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Steady Euler flows on $\mathbb{R}^3$ with wild and universal dynamics

Pierre Berger[*], Anna Florio[†], Daniel Peralta-Salas[‡]

February 6, 2022

Abstract

Understanding complexity in fluid mechanics is a major problem that has attracted the attention of physicists and mathematicians during the last decades. Using the concept of renormalization in dynamics, we show the existence of a locally dense set $\mathscr{G}$ of stationary solutions to the Euler equations in $\mathbb{R}^3$ such that each vector field $X \in \mathscr{G}$ is universal in the sense that any area preserving diffeomorphism of the disk can be approximated (with arbitrary precision) by the Poincaré map of X at some transverse section. We remark that this universality is approximate but occurs at all scales. In particular, our results establish that a steady Euler flow may exhibit any conservative finite codimensional dynamical phenomenon; this includes the existence of horseshoes accumulated by elliptic islands, increasing union of horseshoes of Hausdorff dimension 3 or homoclinic tangencies of arbitrarily high multiplicity. The steady solutions we construct are Beltrami fields with sharp decay at infinity. To prove these results we introduce new perturbation methods in the context of Beltrami fields that allow us to import deep techniques from bifurcation theory: the Gonchenko-Shilnikov-Turaev universality theory and the Newhouse and Duarte theorems on the geometry of wild hyperbolic sets. These perturbation methods rely on two tools from linear PDEs: global approximation and Cauchy-Kovalevskaya theorems. These results imply a strong version of V.I. Arnold's vision on the complexity of Beltrami fields in Euclidean space.

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1 Introduction and statements of the main results

The evolution of an ideal fluid flow in equilibrium is described by an autonomous vector field X (the velocity field of the fluid) and a scalar function P (the hydrodynamic pressure) that satisfy the stationary Euler equations in Euclidean space:

$$\nabla_X X = -\nabla P\,, \qquad \mathrm{div} X = 0\,. \tag{1.1}$$

Here $\nabla_X X$ denotes the covariant derivative of X along itself and div is the divergence operator, everything computed using the Euclidean metric. A vector field X on $\mathbb{R}^3$ that satisfies Equation (1.1) for some P is called a steady Euler flow. We recall that the orbits (or integral curves) of X are the stream lines of the fluid, which in the stationary case coincide with the paths followed by the fluid particles. In terms of the vorticity field $\mathrm{curl}\, X$ and the Bernoulli function $B := P + \frac{1}{2}|X|^2$, the stationary Euler equations can be equivalently written [AK21] as

$$X \times \mathrm{curl}\, X = \nabla B\,, \qquad \mathrm{div} X = 0\,.$$

In 1965 V.I. Arnold [Arn65, Arn66] noticed that B is a first integral of X, and hence it is invariant by its flow. Whenever X is analytic, then either B is constant or displays a critical set of Lebesgue measure zero. In the latter case, the celebrated Arnold's structure theorem showed that any regular level set of the Bernoulli function B is a flat surface on which X acts locally as a translation. Hence, the velocity is integrable in a sense analogous to Liouville integrability in Hamiltonian systems. This can be interpreted as a laminar behavior of steady Euler flows [MYZ95].

Otherwise, $\mathrm{curl}\, X = \lambda \cdot X$, for some real analytic function λ (under the assumption that X is analytic). In [Hén66], Hénon observed that if λ is not constant then the behavior of the flow is laminar. In the case that $\lambda \equiv 0$, then the dynamics X on $\mathbb{R}^3$ must be the gradient of a harmonic function, and its entropy is zero. Hence the only (analytic) solutions of the stationary Euler equations which might have a complicated dynamics are:

Definition 1.1. *A vector field X is λ-***Beltrami** *if there exists a real number $\lambda \neq 0$ such that:*

$$\mathrm{curl}\, X = \lambda \cdot X\,.$$

Note that every Beltrami field is a solution of the steady Euler equations (with constant Bernoulli function). In the context of magnetohydrodynamics, these fields are known as force-free fields, and appear when modeling stellar atmospheres [AK21].

Hénon's numerical experiments with the ABC flows [Hén66] (an explicit family of Beltrami fields on $\mathbb{T}^3$) showed that Beltrami fields might have a complicated dynamics (KAM tori and stochastic sea). In fact, Arnold had the following vision on the complexity of Beltrami flows:

The flow of a vector field satisfying $\operatorname{curl} X = \lambda \cdot X$ *probably displays stream lines with topologies as complicated as those of orbits in celestial mechanics (see [Arn63, Fig. 6]).*

Arnold [Arn65, Rk. P. 90] and [Arn66, P. 347].

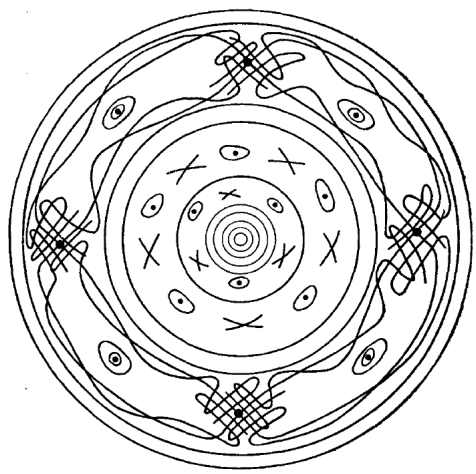

Figure 1: [Arn63, Fig. 6]

As commented around [Arn63, Fig. 6], by "complicated topology", Arnold meant the coexistence of nested (KAM) tori centered at elliptic orbits, such that in between there are elliptic and saddle orbits, the elliptic ones being surrounded by (KAM) tori and the stable and unstable manifolds of the saddle ones intersecting each other to create an intricate network in the "zones of instability". It is relevant to observe that the study of the complexity of Beltrami fields may be of interest to study complex non-stationary solutions of the Navier-Stokes equations. Indeed experimental and numerical observations postulated a phenomenon known as "Beltramization", which states that in turbulent regions the velocity X tends to align to the vorticity $\operatorname{curl} X$ when the Reynolds number is large [FPS01, MRD+06].

As we will discuss in §2, the behavior presented in Figure 1 is well known to be typical among smooth conservative surface dynamics; in contrast, the previous literature on Beltrami fields yields dynamical phenomena far from establishing Arnold's picture. Our main results imply a strong version of Arnold's description. We will show, in particular, the existence of a non-empty open subset $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ of Beltrami fields in which any (families of) conservative dynamics from the disk is arbitrarily well approximated by (families of) Poincaré return maps of generic (families of) Beltrami fields in $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$. The main difficulty of our proof is to design perturbations of the homoclinic tangle to create multiple homoclinic tangencies, combining the very rigid space of Beltrami fields with the wild geometry of the Newhouse domain.

Let us precise the statements of our main results. Denote by $\mathbb{D}$ the closed unit disk of $\mathbb{R}^2$. Let $\mathrm{Diff}^{\infty}_{\mathrm{Leb}+}(\mathbb{D}, \mathbb{R}^2)$ be the set of diffeomorphisms from $\mathbb{D}$ onto their image in $\mathbb{R}^2$ which are infinitely smooth and symplectic. By **symplectic**, we mean orientation and area preserving.

Let $U \subset \mathbb{R}^3$ be an open subset of $\mathbb{R}^3$. A vector field on U is of class $C^{\infty}_{\mathrm{Leb}}$ if it is smooth and volume preserving. Let us denote by $\Gamma^{\infty}_{\mathrm{Leb}}(U)$ the space of $C^{\infty}_{\mathrm{Leb}}$-vector fields on (U, Leb). Let $\mathscr{B}(U) \subset \Gamma^{\infty}_{\mathrm{Leb}}(U)$ be the space of (Euclidean) 1-Beltrami fields on U with sharp decay. These are vector fields X such that $\operatorname{curl} X = X$ and for any multi-index $\alpha \in \mathbb{N}^3$, the norm $\|D^{\alpha} X(x)\|$ is

dominated by $(1+\|x\|)^{-1}$ for every $x \in U$. It is clear that there is no loss of generality in choosing the constant $\lambda = 1$ because the proportionality factor of a Beltrami field can be rescaled by a homothety. In what follows, **Beltrami field** will refer canonically to a 1-Beltrami field.

We recall that a saddle periodic orbit P of a vector field X_0 displays a **homoclinic tangency** if its stable and unstable manifolds are tangent at a point Q. The tangency is **quadratic** if the curvatures of the stable and unstable manifolds at Q are different. An **unfolding** of X_0 is a smooth k-parameter family $(X_p)_{p\in\mathbb{I}^k}$ containing X_0 at $p = 0$, where $\mathbb{I} := [-1, 1]$. If it is a one-parameter family (p varies in an interval $\mathbb{I}$), the unfolding is **non-degenerate** if the relative position of the continuation of the stable and unstable manifolds at the tangency point Q has non-zero derivative at $p = 0$. Then we say that the quadratic homoclinic tangency **unfolds non-degenerately**.

Let $\mathcal{N}_\Gamma(\mathbb{R}^3)$ be the interior of the closure[1] of the C^∞_{Leb}-vector fields on $\mathbb{R}^3$ exhibiting a periodic saddle displaying a homoclinic tangency. A consequence of Newhouse's celebrated theorem [New70] is that $\mathcal{N}_\Gamma(\mathbb{R}^3)$ is non-empty. A consequence of Duarte's Theorem [Dua99] is that the closure of $\mathcal{N}_\Gamma(\mathbb{R}^3)$ is equal to the closure of the set of C^∞_{Leb}-vector fields on $\mathbb{R}^3$ for which there exists a periodic saddle displaying a homoclinic tangency. The open set $\mathcal{N}_\Gamma(\mathbb{R}^3)$ is the so-called **Newhouse domain**. It is formed by dynamics which are extremely rich as we will see in the sequel.

Our first main result is the Beltrami counterpart of these theorems. Let $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ be the interior of the closure[1] of the set of Beltrami fields $X \in \mathscr{B}(\mathbb{R}^3)$ that exhibit a saddle periodic orbit displaying a homoclinic tangency **and** such that the tangency can be non-degenerately unfolded by a family in $\mathscr{B}(\mathbb{R}^3)$. Here is our first main result:

Theorem A. *The set $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ is non-empty.*

The main difficulty of this result is to show the existence of a Beltrami field $X \in \mathscr{B}(\mathbb{R}^3)$ displaying a homoclinic tangency which can be non-degenerately unfolded in $\mathscr{B}(\mathbb{R}^3)$. Then Duarte's theorem implies directly the non-emptiness of $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$. In particular Theorem A provides the **first** example of a Beltrami field in Euclidean space with a **quadratic** homoclinic tangency. Examples of homoclinic tangencies exist on the torus $\mathbb{R}^3/\mathbb{Z}^3$, see e.g. [AK21, ABC flow Example 1.9, Page 76]; yet the manifold is not $\mathbb{R}^3$ and the example is integrable so the tangency is not quadratic. Moreover the displayed non-degenerate quadratic unfolding implies the existence of Beltrami fields with horseshoes (without any computer assisted estimate as in [EPSR20]). Also by Duarte's theorem, this reveals, as a new phenomenon in the Beltrami setting, the coexistence of infinitely many KAM tori which accumulate on a Smale horseshoe. These consequences will be discussed in detail in §2.

Our definition of the Beltrami Newhouse domain $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ asks for the non-degeneracy of the unfolding of quadratic homoclinic tangencies, which is not asked in the standard definition of the smooth Newhouse domain $\mathcal{N}_\Gamma(\mathbb{R}^3)$. Indeed it is well known that a homoclinic tangency in $\Gamma^\infty_{\text{Leb}}(\mathbb{R}^3)$ can be non-degenerately unfolded [BT86]. A natural Beltrami counterpart of this result is:

Conjecture 1.2. *Every homoclinic tangency for a periodic saddle orbit of a Beltrami field $X \in \mathscr{B}(\mathbb{R}^3)$ unfolds non-degenerately in $\mathscr{B}(\mathbb{R}^3)$.*

We believe that this conjecture is the main issue to prove that conservative Kupka-Smale systems are dense among Beltrami fields. Such density has been shown for other rigid classes of dynamics, see [BT86, BHI05].

Now let us shed light on the richness of the generic dynamics in $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$. In short, our second main Theorem B states that a dense subset of $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ is universal: its Poincaré return maps realize a dense subset of conservative dynamics from the disk. This will imply many new phenomena, including Arnold's vision[2].

[1] See §4.1 for the definition of the involved topology.

[2] Theorem B is stronger than Arnold's vision: there is no example of universal dynamics in celestial mechanics.

Definition 1.3. *A* **renormalized iteration** *of a C^∞_{Leb}-vector field X on an open subset $U \subset \mathbb{R}^3$ is a Poincaré return map G at a transverse disk in rescaled coordinates. More precisely, there is an embedding $\phi : \mathbb{R}^2 \hookrightarrow U$, $\Sigma := \phi(\mathbb{R}^2)$ is transverse to X, such that the Poincaré map F from $\check{\Sigma} := \phi(\mathbb{D})$ into Σ is well defined and the map $\phi^{-1} \circ F \circ \phi|_{\mathbb{D}}$ is symplectic and equals G.*

Definition 1.4 (Compare to [BD02, Tur03, GST07])**.** *A C^∞_{Leb}-vector field X on $U \subset \mathbb{R}^3$ is* **universal** *if the set of renormalized iterations of X is dense in $\mathrm{Diff}^\infty_{\mathrm{Leb}+}(\mathbb{D}, \mathbb{R}^2)$. The vector field X is* **compactly universal** *if there exists an open bounded subset $U' \subset U$ such that $X|_{U'}$ is universal.*

Gonchenko-Shilnikov-Turaev's Theorem [GST07] implies that a dense subset of $\mathcal{N}_\Gamma(\mathbb{R}^3)$ is formed by compactly universal dynamics. We propose the counterpart of this result in the much more rigid world of Euclidean Beltrami fields:

Theorem B. *There is a topologically generic*[3] *subset $\mathscr{G}$ of $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ formed by compactly universal vector fields.*

Let us introduce a generalization of the above definition and result for families of dynamics. For $J \geq 0$, let $\mathrm{Diff}^\infty_{\mathrm{Leb}+}(\mathbb{D}, \mathbb{R}^2)_{\mathbb{I}^J}$ denote the space of smooth families $(f_p)_{p \in \mathbb{I}^J}$ of maps $f_p \in \mathrm{Diff}^\infty_{\mathrm{Leb}+}(\mathbb{D}, \mathbb{R}^2)$. We endow this space with the topology given by the canonical inclusion:

$$\mathrm{Diff}^\infty_{\mathrm{Leb}+}(\mathbb{D}, \mathbb{R}^2)_{\mathbb{I}^J} \hookrightarrow C^\infty(\mathbb{I}^J \times \mathbb{D}, \mathbb{R}^2).$$

Definition 1.5. *Let $\mathcal{F}$ be a Fréchet space formed by smooth conservative flows of $\mathbb{R}^3$.*

A subset of $\mathscr{E} \subset \mathcal{F}$ is **J-para-universal** *for $J \geq 0$ if, for any non-empty open subsets $\mathcal{V}$ of $\mathscr{E}$ and $\mathcal{O}_J$ of $\mathrm{Diff}^\infty_{\mathrm{Leb}+}(\mathbb{D}, \mathbb{R}^2)_{\mathbb{I}^J}$, there exists a smooth family $(X_p)_{p \in \mathbb{I}^J}$ formed by vector fields in $\mathcal{V}$ and such that a smooth family of renormalized iterations of $(X_p)_{p \in \mathbb{I}^J}$ is in $\mathcal{O}_J$.*

The subset $\mathscr{E} \subset \mathcal{F}$ is **para-universal** *if it is J-para-universal for every $J \geq 0$.*

The subset $\mathscr{E} \subset \mathcal{F}$ is **compactly para-universal** *if there exists a bounded open subset $U \subset \mathbb{R}^3$ such that $\mathscr{E}$ satisfies the condition of para-universality with orbits of the renomalizations included in U.*

Remark 1.6. If $\mathscr{E}$ is a Baire space and is (resp. compactly) para-universal, then a topologically generic subset $\mathscr{G} \subset \mathscr{E}$ is formed by (resp. compactly) universal vector fields. Indeed, by 0-para-universality of every non-empty open subset of $\mathscr{E}$, for every non-empty open set $\mathcal{O}$ of $\mathrm{Diff}^\infty_{\mathrm{Leb}+}(\mathbb{D}, \mathbb{R}^2)$, there is a dense subset $\mathscr{D}$ of vector fields in $\mathscr{E}$ which display a Poincaré return map in $\mathcal{O}$. As $\mathcal{O}$ is open, $\mathscr{D}$ can be taken open and dense. We conclude by using that $\mathrm{Diff}^\infty_{\mathrm{Leb}+}(\mathbb{D}, \mathbb{R}^2)$ has a countable base of open subsets.

By the latter remark, the following generalizes Theorem B and enables to capture any finite codimensional phenomenon of $\mathrm{Diff}^\infty_{\mathrm{Leb}+}(\mathbb{D}, \mathbb{R}^2)$ in a small open set of Beltrami fields contained in $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$. In particular, this implies that any bifurcation of conservative maps of the disk that is a finite codimensional phenomenon appears in any open subset of $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ as a renormalization.

Theorem C. *The subset $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ is compactly para-universal.*

For the proof of these theorems, we introduce new general perturbation methods in hydrodynamics allowing to join PDE techniques and deep results from bifurcation theory in dynamical systems. These perturbation methods are implemented for 3-dimensional Beltrami flows and start with two tools from PDEs: global approximation and Cauchy-Kovalevskaya theorems. These tools are rather general and appear not only in Beltrami fields but also in any linear second order elliptic or parabolic equation with analytic coefficients [Bro62, EGFPS19].

[3] A topologically generic subset is a set equal to a countable intersection of open and dense subsets.

Let us also mention that the origin of the global approximation theorems in PDEs is Runge's theorem in complex analysis. The latter was used by Buzzard [Buz97] to find the first example of a wild (conservative) polynomial automorphism of $\mathbb{C}^2$. Hence it is natural to think that the methods presented in this work could be (non-trivially) adapted to prove a complex and discrete time counterpart of our main theorem: the local density of compactly universal maps among transcendental conservative complex automorphisms of $\mathbb{C}^2$. In this setting, universal means that any complex and conservative polynomial automorphism can be approximated by a renormalized iteration of the dynamics.

To prove Theorems B and C, we will extract from the Gonchenko-Shilnikov-Turaev theory [GST07] that it suffices to establish the existence of a non-degenerate unfolding of a multiple homoclinic tangency (see Definition 3.10). To show such an existence, we will make use of the geometry of wild hyperbolic sets (see Definition 4.11) unveiled by the seminal works of Newhouse and Duarte [New74, Dua08] to implement a new perturbative method based on the global approximation theorem [EPS12, EPS15] (which is used as a black box). Indeed, the main difficulty will be to duplicate a homoclinic unfolding in the very rigid class of Euclidean Beltrami fields. In order to do so we will have to display disjoint and homoclinically related horseshoes which can be separated by non nested compact surfaces. See Section 6.1.

To prove Theorem A, we will give a self contained extension of the Cauchy-Kovalevskaya theorem for curl of [EPS12] in Appendix B, which we will use (again as a black box) to unfold heteroclinic links (see Section 3.2). Then the main difficulty will be to handle this simultaneously for two heteroclinic links, while interferences appear and might prevent from the creation of the sought unfolding. To perform this we will use a new linear algebra trick applied to an inverse of the Melnikov operator. See Section 5.1.

