = 4nm^4$ candidates $c_1^*, \dots, c_\lambda^*$ that are ranked last by all voters and in the same
 362 order, i.e., $c_1^* \succ_r \dots \succ_r c_\lambda^*$ for all r in \mathcal{P} . We denote the resulting set of candidates by C'
 363 (i.e., $C' = C \cup \{c_1^*, \dots, c_\lambda^*\}$) and the resulting preference profile by \mathcal{P}' . By using Lemma 1,
 364 we will restrict our attention to rankings that rank these additional voters last and in the
 365 same order as the voters, because they are the only possible consensus rankings. Lastly, in
 366 the resulting k -KAP-DEC instance, we set $\tau' = (1 + \sum_{i=1}^{k-2} \binom{4nm^4}{i})(\tau + 1) - 1$.

Given two such rankings r and r' , we have by Equation 1 that $\delta_{\text{KT}}^k(r, r')$ is equal to:

$$\begin{aligned} & \sum_{(c, c') \in C^2} \text{disagree}_{c, c'}(r, r') \sum_{i=0}^{k-2} \binom{|\text{Below}_c(r_C) \cap \text{Below}_{c'}(r'_C)| + \lambda}{i} \\ &= \sum_{(c, c') \in C^2} \text{disagree}_{c, c'}(r, r') \left(1 + \sum_{i=1}^{k-2} \binom{|\text{Below}_c(r_C) \cap \text{Below}_{c'}(r'_C)| + \lambda}{i} \right) \end{aligned}$$

367 because there is no disagreement for $\{c, c'\} \not\subseteq C$ and $\binom{|\text{Below}_c(r_C) \cap \text{Below}_{c'}(r'_C)| + \lambda}{0} = 1$.
From $0 \leq |\text{Below}_c(r_C) \cap \text{Below}_{c'}(r'_C)| < m$, we deduce:

$$\sum_{i=1}^{k-2} \binom{\lambda}{i} \leq \sum_{i=1}^{k-2} \binom{|\text{Below}_c(r_C) \cap \text{Below}_{c'}(r'_C)| + \lambda}{i} < \sum_{i=1}^{k-2} \binom{m + \lambda}{i}.$$

Consequently:

$$\delta_{\text{KT}}(r_C, r'_C) \left(1 + \sum_{i=1}^{k-2} \binom{\lambda}{i} \right) \leq \delta_{\text{KT}}^k(r, r') < \delta_{\text{KT}}(r_C, r'_C) \left(1 + \sum_{i=1}^{k-2} \binom{m + \lambda}{i} \right)$$

because $\sum_{(c, c') \in C^2} \text{disagree}_{c, c'}(r, r') = \delta_{\text{KT}}(r_C, r'_C)$; from which we obtain:

$$\delta_{\text{KT}}(r_C, \mathcal{P}) \left(1 + \sum_{i=1}^{k-2} \binom{\lambda}{i} \right) \leq \delta_{\text{KT}}^k(r, \mathcal{P}') < \delta_{\text{KT}}(r_C, \mathcal{P}) \left(1 + \sum_{i=1}^{k-2} \binom{m + \lambda}{i} \right) \quad (2)$$

368 because $\sum_{r' \in \mathcal{P}'} \delta_{\text{KT}}(r_C, r'_C) = \delta_{\text{KT}}(r_C, \mathcal{P})$ and $\sum_{r' \in \mathcal{P}'} \delta_{\text{KT}}^k(r, r') = \delta_{\text{KT}}^k(r, \mathcal{P}')$, provided that
369 $\text{tail}_\lambda(r) = c_1^* \succ_r \dots \succ_r c_\lambda^*$ and $\text{tail}_\lambda(r') = c_1^* \succ_{r'} \dots \succ_{r'} c_\lambda^* \forall r' \in \mathcal{P}'$.

Now, note that:

$$\binom{m + \lambda}{i} = \frac{\lambda!}{i!(\lambda - i)!} \frac{\prod_{j=1}^m (\lambda + j)}{\prod_{j=1}^m (\lambda - i + j)} = \binom{\lambda}{i} \prod_{j=1}^m \frac{\lambda + j}{\lambda + j - i}. \quad (3)$$

If one sets $\lambda = 4nm^4$, the following inequalities hold:

$$\begin{aligned} \frac{4nm^4 + j}{4nm^4 + j - i} &= 1 + \frac{i}{4nm^4 + j - i} \leq 1 + \frac{i}{4nm^4 - i} \\ &\leq 1 + \frac{2i}{4nm^4} \leq 1 + \frac{1}{2nm^3} \end{aligned}$$

370 where the second inequality follows from $4nm^4/(4nm^4 - i) \leq 2$ for $i \in \llbracket 1, m \rrbracket$.

From Equation 3, we deduce then:

$$\binom{m + 4nm^4}{i} \leq \binom{4nm^4}{i} \left(1 + \frac{1}{2nm^3}\right)^m \leq \binom{4nm^4}{i} \left(1 + \frac{1}{nm^2}\right)$$

371 where the second inequality follows from Lemma 2 with $\varepsilon = \frac{1}{2nm^3}$ because $\frac{1}{2nm^3} < \frac{1}{2m}$ for
372 $m \geq 3$.

Coming back to Equation 2, this implies:

$$\delta_{\text{KT}}(r_C, \mathcal{P}) \leq \frac{\delta_{\text{KT}}^k(r, \mathcal{P}')}{1 + \sum_{i=1}^{k-2} \binom{4nm^4}{i}} < \delta_{\text{KT}}(r_C, \mathcal{P}) + \frac{1}{nm^2} \delta_{\text{KT}}(r_C, \mathcal{P})$$

and therefore:

$$\delta_{\text{KT}}(r_C, \mathcal{P}) \leq \frac{\delta_{\text{KT}}^k(r, \mathcal{P}')}{1 + \sum_{i=1}^{k-2} \binom{4nm^4}{i}} < \delta_{\text{KT}}(r_C, \mathcal{P}) + 1$$

373 because $\delta_{\text{KT}}(r_C, \mathcal{P}) \leq n \binom{m}{2} \leq nm^2$. In particular, $\delta_{\text{KT}}(r_C, \mathcal{P}) \leq \tau$ for an integer τ iff
374 $\delta_{\text{KT}}^k(r, \mathcal{P}') \leq (1 + \sum_{i=1}^{k-2} \binom{4nm^4}{i})(\tau + 1) - 1 = \tau'$. This shows that (\mathcal{P}, τ) is a yes instance of
375 2-KAP-DEC iff (\mathcal{P}', τ') is a yes instance of k -KAP-DEC.

376 It is known that 2-KAP-DEC is NP-complete even if the number n of voters equals 4 [9]
377 and even if the average range of candidates equals 2 [22]. As the reduction above preserves
378 the number of voters and decreases the average range of candidates, the same results hold
379 for k -KAP-DEC. \square

380 Although the above proof uses a conversion from the k -wise Kemeny rule to the standard
381 Kemeny rule, note that it means in no way that both rules are equivalent. There indeed
382 exist instances with arbitrary large sets of candidates such that the k -wise Kemeny rule
383 differs from the Kemeny rule. Consider for instance the election described in Example 1.
384 The only k -wise Kemeny consensus ranking is $c_1 \succ c_2 \succ c_3$, while the only pairwise Kemeny
385 consensus ranking is $c_2 \succ c_3 \succ c_1$. Now modify the preference profile of the previous election
386 by adding candidates c_4 to c_k , with k arbitrarily large such that for any ranking r in the
387 new preference profile, $\text{head}_{k-3}(r) = c_k \succ c_{k-1} \succ \dots \succ c_4$. With this new preference profile,
388 the only k -wise Kemeny consensus ranking is $c_k \succ c_{k-1} \succ \dots \succ c_4 \succ c_1 \succ c_2 \succ c_3$, while
389 the only pairwise Kemeny consensus ranking is $c_k \succ c_{k-1} \succ \dots \succ c_4 \succ c_2 \succ c_3 \succ c_1$.

390 Despite Theorem 1, k -KAP is obviously FPT w.r.t. the number m of candidates, by
391 simply trying the $m!$ rankings in $\mathcal{R}(C)$. We now design a dynamic programming procedure
392 which significantly improves this time complexity.

Proposition 6. *If r^* is an optimal ranking for k -KAP, then $\delta_{\text{KT}}^k(r^*, \mathcal{P}) = d_{\text{KT}}^k(C)$, where, for any subset $S \subseteq C$, $d_{\text{KT}}^k(S)$ is defined by the recursive relation:*

$$\begin{aligned} d_{\text{KT}}^k(S) &= \min_{c \in S} [d_{\text{KT}}^k(S \setminus \{c\}) \\ &\quad + \sum_{r \in \mathcal{P}_S} \sum_{c' \succ_r c} \sum_{i=0}^{k-2} \binom{|S| - \text{rk}(c', r) - 1}{i}] \\ d_{\text{KT}}^k(\emptyset) &= 0. \end{aligned} \tag{4}$$

Proof. Given $S \subseteq C$ and $c \in S$, let us define $\mathcal{R}_c(S)$ as $\{r \in \mathcal{R}(S) \text{ s.t. } \text{top}_r(S) = c\}$. The set $\Delta^k(S)$ can be partitioned into $\Delta_c^k(S) = \{S' \in \Delta^k(S) \text{ s.t. } c \in S'\}$ and $\Delta_c^k(S) = \{S' \in \Delta^k(S) \text{ s.t. } c \notin S'\} = \Delta^k(S \setminus \{c\})$. Given a preference profile \mathcal{P} over C and a ranking $\hat{r} \in \mathcal{R}_c(S)$, the summation defining $\delta_{\text{KT}}^k(\hat{r}, \mathcal{P}_S)$ breaks down as follows:

$$\begin{aligned} \delta_{\text{KT}}^k(\hat{r}, \mathcal{P}_S) &= \sum_{r \in \mathcal{P}_S} \sum_{S' \in \Delta^k(S)} \text{disagree}_{S'}(\hat{r}, r) \\ &= \delta_{\text{KT}}^k(\hat{r}_{S \setminus \{c\}}, \mathcal{P}_{S \setminus \{c\}}) + \sum_{r \in \mathcal{P}_S} \sum_{S' \in \Delta_c^k(S)} \text{disagree}_{S'}(\hat{r}, r). \end{aligned} \tag{5}$$

