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► **To cite this version:**

| Gerrit Coddens. Quantum mechanics with only real numbers. 2022. hal-03559591v2

**HAL Id: hal-03559591**

**<https://hal.science/hal-03559591v2>**

Preprint submitted on 11 Feb 2022

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# Quantum mechanics with only real numbers

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4th February 2022

**Abstract.** We show that every Clifford algebra can be transformed into a formalism that uses only real numbers. The rules we must follow to achieve this are universal. To illustrate the method we construct the real pendant of  $SU(2)$  and we derive a version of the Dirac equation that uses only real numbers. The advantage of the complex formulations is that they reduce the size of the representation matrices by a factor of 2. The reason for the presence of complex numbers in quantum mechanics is that it is formulated in the language of group representation theory.

**PACS.** 03.65.Ta, 03.65.Ud, 03.67.-a

## 1 Introduction

It has recently been claimed that quantum mechanics cannot be formulated without the use of complex numbers [1]. But the proof was built on Bell inequalities and we have shown that an error occurs when we apply these inequalities to physics [2]. This could affect the conclusions reached, but we have not checked this. The proof was further built on the assumption that quantum mechanics is always carried out in a Hilbert space, which is not true [3].

Rather than addressing these possible issues, we will show in this paper that we can write the Dirac equation and its wave function with only real numbers. However, this entails more sizeable, less practical calculations, such that it is only of academic interest. In fact, in the Clifford algebra, the dimension  $D$  of the complex representation matrices for a group of metric or pseudo-metric preserving transformations of  $\mathbb{R}^d$  is given by  $D = 2^{\lfloor d/2 \rfloor}$  while for real representation matrices it is given by  $D = 2^{\lfloor d/2 \rfloor + 1}$ . Here  $\lfloor \cdot \rfloor$  is the largest integer part. For the group of three-dimensional rotations this yields  $2^{\lfloor 3/2 \rfloor} = 2$  and  $2^{\lfloor 3/2 \rfloor + 1} = 4$ , while for the homogeneous Lorentz group of space-time it yields  $2^{\lfloor 4/2 \rfloor} = 4$  and  $2^{\lfloor 4/2 \rfloor + 1} = 8$ . The use of complex numbers permits thus to keep the calculations more compact. It transpires from our work that the reason for the presence of complex numbers in the basic equations of quantum mechanics is that it is based on group representation theory derived from Clifford algebra in this more compact complex form.

In many textbooks one starts introducing complex numbers by simply stating that  $i$  is defined by  $i^2 = -1$ . In other words it is the solution of the equation  $x^2 = -1$ . A normal person should have a conceptual problem with this definition. He should be asking for an existence proof. It looks like defining an elephant with six legs and four wings. Does something like that really exist? Furthermore this definition is ambiguous, because  $-i$  is defined by the same equation  $x^2 = -1$ . Afterwards one introduces numbers  $a + bi$  as polynomials in  $i$  and one teaches how to work with them. One just has to “shut up and calculate!” The latter is rather simple such that the initial objections are soon forgotten. One has learned this way the trick of attitudinizing that one has understood it. Or at least, one can figure that it does not matter that one does not understand it, as one can accomplish all the tasks needed successfully and it churns out useful results.

The following describes the real mathematics of complex numbers (see e.g. [4]). We start by defining a sum and a product on the set  $\mathbb{R}^2$ .

$$\forall (x_1, y_1) \in \mathbb{R}^2, \forall (x_2, y_2) \in \mathbb{R}^2 : (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2). \quad (1)$$

$$\forall (x_1, y_1) \in \mathbb{R}^2, \forall (x_2, y_2) \in \mathbb{R}^2 : (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1). \quad (2)$$

Next we consider the subset  $\mathcal{R} \subset \mathbb{R}^2$  defined by  $\mathcal{R} = \{(x_1, y_1) \in \mathbb{R}^2 \mid y_1 = 0\}$ . This subset is closed under the operations  $+$  and  $\cdot$  of  $\mathbb{R}^2$ :

$$\forall (x_1, 0) \in \mathcal{R}, (x_2, 0) \in \mathcal{R}, (x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) \in \mathcal{R}, (x_1, 0) \cdot (x_2, 0) = (x_1 x_2, 0) \in \mathcal{R}, \quad (3)$$

The function  $f : \mathcal{R} \rightarrow \mathbb{R} : (x_1, 0) \rightarrow f((x_1, 0)) = x_1$  is bijective, and it is an isomorphism between  $(\mathbb{R}, +, \cdot)$  and  $(\mathcal{R}, +, \cdot)$ :

$$f[(x_1, 0) + (x_2, 0)] = f[(x_1, 0)] + f[(x_2, 0)], \quad f[(x_1, 0) \cdot (x_2, 0)] = f[(x_1, 0)] \cdot f[(x_2, 0)]. \quad (4)$$

The isomorphism  $f$  maps also neutral elements onto neutral elements and inverse elements onto inverse elements. We can write this isomorphism as:  $(x_1, 0) \sim x_1$ ,  $\mathcal{R} \sim \mathbb{R}$ . We can thus identify  $\mathcal{R}$  with  $\mathbb{R}$ , just like you mentally identify rotation matrices and rotations. They are “just the same thing” (by isomorphism). Furthermore, we notice that:

$$(0, 1) \cdot (0, 1) = (-1, 0) \sim -1. \quad (5)$$

Hence  $(0, 1)$  acts as though it were a square root of  $-1$ . We introduce the notation  $(0, 1) \sim \iota$ ,  $(x_1, y_1) = (x_1, 0) \cdot (1, 0) + (y_1, 0) \cdot (0, 1) \sim x_1(1, 0) + y_1(0, 1) \sim x_1 + \iota y_1$ .