2 Contributions to the state of the art on complexity of steady Euler flows

Showing complexity in fluid mechanics is a long standing problem. It has inspired many mathematicians and physicists such as Ruelle, Takens [RT71], Newhouse [NRT79], Feigenbaum [Fei79], Coullet, Tresser and Arneodo [CTA80], who have proposed a link between complexity in fluid mechanics and renormalization in dynamical systems. One aspect of this program is to understand the renormalized iterations of a fluid flow. For instance, [RT71] stated that the Navier-Stokes equations in the space of velocity fields could have a renormalization displaying a strange attractor. This also leads to the famous problem on the typicality of the set of dynamical systems which can be obtained after renormalization of a close to the identity map. This was also influential for the works [BD02, Tur03, GST07], which set up the concept of universality in dynamics.

In more flexible spaces than the Euclidean 3-dimensional one, let us mention the work [TdL21] of Torres de Lizaur who constructed an embedding of any smooth flow into the time-dependent evolution of the Euler PDE on high-dimensional Riemannian manifolds. See also [Vak21] for other embedding results in the context of the Oberbeck-Boussinesq model of the Navier-Stokes equations (on 2-dimensional domains) with external parameters (a space-dependent external force and heat source).

As we mentioned in the previous section, we use the most recent mathematical developments in bifurcation theory to establish the universality, and so the extreme complexity, of some steady fluid flows. More generally, the present work is the first which shows the existence of an Euclidean (not forced) fluid motion which is universal (in the sense of Definition 1.4). As we saw, this question goes back, at least, to the foundational works of Arnold [Arn65, Arn66], and was numerically explored

by several authors [Hén66]. Another notion of universality based on Turing completeness has been promoted by the recent Tao's program [Tao18] to address the blow-up problem for the Euler and Navier-Stokes equations: this is a priori independent from the universality studied in the present paper.

Up to now, the previous works on dynamical phenomena exhibited by Beltrami fields in Euclidean space $\mathbb{R}^3$ include:

1. Periodic orbits (elliptic or hyperbolic) and invariant tori of arbitrary topology [EPS12, EPS15].
2. Invariant tori bounding regions that contain any prescribed number of hyperbolic orbits (but possibly not homoclinically related) [ELPS20].
3. Heteroclinic connections and Smale horseshoes, as an auxiliary result of [EPSR20].
4. A universal Turing machine contained in an invariant plane, see [CMPS21].

It is also known that these structures occur with probability 1 as a consequence of the theory of Gaussian random Beltrami fields [EPSR20]. While a union of these objects forms Arnold's picture, we emphasize that, in general, the previous techniques do not allow to embed them in a Beltrami field in such a way that the given relative position are respected (for instance, having the horseshoe inside the torus).

In particular, it was missing to construct an Euclidean Beltrami field exhibiting Arnold's picture shown in Fig. 1 as a Poincaré section. In contrast, this vision is now a trivial consequence of Theorem B. Indeed by KAM theorem, Arnold's picture is persistent by C^∞_{Leb}-perturbations while Theorem B shows that a locally generic Beltrami field has its set of renormalizations which is dense in the space of conservative maps from the disk to the plane.

Moreover Theorem C goes beyond Arnold's vision in the sense that it establishes the existence of parametric families of Beltrami fields whose corresponding families of renormalized iterations are dense in the set of families of conservative diffeomorphisms from the disk into the plane. For instance, as a consequence of Theorems A and C some of the dynamical phenomena that can arise in Beltrami fields are:

1. By Duarte's theorem [Dua08], there exists a topologically generic subset $\mathcal{G}$ of $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ formed by vector fields exhibiting a horseshoe accumulated by non-degenerate elliptic periodic orbits.
2. By Gorodetski's theorem [Gor12] the subset $\mathcal{G}$ can be chosen so that for every vector field in $\mathcal{G}$ there exists an increasing sequence of such horseshoes whose Hausdorff dimension converges to 3.
3. There exists a dense subset $\mathcal{D}$ in $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ formed by vector fields exhibiting homoclinic tangencies of arbitrarily high multiplicity.
4. For any conservative Taylor expansion J there exists a dense subset in $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ formed by vector fields with a renormalized iteration exhibiting a fixed point with Taylor expansion equal to J.
5. There is a dense subset of $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ formed by Beltrami field X with the following property. There exists a saddle orbit O such that for every $k \geq 0$, there exists a non-degenerate elliptic orbit O_k so that $(O_k)_{k\geq 0}$ converges to a compact set containing O and the period of O_k is asymptotic to a certain constant plus k-times the period of O [GS01b, GS03, GG09]. Moreover, there is a 2-codimensional (Fréchet) manifold in $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ on which all the elliptic orbits O_k and the saddle orbit O persist and satisfy the same asymptotic.

Theorem C also implies that any conservative finite codimensional dynamical phenomena appear densely in the Newhouse domain $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ of Beltrami fields. Let us emphasize that Theorem C provides more a lower bound on the richness and complexity of the Beltrami dynamics, rather than a complete understanding of it: a "typical" Beltrami field in $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ encompasses the dynamical complexity of a dense subset of $\mathrm{Diff}^{\infty}_{\mathrm{Leb}+}(\mathbb{D},\mathbb{R}^2)$. Additionally, as universality was used in [BT19] to prove Herman's positive entropy conjecture (see also [Ber21]), our techniques might be useful to address the following open problem:

Problem 2.1. *Show that there exists an Euclidean Beltrami field with positive metric entropy, in the sense that there exists a bounded invariant set of positive Lebesgue measure formed by points with positive Lyapunov exponent.*

Another open problem is that of the statistical complexity of a typical Beltrami field, which is measured by the emergence [Ber17]. This complexity has been shown to be huge for a generic subset of $\mathrm{Diff}^{\infty}_{\mathrm{Leb}+}(\mathbb{D},\mathbb{R}^2)$ [BB21]:

Problem 2.2. *Show that there exists an Euclidean Beltrami field with a high emergence, in the sense that there exists a bounded invariant set of positive Lebesgue measure so that the ergodic decomposition of the flow restricted to this set is infinite dimensional.*

More generally, a natural question that remains open is whether there exists a dynamics of $\mathrm{Diff}^{\omega}_{\mathrm{Leb}+}(\mathbb{D},\mathbb{R}^2)$ which is not a renormalized iteration of an Euclidean Beltrami field. On the other hand, it is well known that not every analytic volume preserving flow is orbit equivalent to a Beltrami field, see e.g. [CV17].

In strong contrast, using the flexibility of contact geometry, it was shown in [CMPSP21] that, if one allows to deform the metric on $\mathbb{R}^3$ (no longer Euclidean), then any dynamics of $\mathrm{Diff}^{\infty}_{\mathrm{Leb}+}(\mathbb{D},\mathbb{R}^2)$ is a renormalized iteration of a Beltrami field for some adapted Riemannian metric. In the context of high-dimensional fluids, a new h-principle for Reeb flows allows one to embed any finite dimensional dynamics (possibly dissipative) as an invariant subset of a Beltrami field for some special Riemannian metric [Car20, CMPSP].

We finish this section recalling that every finite dimensional family of Euclidean Beltrami fields can be approximated by (the localization) of a finite dimensional family of Beltrami fields on the flat torus $\mathbb{T}^3$ (or respectively on the round sphere $\mathbb{S}^3$) [EPSTdL17]. Accordingly, a direct implication of Theorem C is that Beltrami fields on $\mathbb{T}^3$ or $\mathbb{S}^3$ may exhibit (in particular) any finite codimensional phenomena of $\mathrm{Diff}^{\infty}_{\mathrm{Leb}+}(\mathbb{D},\mathbb{R}^2)$.

3 Structure of the proof

3.1 Perturbations techniques in the space of Beltrami fields

So far the study of bifurcations of Beltrami fields were performed using purely analytic methods, at the unfolding of some integrable systems; for example, in [ELPS20] the authors used the subharmonic Melnikov technique to bifurcate hyperbolic periodic orbits from resonant invariant tori. In this work we introduce new geometric methods to perturb systems which may be non-integrable.

A main issue to develop a general perturbation theory of (Euclidean) Beltrami fields is that we cannot localize the perturbations as one does in differentiable dynamical systems. Although this problem appears also for real analytic dynamics, these form a dense subset of differentiable dynamics, and likewise for the corresponding spaces of parameter families. So one can easily transfer any finite codimensional phenomenon from the differentiable setting to the real analytic setting. In contrast, Beltrami fields are not dense in the space of smooth conservative vector fields (the PDE

equation is an infinite codimensional condition). The following tool will be used as an ersatz for local perturbation techniques:

Theorem 3.1 (Global Approximation Theorem [EPS15, Thm 8.3])**.** *Let $K \subset \mathbb{R}^3$ be a compact set whose complement is connected. Let X be a Beltrami field on an open neighborhood of K. Then for every $N \geq 1$, there exists a Beltrami field $\tilde{X}$ with $\tilde{X} \in \mathscr{B}(\mathbb{R}^3)$ such that:*

$$\|D^\alpha X(z) - D^\alpha \tilde{X}(z)\| < \frac{1}{N}\,, \quad \forall \alpha \in \{0, 1, \ldots, N\}^3 \quad \textit{and} \quad \forall z \in K\,.$$

Let us emphasize that the Global Approximation Theorem does not provide any control on the extended vector field $\tilde{X}$ on the complement of a neighborhood of K.

To increase the multiplicity of the unfolding of homoclinic tangencies, we will use the following immediate (although fruitful) consequence of Theorem 3.1:

Lemma 3.2. *Let K_+ and K_- be two disjoint compact subsets of $\mathbb{R}^3$ with connected complement. Let X be a Beltrami field on an open neighborhood of $K = K_+ \sqcup K_-$. Then for every $N \geq 1$, there exist two Beltrami fields X_+ and X_- in $\mathscr{B}(\mathbb{R}^3)$ such that for any $\pm \in \{-, +\}$ and any $\alpha \in \{0, \ldots, N\}^3$:*

$$\|D^\alpha X_\pm(z) - D^\alpha X(z)\| < \frac{1}{N}\,, \quad z \in K_\pm \quad \textit{and} \quad \|D^\alpha X_\pm(z)\| < \frac{1}{N} \quad z \in K \setminus K_\pm\,.$$

Proof. Let V be the domain where the local Beltrami field X is defined. It is an open neighborhood of K and so it can be split into a disjoint union $V =: V_+ \sqcup V_-$, where V_+ and V_- are neighborhoods of K_+ and K_- respectively. For $\pm \in \{-, +\}$, to obtain the desired Beltrami field $X_\pm$, it suffices to apply Theorem 3.1 to the local Beltrami field $\mathbb{1}_{V_\pm} \cdot X$ on V. To apply Theorem 3.1 we only need to observe that K has connected complement because it is the union of two disjoint compact subsets of $\mathbb{R}^3$ with connected complement. As we could not find any reference for this fact, we prove it in Appendix C. □

Interestingly, this corollary will lead us to explore the geometry of wild sets in order to isolate disjoint invariant compact subsets at which the unfolding will be split.

While the Corollary 3.2 creates two perturbations from a local Beltrami field, the next theorem provides a method to create Beltrami fields with a given normal component on a cylindrical surface:

Theorem 3.3. *Let Γ be an analytic surface in $\mathbb{R}^3$ whose closure $\bar{\Gamma}$ is diffeomorphic to $\mathbb{R}/\mathbb{Z} \times \mathbb{I}$. For any nonempty open set $\mathcal{N}$ of $C^\infty(\bar{\Gamma}, \mathbb{R})$, there exists a vector field $\tilde{X} \in \mathscr{B}(\mathbb{R}^3)$ such that the normal component of $\tilde{X}|_\Gamma$ on Γ is in $\mathcal{N}$.*

Theorem 3.3 is proved below using Theorem 3.1 and the following:

Theorem 3.4 (Cauchy-Kovalevskaya's Theorem for curl)**.** *Let Γ be an analytic surface in $\mathbb{R}^3$ whose closure $\bar{\Gamma}$ is diffeomorphic to $\mathbb{R}/\mathbb{Z} \times \mathbb{I}$. For every $g \in C^\omega(\bar{\Gamma}, \mathbb{R})$ there exists a neighborhood Ω of $\bar{\Gamma}$ in $\mathbb{R}^3$ and a Beltrami field $X \in \mathscr{B}(\Omega)$ such that the normal component of $X|_\Gamma$ on Γ is equal to g.*

Theorem 3.4 will be proved in Appendix B using the Cauchy-Kovalevskaya Theorem as in [EPS12, Theorem 3.1].

Proof of Theorem 3.3. By density of the space of analytic maps, there exists $g \in C^\omega(\bar{\Gamma}, \mathbb{R}) \cap \mathcal{N}$. Then we apply Theorem 3.4 with g, which provides a local Beltrami field X, whose normal component on Γ is equal to g, which in turn can be approximated by a Beltrami field $\tilde{X} \in \mathscr{B}(\mathbb{R}^3)$ on account of Theorem 3.1 (obviously, the complement of a neighborhood of $\bar{\Gamma}$ is connected). Observe that the normal component of $X|_{\bar{\Gamma}}$ is close to g and so it belongs to $\mathcal{N}$. □

Theorems 3.1 and 3.3 are the only results from PDE techniques. These perturbation tools will be used as black boxes. The remaining of the proof will be purely dynamical.

3.2 Structure of the proof of Theorem A: existence of a quadratic, non-degenerate homoclinic tangency unfolding in the class of Beltrami fields

We will show that there are quadratic homoclinic tangencies unfolding non-degenerately nearby a Beltrami field that displays a double strong heteroclinic link. Let us define this configuration. Let $U \subset \mathbb{R}^3$ be a bounded open subset of $\mathbb{R}^3$. A vector field $X \in \Gamma^\infty_{\text{Leb}}(U)$ displays a **heteroclinic link** if there are two saddle periodic orbits γ^+ and γ^-, contained in U, such that an unstable branch of γ^- is equal to a stable branch of γ^+ and it is still contained in U. Denote by Γ the intersection of these branches. The heteroclinic link Γ is **strong** if Γ is orientable[4] and the strong stable foliation of γ^+ in Γ coincides with the strong unstable foliation of γ^- in Γ. See Fig. 2. Observe that the dynamics on a strong heteroclinic link is conjugate to a product of a rotation with a gradient on the interval.

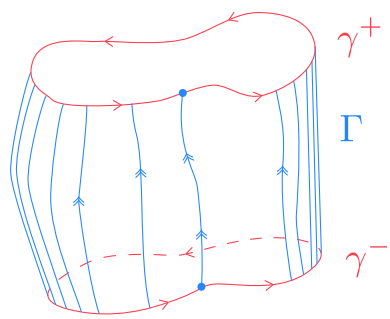

Figure 2: A strong heteroclinic link.

A **double, strong heteroclinic link** is a pair (Γ^+, Γ^-) of strong heteroclinic links between the same saddle periodic orbits (γ^+, γ^-) and so that $\Gamma^+ \subset U$ and $\Gamma^- \subset U$ contain a stable branch of respectively γ^+ and γ^-. See Fig. 3.

Here is the only example available in the literature of a Beltrami field on an open subset of $\mathbb{R}^3$ displaying a **double strong heteroclinic link**.

Example 3.5 ([EPSR20, §5 formula (5.1)]). The following vector field X (written in cylindrical coordinates) is in $\mathscr{B}(\mathbb{R}^3 \setminus \{(0,0)\} \times \mathbb{R})$ and displays a double, strong heteroclinic link:

$$X := \frac{1}{r}\left(\partial_r \psi\, E_z - \partial_z \psi\, E_r + \frac{\psi}{r}\, E_\theta\right), \tag{3.1}$$

where $\psi(z,r) := \cos(z) + 3rJ_1(r)$ with J_1 being the Bessel function of the first kind and order 1. The bounded open set U, with connected complement, can be taken to be any proper subset contained in the intersection of $\mathbb{R}^3 \setminus \{(0,0)\} \times \mathbb{R}$ with a large enough ball.

In order to prove Theorem A we will show that any Beltrami field $X \in \mathscr{B}(U)$, where $U \subset \mathbb{R}^3$ is a bounded open subset of $\mathbb{R}^3$ such that $\mathbb{R}^3 \setminus U$ is connected, displaying a double strong heteroclinic link gives rise to a smooth family $(X_p)_{p\in\mathbb{I}}$ of Beltrami fields in $\mathscr{B}(\mathbb{R}^3)$ which unfolds non-degenerately a quadratic homoclinic tangency. This will be a particular case of Theorem 3.6 where non-degenerate unfoldings of higher multiplicities (see Definition 4.1) are also constructed.

We present now the main tools and the key points for the proof of Theorem 3.6, which states:

Theorem 3.6. *For any $k \geq 0$, there exists a family $(X_p)_{p\in\mathbb{I}^k}$ of Beltrami fields with $X_p \in \mathscr{B}(\mathbb{R}^3)$ that unfolds non-degenerately a homoclinic tangency of multiplicity k at $p = 0$.*

Its proof is divided in four steps.

[4] In particular, the closure $\bar{\Gamma}$ is diffeomorphic to $\mathbb{R}/\mathbb{Z} \times [-1, 1]$.

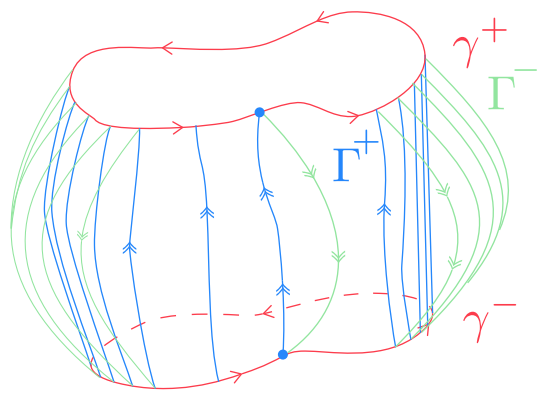

Figure 3: Periodic orbits $\gamma^\pm$ displaying a double (strong) heteroclinic link (Γ^+, Γ^-).

3.2.1 Step 1: The inverse of the Melnikov operator

Let $U \subset \mathbb{R}^3$ be a bounded open subset of $\mathbb{R}^3$ and let $X \in \Gamma^\infty_{\text{Leb}}(U)$ display a strong heteroclinic link $\Gamma \subset U$ between periodic saddle orbits γ^+ and γ^-. We recall that the closure $\bar{L}$ of the half strong stable manifold of a point in γ^+ in Γ is diffeomorphic to a segment. Thus there is a disk $\Sigma \subset U$ which contains $\bar{L}$ and intersects transversally Γ. Up to reducing Σ, assume that Σ is transverse to the vector field X.

Observe that $\Sigma \cap \Gamma$ is a heteroclinic link for the Poincaré map f at Σ. A small perturbation $W \in \Gamma^\infty_{\text{Leb}}(\mathbb{R}^3)$ of the vector field X induces a small perturbation f_W of the Poincaré return map on Σ. In order to study the unfolding of the heteroclinic link Γ, we shall analyze the distance between the stable and unstable manifolds of the perturbed Poincaré return map at Σ using the **Melnikov operator**.

Roughly speaking (see Appendix A.1 for rigorous details), there is a chart of a neighborhood of L in Σ which sends L to $\mathbb{R} \times \{0\}$ and so that seen in this chart the dynamics is the translation by $(1, 0)$. Then for a perturbation of f, the link breaks as two curves which approach (on some fundamental domain) the graphs of $\mathbb{Z}$-periodic functions as the perturbation goes to zero. The asymptotic of the difference between these two functions is the **Melnikov operator** $\mathcal{M} : W \in \Gamma^\infty_{\text{Leb}}(\mathbb{R}^3) \mapsto \mathcal{M}(W) \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$.