Using the same reasoning as in Equation 1 on page 5, the second summand in Equation 5 can be rewritten as follows:

$$\sum_{r \in \mathcal{P}_S} \sum_{c' \succ_r c} \sum_{i=0}^{k-2} \binom{|\text{Below}_c(\hat{r}) \cap \text{Below}_{c'}(r)|}{i}$$

because $\text{top}_{\hat{r}}(S') = c$ for all $S' \in \Delta_c^k(S)$. Note that $\text{Below}_c(\hat{r}) = S \setminus \{c\}$ and $\text{Below}_{c'}(r) = \{c'' \in S \text{ s.t. } c' \succ_r c''\} \subseteq S$, thus $|\text{Below}_c(\hat{r}) \cap \text{Below}_{c'}(r)| = |S| - \text{rk}(c', r) - 1$. Hence, $\delta_{\text{KT}}^k(\hat{r}, \mathcal{P}_S)$ is equal to:

$$\delta_{\text{KT}}^k(\hat{r}_{S \setminus \{c\}}, \mathcal{P}_{S \setminus \{c\}}) + \sum_{r \in \mathcal{P}_S} \sum_{c' \succ_r c} \sum_{i=0}^{k-2} \binom{|S| - \text{rk}(c', r) - 1}{i}. \tag{6}$$

Consider now a ranking $r^* \in \mathcal{R}(S)$ such that $\delta_{\text{KT}}^k(r^*, \mathcal{P}_S) = \min_{r \in \mathcal{R}(S)} \delta_{\text{KT}}^k(r, \mathcal{P}_S)$. We have:

$$\begin{aligned} \delta_{\text{KT}}^k(r^*, \mathcal{P}_S) &= \min_{c \in S} \min_{\hat{r} \in \mathcal{R}_c(S)} \delta_{\text{KT}}^k(\hat{r}, \mathcal{P}_S) \\ &= \min_{c \in S} \left(\min_{\hat{r} \in \mathcal{R}(S \setminus \{c\})} \delta_{\text{KT}}^k(\hat{r}, \mathcal{P}_{S \setminus \{c\}}) \right) \\ &\quad + \sum_{r \in \mathcal{P}_S} \sum_{c' \succ_r c} \sum_{i=0}^{k-2} \binom{|S| - \text{rk}(c', r) - 1}{i} \end{aligned}$$

393 because the second summand in Equation 6 does not depend on \hat{r} (it only depends on c ,
394 which is the argument of the first min operator). If one denotes $\min_{r \in \mathcal{R}(S)} \delta_{\text{KT}}^k(r, \mathcal{P}_S)$ by
395 $d_{\text{KT}}^k(S)$, one obtains Equation 4. This concludes the proof. \square

396 A candidate $c \in S$ that realizes the minimum in Equation 4 can be ranked in first position
397 in an optimal ranking for \mathcal{P}_S . Once $d_{\text{KT}}^k(S)$ is computed for each $S \subseteq C$, a ranking r^*
398 achieving the optimal value $d_{\text{KT}}^k(C)$ can thus be determined recursively starting from $S = C$.
399 The complexity of the induced dynamic programming method is $O(2^m m^2 n)$ as there are 2^m
400 subsets $S \subseteq C$ to consider and each value $d_{\text{KT}}^k(S)$ is computed in $O(m^2 n)$ by Equation 4.
401 The min operation is indeed performed on m values and the sum $\sum_{c' \succ_r c} \sum_{i=0}^{k-2} \binom{|S| - \text{rk}(c', r) - 1}{i}$
402 is computed incrementally in $O(m)$, which entails an $O(mn)$ complexity for the second
403 summand in Equation 4 (the n factor is due to the sum over all $r \in \mathcal{P}_S$). The computation
404 of binomial coefficients $\binom{p}{i}$ for $i \in \{0, \dots, k-2\}$ and $p \in \{i, \dots, m-2\}$ is performed in
405 $O(mk)$ in a preliminary step thanks to Pascal's formula.

406 4. The k -Wise Majority Digraph

407 We now propose and investigate a k -wise counterpart of the pairwise majority digraph,
408 that will be used in a preprocessing procedure for k -KAP.

409 As stated in the introduction, the pairwise Kemeny rule is strongly related to the *pair-*
410 *wise majority digraph*. We denote by $\mathcal{G}_{\mathcal{P}}$ the pairwise majority digraph associated to profile
411 \mathcal{P} . We recall that in this digraph, there is one vertex per candidate, and there is an arc
412 from candidate c to candidate c' if a strict majority of voters prefers c to c' . In the weighted
413 pairwise majority digraph, each arc (c, c') is weighted by $w_{\mathcal{P}}(c, c') := |\{r \in \mathcal{P} \text{ s.t. } c \succ_r c'\}| - |\{r \in \mathcal{P} \text{ s.t. } c' \succ_r c\}|$.
414

415 **Example 5.** Consider a profile \mathcal{P} with 10 voters and 6 candidates such that:

- 416 – 4 voters have preferences $c_1 \succ c_2 \succ c_4 \succ c_3 \succ c_5 \succ c_6$;
- 417 – 4 voters have preferences $c_1 \succ c_3 \succ c_2 \succ c_4 \succ c_5 \succ c_6$;
- 418 – 1 voter has preferences $c_6 \succ c_1 \succ c_2 \succ c_4 \succ c_3 \succ c_5$;
- 419 – 1 voter has preferences $c_6 \succ c_1 \succ c_4 \succ c_3 \succ c_2 \succ c_5$.

420 The weighted pairwise majority digraph $\mathcal{G}_{\mathcal{P}}$ is displayed on the left of Figure 1.

421 From $\mathcal{G}_{\mathcal{P}}$, we can define a set of *consistent* rankings:

422 **Definition 4.** Let \mathcal{G} be a digraph whose vertices correspond to the candidates in C . Let
423 $B_1(\mathcal{G}), \dots, B_{\sigma(\mathcal{G})}(\mathcal{G})$ denote the subsets of C corresponding to the Strongly Connected Com-
424 ponents (SCCs) of \mathcal{G} , and $\mathcal{O}(\mathcal{G})$ denote the set of linear orders $\prec_{\mathcal{G}}$ on $\{1, \dots, \sigma(\mathcal{G})\}$ such
425 that if there exists an arc (c, c') from $c \in B_i(\mathcal{G})$ to $c' \in B_j(\mathcal{G})$ then $i \prec_{\mathcal{G}} j$. Given $\prec_{\mathcal{G}} \in \mathcal{O}(\mathcal{G})$,
426 we say that a ranking r is consistent with $\prec_{\mathcal{G}}$ if the candidates in B_i are ranked before the
427 ones of B_j when $i \prec_{\mathcal{G}} j$.

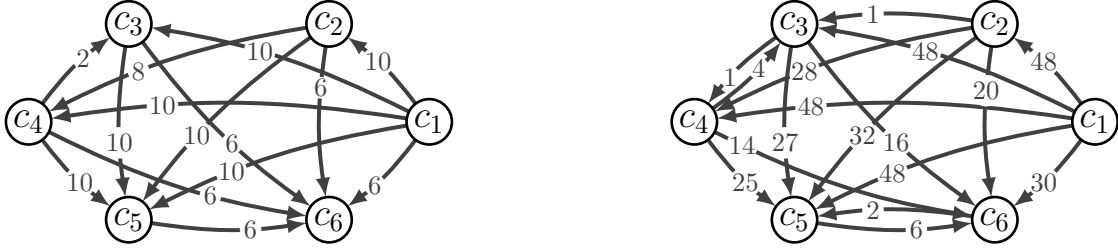


Figure 1: Weighted k -wise majority digraph in Example 5 for $k = 2$ (left) and $k = 3$ (right).

428 The following result states that, for any $\langle \mathcal{G}_{\mathcal{P}} \in \mathcal{O}(\mathcal{G}_{\mathcal{P}})$, there exists a consensus ranking
 429 for δ_{KT} among the rankings consistent with $\langle \mathcal{G}_{\mathcal{P}}$.

Theorem 2 (Theorem 16 in [23], by Charon and Hudry). *Let \mathcal{P} be a profile over C and assume that the SCCs of $\mathcal{G}_{\mathcal{P}}$ are numbered according to a linear order $\langle \mathcal{G}_{\mathcal{P}} \in \mathcal{O}(\mathcal{G}_{\mathcal{P}})$. Consider the ranking r^* , consistent with $\langle \mathcal{G}_{\mathcal{P}}$, obtained by the concatenation of rankings $r_1^*, \dots, r_{t(\mathcal{G}_{\mathcal{P}})}^*$ where $\delta_{\text{KT}}(r_i^*, \mathcal{P}_{B_i(\mathcal{G}_{\mathcal{P}})}) = \min_{r \in \mathcal{R}(B_i(\mathcal{G}_{\mathcal{P}}))} \delta_{\text{KT}}(r, \mathcal{P}_{B_i(\mathcal{G}_{\mathcal{P}})})$. We have:*

$$\delta_{\text{KT}}(r^*, \mathcal{P}) = \min_{r \in \mathcal{R}(C)} \delta_{\text{KT}}(r, \mathcal{P}).$$

430 That is, r^* is a consensus ranking according to the Kemeny rule. Furthermore, if $\mathcal{O}(\mathcal{G}_{\mathcal{P}}) =$
 431 $\{\langle \mathcal{G}_{\mathcal{P}}\}$ and $w_{\mathcal{P}}(c, c') > 0$ for all $c \in B_i(\mathcal{G}_{\mathcal{P}})$ and $c' \in B_j(\mathcal{G}_{\mathcal{P}})$ when $i <_{\mathcal{G}_{\mathcal{P}}} j$, then all consensus
 432 rankings are consistent with $\langle \mathcal{G}_{\mathcal{P}}$.