There is thus an underlying construction that justifies the introduction of  $\iota^2 = -1$ . This underlying construction is so to say the subject matter of Chapter 1 of the textbook. And introducing the notion that  $\iota^2 = -1$  corresponds the rest of the textbook starting from Chapter 2. What most people are taught consists in just skipping Chapter 1, with the result that they feel puzzled by the approach. The construction in this section can be used to translate any complex-valued formalism back to a real-valued one, which by definition it always was. This is a mathematical fact that comes prior to any applications in physics. We will illustrate this below for  $SU(2)$  and the Dirac equation.

The rest of this note depends heavily on references [2]-[3],[5]. It is the development of  $SU(2)$  in [5] that will be used in this note, such that mastering this development is a must. The reader must start from the assumption that he has no clue about its contents. In fact, reference [5] gives a clear explanation of spinors in  $SU(2)$  and uses the understanding gained this way to derive the Dirac equation deductively from scratch. In other words, we are now able to look inside the formalism and know what it means. Dirac and Schrödinger obtained their equations by guessing, with the effect that some of their interpretations of the formalism were not exact. Reference [3] is an addition to [5] and explains why spinors do not form a vector space, as stated in [5]. This is very important and entirely correct, even if this flies in the face of accepted notions.<sup>1</sup> It has dire consequences for what we will develop in this note.

An equally crucial paper is reference [2]. We have shown in [2] that the Bell inequalities cannot be applied to the experiments they are associated with. Therefore entanglement does not exist. It is of course possible to build quantum states that take into account the correlations between two correlated photons that are separated by a large distance. However, such states do not need to be magical because they can now again be seen as due to a common origin. And in deriving Bell inequalities for them one should take care to avoid making the same errors as we pinpointed in [2].

## 2 Replacing the matrices of $SU(2)$ by matrices with real entries

$SU(2)$  is a complex formalism, but it describes geometrical objects, namely rotations, that are entirely real. There is thus no “complex reality” at stake. All the geometry treated by  $SU(2)$  can also be addressed within  $SO(3)$  which is a real-valued formalism and just one of the possible translations of the complex-valued formalism into a real-valued one. We will give here yet another one. This will indicate how by generalization one can develop a real-valued formalism for any development of a Clifford algebra. We can apply then these ideas to the homogeneous Lorentz group and derive a completely real-valued version of the Dirac equation. We might consider the Dirac equation as a foundation for a large part of quantum mechanics.

Let us first introduce this new real-valued formalism for the rotations of  $\mathbb{R}^3$ . We replace the  $2 \times 2$  Pauli matrices:

$$\sigma_x = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} & -\iota \\ \iota & \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \quad (6)$$

by the  $4 \times 4$  matrices:

<sup>1</sup> Many mathematical structures used in physics are not vector spaces but manifolds (or even bundles). A keen example is curved space-time. It can be embedded within a vector space  $\mathbb{R}^5$  but the points of  $\mathbb{R}^5$  which do not belong to space-time do not have physical meaning. This is the reason why general relativity is developed intrinsically on the four-dimensional space-time manifold. Developing differential geometry intrinsically on topological manifolds is much more difficult and elaborate than doing it on embedding vector spaces, but it avoids carrying out meaningless algebra on quantities that do not exist. We have explained in [3] that in complete analogy the spinors of  $SU(2)$  do not form a vector space but a curved manifold and that extrapolating the formalism to a vector space formalism is not meaningful. Hence, a priori the spinors of  $SU(2)$  do not form a Hilbert space.

$$\Sigma_x = \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix}, \quad \Sigma_y = \begin{bmatrix} & -\eta \\ \eta & \end{bmatrix}, \quad \Sigma_z = \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix}, \quad (7)$$

where:

$$\eta = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \quad \text{such that: } \eta^2 = -\mathbb{1}. \quad (8)$$

Hence the real-valued matrix  $\eta$  plays the rôle of the complex number  $\iota$ . Just like one proves:

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk} \mathbb{1}, \quad (9)$$

one can now prove:

$$\Sigma_j \Sigma_k + \Sigma_k \Sigma_j = 2\delta_{jk} \mathbb{1}. \quad (10)$$

To write the proof, it suffices to replace all numbers 1 by  $\mathbb{1}$  and all numbers  $\iota$  by  $\eta$  in the calculus. We define a reflection  $A$  in  $\mathbb{R}^3$  by the unit vector  $\mathbf{a}$  orthogonal to its reflection plane. The  $4 \times 4$  matrix  $\mathbf{A}$  representing the reflection  $A$  is now given by:

$$\mathbf{A} = a_x \Sigma_x + a_y \Sigma_y + a_z \Sigma_z = \begin{bmatrix} a_z \mathbb{1} & a_x \mathbb{1} - a_y \eta \\ a_x \mathbb{1} + a_y \eta & -a_z \mathbb{1} \end{bmatrix} \hat{=} \mathbf{a} \cdot \boldsymbol{\Sigma}. \quad (11)$$

The symbol  $\hat{=}$  serves to remind us that the notation is purely conventional shorthand and does not refer to a true scalar product.  $\mathbf{A}$  represents a vector quantity (see [5]), not a scalar quantity. In  $SU(2)$  the expression for the rotation matrix  $\mathbf{R}$  representing a rotation  $R$ , called the Rodriguez formula, is obtained by calculating the product of two reflections  $A$  and  $B$  with  $2 \times 2$  representation matrices  $\mathbf{a} \cdot \boldsymbol{\sigma}$  and  $\mathbf{b} \cdot \boldsymbol{\sigma}$ :

$$\mathbf{R} = [\mathbf{b} \cdot \boldsymbol{\sigma}] [\mathbf{a} \cdot \boldsymbol{\sigma}] = (\mathbf{a} \cdot \mathbf{b}) \mathbb{1} - \iota (\mathbf{a} \wedge \mathbf{b}) \cdot \boldsymbol{\sigma} = \cos(\varphi/2) \mathbb{1} - \iota \sin(\varphi/2) [\mathbf{s} \cdot \boldsymbol{\sigma}]. \quad (12)$$