Actually, $\mathcal{M}(W)$ depends only on $W|_\Gamma$: thus, passing to the quotient, we obtain an operator $\mathcal{M} : C^\infty(\Gamma, \mathbb{R}^3) \mapsto C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ still denoted by $\mathcal{M}$.

We will show in Appendix A.2 that the Melnikov operator has a right inverse:

Proposition 3.7. *There exists a linear and continuous operator*

$$\mathcal{I} : g \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}) \to \mathcal{I}(g) \in C^\infty(\Gamma, \mathbb{R}^3)$$

such that $\mathcal{M}(\mathcal{I}(g)) = g$ and $\mathcal{I}(g)$ is compactly supported.

To prove this proposition, we will identify Γ with the annulus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$, we will consider a bump function ψ defined on Γ such that the integral of ψ along each X-orbit in Γ equals 1. Then for $g \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, with $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and $p_2 : \Gamma \equiv \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ the canonical projections, we will prove that $\mathcal{I}(g) := \psi \cdot g \circ \pi \circ p_2$ satisfies the sought properties. The idea is inspired by [BT19].

3.2.2 Step 2: Transforming a normal datum on a cylinder into a Beltrami vector field

We would like to complete (an approximation of) the normal vector field obtained in the first step into a Beltrami field on $\mathbb{R}^3$. To this end we use Theorem 3.3 to obtain the following:

Theorem 3.8. *Let $U \subset \mathbb{R}^3$ be an open subset of $\mathbb{R}^3$ and let $X \in \mathscr{B}(U)$ display a strong heteroclinic link $\Gamma \subset U$. Let Σ be a Poincaré section endowed with coordinates, chosen as for Proposition 3.7. This defines the Melnikov operator $\mathcal{M}$. Then for every nonempty open set $\mathcal{U} \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ there exists a Beltrami field $W \in \mathscr{B}(U)$ whose Melnikov function $\mathcal{M}(W)$ belongs to $\mathcal{U}$.*

Proof. Let $g \in \mathcal{U}$. By Proposition 3.7, there is $\mathcal{I}(g) \in C^\infty(\Gamma, \mathbb{R}^3)$ such that $\mathcal{M}(\mathcal{I}(g)) = g$. Applying Theorem 3.3 to $\mathcal{I}(g)$, we obtain a Beltrami field W whose restriction $W|_\Gamma$ is arbitrarily close to $\mathcal{I}(g)$. By continuity of $\mathcal{M}$, we obtain that $\mathcal{M}(W) \equiv \mathcal{M}(W|_\Gamma)$ is close to g and so in $\mathcal{U}$. □

3.2.3 Step 3: Prescribing jets of Melnikov functions on a double strong heteroclinic link

Now we start with a Beltrami field $X \in \mathscr{B}(U)$, for some bounded open set $U \subset \mathbb{R}^3$, displaying a **double** strong heteroclinic link (Γ^+, Γ^-), with $\Gamma^+, \Gamma^- \subset U$. Let $\mathcal{M}_+$ and $\mathcal{M}_-$ be the Melnikov operators, associated, respectively, to Γ^+ and Γ^-.

Our aim is to obtain a Beltrami field unfolding on $\mathbb{R}^3$ which breaks Γ^+ to a transverse heteroclinic intersection and breaks Γ^- to a non-degenerate unfolding of a quadratic heteroclinic tangency.

To do so, we will control the jets of both Melnikov functions $\mathcal{M}_+$ and $\mathcal{M}_-$. Theorem 3.8 can give, separately, when applied to Γ^+, a Beltrami field X_+ whose Melnikov function $\mathcal{M}_+(X_+)$ corresponds to a transverse heteroclinic intersection, and, when applied to Γ^-, a Beltrami field X_- whose Melnikov function $\mathcal{M}_-(X_-)$ corresponds to a (quadratic) heteroclinic tangency. The difficulty is that we cannot just consider the Beltrami field $X_+ + X_-$, because we do not have any control on how X_+ and X_- act, respectively, on Γ^- and Γ^+. Indeed the non-disjointedness of Γ^+ and Γ^- forbids the application of Lemma 3.2.

We overcome this difficulty by obtaining a **non-degenerate unfolding of any multiplicity** of this double strong heteroclinic link. In order to state this precisely, for $g \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, $\alpha \in \mathbb{R}/\mathbb{Z}$ and $k \geq 0$, we identify the k-jet of the function g at α with a vector in $\mathbb{R}^{k+1}$:

$$J^k_\alpha(g) := (a_0, a_1, \dots, a_k) \in \mathbb{R}^{k+1} \quad \text{with} \quad a_i = D^i_\alpha g \ , \quad \forall 0 \leq i \leq k \ .$$

Proposition 3.9. *Fix $k \geq 0$ and let $\alpha \in \mathbb{R}/\mathbb{Z}$. There exist $\beta \in \mathbb{R}/\mathbb{Z}$ and a $(k+3)$-dimensional space $\mathcal{W}$ of Beltrami vector fields defined on U such that the following map is an isomorphism:*

$$W \in \mathcal{W} \mapsto \Big(J^1_\alpha(\mathcal{M}_+(W)), J^k_\beta(\mathcal{M}_-(W)) \Big) \in \mathbb{R}^{k+3} \ . \tag{3.2}$$

The proof of this proposition will be presented in Section 5.1. It uses Theorem 3.8 and a new trick which allows us to avoid a computer-assisted estimate as in [EPSR20]. The trick (see Section 5.1) consists of considering three distinct points θ_1, θ_2, θ_3 in $\mathbb{R}/\mathbb{Z}$. Then by Theorem 3.8, the range of $J^1_\alpha \mathcal{M}_+|_{\mathscr{B}(U)}$ is 2-dimensional. We will show using again Theorem 3.8 that the range of each map $J^k_{\theta_i} \mathcal{M}_-|_{\mathscr{B}(U)}$ is $(k+1)$-dimensional and that their product has a range which is $(3k+3)$-dimensional. Equivalently, the kernel of $J^1_\alpha \mathcal{M}_+|_{\mathscr{B}(U)}$ is two-codimensional while the kernels of the $(J^k_{\theta_i} \mathcal{M}_-|_{\mathscr{B}(U)})_{1 \leq i \leq 3}$ are $(k+1)$-codimensional and in general position. By an elementary algebraic argument, this will imply that the two-codimensional kernel of $J^1_\alpha \mathcal{M}_+|_{\mathscr{B}(U)}$ must be in general position with one of the 3 latter kernel of $J^k_{\theta_i} \mathcal{M}_-|_{\mathscr{B}(U)}$, and so that the product $J^1_\alpha \mathcal{M}_+ \times J^k_{\theta_i} \mathcal{M}_-|_{\mathscr{B}(U)}$ is onto. This implies immediately Proposition 3.9.

3.2.4 Step 4: Existence of homoclinic tangency unfolding non-degenerately in $\mathscr{B}(\mathbb{R}^3)$

By the standard Poincaré-Melnikov Theorem (see Appendix A), the Melnikov operators $\mathcal{M}_+$ and $\mathcal{M}_-$ are the first order approximations of the displacement operators displ_+ and displ_- respectively.

For $\pm \in \{-,+\}$, each displacement operator $\mathrm{displ}_\pm$ associates to a sufficiently small perturbation of X the distance function between the perturbed local stable and unstable manifolds at the heteroclinic link $\Gamma^\pm$, defined over some fundamental domain.

Together with Proposition 3.9, we then deduce that there exists a arbitrarily small neighborhood $\mathcal{N}$ of 0 in $\Gamma^\infty_{\mathrm{Leb}}(\mathbb{R}^3)$ such that the map $\mathcal{N} \cap \mathcal{W} \to \Big(J^1_\alpha(\mathrm{displ}_+(W)), J^k_\beta(\mathrm{displ}_-(W))\Big) \in \mathbb{R}^{k+3}$ is a diffeomorphism onto a neighborhood of 0.

For $k = 2$, we can then select a family of Beltrami fields $(X_p = X + W_p)_{p\in\mathbb{I}}$ with $W_p \in \mathcal{N} \cap \mathcal{W}$ so that each $X_p \in \mathscr{B}(U)$ breaks Γ^+ to a transverse heteroclinic intersection and the family $(X_p)_{p\in\mathbb{I}}$ of Beltrami fields in $\mathscr{B}(U)$ unfolds Γ^- to a quadratic heteroclinic tangency at some parameter p_0.

Thanks to the transverse heteroclinic intersection, we can use the para-inclination Lemma (see [Ber16]) to study the displacement operator between local stable and unstable manifolds of the same saddle point. It will imply that any sufficiently small 2-jet of the displacement function will still be realised at some orbit. Thus, there exists a family of Beltrami vector fields $(X + \tilde{W}_p)_{p\in\mathbb{I}}$ in $\mathscr{B}(U)$, with $\tilde{W}_p \in \mathcal{N} \cap \mathcal{W}$ and the family $(\tilde{W}_p)_{p\in\mathbb{I}}$ C^∞-close to the family $(W_p)_{p\in\mathbb{I}}$, unfolding non-degenerately a homoclinic quadratic tangency at a parameter p_1 close to p_0. Now put $Y := X + \tilde{W}_{p_1}$ and $Z := \partial_p \tilde{W}_{p_1}$. Since unfolding non-degenerately a quadratic homoclinic tangency is a C^2-robust condition on families, the family $(Y + p \cdot Z)_{p\in\mathbb{I}}$ unfolds non-degenerately a homoclinic quadratic tangency at $p = 0$.

Up to shrinking U if necessary, we can assume it is bounded and such that $\mathbb{R}^3 \setminus U$ is connected. Then the Global Approximation Theorem 3.1 asserts the existence of fields $\tilde{Y}$ and $\tilde{Z}$ in $\mathscr{B}(\mathbb{R}^3)$ such that their restrictions on U are close to Y and Z, respectively. Since unfolding non-degenerately a quadratic homoclinic tangency is a robust condition, we conclude that $(\tilde{Y} + p \cdot \tilde{Z})_{p\in\mathbb{I}}$ is the sought family of Beltrami fields in $\mathscr{B}(\mathbb{R}^3)$.

3.3 Structure of the proofs of Theorems B and C: existence and density of (para)-universal dynamics

The strategy to prove Theorems B and C is to find in any non-empty open subset $\mathcal{U} \subset \mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ and for any $k \geq 1$ a family $(X_p)_{p\in\mathbb{I}^k}$ of Beltrami vector fields $X_p \in \mathcal{U}$ which unfolds non-degenerately k orbits of quadratic homoclinic tangencies for k saddle cycles which are homoclinically related. Then we will restate the seminal Gonchenko-Shilnikov-Turaev theory [Tur15, GST07] to show the existence and density of universal and para-universal dynamics, which are the statements of Theorems B and C.

The main technical novelty of the proof of Theorems B and C is to show the existence of a family $(X_p)_{p\in\mathbb{I}^k}$ of Beltrami vector fields $X_p \in \mathcal{U}$ which unfolds non-degenerately k orbits of quadratic homoclinic tangencies for k saddle cycles which are homoclinically related. This will be done by exploring the beautiful work of Duarte to obtain homoclinically related but disjoint wild hyperbolic sets in order to apply perturbative Lemma 3.2.

Before giving more details on these proofs, let us now formalize the statements to be proved, by recalling standard definitions in bifurcation theory of surface diffeomorphisms or 3-dimensional flows.

Definition 3.10. *Let $J \geq 1$ and let $(f_p)_{p\in\mathbb{I}^J}$ be a family of surface diffeomorphisms. Assume that f_0 has a saddle periodic orbit O which displays J different quadratic homoclinic tangencies at $q_j \in W^s(O; f_0) \cap W^u(O; f_0)$ for $1 \leq j \leq J$. This* **J-tuple of homoclinic tangencies unfolds non-degenerately** *at $p = 0$ if, with $\mu_j(p)$ a relative position between $W^s(O; f_p)$ and $W^u(O; f_p)$ at q_j, the following map is a local diffeomorphism at $p = 0$:*

$$p \in \mathbb{I}^J \mapsto (\mu_j(p))_{1\leq j\leq J} \in \mathbb{R}^J \ .$$

Let us state the vector field counterpart of Definition 3.10.

Definition 3.11. *A* **family** $(X_p)_{p\in\mathbb{I}^J}$ **of vector fields displays a** J**-tuple of homoclinic tangencies that unfolds non-degenerately at** $p=0$ *if there is a Poincaré section* Σ *for* X_0 *which defines a family of surface diffeomorphisms* $(f_p)_{p\in(-\epsilon,\epsilon)^J}$ *which displays a saddle point with a* J*-tuple of homoclinic tangencies that unfolds non-degenerately at* $p=0$*.*

We shall show that the proof [GST07] on universal conservative C^r-surface dynamics actually applies to the following broader setting:

Definition 3.12. *Let* $\mathcal{F}$ *be a Fréchet manifold formed by smooth symplectic surface diffeomorphisms or smooth conservative flows of a 3-manifold. The space* $\mathcal{F}$ *is* **GST-wild** *if the following property holds true. For every* $J\geq 1$ *and every non-empty open subset* $\mathscr{U}$ *of* $\mathcal{F}$*, there is a smooth parameter family of dynamics in* $\mathscr{U}$ *parametrized by* $\mathbb{I}^J$ *which displays a* J*-tuple of quadratic homoclinic tangencies that unfolds non-degenerately at* $p=0$*.*

It follows from classical techniques [Kup63, Sma63, BT86] that the Newhouse domains of C^∞_{Leb} or C^ω_{Leb}-surface dynamics and the ones of C^∞_{Leb} or C^ω_{Leb}-flows on $\mathbb{R}^3$ are GST-wild.

The Gonchenko-Shilnikov-Turaev theorem states that universality is generic in GST wild spaces and that they are para-universal. See also [GT10]. These notions of universality are defined in the setting of vector fields in Definitions 1.4 and 1.5 using the notion of renormalized iteration defined in 1.3. These are canonically translated into the discrete setting:

Definition 3.13. *A* **renormalized iteration** G *of a* C^∞_{Leb}*-diffeomorphism* f *of a surface* M *equals to a return map at some disk of* M *in rescaled coordinates. More precisely, there are nested disks* $\check{\Sigma}\Subset\Sigma\subset M$*, an integer* $N\geq 1$ *and coordinates* $\phi:\mathbb{R}^2\to\Sigma$ *sending* $\mathbb{D}$ *onto* $\check{\Sigma}$ *such that* $f^N(\check{\Sigma})\subset\Sigma$ *and* $G=\phi^{-1}\circ f^N\circ\phi|_{\mathbb{D}}$ *is symplectic.*

Definition 3.14 ([BD02, Tur03, GST07])**.** *A* C^∞_{Leb}*-diffeomorphism* f *of a surface* M *is* **universal** *if the set of renormalized iterations of* f *is dense in* $\text{Diff}^\infty_{\text{Leb}+}(\mathbb{D},\mathbb{R}^2)$*. The notions of compactly universal and para-universal are defined similarly.*

Remark 3.15. A surface map f is (resp. compactly) universal iff its suspension defines a vector field which is (resp. compactly) universal.

We will extract from part of the proof of [Tur03, GST07] the following abstract result:

Theorem 3.16 (Gonchenko-Shilnikov-Turaev)**.** *Let* $\mathcal{F}$ *be a Fréchet submanifold of the space of smooth conservative vector fields or the space of smooth symplectic surface diffeomorphisms which is GST wild. Then there exists a topologically generic subset* $\mathcal{G}\subset\mathcal{F}$ *formed by compactly universal dynamics. Moreover* $\mathcal{F}$ *is para-universal.*

This statement did not appear written like this in Gonchenko-Shilnikov-Turaev works, but it can be deduced from a simple combination of their lemmas following a general scheme they use. This scheme consists of extracting from a k-tuple of quadratic homoclinic tangencies a homoclinic tangency of multiplicity k which unfolds non-degenerately [GST07]. Then from this non-degenerate unfolding of homoclinic tangency of multiplicity k, a family of renormalizations equal to the full family of Hénon maps of degree k emerges [GST07]. Finally they use Turaev's Theorem which asserts that any map or family of maps in $\text{Diff}^\infty_{\text{Leb}}(\mathbb{D},\mathbb{R}^2)$ can be approximated by a renormalization of an iterate of a Hénon map of some large degree [Tur03]. This beautiful scheme will be detailed in Section 4.2.

The main issue of the proof of Theorems B and C is to reach the setting of the Gonchenko-Shilnikov-Turaev theory using only perturbative Lemma 3.2, as given by the following:

Theorem D. *The open set $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ in $\mathscr{B}(\mathbb{R}^3)$ is GST wild.*

To show this theorem, we fix a nonempty open subset $\mathscr{U}$ of $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$. By definition of $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$, there are $X_0 \in \mathscr{U}$ and a Beltrami field W_1 such that the family $(X_p = X_0 + p \cdot W_1)_{p\in\mathbb{I}}$ unfolds a quadratic homoclinic tangency at $p = 0$. Then we will explore the powerful work of Duarte [Dua99, Dua08] to construct a hyperbolic basic set with a robust homoclinic tangency. From this we will extract a sub-parameter family which displays a homoclinically related but disjoint hyperbolic set displaying a robust homoclinic tangency. Using the fact that these hyperbolic sets are disjoint, we will cutoff the perturbation W_1 using perturbative Lemma 3.2 to create a new Beltrami field W_2 on $\mathbb{R}^3$. This will create a two parameter family $(X^1_p = X_0 + p_1 \cdot W_1 + p_2 \cdot W_2)_{p=(p_1,p_2)\in\mathbb{I}^2}$ unfolding non-degenerately two quadratic homoclinic tangencies at a parameter p close to 0. We will iterate inductively this construction to obtain the sought k-parameter families. The proof will be done in Section 6.

Now let us remark that Theorem 3.16 and Theorem D immediately imply Theorems B and C:

Proof of Theorems B and C. By Theorem D, the Baire space $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ is GST wild, and so it satisfies the assumptions of Theorem 3.16. Thus a topologically generic subset of $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ is formed by compactly universal vector fields and $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ is compactly para-universal as stated in Theorems B and C. □

4 Elements of bifurcation theory

4.1 Involved topologies

Let us precise the topologies involved in the statements of the main results.

For every $k \geq 0$, let $B_k \subset \mathbb{R}^3$ be the ball centered at zero of radius k. We recall that $\Gamma^\infty_{\text{Leb}}(\mathbb{R}^3)$ endowed with the family $(N_k)_k$ of semi-norms $N_k : X \mapsto \|X|_{B_k}\|_{C^k}$ is a Fréchet space. Equivalently, the topology is defined by the following complete distance:

$$d_{C^\infty} : (X, Y) \in \Gamma^\infty_{\text{Leb}}(\mathbb{R}^3)^2 \mapsto \sum_{k=0}^{\infty} \frac{1}{2^k} \min\left(N_k(X - Y), 1\right) .$$

Note that the subspace of Beltrami fields is closed in $\Gamma^\infty_{\text{Leb}}(\mathbb{R}^3)$, but the subspace of Beltrami fields with sharp decay is not closed.

Hence we endow $\mathscr{B}(\mathbb{R}^3)$ with the following family $(\tilde{N}_\alpha)_{\alpha\in\mathbb{N}^3}$ of norms:

$$\tilde{N}_\alpha : X \mapsto \sup_{x\in\mathbb{R}^3} (1 + \|x\|) \cdot \|D^\alpha X(x)\|$$

with which it is a Fréchet space. Equivalently the topology is defined by the following complete distance:

$$d_{\mathscr{B}} : (X, Y) \in \mathscr{B}(\mathbb{R}^3)^2 \mapsto \sum_{\alpha\in\mathbb{N}^3} 2^{-|\alpha|} \min\left(\tilde{N}_\alpha(X - Y), 1\right)$$

4.2 The Gonchenko-Shilnikov-Turaev theory

4.2.1 Constructing non-degenerate unfoldings of homoclinic tangencies of any multiplicity

A key configuration in the Gonchenko-Shilnikov-Turaev's theory is that of homoclinic tangencies of higher multiplicity. Let us state this notion in the setting of surface diffeomorphism; using Poincaré return maps we will translate it into the context of 3-dimensional flows at the end.