433 This result does not hold anymore if one uses δ_{KT}^k (with $k \geq 3$) instead of δ_{KT} , as shown
 434 by the following example.

435 **Example 6.** *Let us denote by \mathcal{P} the profile of Example 1. The pairwise majority digraph*
 436 *$\mathcal{G}_{\mathcal{P}}$ has three SCCs $B_1(\mathcal{G}_{\mathcal{P}}) = \{c_2\}$, $B_2(\mathcal{G}_{\mathcal{P}}) = \{c_3\}$ and $B_3(\mathcal{G}_{\mathcal{P}}) = \{c_1\}$. In this example,*
 437 *$\mathcal{O}(\mathcal{G}_{\mathcal{P}}) = \{\langle \mathcal{G}_{\mathcal{P}}\}$ where $1 <_{\mathcal{G}_{\mathcal{P}}} 2 <_{\mathcal{G}_{\mathcal{P}}} 3$. The only ranking consistent with $\langle \mathcal{G}_{\mathcal{P}}$ is $c_2 \succ c_3 \succ c_1$*
 438 *while the only consensus ranking w.r.t. the 3-wise Kemeny rule is $c_1 \succ c_2 \succ c_3$.*

439 In order to adapt Theorem 2 to the k -wise Kemeny rule, we now introduce the con-
 440 cept of k -wise majority digraph. Let $\Delta_{c,c'}^k(S) = \{S' \in \Delta^k(S) \text{ s.t. } \{c, c'\} \subseteq S'\}$. If S is not
 441 specified, it is assumed to be C . Given a ranking r , we denote by $\Delta_r^k(S, c, c')$ the set

442 $\{S' \in \Delta_{cc'}^k(S)$ s.t. $\text{top}_r(S') = c\}$. Given a profile \mathcal{P} , we denote by $\phi_{\mathcal{P}}^k(S, c, c')$ the value
443 $\sum_{r \in \mathcal{P}} |\Delta_r^k(S, c, c')|$ and by $w_{\mathcal{P}}^k(S, c, c')$ the difference $\phi_{\mathcal{P}}^k(S, c, c') - \phi_{\mathcal{P}}^k(S, c', c)$. This def-
444 inition implies that $w_{\mathcal{P}}^k(S, c', c) = -w_{\mathcal{P}}^k(S, c, c')$. The value $w_{\mathcal{P}}^k(S, c, c')$ is the net agree-
445 ment loss that would be incurred by swapping c and c' in a feasible solution r of k -KAP
446 where $\text{rk}(c', r) = \text{rk}(c, r) + 1$ and $S = \text{Below}_{c'}(r) \cup \{c, c'\}$. If $\max_{S \in \Delta_{cc'}^m} w_{\mathcal{P}}^k(S, c, c') \geq 0$ (resp.
447 $\min_{S \in \Delta_{cc'}^m} w_{\mathcal{P}}^k(S, c, c') > 0$) then, in a consensus ranking r for δ_{KT}^k where c and c' would be
448 consecutive, it is possible (resp. necessary) that $c \succ_r c'$.

Definition 5. *The k -wise majority digraph associated to a profile \mathcal{P} over a set C of candidates is the digraph $\mathcal{G}_{\mathcal{P}}^k = (\mathcal{V}, \mathcal{A})$, where $\mathcal{V} = C$ and $(c, c') \in \mathcal{A}$ iff:*

$$\exists S \in \Delta_{cc'}^m \text{ s.t. } w_{\mathcal{P}}^k(S, c, c') > 0.$$

In the weighted k -wise majority digraph, each edge (c, c') is weighted by:

$$w_{\mathcal{P}}^k(c, c') := \max_{S \in \Delta_{cc'}^m} w_{\mathcal{P}}^k(S, c, c').$$

449 Note that, if $k \geq 3$, we may obtain edges (c, c') and (c', c) both with strictly positive
450 weights (which is impossible in the pairwise case). For instance, for the profile \mathcal{P} of Exam-
451 ple 5, $w_{\mathcal{P}}^3(c_3, c_4) = w_{\mathcal{P}}^3(\{c_2, c_3, c_4\}, c_3, c_4) = 1$ and $w_{\mathcal{P}}^3(c_4, c_3) = w_{\mathcal{P}}^3(\{c_3, c_4, c_5\}, c_4, c_3) = 4$. For il-
452 lustration, let us explain how $w_{\mathcal{P}}^3(\{c_3, c_4, c_5\}, c_4, c_3)$ yields 4. Table 2 summarizes the number
453 of times c_4 and c_3 appear in top position of $\{c_3, c_4\}$ or $\{c_3, c_4, c_5\}$ for each ranking r in \mathcal{P} . By
454 summing over $r \in \mathcal{P}$: $\phi_{\mathcal{P}}^3(\{c_3, c_4, c_5\}, c_4, c_3) - \phi_{\mathcal{P}}^3(\{c_3, c_4, c_5\}, c_3, c_4) = (4 \times 2 + 2 + 2) - (4 \times 2) = 4$.

Table 2: Number of times c_4 and c_3 appear in top position of $\{c_3, c_4\}$ or $\{c_3, c_4, c_5\}$ for each ranking r in the profile \mathcal{P} of Example 5.

r	$ \Delta_r^3(\{c_3, c_4, c_5\}, c_4, c_3) $	$ \Delta_r^3(\{c_3, c_4, c_5\}, c_3, c_4) $
$c_1 \succ c_2 \succ c_4 \succ c_3 \succ c_5 \succ c_6$ ($\times 4$)	2	0
$c_1 \succ c_3 \succ c_2 \succ c_4 \succ c_5 \succ c_6$ ($\times 4$)	0	2
$c_6 \succ c_1 \succ c_2 \succ c_4 \succ c_3 \succ c_5$ ($\times 1$)	2	0
$c_6 \succ c_1 \succ c_4 \succ c_3 \succ c_2 \succ c_5$ ($\times 1$)	2	0

455 The obtained weighted digraph $\mathcal{G}_{\mathcal{P}}^3$ is shown on the right of Figure 1 (an efficient manner
456 to compute a set S maximizing $w_{\mathcal{P}}^k(S, c, c')$ will be explained later on page 24). Besides,
457 for any \mathcal{P} , $\mathcal{G}_{\mathcal{P}}^2$ is the pairwise majority digraph as $\Delta_{cc'}^2(S) = \{\{c, c'\}\} \forall S \in \Delta_{cc'}^m$. Theorem 2
458 adapts as follows for an arbitrary k :

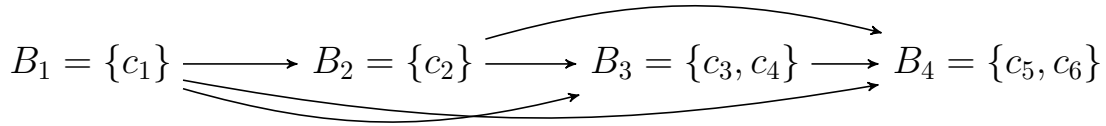


Figure 2: The meta-graph of SCCs of \mathcal{G}_P^3 in Example 5.

459 **Theorem 3.** Let \mathcal{P} be a profile over C and assume that the SCCs of \mathcal{G}_P^k are numbered
 460 according to a linear order $\langle \mathcal{G}_P^k \in \mathcal{O}(\mathcal{G}_P^k) \rangle$. Among the rankings consistent with $\langle \mathcal{G}_P^k \rangle$, there
 461 exists a consensus ranking w.r.t. the k -wise Kemeny rule. Besides, if $\mathcal{O}(\mathcal{G}_P^k) = \{\langle \mathcal{G}_P^k \rangle\}$ and
 462 $\min_{S \in \Delta_{cc'}^m} w_P^k(S, c, c') > 0^1$ for all $c \in B_i(\mathcal{G}_P^k)$ and $c' \in B_j(\mathcal{G}_P^k)$ when $i <_{\mathcal{G}_P^k} j$, then all consensus
 463 rankings are consistent with $\langle \mathcal{G}_P^k \rangle$.

464 *Proof.* Assume that the SCCs of \mathcal{G}_P^k are numbered according to a linear order $\langle \mathcal{G}_P^k \in \mathcal{O}(\mathcal{G}_P^k) \rangle$
 465 and consider a ranking r which is not consistent with $\langle \mathcal{G}_P^k \rangle$. Hence, there exists a pair
 466 (c, c') such that c directly follows c' in r while $c \in B_i(\mathcal{G}_P^k)$ and $c' \in B_j(\mathcal{G}_P^k)$ with $i < j$.
 467 Since $i < j$, there is no arc from c' to c in \mathcal{G}_P^k (i.e., $\forall S \in \Delta_{cc'}^m, w_P^k(S, c, c') \geq 0$). Let S
 468 be the set composed of c' and all candidates placed after c' in r , including c . Then the
 469 ranking $r^{c \leftrightarrow c'}$ obtained from r by exchanging the positions of c and c' verifies $\delta_{\text{KT}}^k(r^{c \leftrightarrow c'}, \mathcal{P}) =$
 470 $\delta_{\text{KT}}^k(r, \mathcal{P}) - w_P^k(S, c, c') \leq \delta_{\text{KT}}^k(r, \mathcal{P})$. The repetition of this argument concludes the proof of
 471 the first claim. The second claim is proved similarly because, in this case, $\delta_{\text{KT}}^k(r^{c \leftrightarrow c'}, \mathcal{P}) =$
 472 $\delta_{\text{KT}}^k(r, \mathcal{P}) - w_P^k(S, c, c') < \delta_{\text{KT}}^k(r, \mathcal{P})$. \square

473 **Example 7.** The meta-graph of SCCs of \mathcal{G}_P^3 in Example 5 is represented in Figure 2.
 474 The above result implies that there exists a consensus ranking among $c_1 \succ c_2 \succ c_3 \succ$
 475 $c_4 \succ c_5 \succ c_6$, $c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_6 \succ c_5$, $c_1 \succ c_2 \succ c_4 \succ c_3 \succ c_5 \succ c_6$ and
 476 $c_1 \succ c_2 \succ c_4 \succ c_3 \succ c_6 \succ c_5$.