In fact, as explained in [5], a rotation in  $\mathbb{R}^3$  can be obtained from two reflections. Here  $\mathbf{s} \parallel \mathbf{a} \wedge \mathbf{b}$  is a unit vector along the rotation axis. By analogy, within the new formalism the Rodriguez formula of  $SU(2)$  becomes now:

$$\mathbf{R} = [\mathbf{b} \cdot \boldsymbol{\Sigma}] [\mathbf{a} \cdot \boldsymbol{\Sigma}] = \cos(\varphi/2) \mathbb{1} - \sin(\varphi/2) [\eta \cdot [\mathbf{s} \cdot \boldsymbol{\Sigma}]], \quad (13)$$

where  $\mathbf{R}$  is a  $4 \times 4$  matrix and multiplying a  $4 \times 4$  matrix with  $\eta$  corresponds to multiplying its  $2 \times 2$  blocks with  $\eta$ . By letting the angle  $\varphi$  vary by putting  $\varphi = \omega_0 \tau$  this matrix can be used to describe the spinning motion of an object or a particle:

$$\mathbf{R}(\tau) = \cos(\omega_0 \tau/2) \mathbb{1} - \sin(\omega_0 \tau/2) [\eta \cdot [\mathbf{s} \cdot \boldsymbol{\Sigma}]]. \quad (14)$$

Instead of  $\mathbf{R}^\dagger \mathbf{R} = \mathbb{1}$  as in  $SU(2)$ , or  $\mathbf{R}^\top \mathbf{R} = \mathbb{1}$  as in  $SO(3)$ , the rotation matrices satisfy now the identity:

$$\mathbf{R}^\vee \mathbf{R} = \mathbb{1}, \quad (15)$$

where  $\mathbf{R}^\vee$  is now obtained from  $\mathbf{R}$  by first performing a block transposition and subsequently replacing  $\eta$  by  $-\eta$  everywhere. The number  $\iota$  in the formalism of  $SU(2)$  is not a generator of rotations as claimed by Hestenes who apparently used the Clifford algebra as a blackbox. Rotations are generated by reflections as rotation groups are subgroups of groups generated by reflections. The utility of complex numbers is that they allow to find one more matrix of a given rank which satisfies Eqs. 9 or 10, and as such keeps the size of the formalism minimal. Formalisms that only allow for real entries double the size of the matrices according to  $D = 2^{\lfloor d/2 \rfloor} \rightarrow D = 2^{\lfloor d/2 \rfloor + 1}$  as neatly illustrated by the example we are working out here.

Following the analogy, the “spinors”  $\boldsymbol{\Xi}$  of the new formalism for rotations in  $\mathbb{R}^3$  would now no longer be matrices of dimensions  $2 \times 1$  but block matrices of block dimensions  $2 \times 1$ , such that they would be no longer embedded in a Hilbert space *stricto sensu*. Here, the  $4 \times 2$  “spinors”  $\boldsymbol{\Xi}$  in  $2 \times 1$  block matrix form would be normalized according to:

$$\boldsymbol{\Xi}^\vee \boldsymbol{\Xi} = \mathbb{1}. \quad (16)$$

If we extend this real-valued block matrix formalism to allow for linear combinations we will introduce quantities that are geometrically meaningless. A non-abelian group is a curved manifold and does not have the structure of a vector space. Operations of summing and multiplying by a scalar are not defined by the group structure. Only the composition of group elements is defined. Introducing linear combinations of group elements can be given *a posteriori* a meaning in terms of sets, but the correctness of the ensuing formalism is then no longer automatically granted for all calculations. The rotation matrix Eq. 14 is:

$$\begin{aligned} \mathbf{R} &= \begin{bmatrix} \cos(\varphi/2)\mathbf{1} - \eta \sin(\varphi/2)s_z & -\eta \sin(\varphi/2)(s_x - \eta s_y) \\ -\eta \sin(\varphi/2)(s_x + \eta s_y) & \cos(\varphi/2)\mathbf{1} + \eta \sin(\varphi/2)s_z \end{bmatrix} \\ &= \begin{bmatrix} \cos(\varphi/2)\mathbf{1} - \eta \sin(\varphi/2)s_z & -\eta \sin(\varphi/2)s_x - s_y \sin(\varphi/2)\mathbf{1} \\ -\eta \sin(\varphi/2)s_x + s_y \sin(\varphi/2)\mathbf{1} & \cos(\varphi/2)\mathbf{1} + \eta \sin(\varphi/2)s_z \end{bmatrix}. \end{aligned} \quad (17)$$

In full this is:

$$\mathbf{R} = \begin{bmatrix} \cos(\varphi/2) & +\sin(\varphi/2)s_z & -\sin(\varphi/2)s_y & +\sin(\varphi/2)s_x \\ -\sin(\varphi/2)s_z & \cos(\varphi/2) & -\sin(\varphi/2)s_x & -\sin(\varphi/2)s_y \\ +\sin(\varphi/2)s_y & +\sin(\varphi/2)s_x & \cos(\varphi/2) & +\sin(\varphi/2)s_z \\ -\sin(\varphi/2)s_x & +\sin(\varphi/2)s_y & -\sin(\varphi/2)s_z & \cos(\varphi/2) \end{bmatrix}. \quad (18)$$