Definition 4.1. *For $k \geq 1$, a saddle periodic point P of a surface diffeomorphism f displays a* **homoclinic tangency of multiplicity** *k, if the local stable and unstable manifolds of P are tangent at a point Q and the order of contact of the tangency is k. More precisely this means that there is a chart of a neighborhood N of Q which identifies $Q \equiv 0$, $W^u_{loc}(P) \cap N \equiv \mathbb{I} \times \{0\}$ and $W^s_{loc}(P) \cap N$ with the graph of a function $w : \mathbb{I} \to \mathbb{R}$ such that:*

$$0 = w(0) = Dw(0) = \cdots = D^k w(0) \quad \text{and} \quad D^{k+1}w(0) = 1 \, .$$

Example 4.2. We notice that a quadratic homoclinic tangency is a homoclinic tangency of multiplicity 1. A cubic homoclinic tangency is a homoclinic tangency of multiplicity 2.

Now let $(f_p)_{p\in\mathbb{I}^k}$ be an unfolding of a surface diffeomorphism $f = f_0$ displaying a homoclinic tangency of multiplicity $k \geq 1$ for a saddle point P. Then the periodic point P persists for p small, and its local stable and unstable manifolds as well. This induces an unfolding of the homoclinic tangency.

Definition 4.3. *The unfolding of the homoclinic tangency of multiplicity k is* **non-degenerate** *if there are parameter dependent coordinates of a neighborhood N of Q which identify $Q \equiv 0$, $W^u_{loc}(P; f_p) \cap N \equiv \mathbb{I} \times \{0\}$ and $W^s_{loc}(P; f_p) \cap N$ with $\{(x, w_p(x)) : x \in \mathbb{I}\}$ where $(w_p)_{p\in\mathbb{I}^k}$ satisfies that:*

- *it holds $w_0(0) = Dw_0(0) = \cdots = D^k w_0(0) = 0$ and $D^{k+1}w_0(0) \neq 0$,*
- *the map $p \in \mathbb{I}^k \mapsto (D^j w_p(0))_{0\leq j\leq k-1} \in \mathbb{R}^k$ is a diffeomorphism onto its image.*

When the multiplicity of the tangency is 1, or equivalently when the homoclinic tangency is quadratic, saying that the unfolding is non-degenerate means roughly speaking that the **relative position** $p \mapsto w_p(0)$ of the continuation of the stable and unstable manifolds at the tangency point has non-zero derivative at $p = 0$.

The following is the first Theorem of the GST-theory. It enables to transform a non-degenerate unfolding of J-quadratic homoclinic tangencies into a non-degenerate unfolding of a homoclinic tangency of multiplicity J:

Theorem 4.4. *Let $(f_p)_{p\in\mathbb{I}^J}$ be a family of surface diffeomorphisms which unfolds non-degenerately at $p = 0$ a J-tuple of (different) quadratic homoclinic tangencies of a saddle orbit O. Then there is $p_1 \in \mathbb{I}^J$ arbitrarily small such that at this parameter, the (hyperbolic continuation of the) saddle O displays a homoclinic tangency of multiplicity J. Moreover this homoclinic tangency of multiplicity J is non-degenerately unfolded by the family $(f_p)_{p\in\mathbb{I}^J}$.*

Proof. We actually apply inductively on $2 \leq k \leq J$ the following lemma:

Lemma 4.5 ([GST07, Lem. 5])**.** *Let $k \geq 2$. Let $(f_p)_{p\in\mathbb{I}^k}$ be a C^∞-family of symplectic surface diffeomorphisms such that f_0 has a saddle orbit O which displays a homoclinic tangency of multiplicity $k-1$ and a quadratic homoclinic tangency. Assume that the first unfolds non-degenerately in the family $(f_p)_{p\in\mathbb{I}^{k-1}\times\{0\}}$ while the second unfolds non-degenerately in the family $(f_p)_{p\in\{0\}\times\mathbb{I}}$. Then there is $\tilde{p} \in \mathbb{I}^k$ arbitrarily small, such that O displays a homoclinic tangency of multiplicity k at $p = \tilde{p}$ which unfolds non-degenerately in $(f_p)_{p\in\mathbb{I}^k}$.*

The mechanism of the proof of Lemma 4.5 is clearly illustrated in Figure 11 in [GST07]. □

4.2.2 Hénon maps as renormalized iterates nearby homoclinic unfoldings

The next key ingredient of the Gonchenko-Shilnikov-Turaev theory involves the conservative Hénon family.

Definition 4.6. *A conservative Hénon map of degree d is a map of the form:*

$$H_g := (x, y) \mapsto (g(x) - y, x);$$

with g a polynomial map of degree d. If $\mathbb{R}_k[X]$ denotes the space of polynomial maps of degree $d \leq k$, the full family of conservative Hénon map is $(H_g)_{g \in \mathbb{R}_k[X]}$.

The second theorem of their theory states that nearby a non-degenerate unfolding of a homoclinic tangency of multiplicity k, there are renormalizations close to the family of conservative Hénon maps of degree $k-1$:

Theorem 4.7. *Let $(f_q)_{q \in \mathbb{I}^k}$ be a smooth family of symplectic diffeomorphisms of a surface S which unfolds non-degenerately a homoclinic tangency of multiplicity k at $q = 0$. For every $M > 0$, let:*

$$\mathbb{B}_k(M) := \{g(X) = \sum_{i=0}^{k-1} a_i \cdot X^i \in \mathbb{R}[X] : a_i \in [-M, M],\ \forall 0 \leq i \leq k-1\}$$

Then for every $\epsilon > 0$, there exist:

- *an embedding $Q : \mathbb{B}_k(M) \hookrightarrow [-\epsilon, \epsilon]^k$ and an integer $N > 0$,*
- *a smooth family $(C_p)_{p \in \mathbb{B}_k(M)}$ of charts $C_p : (-M, M)^2 \hookrightarrow S$ of constant determinant onto a small open set nearby the tangency point,*

such that $(H_g|_{(-M,M)^2})_{g \in \mathbb{B}_k(M)}$ is ϵ-C^∞-close to the family $(\mathcal{R}f_g)_{g \in \mathbb{B}_k(M)}$ formed by $\mathcal{R}f_g := C_g^{-1} \circ f_{Q(g)}^N \circ C_g$.

Proof. This statement is almost the same as Lemma 6 in [GST07], the only difference is that we are dealing with polynomials which do not need to be unitary. Nevertheless, using a rescaling of the phase space by a large factor $\Lambda > 0$, $(x, y) = (\Lambda^{-1}X, \Lambda^{-1}Y)$, the theorem can be immediately deduced from this lemma with the following setting:

Lemma 4.8 (Lemma 6 [GST07])**.** *Under the assumptions of Theorem 4.7, for every $M, \Lambda, \epsilon > 0$, there are:*

- *an embedding $Q : \mathbb{I}^k \hookrightarrow [-\epsilon, \epsilon]^k$ and an integer $N > 0$,*
- *a smooth family $(C_p)_{p \in \mathbb{B}_k(M)}$ of charts $C_p : (-\Lambda^{-1}M, \Lambda^{-1}M)^2 \hookrightarrow S$ of constant determinant onto a small open set nearby the tangency point,*

such that $\mathcal{R}f_p := C_p^{-1} \circ f_{Q(p)}^N \circ C_p$ forms a family $(\mathcal{R}f_p)_{p \in \mathbb{I}^k}$ which is ϵ-C^∞-close to

$$(H_{g_p}|_{(-\Lambda^{-1}M, \Lambda^{-1}M)^2})_{p \in \mathbb{I}^k}$$

with $g_p(x) \mapsto \sum_{i=0}^{k-1} M \cdot \Lambda^{i-1} \cdot p_i \cdot x^i + x^k$ and $p = (p_0, \ldots, p_{k-1})$.

□

4.2.3 Proof of Theorem 3.16

The final result of this theory is due to Turaev. It says that renormalizations of Hénon maps are dense in $\mathrm{Diff}^\infty_{\mathrm{Leb}+}(\mathbb{D},\mathbb{R}^2)$ and likewise for the families.

Theorem 4.9 ([Tur03], Lemma 9 [GST07])**.** *For every $J \geq 0$ and every non-empty open subset $\mathcal{O}_J \subset \mathrm{Diff}^\infty_{\mathrm{Leb}+}(\mathbb{D},\mathbb{R}^2)_{\mathbb{I}^J}$, there exist $k > 0$, a $C^\infty_{\mathrm{Leb}+}$-family of polynomials $(g_p)_{p\in\mathbb{I}^J}$ of degree $< k$, an iteration $N \geq 1$ and an affine change of coordinates ϕ of $\mathbb{R}^2$ such that $(\phi^{-1} \circ H^N_{g_p} \circ \phi|_{\mathbb{D}})_{p\in\mathbb{I}^J}$ is in $\mathcal{O}_J$, where H_{g_p} is the Hénon-like map $(x, y) \mapsto (g_p(x) - y, x)$.*

We are now ready for:

Proof of Theorem 3.16. Let $\mathscr{E}$ be a Baire subset of $\Gamma^\infty_{\mathrm{Leb}}(\mathbb{R}^3)$ which is GST wild. By Remark 1.6, it suffices to show that $\mathscr{E}$ is para-universal. Hence it suffices to prove that for any $J \geq 0$, and any non-empty subsets $\mathscr{U} \subset \mathscr{E}$ and $\mathscr{O}_J \subset \mathrm{Diff}^\infty_{\mathrm{Leb}}(\mathbb{D},\mathbb{R}^2)_{\mathbb{I}^J}$, there exists a smooth J-parameter family in $\mathscr{U}$ which has a renormalization in $\mathscr{O}_J$. By Theorem 4.9, there exist a $k \geq 0$ and a smooth J-parameter family of conservative Hénon maps $(H_{g_p})_{p\in\mathbb{I}^J}$ which displays a renormalization in $\mathscr{O}_J$. By Theorem 4.4 and the definition of the GST wild property, there exists a k-parameter family of dynamics in $\mathscr{U}$ which unfolds non-degenerately a homoclinic tangency of multiplicity k of a saddle point. By Theorem 4.7, we can extract from this k-parameter family a J-parameter subfamily which is arbitrarily close to $(H_{g_p})_{p\in\mathbb{I}^J}$ and so displays a second renormalization in $\mathscr{O}_J$. As a renormalization of a renormalization is a renormalization, we obtained the sought result. □

4.3 Geometry of wild hyperbolic sets after Newhouse and Duarte

We recall that a **hyperbolic basic set** for a diffeomorphism is an invariant compact set Λ which is transitive, hyperbolic and locally maximal (it is the maximal invariant set in one of its neighborhoods $\mathscr{U}$). This generalizes the notion of saddle periodic orbits, which are finite sets, to compact sets. All the points $z \in \Lambda$ have local stable and unstable manifolds, which are injectively immersed submanifolds depending continuously on z in the C^∞-topology. The stable and unstable manifolds of the orbit of z are the union of stable and unstable manifolds of the iterate of z; these are also injectively immersed submanifolds.

By Anosov's Theorem, hyperbolic basic sets are structurally stable: for every smooth (actually C^1 suffices) perturbation of the dynamics, the maximal invariant set $\tilde{\Lambda}$ in $\mathscr{U}$ for the perturbed dynamics is a hyperbolic basic set, and the dynamics on $\tilde{\Lambda}$ is **orbit equivalent** to the one on Λ. There is a homeomorphism $h : \Lambda \to \tilde{\Lambda}$ close to the identity which sends each orbit in Λ to an orbit in $\tilde{\Lambda}$ called its **hyperbolic continuation**.

We say that two hyperbolic basic sets Λ_1 and Λ_2 are **homoclinically related** if there are $z_1 \in \Lambda_1$ and $z_2 \in \Lambda_2$ such that the stable manifold of z_i intersects transversally the unstable manifold of z_j for $1 \leq i \neq j \leq 2$.

We will use many times the following generalization of the classical inclination Lemma:

Lemma 4.10 (Para-inclination lemma [Ber16, Lemma 1.7])**.** *Let $r \geq 1$ and let $U \Subset \mathbb{R}^m$. Suppose that $(f_p)_p$ is a smooth family of diffeomorphisms f_p of U which leaves invariant a basic hyperbolic set Λ_p. Let $z \in \Lambda_0$ and $(D_p)_p$ be a smooth family of C^r manifolds of the same dimension as $W^u_{loc}(z; f_p)$ and such that $(D_p)_p$ intersects transversally $(W^s_{loc}(z; f_p))_p$ at a smooth curve of points $(y_p)_p$. Then, for every $\epsilon > 0$ and for every n large, the ϵ-neighborhood of $f^n_p(y_p)$ in $f^n_p(D_p)$ is a submanifold D^n_p that is C^∞-close to $W^u_\epsilon(f^n_p(z); f_p)$ and $(D^n_p)_p$ is C^∞-close to $(W^u_{loc}(f^n_p(z); f_p))_p$.*

The concept of homoclinic tangency can be generalized to hyperbolic basic sets. We say that a hyperbolic basic set Λ displays a **homoclinic tangency** if there is an orbit $O \subset \Lambda$ whose stable manifold is tangent to its unstable manifold.

Newhouse discovered the first mechanism that yields **wild** hyperbolic basic sets:

Definition 4.11 (Robust quadratic tangency)**.** *Let f be a surface C^2-diffeomorphism leaving invariant a hyperbolic basic set Λ. We say that Λ is* **wild** *if it displays a* **robust quadratic homoclinic tangency***: there are continuous families $(W^s_{loc}(z;f))_{z\in\Lambda}$ and $(W^u_{loc}(z;f))_{z\in\Lambda}$ of local stable and unstable manifolds and a C^2-open neighborhood $\mathcal{N}$ of f satisfying the following property.*

For every $\tilde{f} \in \mathcal{N}$, the hyperbolic continuation $\tilde{\Lambda}$ of Λ is well defined and there exists a point $z \in \Lambda$ and an iteration $n \geq 0$ such that the continuation of the local stable and unstable manifolds $W^s_{loc}(z;\tilde{f})$ and $W^u_{loc}(\tilde{f}^n(z);\tilde{f})$ display a quadratic homoclinic tangency.

Let $(f_p)_{p\in\mathbb{I}}$ be a family containing $f = f_0$. We say that this family **unfolds non-degenerately the homoclinic tangencies of** Λ*, if it unfolds non-degenerately each tangency between $W^s_{loc}(z;f_q)$ and $W^u_{loc}(z';f_q)$, among $z, z' \in \Lambda$ and $q \in \mathbb{I}$ small.*

Proposition 4.12. *Let $(f_p)_{p\in\mathbb{I}}$ be a family unfolding non-degenerately the quadratic homoclinic tangencies of a wild basic hyperbolic set Λ at $p = 0$. Let $P \in \Lambda$ be a saddle periodic point. Then there exists an arbitrarily small parameter at which the orbit of P displays a quadratic homoclinic tangency which unfolds non-degenerately.*

Proof. Let $z \in \Lambda$ and $n \geq 0$ be so that $W^s_{loc}(z;f_0)$ displays a quadratic tangency with $W^u_{loc}(f_0^n(z);f_0)$ which unfolds non-degenerately. Let $N \gg 1$. Let O_P be the orbit of P. We recall that $W^s_{loc}(O_P;f_0)$ intersects transversally $W^u_{loc}(f_0^N(z);f_0)$ and $W^u_{loc}(O_P;f_0)$ intersects transversally $W^s_{loc}(f_0^{n-N}(z);f_0)$. As N is large, by the inclination lemma $W^s(O_P;f_0) \supset f_0^{-N}(W^s_{loc}(O_P;f_0))$ contains a segment which is close to $W^s_{loc}(z;f_0)$ and $W^u(O_P;f_0) \supset f_0^N(W^u_{loc}(O_P;f_0))$ contains a segment which is close to $W^u_{loc}(f_0^n(z);f_0)$. By the para-inclination lemma [Ber16], the hyperbolic continuations of these segments vary with p C^∞-close to the hyperbolic continuations $(W^u_{loc}(f_p^n(z);f_p))_p$ and $(W^s_{loc}(z;f_p))_p$. Thus, as the unfolding is non-degenerate, there exists an arbitrarily small parameter such that a pair of segments in $W^s(O_P;f_p)$ and $W^u(O_P,f_p)$ display a quadratic tangency unfolded non-degenerately. □

Newhouse [New70] gave the first example of a horseshoe displaying a robust homoclinic tangency. Then he proved [New74] that those sets appear at any non-degenerate unfolding of a quadratic homoclinic tangency of a dissipative saddle point of a surface diffeomorphism. The conservative counterpart of this result was proved by Duarte:

Theorem 4.13 (Duarte [Dua08, Theorem B])**.** *Let $(f_p)_p$ be a smooth one-parameter family of symplectic maps in $\mathrm{Diff}^\infty_{\mathrm{Leb}^+}(\Sigma',\Sigma)$. Let P be a periodic hyperbolic orbit displaying a quadratic homoclinic tangency with orbit Q, which unfolds non-degenerately at $p = 0$. Take any small neighborhood U of $P \cup Q$. Then there is a sequence of parameters p_n converging to $p = 0$ for which f_{p_n} leaves invariant a hyperbolic horseshoe*[5] *$\Lambda_n \subset U$ which is homoclinically related to the hyperbolic continuation of P and displays a robust homoclinic tangency.*

We now give a sketch of the proof of Duarte's result, because it will be needed to deduce Corollary 4.14 below.

Sketch of proof. The beautiful proof of Duarte can be divided into three main steps.

[5] A hyperbolic horseshoe is a locally maximal, transitive hyperbolic Cantor set.

(*i*) Refering to the work of Gonchenko and Shilnikov in [GS97] (see also [GST98, GTS93]) and of Mora and Romero in [MR97], a Hénon-like renormalization procedure can be done nearby one point Q_0 of the homoclinic tangency orbit Q at $p = 0$ in the conservative setting. This will be recalled below. The scheme has as limit the **conservative Hénon family**

$$(a, x, y) \mapsto H_a(x, y) := (y, -x + a - y^2)\,.$$

(*ii*) In the framework of the conservative Hénon family, Duarte showed that there exists a parameter a^* close to -1 such that the Hénon map H_{a^*} displays a wild horseshoe Λ_{a^*}, i.e., a hyperbolic basic set with robust homoclinic tangency that unfolds non-degenerately.

(*iii*) These first two steps are used to construct a sequence of parameters $(p_n)_n$ converging to $p = 0$ such that, for every n, the diffeomorphism f_{p_n} leaves invariant a wild horseshoe Λ_n. In [Dua99], it is proved that this sequence $(\Lambda_n)_n$ converges to $P \cup Q$ in the Hausdorff metric and that each Λ_n is homoclinically related to the hyperbolic continuation of P.

We now give more details and references on each step of Duarte's proof.