To take advantage of Theorem 3, one could try 1) to index the SCCs of \mathcal{G}_P^k according to a
 linear order $\langle \mathcal{G}_P^k \in \mathcal{O}(\mathcal{G}_P^k) \rangle$, and then 2) to work on each SCC separately, before concatenating
 the obtained rankings. However, for a consensus ranking consistent with $\langle \mathcal{G}_P^k \rangle$, the relative
 positions of candidates in $B_i(\mathcal{G}_P^k)$ depend on the set of candidates in $B_{>i}(\mathcal{G}_P^k) := B_{i+1}(\mathcal{G}_P^k) \cup$
 $\dots \cup B_{\sigma(\mathcal{G}_P^k)}(\mathcal{G}_P^k)$ (but not on their order). The influence of $B_{>i}(\mathcal{G}_P^k)$ can be captured in the
 dynamic programming procedure by applying a modified version of Equation 4 separately

¹Or, equivalently, $\max_{S \in \Delta_{cc'}^m} w_P^k(S, c', c) < 0$.

for each subset $B_{\sigma(\mathcal{G}_P^k)}(\mathcal{G}_P^k)$ down to $B_1(\mathcal{G}_P^k)$. Formally, if r^* is optimal for k -KAP, then:

$$\delta_{\text{KT}}^k(r^*, \mathcal{P}) = \sum_{i=1}^{\sigma(\mathcal{G}_P^k)} d_{\text{KT}}^k(B_i(\mathcal{G}_P^k))$$

where, for any subset $S \subseteq B_i(\mathcal{G}_P^k)$, $d_{\text{KT}}^k(S)$ is defined by $d_{\text{KT}}^k(\emptyset) = 0$ and ($B_{>i}$ stands for $B_{>i}(\mathcal{G}_P^k)$):

$$d_{\text{KT}}^k(S) = \min_{c \in S} [d_{\text{KT}}^k(S \setminus \{c\}) + \sum_{r \in \mathcal{P}_{S \cup B_{>i}}} \sum_{c' \succ_r c} \sum_{i=0}^{k-2} \binom{|S| + |B_{>i}| - \text{rk}(c', r) - 1}{i}].$$

477 It amounts to replacing S by $S \cup B_{>i}$ in the second summand of Equation 4 to take into
 478 account the existence of a consensus ranking where all the candidates of $B_{>i}$ are ranked
 479 after those of B_i . Let r_i^* be a ranking of $B_i(\mathcal{G}_P^k)$ such that $\delta_{\text{KT}}^k(r_{\geq i}^*, \mathcal{P}_{B_{\geq i}(\mathcal{G}_P^k)}) = d_{\text{KT}}^k(B_i(\mathcal{G}_P^k)) +$
 480 $\dots + d_{\text{KT}}^k(B_{\sigma(\mathcal{G}_P^k)}(\mathcal{G}_P^k))$, where $r_{\geq i}^*$ is the ranking obtained by the concatenation of rankings
 481 $r_i^*, \dots, r_{\sigma(\mathcal{G}_P^k)}^*$ in this order. The ranking $r_{\geq 1}^*$ of C is a consensus ranking w.r.t. the k -wise
 482 Kemeny rule. Given Theorem 3, the k -wise majority digraph thus seems promising to boost
 483 the computation of a consensus ranking. Unfortunately, the following negative result holds:
 484

485 **Theorem 4.** *Given two candidates c and c' in a profile \mathcal{P} , determining if $\max_{S \in \Delta_{cc'}^m} w_{\mathcal{P}}^k(S, c, c') >$
 486 0 is an NP-complete problem for any constant $k \geq 4$.*

487 *Proof.* For the membership part, note that given a set S , we can check in polynomial time
 488 if $w_{\mathcal{P}}^k(S, c, c') > 0$. Indeed, we can enumerate all sets in $\Delta_{cc'}^k(S)$ and for each ranking $r \in \mathcal{P}$
 489 count how many are in $\Delta_r^k(S, c, c')$ or in $\Delta_r^k(S, c', c)$.

490 For the hardness part, we make a reduction from the set cover problem, known to be
 491 NP-complete [24]:

492 SET COVER PROBLEM

493 *Instance:* A set of elements $\mathcal{X} = \{x_1, \dots, x_p\}$, a collection $\mathcal{T} = \{T_1, \dots, T_q\}$ of sets of
 494 elements of \mathcal{X} , and a positive integer b .

495 *Question:* Does there exist a subcollection $\mathcal{K} \subseteq \mathcal{T}$ of at most b sets that covers \mathcal{X} (i.e., such
 496 that $\bigcup_{T \in \mathcal{K}} T = \mathcal{X}$)?

497 We assume that no set in \mathcal{T} contains \mathcal{X} as otherwise the problem is trivial. Furthermore,

498 we assume that no element in \mathcal{X} is contained in all sets of \mathcal{T} as otherwise this element could
 499 be discarded from the instance as any solution would cover this element. We now detail
 500 the preference profile that we create from an instance of the set cover problem.

501 *Set of candidates:* We will create a profile such that $\max_{S \in \Delta_{cc'}^m} w_{\mathcal{P}}^k(S, c, c') > 0$ iff the
 502 answer to the set cover problem is yes for the instance under consideration. More precisely,
 503 we will show that if \mathcal{X} cannot be covered by a subcollection $\mathcal{K} \subseteq \mathcal{T}$ with less than b sets,
 504 then $w_{\mathcal{P}}^k(S, c, c') < 0$ for all S in $\Delta_{cc'}^m$. Otherwise, if set \mathcal{X} can be covered with a subcollection
 505 $\mathcal{K} \subseteq \mathcal{T}$ with less than b sets, then there exists a set S in $\Delta_{cc'}^m$ such that $w_{\mathcal{P}}^k(S, c, c') > 0$. In
 506 addition to candidates c and c' , for each pair $(x, T) \in \mathcal{X} \times \mathcal{T}$ such that $x \in T$ we create a
 507 candidate $c_{x,T}$. Moreover, for each set $T \in \mathcal{T}$ we create a candidate c_T . In the sequel, we
 508 may call candidates $c_{x,T}$ *element candidates* and candidates c_T *set candidates*. This process
 509 yields at most $pq + 2$ candidates $((p-1)q + q + 2)$. The candidates $c_{x,T}$ and c_T will make the
 510 correspondence with the subcollection \mathcal{K} : the candidate c_T will be in the set S iff $T \in \mathcal{K}$
 511 and the candidate $c_{x,T}$ will be in the set S if T is added to \mathcal{K} in order to cover x .

512 *Set of voters:* For each pair $(x, T) \in \mathcal{X} \times \mathcal{T}$ such that $x \in T$, we create $2b$ voters $v_{x,T}^s$
 513 ($s \in \{1, \dots, 2b\}$) with the same ranking $r_{x,T}$ such that $\mathbf{tail}_4(r_{x,T}) = c \succ c_{x,T} \succ c_T \succ c'$. For
 514 each element $x \in \mathcal{X}$, we create $2b$ voters v_x^s ($s \in \{1, \dots, 2b\}$) with the same ranking r_x such
 515 that $\mathbf{tail}_{u+2}(r_x) = c' \succ c_{x,T_{i_1}} \succ \dots \succ c_{x,T_{i_u}} \succ c$ where $T_{i_1}, T_{i_2}, \dots, T_{i_u}$ are the different sets
 516 in \mathcal{T} that contain x . For each set $T \in \mathcal{T}$ we create $2b|T| + 2$ voters v_T^s ($s \in \{1, \dots, 2b|T| + 2\}$)
 517 with the same ranking r_T such that $\mathbf{tail}_3(r_T) = c' \succ c_T \succ c$. Lastly, we create $2q + 1 + 2b$
 518 voters v^s ($s \in \{1, \dots, 2q + 1 + 2b\}$) with the same ranking r such that $\mathbf{tail}_2(r) = c \succ c'$. In
 519 the end, we obtain $O(bpq)$ voters. Note that \mathcal{P} can be build with no unanimity dominance
 520 relationship between two candidates.

521 We now show that, given $k \geq 4$, we have $\max_{S \in \Delta_{cc'}^m} w_{\mathcal{P}}^k(S, c, c') > 0$ iff the answer to the
 522 set cover problem is yes.

First note that when $S = \{c, c'\}$, we have the following values for $\phi_{\mathcal{P}}^k(S, c, c')$ and $\phi_{\mathcal{P}}^k(S, c', c)$:

$$\begin{aligned} \phi_{\mathcal{P}}^k(\{c, c'\}, c, c') &= 2b \sum_{T \in \mathcal{T}} |T| + 2q + 2b + 1 \\ \phi_{\mathcal{P}}^k(\{c, c'\}, c', c) &= 2b \sum_{T \in \mathcal{T}} |T| + 2q + 2bp. \end{aligned}$$

Hence,

$$\begin{aligned} w_{\mathcal{P}}^k(\{c, c'\}, c, c') &= \phi_{\mathcal{P}}^k(\{c, c'\}, c, c') - \phi_{\mathcal{P}}^k(\{c, c'\}, c', c) \\ &= 2b(1 - p) + 1 < 0, \end{aligned}$$

523 as $p \geq 2$ and $b \geq 1$.