Let us take  $\mathbf{s} = \mathbf{e}_z$  and  $\varphi = \omega_0\tau$ . We obtain then the SU(2) spin-up spinor:

$$\boldsymbol{\psi} = \begin{bmatrix} e^{-i\omega_0\tau/2} \\ 0 \end{bmatrix} = e^{-i\omega_0\tau/2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (19)$$

with its corresponding block spinor:

$$\boldsymbol{\Psi} = \begin{bmatrix} \cos(\omega_0\tau/2) & \sin(\omega_0\tau/2) \\ -\sin(\omega_0\tau/2) & \cos(\omega_0\tau/2) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos(\omega_0\tau/2) & \sin(\omega_0\tau/2) \\ -\sin(\omega_0\tau/2) & \cos(\omega_0\tau/2) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \quad (20)$$

as can be seen from Eq. 18. We obtain thus the normal correspondence between complex numbers and  $2 \times 2$  matrices:

$$e^{i\varphi} \leftrightarrow \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}, \quad (21)$$

which both are used to represent rotations in the two-dimensional plane  $\mathbb{C} \equiv \mathbb{R}^2$ . The consequence of this will be that instead of writing in the calculations terms  $e^{-\frac{i}{\hbar}(Et - \mathbf{p} \cdot \mathbf{r})}$  we will have to write terms:

$$\begin{bmatrix} \cos((Et - \mathbf{p} \cdot \mathbf{r})/\hbar) & \sin((Et - \mathbf{p} \cdot \mathbf{r})/\hbar) \\ -\sin((Et - \mathbf{p} \cdot \mathbf{r})/\hbar) & \cos((Et - \mathbf{p} \cdot \mathbf{r})/\hbar) \end{bmatrix}. \quad (22)$$

This is the form we must give to the de Broglie wave when we only use real numbers. We have:

$$\begin{bmatrix} \cos(\omega_0\tau/2) & \sin(\omega_0\tau/2) \\ -\sin(\omega_0\tau/2) & \cos(\omega_0\tau/2) \end{bmatrix} = \cos(\omega_0\tau/2)\mathbf{1} - \sin(\omega_0\tau/2)\eta. \quad (23)$$

Therefore we have:

$$\begin{bmatrix} \cos(\omega_0\tau/2) & \sin(\omega_0\tau/2) \\ -\sin(\omega_0\tau/2) & \cos(\omega_0\tau/2) \end{bmatrix}^\vee = \cos(\omega_0\tau/2)\mathbf{1} + \sin(\omega_0\tau/2)\eta, \quad (24)$$

such that:

$$\boldsymbol{\Psi}^\vee = \begin{bmatrix} \cos(\omega_0\tau/2) & -\sin(\omega_0\tau/2) & 0 & 0 \\ \sin(\omega_0\tau/2) & \cos(\omega_0\tau/2) & 0 & 0 \end{bmatrix}. \quad (25)$$

Note that we have here not transposed but block-transposed. The block-transpose of a  $2 \times 2$  matrix is here the matrix itself. We clearly have  $\boldsymbol{\Psi}^\vee \boldsymbol{\Psi} = \mathbf{1}$ .

The matrix  $\eta$  commutes with all Sigma matrices because it commutes with  $\mathbf{1}$  and  $\eta$ . Hence  $\eta \cdot [\mathbf{s} \cdot \boldsymbol{\Sigma}] = [\mathbf{s} \cdot \boldsymbol{\Sigma}] \cdot \eta$ , such that:

$$\frac{d\mathbf{R}}{d\tau} = -\frac{\omega_0}{2} \sin(\omega_0\tau/2)\mathbf{1} - \frac{\omega_0}{2} \cos(\omega_0\tau/2) [\eta \cdot [\mathbf{s} \cdot \boldsymbol{\Sigma}]] = -\frac{\omega_0}{2} [\eta \cdot [\mathbf{s} \cdot \boldsymbol{\Sigma}]] \mathbf{R}. \quad (26)$$

The solution of this differential equation is:

$$\mathbf{R} = \exp\left(-\frac{\omega_0}{2} [\eta \cdot [\mathbf{s} \cdot \boldsymbol{\Sigma}]] \tau\right). \quad (27)$$

This must of course agree with the expression for  $\mathbf{R}$  we started from. We can perform a simplified check as follows. For  $\mathbf{s} = \mathbf{e}_z$ , we have

$$\eta \cdot [\mathbf{s} \cdot \boldsymbol{\Sigma}] = \eta \cdot \begin{bmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{bmatrix} = \begin{bmatrix} \eta & \\ & -\eta \end{bmatrix}. \quad (28)$$

It is therefore useful to calculate:

$$\exp\left(-\frac{\omega_0}{2}\eta\tau\right) = \exp\left[-\frac{\omega_0}{2}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\tau\right]. \quad (29)$$

We must thus diagonalize:

$$\eta = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & i \\ i & -1 \end{bmatrix} \begin{bmatrix} i & \\ & -i \end{bmatrix} \begin{bmatrix} -1 & -i \\ -i & -1 \end{bmatrix} \frac{1}{\sqrt{2}}. \quad (30)$$

We can now calculate:

$$\begin{aligned} \exp\left[-\frac{\omega_0}{2}\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\tau\right] &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & i \\ i & -1 \end{bmatrix} \begin{bmatrix} \exp(-i\omega_0\tau/2) & \\ & \exp(+i\omega_0\tau/2) \end{bmatrix} \begin{bmatrix} -1 & -i \\ -i & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \\ &= \frac{1}{2} \begin{bmatrix} -1 & i \\ i & -1 \end{bmatrix} \begin{bmatrix} -\exp(-i\omega_0\tau/2) & -i\exp(-i\omega_0\tau/2) \\ -i\exp(+i\omega_0\tau/2) & -\exp(+i\omega_0\tau/2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\omega_0\tau/2) & \sin(\omega_0\tau/2) \\ -\sin(\omega_0\tau/2) & \cos(\omega_0\tau/2) \end{bmatrix}. \end{aligned} \quad (31)$$

This is in agreement with Eq. 23, validates the check and explains the isomorphism we recovered in Eq. 21.