Concerning the renormalization procedure, for every large $n \geq 0$ we can consider a small box $I_n \times B_n$ in $\mathbb{I} \times \Sigma'$ near the point $(0, Q)$, which shrinks to the point itself as $n \to +\infty$ and so that the segments $(I_n)_n$ are disjoint. This small box is chosen so that it is sent sufficiently close to itself by the function $(p, X) \mapsto (p, f_p^n(X))$. In each small box we rescale the coordinates by a map

$$[-30, 30] \times [-4, 4]^2 \ni (a, x, y) \mapsto (\phi_n(a), \psi_{n,a}(x, y)) \in I_n \times B_n$$

which enables to see $f^n_{\phi_n(a)}$ as a map $\mathcal{R}_n f_a := \psi_{n,a}^{-1} \circ f^n_{\phi_n(a)} \circ \psi_{n,a}$ which is C^∞ -close to each Hénon map $H_a : (x, y) \mapsto (y, -x + a - y^2)$ when n is large. In particular, the presence of a C^2-stable wild horseshoe for the Hénon map H_{a^*} for the parameter a^* (that will be proved in the second step) will persist in the renormalized dynamics at the unfolding of the homoclinic tangency, i.e., for f_{p_n} with $p_n := \phi_n(a^*)$, for n large.

In the second step we study the conservative Hénon family of maps $(x, y) \mapsto H_a(x, y) = (y, -x + a - y^2)$. At $a = -1$, the map displays a parabolic fixed point which breaks, for $a > -1$, into a saddle fixed point S and an elliptic fixed point E. For values of $a > -1$ close to -1, the point S exhibits a transverse homoclinic intersection Ω. The asymptotics of the homoclinic angle at Ω is given in [GS01a]. All the points S, E, Ω belong to the axis of symmetry $\{x = y\}$. Moreover, the saddle point S is accumulated by a sequence $(Q_n)_n$ of periodic points of period $2n$. For each n, we can define R_n as the rectangle bounded by the local stable and unstable manifolds of S and Q_n. This is the starting point for Duarte's construction [Dua08, Pages 7-8] of a (positive binary) horsehoe, depicted in Fig. 4. More precisely, let $R_n^0 \subset R_n$ be the rectangle formed by points whose first iteration remains in R_n, while $R_n^1 \subset R_n$ is the rectangle of points which come back to R_n after $2n$ iterations. Denoting by $T_n : R_n^0 \cup R_n^1 \to R_n$ the first return map to[6] R_n, our candidate horseshoe is then the maximal invariant set $\bigcap_{k\in\mathbb{Z}} T_n^k(R_n^0 \cup R_n^1)$. The study of the structure of the aforementioned horseshoe (its thickness) enables him to show that any eventual homoclinic tangency of the horseshoe will persist under perturbation, i.e., the horseshoe will be wild. See [Dua08, Section 5] for the proof. The last remaining key point of Duarte's proof is then showing the existence of quadratic homoclinic tangencies. Duarte analyses the stable and unstable manifolds of S (see [Dua08, Section 6]) and shows that there is a^* close to -1 for which the saddle point S displays a quadratic homoclinic tangency that, moreover, unfolds non-degenerately when

[6] More precisely, the map T_n associates to a point in R_n^0, resp. in R_n^1, its first iterate, resp. its $2n$-th iterate.

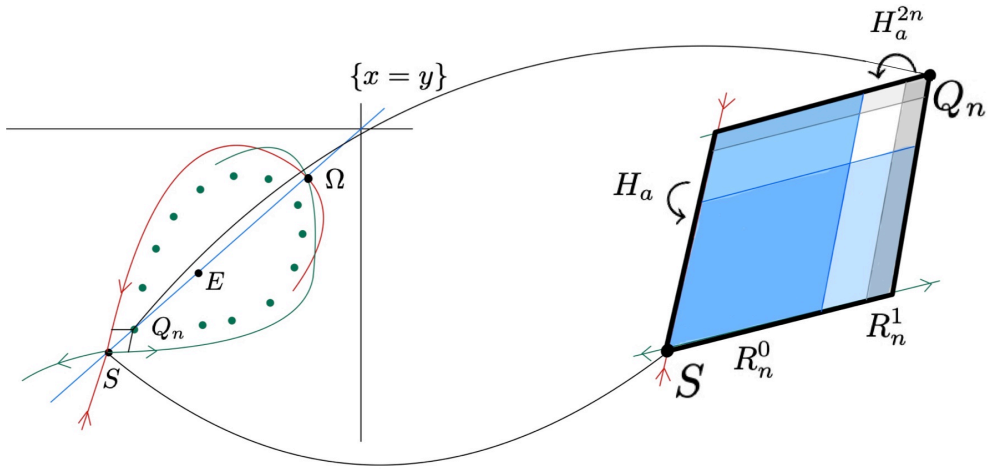

Figure 4: Construction of the horseshoe for the Hénon map.

a varies, see [Dua08, Lemma B].

In the second step Duarte proved the existence of a wild horseshoe $\check{\Lambda}$ left invariant by the Hénon map H_{a^*}. As $\mathcal{R}_n f_{a^*}$ is C^2-close to H_{a^*} when n is large, the continuation of $\check{\Lambda}$ remains wild for the renormalization $\mathcal{R}_n f_{a^*}$ of f_{p_n}. Thus the f_{p_n}-orbit Λ_n of $\psi_{n,a^*}(\check{\Lambda})$ is a wild horseshoe for f_{p_n}. In [Dua99, Section 6], Duarte detailed the proof that, for any neighborhood U of $P \cup Q$, the horseshoe Λ_n is contained in U for n sufficiently large. Moreover, in [Dua99, Lemma 7.1], he showed that each horseshoe Λ_n is homoclinically related to the hyperbolic continuation of P. □

From items (i) and (ii) of Duarte's proof, we have the following corollary:

Corollary 4.14. *Under the assumption of Theorem 4.13, the unfolding of Λ_n in $(f_p)_p$ at $p = p_0$ is non-degenerate. Moreover, for every point Q_0 in the homoclinic orbit Q and for every neighborhood U_0 of Q_0, for large enough n the hyperbolic set Λ_n satisfies the following property: every $z \in \Lambda_n$ has an iterate in U_0.*

5 Proof of Theorem A

To prove Theorem A, we are going to show the following stronger result:

Theorem 3.6. For any $k \geq 0$, there exists a family $(X_p)_{p\in\mathbb{I}^k}$ of Beltrami fields with $X_p \in \mathscr{B}(\mathbb{R}^3)$ that unfolds non-degenerately a homoclinic tangency of multiplicity k at $p = 0$.

Indeed, we can easily show that Theorem 3.6 implies Theorem A using Duarte's Theorem:

Proof of Theorem A. By Theorem 3.6, there exists a family $(X_p)_{p\in\mathbb{I}}$ of Beltrami fields with $X_p \in \mathscr{B}(\mathbb{R}^3)$ that unfolds non-degenerately a quadratic homoclinic tangency at $p = 0$. Consider a Poincaré section for $(X_p)_p$ around $p = 0$ such that the associated family of return maps $(f_p)_{p\in\mathbb{I}}$ is a family of symplectic surface diffeomorphisms unfolding non-degenerately a homoclinic tangency for a saddle point P. By Duarte's theorem [Dua08, Theorem B] (here Theorem 4.13), at some p_0 small, the point P belongs to a wild hyperbolic horseshoe which is non-degenerately unfolded.

Then every $\tilde{X}$ in a neighborhood $\mathcal{V} \subset \mathscr{B}(\mathbb{R}^3)$ of X_{p_0} induces a Poincaré map $\tilde{f}_{p_0}$ which displays a wild horseshoe. Furthermore, if the neighborhood $\mathcal{V}$ is small enough, then the family $(\tilde{X} - X_{p_0} + X_p)_p$ is sufficiently close to $(X_p)_p$ to unfold non-degenerately the wild hyperbolic horseshoe at p_0.

Then by Proposition 4.12, there is p_1 arbitrarily close to p_0 so that $(\tilde{X} - X_{p_0} + X_p)_p$ unfolds non-degenerately a quadratic homoclinic tangency at $p = p_1$. Hence $\tilde{X}$ belongs to the closure $H_{\mathscr{B}}(\mathbb{R}^3)$ of the set of Beltrami fields that exhibit a saddle periodic orbit displaying a homoclinic tangency and such that the tangency can be non-degenerately unfolded by a family in $\mathscr{B}(\mathbb{R}^3)$. Thus $\mathcal{V}$ is included in $H_{\mathscr{B}}(\mathbb{R}^3)$ and so the interior $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ of $H_{\mathscr{B}}(\mathbb{R}^3)$ is not empty. □

The proof of Theorem 3.6 follows exactly the strategy presented in Section 3.2.

5.1 Beltrami vector fields realizing prescribed jets of Melnikov functions at a strong, double heteroclinic link

We recall that in Section 3.2.3, we identified the k-jet of a real function g at some point θ, denoted by $J^k_\theta(g)$, to a vector in $\mathbb{R}^{k+1}$. Let $U \subset \mathbb{R}^3$ be a bounded open subset of $\mathbb{R}^3$. A consequence of Theorem 3.8, which states that there exists an approximation of the right inverse of the Melnikov operator in the space $\mathscr{B}(U)$ of Beltrami fields defined on U, is:

Corollary 5.1. *Let $X \in \mathscr{B}(U)$ display a strong heteroclinic link $\Gamma \subset U$ and let $\Sigma \subset U$ be a Poincaré section endowed with coordinates, chosen as in Theorem 3.8. For any $n \in \mathbb{N}$ and $N \in \mathbb{N}^*$, for every N-uplet of different points $\theta_1, \cdots, \theta_N \in \mathbb{R}/\mathbb{Z}$, the following map is onto:*

$$W \in \mathscr{B}(U) \mapsto (J^n_{\theta_i}(\mathcal{M}(W)))_{1\leq i\leq N} \in \mathbb{R}^{N\cdot(n+1)} .$$

Proof. As the map $S : g \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}) \mapsto (J^n_{\theta_i}(g))_{1\leq i\leq N} \in \mathbb{R}^{N\cdot(n+1)}$ is onto, there exists an $N \cdot (n+1)$-vector subspace $E \subset C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ such that $S(E) = \mathbb{R}^{N\cdot(n+1)}$. Now choose a basis $(g_i)_{1\leq i\leq N\cdot(n+1)}$ of E. For every $1 \leq i \leq N \cdot (n+1)$, by Theorem 3.8, there exists a Beltrami field $W_i \in \mathscr{B}(U)$ such that $\mathcal{M}(W_i)$ is C^∞-close to g_i. The image of the family $(W_i)_{1\leq i\leq N\cdot(n+1)}$ by the linear map

$$W \in \mathscr{B}(U) \mapsto (J^n_{\theta_i}(\mathcal{M}(W)))_{1\leq i\leq N} \in \mathbb{R}^{N\cdot(n+1)}$$

is then close to the basis $(S(g_i))_{1\leq i\leq N\cdot(n+1)}$ and so forms a basis of $\mathbb{R}^{N\cdot(n+1)}$. This implies that the latter map is onto. □

Let now $X \in \mathscr{B}(U)$ display a double, strong heteroclinic link (Γ^+, Γ^-) in $U \subset \mathbb{R}^3$. We choose two Poincaré sections $\Sigma_+ \subset U$ and $\Sigma_- \subset U$ of the dynamics nearby Γ^+ and Γ^-, endowed with coordinates, as it is considered in Theorem 3.8. We fix reference points $0 \equiv p^+ \in \Gamma^+$ and $0 \equiv p^- \in \Gamma^-$ to define the associated Melnikov operators $\mathcal{M}_+$ and $\mathcal{M}_-$.

Let $k \geq 1$. In order to prove Proposition 3.9, we want to control the 1-jet of the Melnikov function at some point of Γ^+ **and** the k-jet of the Melnikov function at some point of Γ^-. We recall that it is not possible to simply apply Corollary 5.1 separately on Γ^+ and Γ^- and then sum the resulting Beltrami fields. Indeed, if we apply Corollary 5.1 on Γ^+, we do not have any control on the behavior of the corresponding Beltrami vector field on Γ^- and vice versa: each Beltrami field could possibly destroy the profile of the Melnikov function of the other one. We will overcome this issue through a new trick based on elementary linear algebra.

Proof of Proposition 3.9. By Corollary 5.1 applied at the heteroclinic link Γ^+ with $n = N = 1$ and $\theta_1 = \alpha \in \mathbb{R}/\mathbb{Z}$, the following map is onto:

$$J_+ := W \in \mathscr{B}(U) \mapsto J^1_\alpha(\mathcal{M}_+(W)) \in \mathbb{R}^2 .$$

Again by Corollary 5.1 applied at the heteroclinic link Γ^- with $n = k$ and $N = 3$ different points $\theta_1, \theta_2, \theta_3 \in \mathbb{R}/\mathbb{Z}$, the following map is onto:

$$J_- := W \in \mathscr{B}(U) \mapsto (J_1, J_2, J_3)(W) \in \mathbb{R}^{3k+3} \text{ with } J_i(W) := J^k_{\theta_i}(\mathcal{M}_-(W)) \in \mathbb{R}^{k+1} .$$

Equivalently, the family $(\ker J_1, \ker J_2, \ker J_3)$ is in general position:

$$\operatorname{codim} \bigcap_i \ker J_i = \sum_i \operatorname{codim} \ker J_i \ .$$

To prove the Proposition 3.9, it suffices to show the existence of $1 \leq i \leq 3$ such that the product map $J_- \times J_i$ is onto $\mathbb{R}^{k+3}$. Otherwise, for every i, we have:

$$\operatorname{codim} \ker J_+ + \operatorname{codim} \ker J_i > \operatorname{codim}(\ker J_+ \cap \ker J_i)$$

Thus $\ker J_+ + \ker J_i \neq \mathscr{B}(U)$. Let F_i be a codimension 1 space containing $\ker J_+ + \ker J_i$. The family of spaces (F_1, F_2, F_3) must be in general position since its elements include those of the family $(\ker J_1, \ker J_2, \ker J_3)$ which are in general position. Thus $\operatorname{codim}(F_1 \cap F_2 \cap F_3) = 3$. This is a contradiction with $F_1 \cap F_2 \cap F_3 \supset \ker J_+$ which is 2-codimensional. □

5.2 Families of Beltrami fields unfolding non-degenerately a homoclinic tangency of multiplicity k

Recall that $U \subset \mathbb{R}^3$ is a bounded open subset of $\mathbb{R}^3$. Let $X \in \mathscr{B}(U)$ display a strong heteroclinic link $\Gamma \subset U$. Let $\Sigma \subset U$ be a Poincaré section chosen as in Theorem 3.8. The Melnikov operator $\mathcal{M}$ was introduced to study the **displacement** operator, as explained in Appendix A. Choose a reference point in $\Gamma \cap \Sigma$. Then, the displacement operator, denoted by displ, is defined on a neighborhood $\mathcal{U}$ of $0 \in \Gamma^\infty_{\text{Leb}}(U)$, and for every perturbation $W \in \mathcal{U}$ the function

$$t \in [0,1] \mapsto \operatorname{displ}(W)(t) \in \mathbb{R}$$

corresponds to the distance between the perturbed local stable and unstable manifolds over a fundamental domain, where this domain is parametrized by t.

Assume now that $U \subset \mathbb{R}^3$ is so that $\mathbb{R}^3 \setminus U$ is connected. Let $X \in \mathscr{B}(U)$ display a double, strong heteroclinic link (Γ^+, Γ^-) contained in U[7]. Denote by displ_+ and displ_- the displacement operators corresponding to the heteroclinic links Γ^+ and Γ^- respectively. From Proposition 3.9 and since the Melnikov operator is the first order approximation of the displacement operator, we deduce the following

Corollary 5.2. *Let $k \geq 0$. There exist $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$, a $(k+3)$-vector subspace $\mathcal{W} \subset \mathscr{B}(U)$ and a neighborhood $\mathcal{U}$ of 0 in $\mathcal{W}$ such that the map*

$$W \in \mathcal{U} \mapsto \left(J^1_\alpha(\operatorname{displ}_+(W)), J^{k+1}_\beta(\operatorname{displ}_-(W)) \right) \in \mathbb{R}^{k+4}$$

is a diffeomorphism onto a neighborhood of $0 \in \mathbb{R}^{k+3}$.

Proof. Let $\alpha, \beta \in [0,1]$ and $\mathcal{W} \subset \mathscr{B}(U)$ be given by Proposition 3.9. The latter states that the following Melnikov operator is an isomorphism:

$$W \in \mathcal{W} \mapsto \left(J^1_\alpha(\mathcal{M}_+(W)), J^k_\beta(\mathcal{M}_-(W)) \right) \in \mathbb{R}^{k+3} \, .$$

By the Poincaré-Melnikov Theorem (see Appendix A), the Melnikov operator is the differential of the displacement operator at 0. Thus by the local inversion theorem, the displacement operator is a diffeomorphism from an open neighborhood $\mathcal{U}$ of 0 onto an open neighborhood of $0 \in \mathbb{R}^{k+3}$. □

[7] This is the case of Example 3.5, where U is the intersection of $\mathbb{R}^3 \setminus \{(0,0)\} \times \mathbb{R}$ with a large ball.

The proof of Theorem 3.6 follows then from Corollary 5.2 and the para-inclination Lemma.

Proof of Theorem 3.6. First we extract from the family $(X+W)_{W\in\mathcal{U}}$ a k-parameter family $(X+W)_{W\in\mathcal{U}'}$ such that $W^u_{loc}(\gamma^-, X+W)$ intersects transversally $W^s_{loc}(\gamma^+, X+W)$ for every $W\in\mathcal{U}'$ and $(W^s_{loc}(\gamma^-, X+W))_{W\in\mathcal{U}'}$ unfolds non-degenerately a heteroclinic tangency of multiplicity k with $(W^u_{loc}(\gamma^+, X+W))_{W\in\mathcal{U}'}$. In order to do so, for a small $\delta>0$, we consider the preimage, denoted by $\mathcal{U}'\subset\mathcal{U}$, of $\{0\}\times\{\delta\}\times(-\delta,\delta)^k\times\{0\}\times\{\delta\}$ by the diffeomorphism of Corollary 5.2. Note that for δ sufficiently small the map

$$W\in\mathcal{U}'\mapsto J^{k+1}_\beta(\mathrm{displ}_(W))\in\mathbb{R}^k\times\{0\}\times\{\delta\} \tag{5.1}$$

is still a diffeomorphism onto its image, which contains $\{0\}^k\times\times\{0\}\times\{\delta\}$. Thus there is indeed a non-degenerate unfolding of a heteroclinic tangency of multiplicity k between $(W^s_{loc}(\gamma^-, X+W))_{W\in\mathcal{U}'}$ and $(W^u_{loc}(\gamma^+, X+W))_{W\in\mathcal{U}'}$.