524 Now let us look at how adding an element to S starting from $S = \{c, c'\}$ modifies
 525 values $\phi_{\mathcal{P}}^k(S, c, c')$ and $\phi_{\mathcal{P}}^k(S, c', c)$. We will assume without loss of generality that we add
 526 set candidates before element candidates.

- 527 • If we assume S is only composed of c, c' and set candidates, then adding a candidate
 528 c_T to it results in adding $2b|T|$ to $\phi_{\mathcal{P}}^k(S, c, c')$ and $2b|T| + 2$ to $\phi_{\mathcal{P}}^k(S, c', c)$.
- 529 • If we add a candidate $c_{x,T}$ to S , then we add $2b$ to $\phi_{\mathcal{P}}^k(S, c, c')$ if $c_T \notin S$ and $4b$
 530 otherwise. Additionally, we add $2b$ to $\phi_{\mathcal{P}}^k(S, c', c)$ if there is no other $c_{x,T'} \in S$,
 531 otherwise we add to it something that is greater than or equal to $4b$. Note that we
 532 have used the fact that $k \geq 4$, as we would only add $2b$ to both values if k was equal
 533 to 3.

534 From these observations, we can derive the following rules:

- 535 1. If $\{c_{x,T}, c_{x,T'}\} \subset S$ with $T \neq T'$, then $w_{\mathcal{P}}^k(S, c, c') \leq w_{\mathcal{P}}^k(S \setminus \{c_{x,T'}\}, c, c')$.
- 536 2. If $c_{x,T} \in S$ and $c_T \notin S$, then $w_{\mathcal{P}}^k(S, c, c') \leq w_{\mathcal{P}}^k(S \setminus \{c_{x,T}\}, c, c')$.
- 537 3. If $c_T \in S$ and $\forall x \in T, c_{x,T} \notin S$, then $w_{\mathcal{P}}^k(S, c, c') \leq w_{\mathcal{P}}^k(S \setminus \{c_T\}, c, c')$.

538 Stated differently, while rule 1 states that, to maximize $w_{\mathcal{P}}^k(S, c, c')$, we should keep no more
 539 than one candidate $c_{x,T}$ per element x , rules 2 and 3 state that there should be a candidate
 540 $c_{x,T}$ in S iff there should also be candidate c_T .

541

Let us now consider a set $S \in \Delta_{cc'}^m$ verifying rules 1, 2 and 3 that includes one candidate
 $c_{x,T}$ for s elements $\{x_{i_1}, \dots, x_{i_s}\}$ and v set candidates. Then:

$$\begin{aligned} w_{\mathcal{P}}^k(S, c, c') &= 2b(1 - p) + 1 - 2v + 2bs \\ &= 2b(s - p) + 2(b - v) + 1. \end{aligned}$$

542 Note that if $S \neq \{c, c'\}$, then $v \geq 1$ and that by rule 1, $s \leq p$. If $s < p$, then
 543 $w_{\mathcal{P}}^k(S, c, c') \leq -2b + 2(b - v) + 1 = 1 - 2v < 0$. Hence, if $w_{\mathcal{P}}^k(S, c, c') \geq 0$ then $s = p$,
 544 which further implies that $v \leq b$. In this case, it is easy to see that the v sets corresponding
 545 to the set candidates in S form a valid set cover of \mathcal{X} as $\{x_{i_1}, \dots, x_{i_s}\} = \mathcal{X}$, $v \leq b$ and
 546 S verifies rule 2. To summarize, making the assumption that $w_{\mathcal{P}}^k(S, c, c') \geq 0$, (In fact in
 547 this case $w_{\mathcal{P}}^k(S, c, c') > 0$) we have showed that we could build a valid set cover of \mathcal{X} from
 548 S . Consequently, this implies that if the set cover instance admits no valid set cover, then
 549 $\max_{S \in \Delta_{cc'}^m} w_{\mathcal{P}}^k(S, c, c') < 0$.

550

Conversely, let us assume that there exists a subcollection $\mathcal{K} = \{T_{i_1}, \dots, T_{i_v}\}$ with $v \leq b$
 sets that covers \mathcal{X} . We consider a set S such that $\{c_{T_{i_1}}, \dots, c_{T_{i_v}}, c, c'\} \subset S$ and such that

for each $x \in \mathcal{X}$, S contains exactly one candidate $c_{x,T}$ where $x \in T$ and $T \in \mathcal{K}$. Then, simple calcula show that:

$$\begin{aligned}\phi_{\mathcal{P}}^k(S, c, c') &= 2b \sum_{T \in \mathcal{T}} |T| + 2b \sum_{T \in \mathcal{K}} |T| + 4bp + 2q + 1 + 2b, \\ \phi_{\mathcal{P}}^k(S, c', c) &= 4bp + 2b \sum_{T \in \mathcal{T}} |T| + 2q + 2b \sum_{T \in \mathcal{K}} |T| + 2v.\end{aligned}$$

551 Hence, $w_{\mathcal{P}}^k(S, c, c') \geq 1$ and $\max_{S \in \Delta_{cc}^m} w_{\mathcal{P}}^k(S, c, c') > 0$. □

552 Hence, computing $\mathcal{G}_{\mathcal{P}}^k$ from \mathcal{P} is NP-hard for $k \geq 4$. In contrast, $\mathcal{G}_{\mathcal{P}}^3$ can be computed
553 in polynomial time. Indeed, given a set $S \subset C$ such that $\{c, c'\} \subseteq S$, adding an element
554 $x \notin S$ to S increases $\phi_{\mathcal{P}}^3(S, c, c')$ by one for each $r \in \mathcal{P}$ such that $c \succ_r c'$ and $c \succ_r x$.
555 Let us denote by $\mathcal{P}_{c \succ c'}$ the set $\{r \in \mathcal{P} \text{ s.t. } c \succ_r c'\}$. A set S^* maximizing $w_{\mathcal{P}}^3(S, c, c')$ is
556 $S^* := \{c, c'\} \cup \{x \in C \text{ such that } |\mathcal{P}_{c \succ c'} \cap \mathcal{P}_{c \succ x}| > |\mathcal{P}_{c' \succ c} \cap \mathcal{P}_{c' \succ x}|\}$. For instance, coming
557 back to the weighted digraph $\mathcal{G}_{\mathcal{P}}^3$ shown on the right of Figure 1, a set S^* maximizing
558 $w_{\mathcal{P}}^3(S, c_4, c_3)$ in Example 5 is $\{c_3, c_4, c_5\}$, which yields $w_{\mathcal{P}}^3(S^*, c_4, c_3) = 4$ as reported previ-
559 ously.

560

561 Note that one can take advantage of the meta-graph of SCCs to trim the graph $\mathcal{G}_{\mathcal{P}}^k$ if
562 one looks for a consensus ranking r^* consistent with a specific order $\langle_{\mathcal{G}_{\mathcal{P}}^k} \in \mathcal{O}(\mathcal{G}_{\mathcal{P}}^k)$. It may
563 indeed happen that, for an edge (c, c') , the weight $w_{\mathcal{P}}^k(c, c') = w_{\mathcal{P}}^k(S, c, c') > 0$ corresponds to
564 a set S which contains candidates that will never be below c in r^* . Conversely, the set S
565 may omit candidates that are necessarily below c in r^* . These constraints can be induced
566 by either unanimity dominance relations or by $\langle_{\mathcal{G}_{\mathcal{P}}^k}$. The following example illustrates this
567 idea.

568 **Example 8.** *Let us refine the digraph $\mathcal{G}_{\mathcal{P}}^3$ previously obtained for the profile \mathcal{P} of Example 5.*
569 *The SCCs are $B_1 = \{c_1\}$, $B_2 = \{c_2\}$, $B_3 = \{c_3, c_4\}$ and $B_4 = \{c_5, c_6\}$ and $\mathcal{O}(\mathcal{G}_{\mathcal{P}}^k) = \{\langle_{\mathcal{G}_{\mathcal{P}}^k}\}$,*
570 *where $1 \langle_{\mathcal{G}_{\mathcal{P}}^k} 2 \langle_{\mathcal{G}_{\mathcal{P}}^k} 3 \langle_{\mathcal{G}_{\mathcal{P}}^k} 4$. A set maximizing $w_{\mathcal{P}}^3(S, c_3, c_4)$ is $S = \{c_2, c_3, c_4\}$. This set*
571 *contains c_2 while it is necessarily above c_3 in a consistent ranking. Conversely, candidates*
572 *c_5 and c_6 are omitted while they are necessarily below c_3 . By taking into account these*
573 *constraints, we obtain that a set maximizing $w_{\mathcal{P}}^3(S, c_3, c_4)$ is $S = \{c_3, c_4, c_5, c_6\}$, for which*
574 *$w_{\mathcal{P}}^3(S, c_3, c_4) = -4$. Hence, we can remove the arc (c_3, c_4) from $\mathcal{G}_{\mathcal{P}}^3$. Similarly, it is possible*
575 *to show that the arc (c_6, c_5) can be removed from $\mathcal{G}_{\mathcal{P}}^3$. Thanks to these refinement steps, we*
576 *can conclude that a consensus ranking is $r^* = c_1 \succ c_2 \succ c_4 \succ c_3 \succ c_5 \succ c_6$.*

577 **5. A polynomial time 2-approximation algorithm**

578 In this section, we provide a polynomial time 2-approximation algorithm for problem
 579 k -KAP, in the same spirit as the 2-approximation algorithm by Dwork et al. [9] for the
 580 Kemeny aggregation problem. This latter algorithm relies on the Spearman distance.