Instead of a  $4 \times 2$  block matrix, we can still take the first column of Eq. 18 to specify the rotation matrix unambiguously and use it as a  $4 \times 1$  spinor:

$$\boldsymbol{\xi} = \begin{bmatrix} \cos(\varphi/2) \\ -\sin(\varphi/2)s_z \\ +\sin(\varphi/2)s_y \\ -\sin(\varphi/2)s_x \end{bmatrix}. \quad (32)$$

This spinor is then a stenographic notation for the  $4 \times 4$  rotation matrix, just like in  $SU(2)$  a  $2 \times 1$  spinor is a shorthand for a  $2 \times 2$  rotation matrix. The new real-valued  $4 \times 1$  spinors are obviously normalized to 1 according to:

$$\boldsymbol{\xi}^\top \boldsymbol{\xi} = 1, \quad (33)$$

such that the Born rule applies (as explained in [5]). In the absence of complex numbers,  $\boldsymbol{\xi}^\top = \boldsymbol{\xi}^\dagger$ . But, as already explained, extending the calculus of these  $4 \times 1$  spinors to a Hilbert space formalism of ‘‘column vectors’’ will introduce objects that are geometrically meaningless. We may finally note that the homogeneous Lorentz group can be represented by  $SL(2, \mathbb{C})$  matrices, whose columns do no longer contain the full information about the group elements. It is possible to write a Dirac-like equation of motion using such  $SL(2, \mathbb{C})$  matrices but they do not form a Hilbert space.

We see thus that it is not necessary to use complex numbers, that it is not necessary to use spinors under the form of column matrices, and that the spinors do not build a vector space. The foundational basis for quantum mechanics is not about moulding it into a formal straightjacket under the form of a set of some abstract rules which define a Hilbert space formalism, only to render it more unassailable to any physical intuition. The foundational basis of quantum mechanics consists in establishing the meaning of the group representation theory used in it as explained in reference [5]. This permits to see what is going on behind the scenes of a formalism which otherwise looks completely abstruse and mysterious. It is because this group representation theory uses complex numbers that we use complex numbers in quantum mechanics.

### 3 The new Dirac matrices

We will now derive the Dirac equation in a formalism that contains only real numbers. The traditional  $4 \times 4$  Dirac matrices are:

$$\gamma_x = \begin{bmatrix} & \sigma_x \\ -\sigma_x & \end{bmatrix}, \quad \gamma_y = \begin{bmatrix} & \sigma_y \\ -\sigma_y & \end{bmatrix}, \quad \gamma_z = \begin{bmatrix} & \sigma_z \\ -\sigma_z & \end{bmatrix}, \quad \gamma_t = \begin{bmatrix} & \mathbf{1} \\ \mathbf{1} & \end{bmatrix}. \quad (34)$$

We have also:

$$\gamma_5 = \begin{bmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{bmatrix}. \quad (35)$$

Explicitly this is:

$$\begin{aligned} \gamma_x &= \begin{bmatrix} & & +1 & \\ & -1 & & \\ -1 & & & \\ & & & +1 \end{bmatrix}, & \gamma_y &= \begin{bmatrix} & & & -i \\ & +i & & \\ & & +i & \\ -i & & & \end{bmatrix}, \\ \gamma_z &= \begin{bmatrix} & & +1 & \\ -1 & & & -1 \\ & & & \\ & +1 & & \end{bmatrix}, & \gamma_t &= \begin{bmatrix} & & +1 & \\ +1 & & & +1 \\ & & & \\ & +1 & & \end{bmatrix}. \end{aligned} \quad (36)$$

The new real  $8 \times 8$  Dirac matrices are therefore:

$$\Gamma_x = \begin{bmatrix} & \Sigma_x \\ -\Sigma_x & \end{bmatrix}, \quad \Gamma_y = \begin{bmatrix} & \Sigma_y \\ -\Sigma_y & \end{bmatrix}, \quad \Gamma_z = \begin{bmatrix} & \Sigma_z \\ -\Sigma_z & \end{bmatrix}, \quad \Gamma_t = \begin{bmatrix} \mathbf{1} & \\ & \mathbf{1} \end{bmatrix}, \quad (37)$$

with:

$$\Gamma_5 = \begin{bmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{bmatrix}. \quad (38)$$

Here the blocks  $\mathbf{1}$ ,  $\Sigma_x$ ,  $\Sigma_y$ ,  $\Sigma_z$  are  $4 \times 4$  matrices. Hence, in terms of  $2 \times 2$ -sized blocks:

$$\begin{aligned} \Gamma_x &= \begin{bmatrix} & & +\mathbf{1} & \\ & -\mathbf{1} & & \\ -\mathbf{1} & & & \\ & & & +\mathbf{1} \end{bmatrix}, & \Gamma_y &= \begin{bmatrix} & & & -\eta \\ & +\eta & & \\ -\eta & & +\eta & \\ & & & \end{bmatrix}, \\ \Gamma_z &= \begin{bmatrix} & & +\mathbf{1} & \\ -\mathbf{1} & & & -\mathbf{1} \\ & & & \\ & +\mathbf{1} & & \end{bmatrix}, & \Gamma_t &= \begin{bmatrix} & & +\mathbf{1} & \\ +\mathbf{1} & & & +\mathbf{1} \\ & & & \\ & +\mathbf{1} & & \end{bmatrix}, \end{aligned} \quad (39)$$

with:

$$\Gamma_5 = \begin{bmatrix} +\mathbf{1} & & & \\ & +\mathbf{1} & & \\ & & -\mathbf{1} & \\ & & & -\mathbf{1} \end{bmatrix}. \quad (40)$$

Explicitly:

$$\Gamma_x = \begin{bmatrix} & & & & +1 & & \\ & & & & & +1 & \\ & & & +1 & & & \\ & & -1 & & +1 & & \\ -1 & & & -1 & & & \\ & -1 & & & & & \end{bmatrix}. \quad (41)$$

$$\Gamma_y = \begin{bmatrix} & & & & & & -1 \\ & & & & & +1 & \\ & & & & -1 & & \\ & & & +1 & & & \\ & & -1 & & & & \\ -1 & & +1 & & & & \end{bmatrix}. \quad (42)$$

$$\Gamma_z = \begin{bmatrix} & & & & +1 & & & \\ & & & & & +1 & & \\ & & & & & & -1 & \\ -1 & & & & & & & -1 \\ & -1 & & & & & & \\ & & +1 & & & & & \\ & & & +1 & & & & \end{bmatrix}. \quad (43)$$

$$\Gamma_t = \begin{bmatrix} & & & & +1 & & & \\ & & & & & +1 & & \\ & & & & & & +1 & \\ +1 & & & & & & & +1 \\ & +1 & & & & & & \\ & & +1 & & & & & \\ & & & +1 & & & & \end{bmatrix}. \quad (44)$$

$$\Gamma_5 = \begin{bmatrix} +1 & & & & & & & \\ & +1 & & & & & & \\ & & +1 & & & & & \\ & & & +1 & & & & \\ & & & & +1 & & -1 & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \end{bmatrix}. \quad (45)$$