Up to shrinking $\mathcal{U}'$, view in the Poincaré section Σ defined above, we obtained a family of Poincaré maps $(f_W)_{W\in\mathcal{U}'}$ which displays two saddle fixed points q^+ and q^- such that $W^u_{loc}(q^-, f_W)$ intersects transversally $W^s_{loc}(q^+, f_W)$ for every $W\in\mathcal{U}'$, and so that there is a non-degenerate unfolding of a heteroclinic tangency of multiplicity k between $(W^s_{loc}(q^-, f_W))_{W\in\mathcal{U}'}$ and $(W^u_{loc}(q^+, f_W))_{W\in\mathcal{U}'}$. Let $(S_W)_{W\in\mathcal{U}'}$ be a family of segments contained in $(W^u_{loc}(q^-, f_W))_{W\in\mathcal{U}'}$ intersecting transversally $(W^s_{loc}(q^+, f_W))_{W\in\mathcal{U}'}$. By the para-inclination Lemma 4.10, we can tune $(S_W)_{W\in\mathcal{U}'}$ such that it is sent by a large iterate $(f^n_W)_{W\in\mathcal{U}'}$ to a family $(f^n_W(S_W))_{W\in\mathcal{U}'}$ which is C^∞-close to $(W^u_{loc}(q^+, f_W))_{W\in\mathcal{U}'}$. As the non-degeneracy of an unfolding is an open property, it comes that $(f^n_W(S_W))_{W\in\mathcal{U}'}$ and $(W^s_{loc}(q^-, f_W))_{W\in\mathcal{U}'}$ unfold non-degenerately a tangency of multiplicity k. As $S_W\subset W^u_{loc}(q^-, f_W)$ for every $W\in\mathcal{U}'$, we showed that $(f_W)_{W\in\mathcal{U}'}$ unfolds non-degenerately a homoclinic tangency of multiplicity k for q^-. This implies that there exists a k-parameter family of Beltrami vector fields $(X+W_p)_{p\in\mathbb{I}^k}$ in $\mathscr{B}(U)$, where each $W_p\in\mathcal{U}'$, which unfolds non-degenerately a homoclinic tangency of multiplicity k for γ^- at a parameter $p_0\in\mathbb{I}^k$. Denote by $Y:=X+W_{p_0}$, while $Z_i:=\partial_i W_{p_0}$ is the partial derivative along the direction of the i-th coordinate of $p\in\mathbb{I}^k$, for $i=1,\dots,k$. Since unfolding non-degenerately a homoclinic tangency of multiplicity k is a robust condition, the family $(Y+p_1\cdot Z_1+\cdots+p_k\cdot Z_k)_{(p_1,\dots,p_k)\in\mathbb{I}^k}$ unfolds non-degenerately a homoclinic tangency of multiplicity k at some parameter in $\mathbb{I}^k$ close to 0. Since $\mathbb{R}^3\backslash U$ is connected, Theorem 3.1 provides the existence of vector fields $\tilde{Y},\tilde{Z}_1,\dots,\tilde{Z}_k$ in $\mathscr{B}(\mathbb{R}^3)$ such that their restrictions to U are close to $Y,Z_1,\dots,Z_k$ respectively. Again by the robustness of a non-degenerate unfolding of a homoclinic tangency of multiplicity k, the family of Beltrami fields $(\tilde{Y}+p_1\cdot\tilde{Z}_1+\cdots+p_k\cdot\tilde{Z}_k)_{(p_1,\dots,p_k)\in\mathbb{I}^k}$ is the sought family in $\mathscr{B}(\mathbb{R}^3)$.

□

6 Proof of Theorem D

We are going to prove that $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$ is GST wild. In other words, we will prove that for every $J\geq 1$ and every non-empty open subset $\mathscr{U}$ of $\mathcal{N}_{\mathscr{B}}(\mathbb{R}^3)$, the following property holds true:

($GST_{\mathscr{U},J}$) there is a family $(X_p)_{p\in\mathbb{I}^J}$ of vector fields $X_p\in\mathscr{U}$ such that X_0 has a saddle periodic orbit O which displays J different quadratic homoclinic tangencies unfolded non-degenerately by the family.

To this end we start with a family $(X_p)_{p\in\mathbb{I}}$ of vector fields in $\mathscr{U}$ which unfolds a quadratic homoclinic tangency of a saddle periodic orbit O at $p=0$. As the space of Beltrami fields is a vector space, we can assume that $X_p=X_0+p\cdot X_1$ with $X_1:=\partial_p X_p|_{\{p=0\}}$.

We are going to use Duarte's Theorem 4.13 to create robust homoclinic tangencies for disjoint but homoclinically related hyperbolic basic sets. Actually we would like moreover that their unions C_p and C'_p with their orbits of homoclinic tangency are disjoint. Then the perturbative Lemma 3.2 enables to create a new vector field X_2 which is small on C_p but close to X_1 on C'_p, and so the family $(X_0+p_1\cdot X_1+p_2\cdot X_2)_{p=(p_1,p_2)\in\mathbb{I}^2}$ unfolds these two basic sets independently. To obtain that C_p and C'_p are disjoint we will use Corollary 4.14 of Duarte's proof.

For the sake of simplicity, all along the proof, the hyperbolic continuation of an f-orbit O for a perturbation $\tilde{f}$ of f will still be denoted by O.

6.1 Case $J = 2$

From the latter discussion, to prove $(GST_{\mathscr{U},2})$, it suffices to show:

Proposition 6.1. *Let (X_0, X_1) be Euclidean Beltrami fields and let $X_p := X_0 + p\cdot X_1$. Assume that $(X_p)_{p\in\mathbb{R}}$ unfolds non-degenerately a quadratic homoclinic tangency Γ_0 of a saddle orbit O at $p = 0$. Then there exist an arbitrarily small parameter $p_0 \in \mathbb{I}$, an Euclidean Beltrami field X_2 and a parameter $\tilde{p}_0 \in \mathbb{I}^2$ arbitrarily close to $(p_0, 0) \in \mathbb{I}^2$ such that the family $(\tilde{X}_p)_{p\in\mathbb{R}^2}$ formed by $\tilde{X}_p := X_0 + p_1\cdot X_1 + p_2\cdot X_2$ unfolds non-degenerately 2 different quadratic homoclinic tangencies of the saddle orbit O at the parameter $\tilde{p}_0$.*

Proof of Proposition 6.1. For $p \in \mathbb{I}$, we recall that $X_p = X_0+p\cdot X_1$, where X_0 and X_1 are Beltrami fields. Also $(X_p)_{p\in\mathbb{R}}$ unfolds non-degenerately a quadratic homoclinic tangency Γ_0 of a saddle periodic orbit O at $p = 0$. Let Σ be a disk transverse to X_p for every p small, and intersecting O at a unique point P. We assume that

$$(cl(\Sigma)\setminus\Sigma)\cap(\Gamma_0\cup O) = \emptyset\ .$$

Then the return time to Σ of the flow of X_0 is smooth at some neighborhood $\check{\Sigma}\subset\Sigma$ of $\Sigma\cap(\Gamma_0\cup O)$. For every $\epsilon > 0$ small, up to taking $\check{\Sigma}$ slightly smaller, the same occurs for the flows of X_p for any $|p| < \epsilon$.

Let $f_p : \check{\Sigma}\to\Sigma$ be the induced Poincaré map. Note that $(f_p)_{-\epsilon\le p\le\epsilon}$ is a C^∞ family of maps in $\mathrm{Diff}^\infty(\check{\Sigma},\Sigma)$. It is standard that the map f_p leaves invariant a symplectic form ω_p which depends smoothly on p. By Darboux's Theorem, there are smooth coordinates $\phi_p : \mathbb{D}\to\Sigma$ such that ϕ_p sends Leb into $\mathbb{R}\cdot\omega_p$. Then $F_p := \phi_p^{-1}\circ f_p\circ\phi_p$ forms a smooth family of symplectic maps. Moreover:

Fact 6.2. *The family $(F_p)_{-\epsilon\le p\le\epsilon}$ satisfies the assumptions of Duarte's Theorem 4.13.*

Hence, for ϵ arbitrarily small, there exist a non trivial interval $I_0\subset(-\epsilon,\epsilon)$ and a continuation $(\Lambda_p)_{p\in I_0}$ of a hyperbolic horseshoe for $(f_p)_{p\in I_0}$ such that for every $p\in I_0$:

- Λ_p is homoclinically related to the saddle fixed point P.
- Λ_p displays a robust homoclinic tangency.

Let $W^s_{loc}(\Lambda_p; f_p) = \bigcup_{z\in\Lambda_p} W^s_{loc}(z; f_p)$ and $W^u_{loc}(\Lambda_p; f_p) = \bigcup_{z\in\Lambda_p} W^u_{loc}(z; f_p)$ be the continuous families of local stable and unstable manifolds which are robustly tangent. Moreover by Corollary 4.14, we can assume that this unfolding is non-degenerate.

For any $p\in I_0$, there exists a point $z\in\Lambda_p$ such that $W^u_{loc}(z; f_p)$ is tangent to $W^s_{loc}(z; f_p)$. Then there are segments of $W^u(P; f_p)$ and $W^s(P; f_p)$ which are arbitrarily close to $W^u_{loc}(z; f_p)$ and $W^s_{loc}(z; f_p)$. As the unfolding of the tangency between $W^u_{loc}(z; f_p)$ and $W^s_{loc}(z; f_p)$ is non-degenerate, there exists p_0 in the interior of I_0 such that P displays an orbit of homoclinic tangency Q at $p = p_0$

which unfolds non-degenerately in $(f_p)_{p\in I_0}$. Let Q_0 be a point of Q and let U_0 be a neighborhood of Q_0 which does not intersect Λ_{p_0}.

Now we apply a second time Duarte's Theorem 4.13 and moreover Corollary 4.14 for the homoclinic orbit of Q and the point $Q_0 \in Q$. This gives the existence of a non trivial segment $I_1 \Subset I_0$ close to p_0 and a continuation of $(\Lambda'_p)_{p\in I_0}$ of another hyperbolic horseshoe such that for every $p \in I_1$:

(i) The set Λ'_p is homoclinically related to P and so that any point of Λ'_p has an iterate in U_0.

(ii) The set Λ'_p displays a robust homoclinic tangency which unfolds non-degenerately.

As I_1 is close to p_0, Λ_p is close to Λ_{p_0} and so does not intersect U_0. Hence none of the point Λ_p has an iterate in U_0. By (i), this implies that Λ_p and Λ'_p are disjoint for every $p \in I_1$. Let $W^s_{loc}(\Lambda'_p; f_p) = \bigcup_{z\in\Lambda'_p} W^s_{loc}(z; f_p)$ and $W^u_{loc}(\Lambda'_p; f_p) = \bigcup_{z\in\Lambda'_p} W^u_{loc}(z; f_p)$ be the continuous families of local stable and unstable manifolds which are robustly tangent and whose unfolding is non-degenerate. As Λ'_p and $\Lambda_p \cup \{P\}$ are disjoint for every $p \in I_1$, we have:

- $W^s_{loc}(\Lambda'_p; f_p)$ is disjoint from $W^s_{loc}(\Lambda_p \cup \{P\}; f_p)$,
- $W^u_{loc}(\Lambda'_p; f_p)$ is disjoint from $W^u_{loc}(\Lambda_p \cup \{P\}; f_p)$.

We can assume that all the local stable manifolds are compact; thus all the above sets are compact. Thus for every $p \in I_1$:

- $K'^0_p := W^s_{loc}(\Lambda'_p; f_p) \cap W^u_{loc}(\Lambda'_p; f_p)$ and $K^0_p := W^s_{loc}(\Lambda_p; f_p) \cap W^u_{loc}(\Lambda_p; f_p)$ are disjoint.
- more generally $f^n_p(K'^0_p)$ and $f^m_p(K^0_p)$ are disjoint for every $n, m \in \mathbb{Z}$.

As $f^n_p(K'^0_p) \to \Lambda'_p$ and $f^n_p(K^0_p) \to \Lambda_p$ when $n \to \pm\infty$ we obtain:

Fact 6.3. *For any $p \in I_1$, $K_p := \bigcup_{n\in\mathbb{Z}} f^n_p(K^0_p)$ and $K'_p := \bigcup_{n\in\mathbb{Z}} f^n_p(K'^0_p)$ are disjoint compact sets.*

Let us precise the topology of these sets:

Fact 6.4. *The subsets K'_p and K_p are totally disconnected.*

Proof. Let us prove that K_p is totally disconnected; the proof for K'_p is identical. First, Λ_p is totally disconnected since it is a horseshoe and so a Cantor set.

Let us show that $K^0_p \setminus \Lambda_p$ is totally disconnected. Indeed, $K^0_p \setminus \Lambda_p$ is formed by the quadratic tangency points between $W^s_{loc}(\Lambda_p; f_p)$ and $W^u_{loc}(\Lambda_p; f_p)$. These are a totally disconnected union of leaves (Λ_p is a horseshoe). Hence the set of quadratic tangencies between the leaves of $W^s_{loc}(\Lambda_p; f_p)$ and $W^u_{loc}(\Lambda_p; f_p)$ is totally disconnected. Thus $K_p := \Lambda_p \cup \bigcup_{n\in\mathbb{Z}} f^n_p(K^0_p \setminus \Lambda_p)$ is a compact set equal to a countable union of totally disconnected compact subsets. By [HW41, II.4.A] any compact set is totally disconnected iff it is zero dimensional in the sense of [HW41, Def. II.1]: any of its point has an arbitrarily small clopen neighborhood. By [HW41, Thm II.2.], a countable union of closed zero-dimensional sets is still zero-dimensional; thus the compact set K_p is zero-dimensional and so totally disconnected [HW41]. □

We have furthermore:

Lemma 6.5. *The maps $p \in I_1 \mapsto K_p$ and $p \in I_1 \mapsto K'_p$ are upper semi-continuous: for every $p_1 \in I_1$, for every neighborhood U of K_{p_1} (resp. K'_{p_1}), the compact set K_p (resp. K'_p) is included in U for p close to p_1.*

Proof. We prove the statement for $p \in I_1 \mapsto K_p$; the proof for $p \in I_1 \mapsto K'_p$ is the same. It suffices to notice that $\bigcup_{p\in I_1}\{p\}\times K_p$ is compact. This is indeed the case since the local stable and unstable sets of Λ_p vary continuously with p. So $\bigcup_{p\in I_1}\{p\}\times W^s_{loc}(K_p;f_p)$ and $\bigcup_{p\in I_1}\{p\}\times W^u_{loc}(K_p;f_p)$ are compact, and their intersection $\bigcup_{p\in I_1}\{p\}\times K^0_p$ is compact. Hence $p\mapsto K^0_p$ is upper continuous. Thus $p\mapsto f^n(K^0_p)$ is upper semi continuous and $p\mapsto \Lambda_p\cup\bigcup_{-N\le n\le N} f^n(K^0_p)$ is upper semi continuous for every N. As this sequence of functions converges uniformly to $p\mapsto K_p$ when $|n|\to\infty$, the latter is upper semi-continuous. □

We recall that K_p and K'_p are included in $\check{\Sigma}$ for every $p\in I_1$. Also the return time τ_p from $\check{\Sigma}$ to Σ is smooth. Let $(\phi^t_p)_{t\in\mathbb{R}}$ be the flow of X_p. We define for $p\in I_1$:

$$C_p := \bigcup_{z\in K_p}\{\phi^t_p(z) : 0\le t\le \tau_p(z)\} \quad \text{and} \quad C'_p := \bigcup_{z\in K'_p}\{\phi^t_p(z) : 0\le t\le \tau_p(z)\}\,.$$

By Fact 6.3 it holds:

Fact 6.6. *For every $p\in I_1$ the compact subsets C_p and C'_p of $\mathbb{R}^3$ are disjoint.*

By Lemma 6.5 and the smoothness of τ, we have:

Fact 6.7. *The maps $p\in I_1\mapsto C_p$ and $p\in I_1\mapsto C'_p$ are upper semi-continuous.*

Now we prove:

Lemma 6.8. *Let $p_0\in I_1$. The compact subsets C_{p_0} and C'_{p_0} of $\mathbb{R}^3$ have connected complement in $\mathbb{R}^3$.*

Proof. By symmetry, we show the lemma only for C_{p_0}. Assume by contradiction that $\mathbb{R}^3\setminus C_{p_0}$ has a bounded component F. As the flow of X_{p_0} leaves invariant C_{p_0}, it leaves also the component F of $\mathbb{R}^3\setminus C_{p_0}$ invariant. Thus the component F intersects $\Sigma\setminus K_{p_0}$. Recall that C_{p_0} is included in the suspension of $\check{\Sigma}\Subset\Sigma$, which is isotopic to an embedded solid torus and so its complement is connected. Hence $F\cap\Sigma$ cannot be connected to the boundary of Σ by a path in $\Sigma\setminus K_{p_0}$. This implies that $\Sigma\setminus K_{p_0}$ is not connected. This is a contradiction with the connectedness [Moi77, Theorem 4, Page 93] of the complement in a disk Σ of any totally disconnected compact set, such as K_{p_0}, by Fact 6.4. □

Now we can apply the Global Approximation Theorem to construct X_2. More precisely, we apply Lemma 3.2 with C_{p_0} and C'_{p_0} as respectively K_+ and K_-. They are disjoint compact subsets of $\mathbb{R}^3$ by Fact 6.6 and they both have connected complement in $\mathbb{R}^3$ by Lemma 6.8. Thus Lemma 3.2 provides the existence of a Beltrami field X_2 whose restriction to $C_{p_0}\sqcup C'_{p_0}$ is arbitrarily close to

$$\tilde{X}_2 : z\in V\sqcup V' \mapsto \begin{cases} 0 & \text{if } z\in V,\\ X_1(z) & \text{if } z\in V'\,,\end{cases}$$

where V and V' are disjoint neighborhoods of respectively C_{p_0} and C'_{p_0}. We recall that for a dense subset of parameters $p_0\in I_1$, the vector field $X_0+p_0\cdot X_1$ displays a quadratic homoclinic tangency for a saddle periodic orbit $O_1\in\Lambda_{p_0}$ which unfolds non-degenerately. Also for p'_0 arbitrarily small, the vector field $X_0+(p_0+p'_0)\cdot X_1$ unfolds non-degenerately a quadratic homoclinic tangency for a saddle periodic orbit $O_2\in\Lambda'_{p_0+p'_0}$. Note that $X_0+p_0\cdot X_1+p'_0\cdot\tilde{X}_2$ displays two quadratic homoclinic tangencies with the orbits O_1 and O_2 and which are non-degenerately unfolded in the family $(X_0+p_1\cdot X_1+p_2\cdot\tilde{X}_2)_{(p_1,p_2)\in\mathbb{I}^2}$. Indeed by Fact 6.7, the closure of the orbit of the tangencies C'_p of Λ'_p is in V' for p close to p_1.

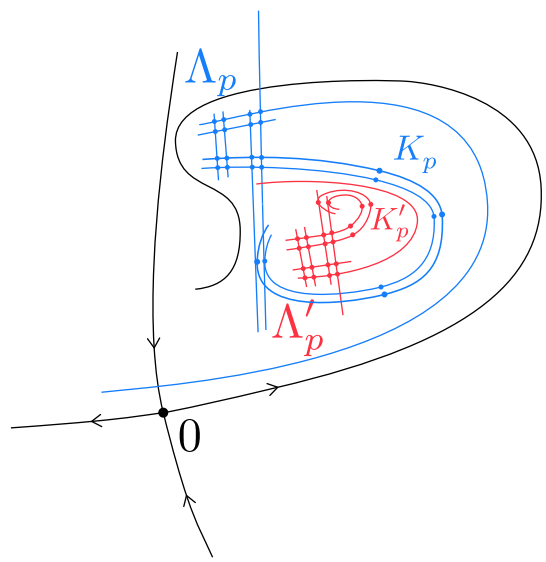

Figure 5: Basic hyperbolic sets Λ_p and Λ'_p displaying disjoint homoclinic sets K_p and K'_p.

As this is a robust condition, for X_2 close enough to $\tilde{X}_2$, there exists $\tilde{p}'_0$ nearby (p_0, p'_0) (and so nearby $(p_0, 0)$) such that the family $(\tilde{X}_p)_{p=(p_1,p_2)\in\mathbb{I}^2}$ formed by $\tilde{X}_p := X_0 + p_1 \cdot X_1 + p_2 \cdot X_2$ displays at $p = \tilde{p}'_0$ two quadratic homoclinic tangencies which unfold non-degenerately for, respectively, the hyperbolic continuations of O_1 and O_2.