Definition 6. *The Spearman distance $\delta_S(r, r')$ between two rankings r and r' is defined by*

$$\delta_S(r, r') = \sum_{c \in C} |\mathbf{rk}(c, r) - \mathbf{rk}(c, r')|.$$

The *Diaconis-Graham inequality* [25] states that the Kendall tau and Spearman distances remain within a constant factor of each other for all pairs of rankings, namely:

$$\delta_{\text{KT}}(r, r') \leq \delta_S(r, r') \leq 2 \delta_{\text{KT}}(r, r').$$

581 Note that the right bound is tight, as witnessed by the two rankings $c_1 \succ_r c_2$ and $c_2 \succ_{r'} c_1$,
 582 for which $\delta_{\text{KT}}(r, r') = 1$ and $\delta_S(r, r') = 2$.

Dwork et al. [9] have shown that a ranking r_S minimizing the sum of Spearman distances to the rankings in a preference profile \mathcal{P} (i.e., $\sum_{r' \in \mathcal{P}} \delta_S(r_S, r') = \min_r \sum_{r' \in \mathcal{P}} \delta_S(r, r')$) can be computed in polynomial time via a minimum cost matching algorithm. As a consequence of the Diaconis-Graham inequality, the sum of Kendall tau distances between r_S and the rankings in \mathcal{P} is in the worst case twice that of an optimal ranking for the Kemeny rule. Denoting by r_{KT} an optimal ranking for the Kemeny rule on \mathcal{P} , we have indeed:

$$\sum_{r' \in \mathcal{P}} \delta_{\text{KT}}(r_S, r') \leq \sum_{r' \in \mathcal{P}} \delta_S(r_S, r') \leq \sum_{r' \in \mathcal{P}} \delta_S(r_{\text{KT}}, r') \leq 2 \sum_{r' \in \mathcal{P}} \delta_{\text{KT}}(r_{\text{KT}}, r').$$

583 In order to obtain the same kind of result for the k -wise Kendall tau distance, we
 584 introduce a k -wise variant of the Spearman distance.

Definition 7. *The k -wise Spearman distance $\delta_S^k(r, r')$ between two rankings r and r' is defined by:*

$$\delta_S^k(r, r') = \sum_{c \in C} \sum_{i=\min\{\mathbf{rk}(c, r), \mathbf{rk}(c, r')\}}^{\max\{\mathbf{rk}(c, r), \mathbf{rk}(c, r')\}-1} N_{k-2}^{m-i-1}$$

585 where $N_k^p = \sum_{i=0}^k \binom{p}{i}$ is the number of subsets of size less than or equal to k within a set of
 586 size p .

This definition is motivated by the fact that N_{k-2}^{m-i-1} corresponds to the k -wise Kendall tau distances between a ranking and the ranking obtained by swapping candidates of ranks i and $i + 1$. Note that $\delta_S^2 = \delta_S$. We have indeed:

$$\begin{aligned} \sum_{i=\min\{\mathbf{rk}(c, r), \mathbf{rk}(c, r')\}}^{\max\{\mathbf{rk}(c, r), \mathbf{rk}(c, r')\}-1} N_0^{m-i-1} &= \max\{\mathbf{rk}(c, r), \mathbf{rk}(c, r')\} - \min\{\mathbf{rk}(c, r), \mathbf{rk}(c, r')\} \\ &= |\mathbf{rk}(c, r) - \mathbf{rk}(c, r')|. \end{aligned}$$

587 because $N_0^{m-i-1} = 1$ for all i in the summation.

588 We now state and prove the extension of the Diaconis-Graham inequality to our k -wise
589 variants:

590 **Lemma 3.** *For any two rankings r and r' : $\delta_{\text{KT}}^k(r, r') \leq \delta_{\text{S}}^k(r, r') \leq 2\delta_{\text{KT}}^k(r, r')$.*

Proof. We first prove that $\delta_{\text{KT}}^k(r, r') \leq \delta_{\text{S}}^k(r, r')$. We recall that:

$$\delta_{\text{KT}}^k(r, r') = |\{S \in \Delta^k : \text{top}_r(S) \neq \text{top}_{r'}(S)\}|$$

which can be reformulated:

$$\delta_{\text{KT}}^k(r, r') = \sum_{c \in C} |\{S \in \Delta^k : \text{top}_r(S) = c \text{ and } \text{top}_{r'}(S) \neq c\}|.$$

591 Let us denote by $\Delta_c^k(r, r')$ the set $\{S \in \Delta^k : \text{top}_r(S) = c \text{ and } \text{top}_{r'}(S) \neq c\}$. For any
592 $S \in \Delta_c^k(r, r')$, one of the followings holds true:

- 593 • $\text{rk}(\text{top}_r(S), r) \leq \text{rk}(\text{top}_{r'}(S), r') < \text{rk}(\text{top}_r(S), r')$,
- 594 • or $\text{rk}(\text{top}_{r'}(S), r) > \text{rk}(\text{top}_r(S), r) \geq \text{rk}(\text{top}_{r'}(S), r')$,
- 595 • or both (if $\text{rk}(\text{top}_r(S), r) = \text{rk}(\text{top}_{r'}(S), r')$).

$$\begin{aligned} \text{Consequently: } \delta_{\text{KT}}^k(r, r') &\leq \sum_{c \in C} |\{S \in \Delta_c^k(r, r') : \text{rk}(c, r) \leq \text{rk}(\text{top}_{r'}(S), r') < \text{rk}(c, r')\}| \\ &\quad + \sum_{c \in C} |\{S \in \Delta_c^k(r, r') : \text{rk}(\text{top}_{r'}(S), r) > \text{rk}(c, r) \geq \text{rk}(\text{top}_{r'}(S), r')\}| \end{aligned}$$

596 because the union of the two sets is $\Delta_c^k(r, r')$ and $|A \cup B| \leq |A| + |B|$ for any two sets A
597 and B .

On the one hand, we have:

$$\begin{aligned} &\sum_{c \in C} |\{S \in \Delta_c^k(r, r') : \text{rk}(c, r) \leq \text{rk}(\text{top}_{r'}(S), r') < \text{rk}(c, r')\}| \\ &\leq \sum_{\substack{c \in C \\ \text{rk}(c, r) < \text{rk}(c, r')}} \sum_{i=\text{rk}(c, r)}^{\text{rk}(c, r')-1} N_{k-2}^{m-i-1}. \end{aligned}$$

598 The upper bound results from the fact that each candidate c' such that $\text{rk}(c, r) \leq \text{rk}(c', r') <$
599 $\text{rk}(c, r')$ belongs to at most $N_{k-2}^{m-\text{rk}(c', r')-1}$ subsets $S \in \Delta_c^k(r, r')$ such that $\text{top}_{r'}(S) = c'$.

On the other hand, we have:

$$\begin{aligned}
& \sum_{c \in C} |\{S \in \Delta_c^k(r, r') : \mathbf{rk}(\mathbf{top}_{r'}(S), r) > \mathbf{rk}(c, r) \geq \mathbf{rk}(\mathbf{top}_{r'}(S), r')\}| \\
&= \sum_{c' \in C} |\{S \in \Delta_{c'}^k(r', r) : \mathbf{rk}(c', r) > \mathbf{rk}(\mathbf{top}_r(S), r) \geq \mathbf{rk}(c', r')\}| \\
&\leq \sum_{\substack{c' \in C \\ \mathbf{rk}(c', r) > \mathbf{rk}(c', r')}} \sum_{i=\mathbf{rk}(c', r')}^{\mathbf{rk}(c', r)-1} N_{k-2}^{m-i-1} = \sum_{\substack{c \in C \\ \mathbf{rk}(c, r) > \mathbf{rk}(c, r')}} \sum_{i=\mathbf{rk}(c, r')}^{\mathbf{rk}(c, r)-1} N_{k-2}^{m-i-1}.
\end{aligned}$$

600 The first equality results from the fact that varying c and considering subsets S such that
601 $\mathbf{top}_r(S) = c$ and $\mathbf{rk}(\mathbf{top}_{r'}(S), r) > \mathbf{rk}(c, r) \geq \mathbf{rk}(\mathbf{top}_{r'}(S), r')$ is equivalent to varying c' and
602 considering subsets S such that $\mathbf{top}_{r'}(S) = c'$ and $\mathbf{rk}(c', r) > \mathbf{rk}(\mathbf{top}_r(S), r) \geq \mathbf{rk}(c', r')$.

Hence:

$$\begin{aligned}
\delta_{\text{KT}}^k(r, r') &\leq \sum_{\substack{c \in C \\ \mathbf{rk}(c, r) < \mathbf{rk}(c, r')}} \sum_{i=\mathbf{rk}(c, r)}^{\mathbf{rk}(c, r)-1} N_{k-2}^{m-i-1} + \sum_{\substack{c \in C \\ \mathbf{rk}(c, r) > \mathbf{rk}(c, r')}} \sum_{i=\mathbf{rk}(c, r')}^{\mathbf{rk}(c, r)-1} N_{k-2}^{m-i-1} \\
&= \sum_{c \in C} \sum_{i=\min\{\mathbf{rk}(c, r), \mathbf{rk}(c, r')\}}^{\max\{\mathbf{rk}(c, r), \mathbf{rk}(c, r')\}-1} N_{k-2}^{m-i-1} = \delta_{\text{S}}^k(r, r').
\end{aligned}$$

603 We now prove that $\delta_{\text{S}}^k(r, r') \leq 2\delta_{\text{KT}}^k(r, r')$. In this purpose, we consider a sequence $r_0 =$
604 $r, r_1, \dots, r_\delta = r'$ of rankings, where r_{j+1} is obtained from r_j by swapping in r_j the candidate
605 $c'_j = \mathbf{top}_{r'}(\{c \in C : \mathbf{rk}(c, r_j) \neq \mathbf{rk}(c, r')\})$ with the previous candidate in r_j , denoted by c_j .
606 This is like doing an in-place selection sort w.r.t. the order defined by r' , moving c'_j to its
607 place in r' by successive swaps. Note the following things:

- 608 • at each step $\mathbf{rk}(c'_j, r') < \mathbf{rk}(c'_j, r_j)$ by definition of c'_j ;
- 609 • $\delta = \delta_{\text{KT}}(r, r')$ because each swap of consecutive candidates in r_j decreases by exactly
610 one the number of pairwise disagreements between r_j and r' ($c'_j \succ_{r'} c_j$ by definition
611 of c'_j).