The  $8 \times 8$  reflection matrix  $\mathbf{A}$  is now given by:

$$\mathbf{A} = \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ a_t \mathbb{1} - \mathbf{a} \cdot \boldsymbol{\Sigma} & & & & & & & \\ & a_t \mathbb{1} + \mathbf{a} \cdot \boldsymbol{\Sigma} & & & & & & \end{bmatrix} = \begin{bmatrix} \mathbf{A}^* & \mathbf{A} \end{bmatrix} = \begin{bmatrix} & & & & a_t + a_z & & a_x - a_y \eta & \\ & & & & a_x + a_y \eta & & a_t - a_z & \\ a_t - a_z & & & & -a_x + a_y \eta & & & \\ -a_x - a_y \eta & & & & a_t + a_z & & & \end{bmatrix}. \quad (46)$$

Here  $\mathbf{A}$  and  $\mathbf{A}^*$  are  $4 \times 4$  matrices. It is tacitly assumed that the scalars like  $a_t$  in the last matrix are actually a shorthand for  $a_t \mathbb{1}$ , where  $\mathbb{1}$  is the  $2 \times 2$  unit matrix. The  $2 \times 2$  matrix  $\eta$  commutes with all reflection matrices because these matrices contain only  $\mathbb{1}$  and  $\eta$ . A Lorentz transformation is of the form  $\mathbf{L} = \mathbf{B}\mathbf{A}$ . Furthermore:

$$\mathbf{A}^\vee = \begin{bmatrix} a_t + a_z & a_x - a_y \eta \\ a_x + a_y \eta & a_t - a_z \end{bmatrix}^\vee = \begin{bmatrix} a_t + a_z & a_x - a_y \eta \\ a_x + a_y \eta & a_t - a_z \end{bmatrix} = \mathbf{A}, \quad \mathbf{A}^{*\vee} = \mathbf{A}^*, \quad (47)$$

because one must first transpose and then replace  $\eta$  by  $-\eta$  everywhere. Now:

$$\mathbf{A}^2 = \mathbb{1} = \begin{bmatrix} \mathbf{A}\mathbf{A}^* & \\ & \mathbf{A}^*\mathbf{A} \end{bmatrix}, \quad (48)$$

such that  $\mathbf{A}^* = \mathbf{A}^{-1}$ . We have assumed here that  $\det \mathbf{A} = a_t^2 - \mathbf{a}^2 = 1$ . This may not be general, but the modifications required are then simple. We have discussed in [5] that in the complex Dirac formalism we always must use  $2 \times 2$  Lorentz matrices  $\mathbf{L}$  with  $\det \mathbf{L} = 1$ , such that  $\mathbf{L}$  belongs then to  $SL(2, \mathbb{C})$ . This can now be generalized to the  $4 \times 4$  matrices  $\mathbf{L}$  in Eq. 49 below. Here  $\mathbf{L}$  is defined by  $\mathbf{L} = \mathbf{A}\mathbf{B}^{-1}$ . A Lorentz transformation is then given by:

$$\mathbf{L} = \mathbf{A}\mathbf{B} = \begin{bmatrix} \mathbf{A}\mathbf{B}^* & \\ & \mathbf{A}^*\mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{B}^{-1} & \\ & \mathbf{A}^{-1}\mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{L} & \\ & [\mathbf{L}^\vee]^{-1} \end{bmatrix}, \quad (49)$$

because  $\mathbf{A}^{-1}\mathbf{B} = [\mathbf{B}^{-1}\mathbf{A}]^{-1} = [[\mathbf{A}\mathbf{B}^{-1}]^\vee]^{-1} = [\mathbf{L}^\vee]^{-1}$ . Furthermore:

$$\mathbf{L}^{-1} = \mathbf{B}\mathbf{A} = \begin{bmatrix} \mathbf{B}\mathbf{A}^* & \\ & \mathbf{B}^*\mathbf{A} \end{bmatrix} = \begin{bmatrix} \mathbf{B}\mathbf{A}^{-1} & \\ & \mathbf{B}^{-1}\mathbf{A} \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{-1} & \\ & \mathbf{L}^\vee \end{bmatrix}. \quad (50)$$



## 4 The new Dirac equation

To obtain the new Dirac equation we just have to transcribe the derivation in [5]. The calculation is therefore analogous to the calculation in [5]. For guidance we suggest to inspect the two calculations side by side. This way we can avoid copying almost verbatim all the explanations from [5] to the present calculation. We can then just list the modified equations. In essence, it corresponds to substituting consistently  $\iota \rightarrow \eta$  and introducing capital letters for the new Pauli and Dirac matrices. The time derivative of  $\mathbf{R}(\mathbf{s}, \omega_0\tau)$  yields:

$$\frac{d\mathbf{R}}{d\tau} = -(\omega_0/2) [\eta \cdot [\mathbf{s} \cdot \boldsymbol{\Sigma}]] \mathbf{R}, \quad \text{and:} \quad \frac{d\boldsymbol{\Xi}}{d\tau} = -(\omega_0/2) [\eta \cdot [\mathbf{s} \cdot \boldsymbol{\Sigma}]] \boldsymbol{\Xi}, \quad (51)$$

where the  $2 \times 1$  block spinor  $\boldsymbol{\Xi}$  is the first block column of  $\mathbf{R}(\mathbf{s}, \omega_0\tau)$ . We will not use  $\boldsymbol{\Xi}$  and formulate everything with the  $8 \times 8$  representation matrices. We can then write the following differential equation  $\forall(\mathbf{r}, \tau) \in \mathbb{R}^4$ :