It remains to prove the existence of two quadratic homoclinic tangencies which unfold non-degenerately for the same periodic saddle O. To this end, we recall that $O_1 \in C_{\tilde{p}_0}$ and $O_2 \in C'_{\tilde{p}_0}$ are homoclinically related to O. This implies that for every $1 \leq k \leq 2$, there are segments S_k and U_k of $W^s(O; f_{\tilde{p}'_0})$ and $W^u(O; f_{\tilde{p}'_0})$ which are close to $W^s_{loc}(O_k; f_{\tilde{p}'_0})$ and $W^u_{loc}(O_k; f_{\tilde{p}'_0})$. Moreover, by the para-inclination lemma [Ber16], when p varies, the hyperbolic continuation of these segments forms families which are C^2-close to the hyperbolic continuation of $W^s_{loc}(O_k; f_{\tilde{p}'_0})$ and $W^u_{loc}(O_k; f_{\tilde{p}'_0})$. In particular the relative positions are C^1-close. Thus by the local inversion Theorem there is $\tilde{p}_0$ arbitrarily close to $\tilde{p}'_0$ such that the continuation of S_k has a quadratic tangency with U_k for every $1 \leq k \leq 2$, and these quadratic homoclinic tangencies are unfolded non-degenerately when p varies. □

6.2 Case $J \geq 3$

To obtain $(GST_{\mathscr{U},J})$ for every $J \geq 1$, we will apply inductively on k the following generalization of Proposition 6.1:

Proposition 6.9. *Let $k \geq 1$. Let $(X_0, X_1, \ldots, X_k)$ be Euclidean Beltrami fields and for every $p = (p_1, \ldots, p_k) \in \mathbb{R}^k$, let $X_p := X_0 + p_1 \cdot X_1 + \cdots + p_k \cdot X_k$. Assume that there exists a saddle periodic orbit O for X_0 which displays k quadratic homoclinic tangencies which unfold non-degenerately in the family $(X_p)_{p\in\mathbb{R}^k}$.*

Then there exist an arbitrarily small parameter $p_0 \in \mathbb{R}^k$, an Euclidean Beltrami field X_{k+1} and a parameter $\tilde{p}_0 \in \mathbb{R}^{k+1}$ arbitrarily close to $(p_0, 0) \in \mathbb{R}^{k+1}$ such that the family $(\tilde{X}_p)_{p\in\mathbb{I}^{k+1}}$ formed by $\tilde{X}_p := X_0 + p_1 \cdot X_1 + \cdots + p_{k+1} \cdot X_{k+1}$ unfolds non-degenerately $k+1$ different quadratic homoclinic tangencies to O at the parameter $\tilde{p}_0$.

Proof of Proposition 6.9. We proved the case $k=1$ in Proposition 6.1. Now assume $k \geq 2$.

By non-degeneracy of the unfolding, there is a path $t \in \mathbb{I} \mapsto p(t) \in \mathbb{I}^k$ with $p(0)=0$ so that the family $(X_{p(t)})_{t\in\mathbb{I}}$ unfolds non-degenerately the kth homoclinic tangency Γ_k but leaves the other tangencies Γ_i persistent for $1 \leq i < k$.

Then we proceed similarly to the case $k=1$. For t_0' arbitrarily small, at $p_0' = p(t_0')$, Duarte's theorem provides a hyperbolic set Λ which is homoclinically related to O and displays a robust quadratic homoclinic tangency which unfolds non-degenerately. Up to perturbing p_0', we can assume that one of these new homoclinic tangencies is given by a periodic orbit $O' \neq O$. Hence we obtain for t_0' arbitrarily small, the existence of a new orbit of quadratic homoclinic tangency Γ' to a periodic saddle $O' \neq O$ which unfolds non-degenerately at $p_0' = p(t_0')$, and so that O' is homoclinically related to O.

Observe that $\Gamma' \cup O'$ is a compact set which is disjoint from $\bigcup_{1\leq i<k} O \cup \Gamma_i$. Observe that both $\Gamma' \cup O'$ and $\bigcup_{1\leq i<k} O \cup \Gamma_i$ have connected complement in $\mathbb{R}^3$. Let U' and U be small disjoint neighborhoods of respectively $\Gamma' \cup O'$ and $\bigcup_{1\leq i<k} O \cup \Gamma_i$.

We apply Proposition 6.1 to the one parameter family $(X_{p(t)})_{t\in\mathbb{I}}$. It provides $p_0'' = p(t_0'') \in \mathbb{R}^k$ close to $p_0' = p(t_0')$ (and so small), a vector field X_{k+1}, and $(t_0''', s_0) \in \mathbb{R}^2$ arbitrarily close to $(t_0'', 0)$, such that at $(t,s) = (t_0''', s_0)$, the family $(X_{p(t)} + s \cdot X_{k+1})_{(t,s)\in\mathbb{R}^2}$ unfolds non-degenerately two different orbits of homoclinic tangency at O' that are contained in U'.

At $(p,s) = (p(t_0'''), s_0)$, the family $(X_p|_{U\sqcup U'} + s \cdot 1\!\!1_{U'} \cdot X_{k+1})_{(p,s)\in\mathbb{R}^k\times\mathbb{R}}$ displays $(k-1)$-orbits of quadratic tangency to O and 2 orbits of homoclinic tangency to O'. They unfold non-degenerately when (p,s) varies in $\mathbb{R}^{k+1}$.

Using then Lemma 3.2 with $\Gamma' \cup O'$ and $\bigcup_{1\leq i<k} O \cup \Gamma_i$ as respectively K_+ and K_-, and with $1\!\!1_{U'} \cdot X_{k+1}$ as Beltrami field on the neighborhood $U \sqcup U'$ of $K_+ \sqcup K_-$, we deduce the existence of a Beltrami field $\tilde{X}_{k+1} \in \mathscr{B}(\mathbb{R}^3)$ arbitrarily close to $1\!\!1_{U'} \cdot X_{k+1}$ on $U \sqcup U'$. As having $(k+1)$-homoclinic tangencies which unfold non-degenerately is an open condition on families, this property persists. Up to rename $\tilde{X}_{k+1}$ as X_{k+1}, that is why we can assume X_{k+1} defined on $\mathbb{R}^3$.

Finally we use again the (para)-inclination lemma and the fact that O and O' are homoclinically related to deduce from the non-degenerate unfolding of the two quadratic homoclinic tangencies to O', that two new quadratic homoclinic tangencies to O are obtained together with the $(k-1)$-previous quadratic homoclinic tangencies to O, and they unfold non-degenerately. □

Appendices

A Proof of Proposition 3.7: right inverse of the Melnikov operator

The aim of this appendix is to construct an inverse of the Melnikov operator for a vector field $X \in \Gamma^\infty_{\text{Leb}}(U)$ displaying a strong heteroclinic link in U, for some open subset U of $\mathbb{R}^3$.

A.1 Displacement and Melnikov operators

The aim of this section is to introduce the definitions of displacement operator, Melnikov operator and explain their relation.

Let $\Sigma \subset \mathbb{R}^2$ and let Ω be the standard symplectic, smooth 2-form on $\Sigma \subset \mathbb{R}^2$. Let $f \in C^\infty(\Sigma, \mathbb{R}^2)$ be a smooth, symplectic diffeomorphisms from Σ onto its image. We assume that f displays two hyperbolic fixed point p^+, p^- and a heteroclinic link L from p^- to p^+. The **displacement operator** will describe the unfolding of L when we consider a perturbation of f.

Let $\mathcal{N} \subset C^\infty(\Sigma, \mathbb{R}^2)$ be a neighborhood of 0. Assume that $\mathcal{N}$ is small enough such that, for every $\epsilon \in \mathcal{N}$, the diffeomorphism $f+\epsilon$ admits unique hyperbolic fixed points $p_\epsilon^\pm$ nearby $p^\pm$, called

their hyperbolic continuations. Let $\ell : \mathbb{R} \to L$ be a smooth parametrization of the heteroclinic connection L such that

$$\lim_{t\to\pm\infty} \ell(t) = p^{\pm} \quad \text{and} \quad f(\ell(t)) = \ell(t+1).$$

Fix local unstable and local stable manifolds $W^u_{loc}(p^-; f)$ and $W^s_{loc}(p^+; f)$, of respectively p^- and p^+, which both contain $\ell([0,1])$.

For a fixed $\epsilon \in C^\infty(\Sigma, \mathbb{R}^2)$ small, consider the smooth immersed submanifold $W^s_{loc}(p^+_\epsilon; f+\epsilon)$. We can find a smooth diffeomorphism $\phi^s_\epsilon : W^s_{loc}(p^+; f) \to W^s_{loc}(p^+_\epsilon; f+\epsilon)$. Similarly, we find a smooth diffeomorphism $\phi^u_\epsilon : W^u_{loc}(p^-; f) \to W^u_{loc}(p^-_\epsilon; f+\epsilon)$. By Irwin's proof [Irw72, Theorem 28] of the stable/unstable manifold theorem (which is based on the implicit function theorem), both ϕ^s_ϵ and ϕ^u_ϵ can be chosen so that they depend smoothly on ϵ. Up to a smooth reparametrization, we can assume that

$$(f+\epsilon)\phi^u_\epsilon(\ell(0)) = \phi^u_\epsilon(\ell(1)) \qquad \text{and} \qquad (f+\epsilon)\phi^s_\epsilon(\ell(0)) = \phi^s_\epsilon(\ell(1))\,. \tag{A.1}$$

We define, for $t \in [0,1]$,[8]

$$\mathrm{displ}(\epsilon)(t) = \Omega(\partial_t \ell(t), \phi^u_\epsilon(\ell(t)) - \phi^s_\epsilon(\ell(t))). \tag{A.2}$$

Roughly speaking, $\mathrm{displ}(\epsilon)(t)$ is the algebraic distance, along an affine line passing through $\ell(t)$ and normal to L, between the perturbed local stable and unstable manifolds. See Fig. 6.

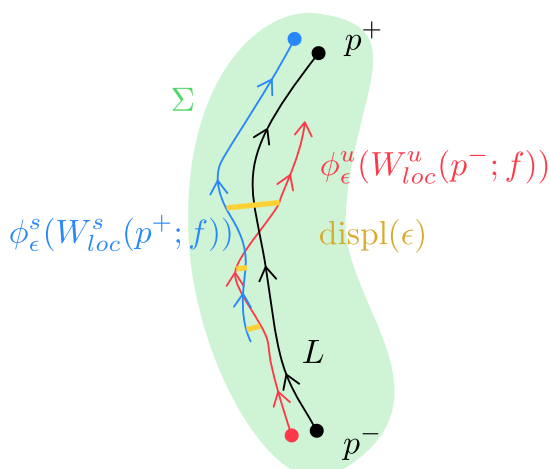

Figure 6: The displacement function is the algebraic distance between perturbed local stable and unstable manifolds.

By Eq. (A.1), $\phi^u_\epsilon(\ell([0,1]))$ and $\phi^s_\epsilon(\ell([0,1]))$ are fundamental domains. Then, the function $\mathrm{displ}(\epsilon)$ vanishes if and only if $\phi^s_\epsilon(\ell(t)) = \phi^u_\epsilon(\ell(t))$ for every $t \in [0,1]$. Thus, the heteroclinic link persists if and only if $\mathrm{displ}(\epsilon) \equiv 0$.

Definition A.1. *The* **displacement operator** *is the following map from a neighborhood $\mathcal{N}$ of* 0 *in $C^\infty(\Sigma, \mathbb{R}^2)$:*

$$\mathrm{displ} : \mathcal{N} \to C^\infty([0,1], \mathbb{R})$$
$$\epsilon \mapsto \mathrm{displ}(\epsilon),$$

where the function $\mathrm{displ}(\epsilon)$ *is defined in Eq.* (A.2).

[8] Equivalently, we can express the displacement function as the vector product $\partial_t \ell(t) \wedge (\phi^u_\epsilon(\ell(t)) - \phi^s_\epsilon(\ell(t)))$.

Remark A.2. The displacement operator is not uniquely defined: it depends on the parametrization ℓ of L and on the families $(\phi^u_\epsilon)_\epsilon$ and $(\phi^s_\epsilon)_\epsilon$.

We recall that ϕ^s_ϵ and ϕ^u_ϵ depend smoothly on ϵ, so the following proposition holds.

Proposition A.3. *The map* $\mathrm{displ} : \mathcal{N} \to C^\infty([0,1],\mathbb{R})$ *is smooth.*

The distance between the perturbed stable and unstable manifolds can be studied through the so-called Melnikov functions. We present here the main definitions and properties and refer for detailed proofs to [LMRR08], [DRR97], [DRR96], [GPB89]; the historical references are [Poi99] and [Mel63].

Definition A.4. *Let* $f \in C^\infty(\Sigma,\mathbb{R}^2)$ *be a symplectic diffeomorphism, which displays a heteroclinic link* L*, parametrized by* $\ell : \mathbb{R} \to L$*. The* **Melnikov function** *associated to a smooth family* $r \mapsto \epsilon(r) \in C^\infty(\Sigma,\mathbb{R}^2)$*, with* $\epsilon(0) = 0$*, is:*

$$\begin{aligned} &\mathcal{M}(\epsilon(s)) : \mathbb{R} \to \mathbb{R} \\ &t \mapsto \mathcal{M}(\epsilon(s))(t) := \sum_{k\in\mathbb{Z}} \Omega\left(\partial_t \ell(t), (f^*)^k(\partial_r \epsilon(r)|_{r=0} \circ f^{-1}) \circ \ell(t)\right), \end{aligned} \tag{A.3}$$

where $\partial_r \epsilon(r)|_{r=0} := \lim_{r\to 0} \frac{\epsilon(r)}{r}$.

Remark A.5. In Eq. (A.3) the presence of $\partial_r \epsilon(r)|_{r=0} \circ f^{-1}$ is due to the fact that, for the given family $(f+\epsilon(r))_r$, it is the vector field $X = \partial_r \epsilon(r)|_{r=0} \circ f^{-1}$ which satisfies $\frac{d}{dr}(f+\epsilon(r))|_{r=0} = X \circ f$. See [LMRR08, Section 2].

Recall that f is symplectic and, since $f(\ell(t)) = \ell(t+1)$, $\partial_t \ell(t) = Df^{-k}(\ell(t+k))\partial_t \ell(t+k)$. Thus, it holds

$$\mathcal{M}(\epsilon(s))(t) = \sum_{k\in\mathbb{Z}} \Omega\left(\partial_t \ell(t+k), \partial_s \epsilon|_{s=0}(\ell(t+k-1))\right). \tag{A.4}$$

Note that the Melnikov function belongs to the space $C^\infty(\mathbb{R}/\mathbb{Z},\mathbb{R})$ of smooth 1-periodic functions.

Definition A.6. *Let* f *be a symplectic diffeomorphism in* $C^\infty(\Sigma,\mathbb{R}^2)$ *displaying a heteroclinic link* L*, parametrized by* $\ell : \mathbb{R} \to L$*. The* **Melnikov operator** *is a linear and continuous map*

$$\mathcal{M} : C^\infty(\Sigma,\mathbb{R}^2) \ni \epsilon \mapsto \mathcal{M}(\epsilon) \in C^\infty(\mathbb{R}/\mathbb{Z},\mathbb{R}),$$

where $\mathcal{M}(\epsilon)$ *is the Melnikov function with respect to the smooth family* $r \mapsto r\epsilon$*.*

Remark A.7. Observe that $\mathcal{M}(\epsilon)$ depends only on $\epsilon|_L$. Thus, we can define $\mathcal{M}(\epsilon)$ for $\epsilon \in C^\infty(L,\mathbb{R}^2)$.

Theorem A.8 (Poincaré-Melnikov, [Poi99]-[Mel63])**.** *Let* f *be a symplectic diffeomorphism in* $C^\infty(\Sigma,\mathbb{R}^2)$*, displaying a heteroclinic link* L*, parametrized by* $\ell : \mathbb{R} \to L$*. Then, the partial differential of the displacement operator at* 0 *is equal to the Melnikov operator at* ϵ*, i.e.*

$$D_0\mathrm{displ} = \mathcal{M}. \tag{A.5}$$

Thus, for any $r \geq 1$*, in the* C^r*-uniform norm it holds*

$$\mathrm{displ}(\epsilon) = \mathrm{displ}(0) + D_0\mathrm{displ}(\epsilon) + O(\epsilon^2) = \mathcal{M}(\epsilon) + O(\epsilon^2).$$

For a detailed proof, we refer for example to [DRR97] and [GPB89].

A.2 Inverse Melnikov operator for flows in $\mathbb{R}^3$

Let U be an open subset of $\mathbb{R}^3$. Let $X \in \Gamma^\infty_{\text{Leb}}(U)$ display a **strong heteroclinic link** $\Gamma \subset U$ between two saddle periodic orbits $\gamma^\pm \subset U$. In particular, the closure of the link $\bar{\Gamma}$ is diffeomorphic to $\mathbb{R}/\mathbb{Z} \times [-1, 1]$.

Let $\bar{L}$ be the closure of the half strong stable manifold of a point in γ^+ in Γ. Let $\hat{\Sigma} \subset \mathbb{R}^3$ be a disk which contains $\bar{L}$ and that intersects transversally Γ. Up to reducing $\hat{\Sigma}$, it is transverse to the vector field X. Let $\Sigma \Subset \hat{\Sigma}$ be a neighborhood of L in $\hat{\Sigma}$ such that for every point $z \in \Sigma$, there exists $\tau(z) > 0$ minimal such that the orbit of z intersect transersally $\hat{\Sigma}$ at time $\tau(z)$ at a point $f(z)$. From the definition of strong heteroclinic link, the first return time $\tau|_{\Sigma\cap\Gamma}$ is constant, in particular γ^+ and γ^- have the same period. Indeed, since $\Sigma \cap \Gamma = L$ is a half strong stable manifold of a point in γ^+, it is $T(\gamma^+)$-periodic, where $T(\gamma^+)$ denotes the period of γ^+ (similarly, for a half strong unstable manifold of a point if γ^-). The disks $\Sigma \subset \hat{\Sigma}$ are called Poincaré sections and f is called a Poincaré first return map. The maps f and τ are in, respectively, $C^\infty(\Sigma, \hat{\Sigma})$ and $C^\infty(\Sigma, \mathbb{R}^+)$. Moreover, it is standard that there exists a smooth symplectic form Ω in $\hat{\Sigma}$ which is invariant by the action of $f|_\Sigma$. By Darboux's Theorem, we can identify Σ with a disk of $\mathbb{R}^2$ so that Ω is the standard symplectic form.

Let $W \in \Gamma^\infty_{\text{Leb}}(U)$ be a small perturbation of X: it induces a small perturbation $\epsilon \in C^\infty(\Sigma, \mathbb{R}^2)$ of the Poincaré map f. So we can study the unfolding of Γ using the tools of Melnikov operator for the heteroclinic link $L = \hat{\Sigma} \cap \Gamma$ presented above.

Fix a parametrization $\ell : \mathbb{R} \to L$ of L, such that $\lim_{t\to\pm\infty} \ell(t) = p^\pm$, where $p^\pm := \hat{\Sigma} \cap \gamma^\pm$, and $f(\ell(t)) = \ell(t + 1)$. Fix also $(\phi^s_\epsilon)_\epsilon$ and $(\phi^u_\epsilon)_\epsilon$ as in A.1. Let $\mathcal{M}$ be the Melnikov operator associated to this setting.

Definition A.9. *The* **Melnikov operator** *is the following linear and continuous map from a neighborhood $\mathcal{N}$ of 0 in $\Gamma^\infty_{\text{Leb}}(U)$*

$$\mathcal{M} : \mathcal{N} \ni W \mapsto \mathcal{M}(W) := \mathcal{M}(\epsilon) \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}) ,$$

where $\epsilon \in C^\infty(\Sigma, \mathbb{R}^2)$ is the perturbation of the first return map f associated to the vector field W.