612 The proof is in two steps:

(i) We show that $\delta_{\text{KT}}^k(r_j, r') = \delta_{\text{KT}}^k(r_j, r_{j+1}) + \delta_{\text{KT}}^k(r_{j+1}, r')$. By induction, an immediate consequence is that:

$$\delta_{\text{KT}}^k(r, r') = \sum_{j=0}^{\delta-1} \delta_{\text{KT}}^k(r_j, r_{j+1}).$$

613 (ii) We define $D_j = \delta_{\text{S}}^k(r_j, r')$. We show that $D_j - D_{j+1} \leq 2\delta_{\text{KT}}^k(r_j, r_{j+1})$, which implies:
614 $\delta_{\text{S}}^k(r, r') = \sum_{j=0}^{\delta-1} (D_j - D_{j+1}) \leq 2 \sum_{j=0}^{\delta-1} \delta_{\text{KT}}^k(r_j, r_{j+1}) = 2\delta_{\text{KT}}^k(r, r')$.

Proof of (i). We have the following sequence of equalities:

$$\begin{aligned}
& \delta_{\text{KT}}^k(r_j, r') - \delta_{\text{KT}}^k(r_{j+1}, r') \\
&= |\{S \in \Delta^k : \text{top}_{r_j}(S) \neq \text{top}_{r'}(S)\}| - |\{S \in \Delta^k : \text{top}_{r_{j+1}}(S) \neq \text{top}_{r'}(S)\}| \\
&= |\{S \in \Delta^k : \text{top}_{r_j}(S) = c_j \text{ and } \text{top}_{r'}(S) = c'_j\}| \\
&\text{(because } r_j \text{ and } r_{j+1} \text{ only differ in the order of } c_j \text{ and } c'_j\text{)} \\
&= |\{S \in \Delta^k : \{c_j, c'_j\} \subseteq S \text{ and } S \setminus \{c_j, c'_j\} \subseteq \text{Below}_{c'_j}(r_j)\}| \\
&\text{(because } c'_j \succ_{r'} c, \forall c \in \text{Below}_{c'_j}(r_j)\text{, which implies that } \text{Below}_{c_j}(r_j) \subseteq \text{Below}_{c'_j}(r')\text{)} \\
&= |\{S \in \Delta^k : \text{top}_{r_j}(S) \neq \text{top}_{r_{j+1}}(S)\}| \\
&\text{(because } r_j \text{ and } r_{j+1} \text{ only differ in the order of } c_j \text{ and } c'_j\text{)} \\
&= \delta_{\text{KT}}^k(r_j, r_{j+1}).
\end{aligned}$$

615 The result follows.

Proof of (ii). As r_j and r_{j+1} only differ in the ranks of c_j and c'_j , the difference $D_j - D_{j+1}$ is equal to:

$$\sum_{c \in \{c_j, c'_j\}} \left(\sum_{i=\min\{\text{rk}(c, r_j), \text{rk}(c, r')\}}^{\max\{\text{rk}(c, r_j), \text{rk}(c, r')\}-1} N_{k-2}^{m-i-1} - \sum_{i=\min\{\text{rk}(c, r_{j+1}), \text{rk}(c, r')\}}^{\max\{\text{rk}(c, r_{j+1}), \text{rk}(c, r')\}-1} N_{k-2}^{m-i-1} \right).$$

Note that $\text{rk}(c_j, r_{j+1}) = \text{rk}(c_j, r_j) + 1$ and $\text{rk}(c'_j, r_{j+1}) = \text{rk}(c'_j, r_j) - 1$ and $\text{rk}(c'_j, r') \leq \text{rk}(c'_j, r_{j+1})$. By swapping c_j and c'_j in r_j , the k -wise Spearman distance to r' will thus decrease with respect to c'_j by $N_{k-2}^{m-\text{rk}(c'_j, r_j)}$. Regarding c_j , it may increase by $N_{k-2}^{m-\text{rk}(c_j, r_j)-1}$ if $\max\{\text{rk}(c_j, r_j), \text{rk}(c_j, r')\} = \text{rk}(c_j, r_j)$, or decrease by $N_{k-2}^{m-\text{rk}(c_j, r_j)-1}$ otherwise. Hence, the largest possible decrease in δ_{S}^k occurs in this latter case. We derive from this short analysis:

$$\begin{aligned}
D_j - D_{j+1} &\leq N_{k-2}^{m-\text{rk}(c'_j, r_j)} + N_{k-2}^{m-\text{rk}(c_j, r_j)-1} \\
&= 2N_{k-2}^{m-\text{rk}(c'_j, r_j)} \text{ (as } \text{rk}(c'_j, r_j) = \text{rk}(c_j, r_j) + 1\text{)} \\
&= 2\delta_{\text{KT}}^k(r_j, r_{j+1})
\end{aligned}$$

616 The last line results from the fact that, as stated previously, N_{k-2}^{m-i-1} corresponds to the
617 k -wise Kendall tau distance between a ranking and the ranking obtained by swapping
618 candidates of ranks i and $i + 1$. \square

619 The following result states that a consensus ranking for the k -wise Spearman distance
620 can be computed in polynomial time in the numbers n of voters and m of candidates:

621 **Lemma 4.** *A consensus ranking for the k -wise Spearman distance can be computed in time*
 622 *$O(nm^3)$, for a preference profile with n voters and m candidates.*

623 *Proof.* Consider the complete bipartite graph with two independent sets V_1 and V_2 , where:

- 624 • each vertex in V_1 corresponds to a candidate in C ,
- 625 • each vertex in V_2 corresponds to a position in $\{1, \dots, m\}$.

The edge $\{c, p\}$ between $c \in V_1$ and $p \in V_2$ is weighted by $\sum_{r' \in \mathcal{P}} \sum_{i=\min\{p, \text{rk}(c, r')\}}^{\max\{p, \text{rk}(c, r')\}-1} N_{k-2}^{m-i-1}$. Each perfect matching corresponds to a ranking r , where $\text{rk}(c, r) = p$ if edge $\{c, p\}$ belongs to the matching. Determining a minimum weight matching in this graph thus amounts to solve the following minimization problem:

$$\min_r \sum_{c \in C} \sum_{r' \in \mathcal{P}} \sum_{i=\min\{\text{rk}(c, r), \text{rk}(c, r')\}}^{\max\{\text{rk}(c, r), \text{rk}(c, r')\}-1} N_{k-2}^{m-i-1}.$$

The two first sum operators can be swapped, which yields:

$$\min_r \sum_{r' \in \mathcal{P}} \sum_{c \in C} \sum_{i=\min\{\text{rk}(c, r), \text{rk}(c, r')\}}^{\max\{\text{rk}(c, r), \text{rk}(c, r')\}-1} N_{k-2}^{m-i-1} = \min_r \sum_{r' \in \mathcal{P}} \delta_S^k(r, r').$$

626 Thus a minimum weight matching corresponds to a consensus ranking for the k -wise Spearman
 627 distance. Such a matching can be computed in time $O(|V_1|^3)$ by the Hungarian algo-
 628 rithm, thus in $O(m^3)$ as $|V_1| = m$. The computation of the complete bipartite graph itself
 629 is performed in $O(nm^3)$ (time required for computing the weights of edges). The overall
 630 complexity is therefore $O(nm^3)$. □

631 From Lemma 3, which implies that any consensus ranking for the k -wise Spearman
 632 distance is a 2-approximation of an optimal consensus ranking for the k -wise Kemeny rule,
 633 and Lemma 4, which establishes that a consensus ranking for the k -wise Spearman distance
 634 can be computed in $O(nm^3)$, we deduce the main result of this section:

635 **Theorem 5.** *There exists a 2-approximation algorithm for k -KAP which runs in $O(nm^3)$*
 636 *time.*

637 Note that determining whether the bound 2 is tight remains an open problem (as is the
 638 case for $k=2$, to our knowledge).

639 **6. Numerical Tests**

640 The numerical tests² we carried out had several objectives:

- 641 • to evaluate the computational performance of the dynamic programming approach of
642 Section 3,
- 643 • to evaluate the impact of parameter k on the set of consensus rankings,
- 644 • to assess the efficiency of the preprocessing technique of Section 4,
- 645 • to study the practical approximation ratio of the polynomial-time 2-approximation
646 algorithm proposed in Section 5.

647 *Generation of preference profiles.* The preference profiles are generated according to the
648 Mallows model [26], using the Python package PrefLib-Tools [27] in most tests (the PerMal-
649 lows R package [28] was used for the tests of the 2-approximation algorithm). This model
650 takes two parameters as input: a reference ranking σ (the mode of the distribution) and a
651 dispersion parameter $\phi \in (0, 1)$. Given these inputs, the probability of generating a ranking
652 r is proportional to $\phi^{\delta_{\text{KT}}(r, \sigma)}$. The more ϕ tends towards 0 (resp. 1), the more the preference
653 rankings become correlated and resemble σ (resp. become equally probable, i.e., we are
654 close to the *impartial culture assumption*). This model enables us to measure in a simple
655 way how the level of correlation in the input rankings impacts our results. In all tests, the
656 number n of voters is set to 50 and the ranking σ is set arbitrarily as the k -wise Kemeny rule
657 is neutral. For each triple (m, k, ϕ) considered, the results are averaged over 50 preference
658 profiles.

659 *Practicability of the dynamic programming approach.* We first evaluate our dynamic pro-
660 gramming approach on instances with different values for m and k . Note that the com-
661 putational performance measured here is not impacted by the level of correlation in the
662 input rankings as it does not change the number of states in dynamic programming nor the
663 computation time to determine the optimal value in each state. Hence, we only consider
664 instances generated under the impartial culture assumption, i.e., with $\phi \approx 1$.