$$\begin{aligned} \left[ \begin{array}{cc} \frac{d}{d\tau} \mathbb{1} & \\ & \frac{d}{d\tau} \mathbb{1} \end{array} \right] \left[ \begin{array}{cc} \mathbf{R} & \\ & \mathbf{R} \end{array} \right] &= -\frac{\omega_0}{2} \eta \cdot \left[ \begin{array}{cc} & [\mathbf{s} \cdot \boldsymbol{\Sigma}] \\ [\mathbf{s} \cdot \boldsymbol{\Sigma}] & \end{array} \right] \left[ \begin{array}{cc} \mathbf{R} & \\ & \mathbf{R} \end{array} \right] \\ &= -\frac{\omega_0}{2} \eta \cdot \left[ \begin{array}{cc} & [\mathbf{s} \cdot \boldsymbol{\Sigma}] \\ -[\mathbf{s} \cdot \boldsymbol{\Sigma}] & \end{array} \right] \left[ \begin{array}{cc} \mathbf{R} & \\ & \mathbf{R} \end{array} \right]. \end{aligned} \quad (52)$$

All blocks like  $\mathbb{1}$ ,  $\mathbf{R}$  and  $[\mathbf{s} \cdot \boldsymbol{\Sigma}]$  are here  $4 \times 4$  matrices. Under a general Lorentz transformation  $\mathbf{L}$  we have:

$$\begin{aligned} &\mathbf{L} \left[ \begin{array}{cc} \frac{d}{d\tau} \mathbb{1} & \\ & \frac{d}{d\tau} \mathbb{1} \end{array} \right] \mathbf{L}^{-1} \cdot \mathbf{L} \left[ \begin{array}{cc} \mathbf{R} & \\ & \mathbf{R} \end{array} \right] \\ &= -\frac{\omega_0}{2} \eta \cdot \mathbf{L} \left[ \begin{array}{cc} & [\mathbf{s} \cdot \boldsymbol{\Sigma}] \\ -[\mathbf{s} \cdot \boldsymbol{\Sigma}] & \end{array} \right] \mathbf{L}^{-1} \cdot \mathbf{L} \left[ \begin{array}{cc} \mathbf{R} & \\ & \mathbf{R} \end{array} \right], \end{aligned} \quad (53)$$

as multiplication with  $\eta$  commutes with the  $8 \times 8$  matrix  $\mathbf{L}$ . We define:

$$\mathbf{L} \left[ \begin{array}{cc} \mathbf{R} & \\ & \mathbf{R} \end{array} \right] = \left[ \begin{array}{c} \phi \\ [\phi^\vee]^{-1} \end{array} \right] = \Phi. \quad (54)$$

We have used here Eq. 49. The result of carrying out the Lorentz transformations  $\mathbf{L}$  in Eq. 53 is:

$$\begin{aligned} &\left[ \begin{array}{cc} \frac{\partial}{\partial t} \mathbb{1} + c[\nabla \cdot \boldsymbol{\Sigma}] & \frac{\partial}{\partial t} \mathbb{1} - c[\nabla \cdot \boldsymbol{\Sigma}] \end{array} \right] \Phi \\ &= -\frac{\omega_0}{2} \eta \cdot \left[ \begin{array}{cc} & s_t \mathbb{1} + [\mathbf{s}' \cdot \boldsymbol{\Sigma}] \\ -[s_t \mathbb{1} - [\mathbf{s}' \cdot \boldsymbol{\Sigma}]] & \end{array} \right] \Phi. \end{aligned} \quad (55)$$

Eq. 55 can also be written as:

$$\left[ \Gamma_t \frac{\partial}{\partial t} - c \nabla \cdot \boldsymbol{\Gamma} \right] \Phi = -\frac{\omega_0}{2} \eta \cdot \Gamma_5 [s_t \gamma_t + \mathbf{s} \cdot \boldsymbol{\Gamma}] \Phi, \quad (56)$$

where we have dropped the accent on  $\mathbf{s}'$ , in order to write the covariant equation in its standard form. We introduce the  $8 \times 8$  matrix:

$$\mathbf{S} = \left[ \begin{array}{cc} & s_t \mathbb{1} + [\mathbf{s} \cdot \boldsymbol{\Sigma}] \\ -[s_t \mathbb{1} - [\mathbf{s} \cdot \boldsymbol{\Sigma}]] & \end{array} \right] \Rightarrow \mathbf{S}^2 = \mathbb{1}, \quad (57)$$

by covariance, as in the rest frame  $s_t^2 - \mathbf{s}^2 = -1$ . We define:

$$\boldsymbol{\Psi} = (\mathbb{1} + \mathbf{S})\Phi, \quad (58)$$

such that:

$$\mathbf{S}\boldsymbol{\Psi} = \mathbf{S}(\mathbb{1} + \mathbf{S})\Phi = (\mathbf{S} + \mathbb{1})\Phi = \boldsymbol{\Psi}. \quad (59)$$

The covariance of Eq. 59 follows from:

$$\mathbf{L}\mathbf{S}\mathbf{L}^{-1} \cdot \mathbf{L}(\mathbb{1} + \mathbf{S})\mathbf{L}^{-1} \cdot \mathbf{L}\Phi = \mathbf{L}(\mathbf{S} + \mathbb{1})\mathbf{L}^{-1} \cdot \mathbf{L}\Phi \Rightarrow \mathbf{S}'(\mathbb{1} + \mathbf{S}')\Phi' = (\mathbf{S}' + \mathbb{1})\Phi'. \quad (60)$$

As we have assumed that  $\mathbf{s}$  does not vary with time:

$$\begin{bmatrix} \frac{d}{d\tau} \mathbf{1} & \frac{d}{d\tau} \mathbf{1} \\ -[-[\mathbf{s} \cdot \boldsymbol{\Sigma}]] & [\mathbf{s} \cdot \boldsymbol{\Sigma}] \end{bmatrix} = \mathbf{0}. \quad (61)$$

By covariance we have then:

$$\mathbf{L} \begin{bmatrix} \frac{d}{d\tau} \mathbf{1} & \frac{d}{d\tau} \mathbf{1} \\ -[-[\mathbf{s} \cdot \boldsymbol{\Sigma}]] & [\mathbf{s} \cdot \boldsymbol{\Sigma}] \end{bmatrix} \mathbf{L}^{-1} = \mathbf{L} \mathbf{0} \mathbf{L}^{-1} = \mathbf{0}, \quad (62)$$

such that:

$$\begin{aligned} & \begin{bmatrix} \frac{\partial}{\partial t} \mathbf{1} + c[\nabla \cdot \boldsymbol{\Sigma}] & \frac{\partial}{\partial t} \mathbf{1} - c[\nabla \cdot \boldsymbol{\Sigma}] \end{bmatrix} \mathbf{S} \\ = & \begin{bmatrix} \frac{\partial}{\partial t} \mathbf{1} + c[\nabla \cdot \boldsymbol{\Sigma}] & \frac{\partial}{\partial t} \mathbf{1} - c[\nabla \cdot \boldsymbol{\Sigma}] \end{bmatrix} \begin{bmatrix} -[s_t \mathbf{1} - [\mathbf{s} \cdot \boldsymbol{\Sigma}]] & s_t \mathbf{1} + [\mathbf{s} \cdot \boldsymbol{\Sigma}] \end{bmatrix} \\ = & \mathbf{0}. \end{aligned} \quad (63)$$

Hence:

$$\begin{aligned} & \begin{bmatrix} \frac{\partial}{\partial t} \mathbf{1} + c[\nabla \cdot \boldsymbol{\Sigma}] & \frac{\partial}{\partial t} \mathbf{1} - c[\nabla \cdot \boldsymbol{\Sigma}] \end{bmatrix} (\mathbf{1} + \mathbf{S}) \Phi \\ = & (\mathbf{1} + \mathbf{S}) \begin{bmatrix} \frac{\partial}{\partial t} \mathbf{1} + c[\nabla \cdot \boldsymbol{\Sigma}] & \frac{\partial}{\partial t} \mathbf{1} - c[\nabla \cdot \boldsymbol{\Sigma}] \end{bmatrix} \Phi. \end{aligned} \quad (64)$$

Using Eq. 55 this leads to:

$$\begin{aligned} & \begin{bmatrix} \frac{\partial}{\partial t} \mathbf{1} + c[\nabla \cdot \boldsymbol{\Sigma}] & \frac{\partial}{\partial t} \mathbf{1} - c[\nabla \cdot \boldsymbol{\Sigma}] \end{bmatrix} (\mathbf{1} + \mathbf{S}) \Phi \\ = & -\frac{\omega_0}{2} \boldsymbol{\eta} \cdot (\mathbf{1} + \mathbf{S}) \mathbf{S} \Phi = -\frac{\omega_0}{2} \boldsymbol{\eta} \cdot \mathbf{S} (\mathbf{1} + \mathbf{S}) \Phi. \end{aligned} \quad (65)$$

With the aid of Eq. 59 this implies:

$$\begin{bmatrix} \frac{\partial}{\partial t} \mathbf{1} + c[\nabla \cdot \boldsymbol{\Sigma}] & \frac{\partial}{\partial t} \mathbf{1} - c[\nabla \cdot \boldsymbol{\Sigma}] \end{bmatrix} \Psi = -\frac{\omega_0}{2} \boldsymbol{\eta} \cdot \mathbf{S} \Psi = -\frac{\omega_0}{2} \boldsymbol{\eta} \cdot \Psi. \quad (66)$$

In summary:

$$\begin{bmatrix} \frac{\partial}{\partial t} \mathbf{1} + c[\nabla \cdot \boldsymbol{\Sigma}] & \frac{\partial}{\partial t} \mathbf{1} - c[\nabla \cdot \boldsymbol{\Sigma}] \end{bmatrix} \Psi = -\frac{\omega_0}{2} \boldsymbol{\eta} \cdot \Psi. \quad (67)$$

which after substituting  $\hbar\omega_0/2 = m_0c^2$ , yields the new form of the Dirac equation:

$$\boldsymbol{\eta} \cdot \begin{bmatrix} \hbar \frac{\partial}{\partial t} \mathbf{1} + c\hbar[\nabla \cdot \boldsymbol{\Sigma}] & \hbar \frac{\partial}{\partial t} \mathbf{1} - c\hbar[\nabla \cdot \boldsymbol{\Sigma}] \end{bmatrix} \Psi = m_0c^2 \Psi. \quad (68)$$

The quantum operators are now:

$$\hat{E} = \hbar[\boldsymbol{\eta} \cdot \boldsymbol{\Gamma}_t] \frac{\partial}{\partial t}, \quad \hat{\mathbf{p}} = -\hbar[\boldsymbol{\eta} \cdot \boldsymbol{\Gamma}] \cdot \nabla. \quad (69)$$

This is a consequence of the universal substitution rule  $i \rightarrow \boldsymbol{\eta}$ . Due to the definition of  $\Phi$  in Eq. 54,  $\Psi$  is here defined as:

$$\Psi = (\mathbf{1} + \mathbf{S}) \Phi = \begin{bmatrix} \phi & (\mathbf{1}s_t + [\mathbf{s} \cdot \boldsymbol{\Sigma}])[\phi^\vee]^{-1} \\ -(\mathbf{1}s_t - [\mathbf{s} \cdot \boldsymbol{\Sigma}])\phi & [\phi^\vee]^{-1} \end{bmatrix}. \quad (70)$$

We can thus write the Dirac equation without using complex numbers, but this is much more elaborate because the matrices are now of the size  $8 \times 8$ . We can use here a formalism with  $8 \times 2$  block spinors, but we can also use the  $8 \times 1$  spinors.

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