665 Table 3 (Rows 3-5) displays the average, max and min running times obtained for some
666 representative (m, k) values. As expected, the running times increase exponentially with
667 m . Conversely, parameter k seems to have a moderate impact on the running times. The
668 dynamic programming approach enables us to solve k -KAP in a time of up to 3 sec. (resp.
669 76 sec.) for $m \leq 14$ (resp. $m \leq 18$).

²Implementation in C++, except for the polynomial-time 2-approximation algorithm implemented in Python3. [Unless otherwise stated](#), all times are CPU seconds on an Intel Core I7-8700 3.20 GHz processor with 16GB of RAM.

670 *Influence of k on the set of consensus rankings.* Second, we study the impact of k on the
671 set of optimal solutions to k -KAP. Indeed, one criticism for the Kemeny rule is that there
672 exists instances for which the set of consensus rankings is compounded of many solutions
673 which are quite different from one another. Thus, we investigate if increasing k helps
674 in mitigating this issue. For this purpose, we consider the same instances as before and
675 compute the average number of consensus rankings denoted by $|\mathcal{R}^*|_{\text{avg}}$. The results are
676 displayed in the sixth row of Table 3. Interestingly, this measure decreases quickly with
677 k . For instance, when $m = 18$, $|\mathcal{R}^*|_{\text{avg}}$ is divided by 5 when k increases from 2 to 9 and is
678 below 2 when $k = m$. The intuition is that δ_{KT}^k becomes more fine-grained as k increases.

Table 3: Average, max and min CPU times in seconds of the dynamic programming approach of Section 3 for varying values of m and k (Rows 3 to 5). Average number of consensus rankings for increasing values of m and k (Row 6).

m	6			10			14			18		
k	2	3	6	2	5	10	2	7	14	2	9	18
Average time	<0.01	<0.01	<0.01	0.07	0.08	0.08	2.52	2.54	2.61	70.93	72.26	74.95
Max time	<0.01	<0.01	<0.01	0.80	0.08	0.09	2.64	2.60	2.64	71.57	73.57	75.38
Min time	<0.01	<0.01	<0.01	0.70	0.07	0.08	2.49	2.49	2.57	70.27	71.91	74.33
$ \mathcal{R}^* _{\text{avg}}$	3.00	1.20	1.05	3.84	1.24	1.10	5.36	2.36	1.16	19.70	4.12	1.47

679 *Impact of the 3-wise majority graph.* We now study the impact of the preprocessing method
680 proposed in Section 4 for $k = 3$. This preprocessing uses the k -wise majority digraph
681 to divide k -KAP into several subproblems which can be solved separately by dynamic
682 programming. Hopefully, when voters' preferences are correlated (i.e., for "small" ϕ values),
683 these subproblems become smaller and more numerous, making the preprocessing more
684 efficient. The results are shown in Table 4, where the results obtained without preprocessing
685 are also given in the last column. The obtained running times are highly dependent on ϕ .
686 For instance, with $m = 18$, the average running time for solving 3-KAP is above 1 minute
687 if $\phi = 0.95$ while it is below 1 second if $\phi \leq 0.85$. This gap is necessarily related to the
688 preprocessing step, since ϕ has no impact on the running time of the dynamic programming
689 approach. To explain this significant speed-up, we display in Table 5 the average size
690 of the largest SCC of the 3-wise majority digraph at the end of the preprocessing step.
691 Unsurprisingly, this average size turns out to be correlated with ϕ : when $\phi \leq 0.5$, the size of
692 the largest SCC is almost always 1. Hence, the preprocessing step is likely to yield directly
693 a consensus ranking. In contrast, when $\phi = 0.95$, the average size of the largest SCC is close
694 to m , thus the impact of the preprocessing is low.

695

Table 4: Average, max and min CPU times (in seconds) for the 3-wise Kemeny rule with preprocessing.

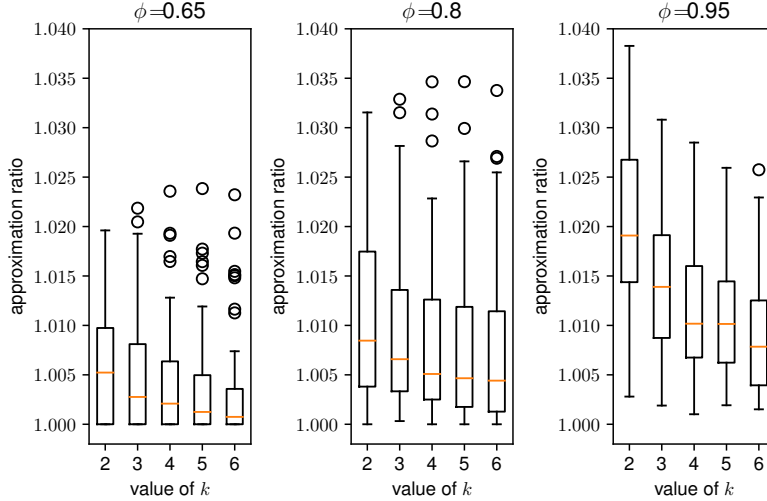
m	ϕ	0.5	0.8	0.85	0.9	0.95	w/o preproc.
6	Avg time	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
	Max time	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
	Min time	<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
10	Avg time	0.03	0.03	0.04	0.07	0.10	0.07
	Max time	0.03	0.03	0.10	0.15	0.17	0.08
	Min time	0.03	0.03	0.03	0.03	0.03	0.07
14	Avg time	0.09	0.09	0.11	0.94	2.21	2.52
	Max time	0.09	0.13	0.25	3.14	3.26	2.59
	Min time	0.08	0.08	0.09	0.10	0.26	2.49
18	Avg time	0.20	0.20	0.55	14.87	61.72	71.17
	Max time	0.31	0.21	8.46	79.87	80.11	71.61
	Min time	0.19	0.19	0.19	0.22	6.02	71.02

Table 5: Average size of the largest SCC after preprocessing.

$m \setminus \phi$	0.47	0.81	0.85	0.88	0.95
6	<1.10	1.84	1.88	2.72	3.28
10	<1.10	1.64	3.28	5.32	8.20
14	<1.10	2.68	3.84	9.12	12.91
18	<1.10	2.84	4.27	9.80	17.44

696 *Polynomial-time 2-approximation algorithm.* Lastly, we study the practical approximation
697 ratio of the proposed polynomial-time 2-approximation algorithm. The results are averaged
698 over 50 instances randomly generated according to the Mallows model, for 50 voters and
699 12 candidates. Preliminary tests not reported here tended to show that this approximation
700 ratio does not depend on the number of candidates. Furthermore, the approximation ratio
701 becomes better when the number of voters increases which can be explained by the fact
702 that the law of large numbers make dominant the reference ranking in the Mallows model,
703 regardless of the aggregation rule. We report in Figure 3 the box plots obtained by varying
704 the values of k and ϕ (the closer to 1, the closer to the impartial culture assumption). **Note**
705 **that the CPU time to compute a consensus ranking for the k -wise Spearman distance (for**
706 **50 voters and 12 candidates) is below 12 milliseconds whatever the values of k and ϕ , using**
707 **the `linear_sum_assignment` function of the SciPy Python library on an Intel Core i5 2.3**
708 **GHz dual core processor with 8GB of RAM.** It is interesting to observe that the practical
709 approximation ratio is much better than 2: the worst ratio over all generated instances is
710 1.04. For the same reason as for the number of voters, the more correlated the preferences
711 in the profile, the better the approximation ratio. Furthermore, the greater the value of k ,
712 the better the approximation ratio, as is clear from the curves obtained.

Figure 3: Practical approximation ratio of the 2-approximation algorithm.



7. Conclusion

In this paper, we advocate using the results of *setwise* contests between candidates to design social welfare functions that are less myopic than those only based on pairwise comparisons. One natural such social welfare function is a k -wise generalization of the Kemeny rule, which returns a ranking minimizing the number of disagreements on top candidates of sets of cardinality lower than or equal to k . We have studied this k -wise Kemeny rule from both axiomatic and algorithmic viewpoints. In more detail, we established that determining a consensus ranking is NP-hard for any $k \geq 3$. Then, after proposing a dynamic programming procedure, we have investigated a k -wise variant of the majority graph, from which we developed a preprocessing step. Computing this graph is a polynomial time problem for $k = 3$ but becomes NP-hard for $k \geq 4$. The numerical tests show the practicability of the approach for up to 18 candidates. Lastly, we have designed a 2-approximation algorithm for the k -wise Kemeny aggregation problem. This approximation algorithm in fact returns a ranking minimizing a k -wise variant of the Spearman distance, and the worst case approximation ratio is then derived from an adaptation of the Diaconis-Graham inequality. The numerical tests suggest that, in practice, the k -wise Kemeny score of the returned ranking is often much better than a 2-approximation of the optimal score.

Several research directions could be further investigated. One of them would be to investigate the complexity of determining a consensus ranking for δ_{KT}^k when $k = m$, because our hardness result only holds for *fixed* values of k . Secondly, note that δ_{KT}^k focuses on “small” sets as we count disagreements on sets of size lower than or equal to k . An opposite

734 viewpoint would be to consider “large” sets by counting disagreements on sets of size greater
735 than or equal to $m - k$. Note that for $k = 0$ this would lead to a rule similar to plurality.
736 Another direction is to study the complexity of recognition problems related to the k -wise
737 Kemeny rule [29], i.e., deciding if a given ranking is a consensus ranking for the k -wise
738 Kemeny rule. Alternative definitions of k -wise majority graphs that are easier to compute
739 for $k > 3$ are also worth investigating. Finally, other social welfare functions based on the
740 results of setwise contests are worth investigating in our opinion, both from the axiomatic
741 and the computational points of view.

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