Remark A.10. Observe that, since $\mathcal{M}(\epsilon)$ depends only on $\epsilon|_L$ (see Remark A.7), and so on $W|_\Gamma$, the operator $\mathcal{M}$ can be defined for $W \in C^\infty(\Gamma, \mathbb{R}^3)$.

The aim of this subsection is to give the proof of Proposition 3.7: such a proposition gives an inverse operator of the Melnikov one.

Proof of Proposition 3.7. We recall that Γ is an invariant cylinder smoothly embedded in $\mathbb{R}^3$, and that $L \subset \Gamma$ is a line transverse to the flow whose return time is a constant $T > 0$. As the statement is invariant by reparametrizing the flow, we can assume that $T = 1$.

This defines a canonical diffeomorphism $\Gamma \to L \times \mathbb{R}/\mathbb{Z}$ which maps L to $L \times \{0\}$. Recall that $\ell : \mathbb{R} \to L$ is a parametrization of L such that $f \circ \ell(t) = \ell(t + 1)$ for every $t \in \mathbb{R}$, where f is the first return map to Σ. Then using ℓ, we identify Γ to $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$ and L to $\mathbb{R} \times \{0\}$:

$$\Gamma \equiv \mathbb{R} \times \mathbb{R}/\mathbb{Z} \quad \text{and} \quad L \equiv \mathbb{R} \times \{0\} ,$$

such that, with $p_2 : \Gamma \to \mathbb{R}/\mathbb{Z}$ the second coordinate projection, the flow $(\Phi^t_X)_{t\in\mathbb{R}}$ of X satisfies:

$$p_2 \circ \Phi^t_X(s, \theta) = \theta + t \mod 1 , \quad \forall (s, \theta) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z} \equiv \Gamma .$$

Lemma A.11. *There is a compactly supported function $\Psi \in C^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, \mathbb{R}^+)$ such that:*

$$\int_{\mathbb{R}} \Psi \circ \Phi^t_X(s, 0)dt = 1 , \quad \forall s \in \mathbb{R}.$$

Proof. Let $\Psi_0 \in C^\infty(\mathbb{R}\times\mathbb{R}/\mathbb{Z},\mathbb{R}^+)$ be a smooth, compactly supported, non-negative function which is positive on $[\frac{1}{2},\frac{3}{2}]\times\{0\}$. For every $(s,\theta)\in\mathbb{R}\times\mathbb{R}/\mathbb{Z}\equiv\Gamma$ define

$$C(s,\theta) := \int_{\mathbb{R}} \Psi_0 \circ \Phi_X^t(s,\theta)dt\,.$$

Note that every orbit intersects $[\frac{1}{2},\frac{3}{2}]\times\{0\}$, so C is positive. Also the function C is Φ_V^t-invariant and smooth. Actually, $C(s,\theta)$ is equal to some constant C, because $\Phi_X^t|_\Gamma$ does not have any non trivial continuous first integral. Define then the function $\Psi\in C^\infty(\mathbb{R}\times\mathbb{R}/\mathbb{Z},\mathbb{R}^+)$ as

$$(s,\theta)\in\mathbb{R}\times\mathbb{R}/\mathbb{Z}\mapsto\Psi(s,\theta):=\frac{\Psi_0(s,\theta)}{C}\in\mathbb{R}^+\,.$$

It is obvious that the mean of Ψ along the orbits is one, as desired. □

Note that for every $(s,\theta)\in\mathbb{R}\times\mathbb{R}/\mathbb{Z}$, the orbit of (s,θ) intersects L at a set $(\varphi(s,\theta)+\mathbb{Z})\times\{0\}$. This defines a Φ_X^t-invariant map:

$$\varphi:\mathbb{R}\times\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}.$$

Note that for every $s,t\in\mathbb{R}$ it holds

$$\varphi\circ\Phi_X^t(s,0)=s \mod 1\,. \tag{A.6}$$

We recall that Σ is a Poincaré section containing L and transverse to Γ and that Ω denote an invariant symplectic form for the first return map f. Let $N:\mathbb{R}\times\mathbb{R}/\mathbb{Z}\equiv\Gamma\to\mathbb{R}^3$ be a vector field such that:

$$T\Gamma\oplus\mathbb{R}N=\mathbb{R}^3 \quad\text{and}\quad \Omega(\partial_t\ell(s),N_\Sigma\circ\ell(s))=1 \quad \forall s\in\mathbb{R}\,, \tag{A.7}$$

where N_Σ denotes the projection of $N|_\Gamma$ into the section Σ.

Now for every $g\in C^\infty(\mathbb{R}/\mathbb{Z},\mathbb{R})$ we define the smooth vector field $\mathcal{I}(g)\in C^\infty(\mathbb{R}\times\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$ as:

$$\mathcal{I}(g):=(s,\theta)\in\Gamma\mapsto g\circ\varphi(s,\theta)\cdot\Psi(s,\theta)\cdot N(s,\theta)\in\mathbb{R}^3. \tag{A.8}$$

We shall prove that the operator which associates to g the Melnikov function of $\mathcal{I}(g)$ is the identity. In order to do this, let us go back to the two dimensional setting. Let $\epsilon(g)\in C^\infty(\Sigma,\mathbb{R}^2)$ be the perturbation induced by the vector field $X+\mathcal{I}(g)$ on the first return map f. Consider the smooth family $r\mapsto\epsilon(rg)$. Then the partial derivative $\partial_r\epsilon|_{r=0}$ is:

$$\partial_r\epsilon(rg)|_{r=0}=\lim_{r\to0}\frac{1}{r}\epsilon(r\cdot g)\,.$$

Lemma A.12. *For every $s\in\mathbb{R}$ it holds:*

$$\Omega(\partial_t\ell(s+1),\partial_r\epsilon(rg)|_{r=0}\circ\ell(s))=g(s)\cdot\left(\int_0^1\Psi\circ\Phi_X^t(s,0)dt\right)\,,$$

where g is identified with a 1-periodic real function.

Proof. A flow-box coordinate neighborhood V of Γ is diffeomorphic to $\Gamma\times(-1,1)$. This diffeomorphism induces an identification of V with $\mathbb{R}\times\mathbb{R}/\mathbb{Z}\times(-1,1)$ and can be chosen so that N is constantly equal to $(0,0,1)$, $\Sigma\cap V$ is identified with $\mathbb{R}\times\{0\}\times\mathbb{R}$ and X has zero third component.

Let $X_g := X + \mathcal{I}(g)$ and denote by $(\Phi^t_{X_g})_{t\in\mathbb{R}}$ its flow. Let $\tau_g : \Sigma \to \mathbb{R}$ be the first return time on Σ associated to X_g. In the coordinates given by the identification, we have:

$$X_g(s,t) = X(s,\theta) + (0,0,g\circ\varphi(s,\theta)\cdot\Psi(s,\theta))\,, \quad \forall (s,\theta)\in\mathbb{R}\times\mathbb{R}/\mathbb{Z}\,.$$

Let $z = \ell(s)$ and express the perturbation $\epsilon(g)$ associated to $\mathcal{I}(g)$ as

$$\epsilon(g)(z) = ((f+\epsilon(g))(z) - z) - (f(z) - z)\,.$$

In the coordinates of the tubular neighborhood of Γ we have:

$$(0,\epsilon(g)(z)) = \int_0^{\tau_g(z)} (X+\mathcal{I}(g))\circ\Phi^t_{X+\mathcal{I}(g)}(z)dt - \int_0^1 X\circ\Phi^t_X(z)dt\,,$$

where the coordinate 0 is the coordinate in $\mathbb{R}/\mathbb{Z}$ (we have changed the ordering of the coordinates for the ease of notation).

Hence for every $r\geq 1$:

$$(0,\epsilon(g)(z)) = \int_1^{\tau_g(z)} X\circ\Phi^t_X(z)dt + \int_0^1 \mathcal{I}(g)\circ\Phi^t_X(z)dt + \int_0^1 \left(X\circ\Phi^t_{X+\mathcal{I}(g)}(z) - X\circ\Phi^t_X(z)\right)dt + O(\|g\|^2_{C^r})\,,$$

where we have used that $\int_1^{\tau_g(z)} X\circ\Phi^t_{X+\mathcal{I}(g)}(z)dt = \int_1^{\tau_g(z)} X\circ\Phi^t_X(z)dt + O(\|g\|^2_{C^r})$. Since, with $p_3 : \mathbb{R}\times\mathbb{R}/\mathbb{Z}\times(-1,1)\to(-1,1)$ the third coordinate projection, it holds $p_3(X) = 0$ on the neighborhood V, the first and third terms have their third components null while the second term has its first and second components null. Thus, it holds:

$$p_3(0,\epsilon(g)(z)) = p_3\left(\int_0^1 \mathcal{I}(g)\circ\Phi^t_X(z)dt\right) + O(\|g\|^2_{C^r})\,.$$

Now we plug Eq. (A.8) to obtain:

$$p_3(0,\epsilon(g)(z)) = p_3\left(\int_0^1 g\circ\varphi\circ\Phi^t_X(z)\cdot\Psi\circ\Phi^t_X(z)\cdot N\circ\Phi^t_X(z)dt\right) + O(\|g\|^2_{C^r})\,.$$

Now we use that $p_3\circ N\equiv 1$ and that, from (A.7), $\Omega(\partial_t\ell(s+1),\epsilon(g)\circ\ell(s)) = p_3(0,\epsilon(g)(z))$, to obtain:

$$\Omega(\partial_t\ell(s+1),\epsilon(g)\circ\ell(s)) = \int_0^1 g\circ\varphi\circ\Phi^t_X(z)\cdot\Psi\circ\Phi^t_X(z)dt + O(\|g\|^2_{C^r})\,. \tag{A.9}$$

Using the invariance of φ by the flow, we have $g\circ\varphi\circ\Phi^t_X(z) = g\circ\ell(s)$ for every t. Using Eq.(A.9) with the perturbation $\epsilon(rg)$, taking its derivative with respect to r and evaluating at $r=0$ yield the sought equality. □

The Melnikov function associated to $\mathcal{I}(g)$ is

$$\mathcal{M}(\mathcal{I}(g))(s) = \mathcal{M}(\epsilon(g))(s) = \sum_{k\in\mathbb{Z}} \Omega(\partial_t\ell(s+k),\partial_r\epsilon(rg)|_{r=0}(\ell(s+k-1))).$$

By Lemma A.12, since the function g is 1-periodic and by Lemma A.11, we deduce that

$$\mathcal{M}(\mathcal{I}(g))(s) = g(s)\cdot\sum_{k\in\mathbb{Z}}\left(\int_0^1 \Psi\circ\Phi^t_X(s+k,0)\,dt\right)$$

and, since $\Phi^t_X(s+k,0) = \Phi^{t+k}_X(s,0)$ for every $k\in\mathbb{Z}$, we conclude that

$$\mathcal{M}(\mathcal{I}(g))(s) = g(s)\cdot\int_{\mathbb{R}}\Psi\circ\Phi^t_X(s,0)dt = g(s)\,.$$

□

B Proof of Theorem 3.4: Cauchy-Kovalevskaya's Theorem for curl

B.1 Application of Cauchy-Kovalevskaya's Theorem for curl

Let $\Sigma \subset \mathbb{R}^3$ be a bounded, oriented, analytic surface that can be analytically extended in a neighborhood. Given a vector field W on Σ, we can decompose it as follows:

$$W = W_T + (W \cdot N)N,$$

where N is a unit normal to Σ, while W_T is the component of W tangent to Σ. We refer to W as the Cauchy datum for the Cauchy-Kovalevskaya theorem. Denote as $g := W \cdot N$ the normal component of the Cauchy datum.

We will be interested in W satisfying the following condition:

$$(*) \qquad d(W_T^\flat) = g\sigma,$$

where $W_T^\flat$ is the 1-form dual to W_T, push-forwarded on Σ, and σ is the area form on Σ induced from the ambient space $\mathbb{R}^3$. The following generalizes Theorem 3.1 in [EPS12].

Theorem B.1 (Cauchy-Kovalevskaya's Theorem for curl)**.** *Let $\Sigma \subset \mathbb{R}^3$ be a surface as above and let W be the analytic Cauchy datum in $C^\omega(\bar{\Sigma}, \mathbb{R}^3)$. The equation*

$$\text{(B.1)} \qquad \operatorname{curl} X = X,$$

$$\text{(B.2)} \qquad X|_\Sigma = W$$

has a unique, analytic solution in a neighborhood Ω of Σ if and only if W fulfills condition $(*)$.

This theorem implies Theorem 3.4

Proof of Theorem 3.4. Let Γ be an analytic surface in $\mathbb{R}^3$ whose closure $\bar{\Gamma}$ is diffeomorphic to $\mathbb{I} \times \mathbb{R}/\mathbb{Z}$. We are going to show that for every $g \in C^\omega(\bar{\Gamma}, \mathbb{R})$, there exists a neighborhood Ω of $\bar{\Gamma}$ in $\mathbb{R}^3$ and a Beltrami field $X \in C^\omega(\Omega, \mathbb{R}^3)$ such that the normal component to Γ of $X|_\Gamma$ equals g. To be precise, with N a unit normal vector field on Γ, this means that $(X|_\Gamma) \cdot N = g$.

In order to show this, it suffices to observe that, given the normal component $g \in C^\omega(\bar{\Gamma}, \mathbb{R})$ of a Cauchy datum on Γ, there are many Cauchy data W satisfying condition $(*)$ and such that $W \cdot N = g$. Endow $\bar{\Gamma} \simeq \mathbb{I} \times \mathbb{R}/\mathbb{Z}$ with coordinates $(z, \theta) \in \mathbb{I} \times \mathbb{R}/\mathbb{Z}$ so that the area form σ is $dz \wedge d\theta$. We are looking for a Cauchy datum W such that

$$d(W_T^\flat) = g dz \wedge d\theta.$$

Defining

$$\tilde{g}(z, \theta) := \int_{-1}^{z} g(s, \theta) ds,$$

we set

$$W_T^\flat := \tilde{g}(z, \theta) d\theta.$$

Thus, it clearly holds that $d(W_T^\flat) = g(z, \theta) dz \wedge d\theta$, that is W satisfies condition $(*)$. By Theorem B.1 applied at Γ, we conclude that there exists a neighborhood Ω of Γ in $\mathbb{R}^3$ and an Euclidean Beltrami field $X(g) \equiv X \in C^\omega(\Omega, \mathbb{R}^3)$, defined as

$$X|_\Gamma := gN + (\tilde{g}(z, \theta) d\theta)^\#,$$

whose normal component is g. $\square$

Remark B.2. Any other $W_T^\flat$ satisfying condition $(*)$ is of the form $\tilde{g}(z,\theta)d\theta+\alpha$, where α is a closed 1-form on Σ. In the following, we will always choose $\alpha = 0$, which implies the uniqueness of the Beltrami field with normal datum g.

Remark B.3. Let $g \in C^\omega(\bar{\Sigma},\mathbb{R}) \mapsto X(g) \in C^\omega(\Omega,\mathbb{R}^3)$ be the map that associates to every normal datum g the corresponding Beltrami field $X(g)$ given by Theorem 3.4. Such a map is linear. Indeed, let g_1, g_2 be normal data and $c_1, c_2 \in \mathbb{R}$. Then, for $i = 1, 2$, there exists a neighborhood Ω_i of Σ and a Beltrami field $X^i := X(g_i) \in C^\omega(\Omega_i,\mathbb{R}^3)$ such that

$$X^i|_\Sigma = W_T^i + g_i N$$

where $W_T^i := (\tilde{g}_i(z,\theta)d\theta)^\#$. Thus, the Beltrami field $X(c_1g_1 + c_2g_2) \in C^\omega(\Omega_1 \cap \Omega_2,\mathbb{R}^3)$ satisfies

$$X(c_1g_1 + c_2g_2)|_\Sigma =: X =$$

$$(c_1\tilde{g}_1(z,\theta)d\theta)^\# + (c_2\tilde{g}_2(z,\theta)d\theta)^\# + (c_1g_1 + c_2g_2)N = (c_1W_T^1 + c_2W_T^2) + (c_1g_1 + c_2g_2)N =$$
$$c_1(W_T^1 + g_1N) + c_2(W_T^2 + g_2N) = c_1X^1 + c_2X^2.$$

Since the equation $\operatorname{curl} X = X$ is linear, by uniqueness (see Remark B.2) we have that X associated to the normal datum $c_1g_1 + c_2g_2$ is

$$X = c_1X^1|_{\Omega_1\cap\Omega_2} + c_2X^2|_{\Omega_1\cap\Omega_2}.$$

B.2 Proof of Theorem B.1

Let us first consider a toy model to understand Theorem B.1 and Condition $(*)$, following [EPS20, Subsection 3.2]. Let us consider $\Sigma = \{x_3 = 0\}$ with coordinates (x_1, x_2, x_3) in $\mathbb{R}^3$. Fix an analytic Cauchy datum

$$W = W_1(x_1,x_2)\partial_{x_1} + W_2(x_1,x_2)\partial_{x_2} + W_3(x_1,x_2)\partial_{x_3}.$$

In coordinates, the Beltrami equation $\operatorname{curl} X = X$ reads as

$$\begin{cases} \dfrac{\partial X_1}{\partial x_3} = X_2 + \frac{\partial X_3}{\partial x_1} & \text{(B.3)} \\ \dfrac{\partial X_2}{\partial x_3} = -X_1 + \frac{\partial X_3}{\partial x_2} & \text{(B.4)} \\ \dfrac{\partial X_3}{\partial x_3} = -\frac{\partial X_1}{\partial x_1} - \frac{\partial X_2}{\partial x_2} & \text{(B.5)} \end{cases}.$$

By the standard Cauchy-Kovaleskaya's Theorem applied at $\Sigma = \{x_3 = 0\}$, there exists a unique analytic solution to this system in a neighborhood Ω of Σ, with Cauchy datum W.

Considering $\frac{\partial}{\partial x_1}$(B.4)$-\frac{\partial}{\partial x_2}$(B.3) and using (B.5), we obtain

$$\frac{\partial}{\partial x_3}\left(\frac{\partial X_2}{\partial x_1} - \frac{\partial X_1}{\partial x_2} - X_3\right) = 0.$$

Thus

$$\frac{\partial X_2}{\partial x_1} - \frac{\partial X_1}{\partial x_2} = X_3 + f(x_1,x_2), \tag{B.6}$$

in Ω, for some analytic function f. Evaluating (B.6) at $\Sigma = \{x_3 = 0\}$, we obtain

$$\frac{\partial W_2}{\partial x_1} - \frac{\partial W_1}{\partial x_2} = W_3 + f(x_1,x_2).$$

The vector field $X = (X_1, X_2, X_3)$ is Beltrami in Ω if and only if the Cauchy datum satisfies the constraint

$$\frac{\partial W_2}{\partial x_1} - \frac{\partial W_1}{\partial x_2} = W_3. \tag{B.7}$$

In terms of the dual 1-form $\beta := W_1 dx_1 + W_2 dx_2$, this is equivalent to

$$d\beta = W_3 dx_1 \wedge dx_2. \tag{B.8}$$

We have to choose the tangent part β of the Cauchy datum so that (B.8) is fullfilled. If Σ is open, given W_3, this is always possible.

We now prove Theorem B.1, whose proof follows closely that of Theorem 3.1 in [EPS12]. Observe that Condition ($*$) reduces to the condition of [EPS12, Theorem 3.1] when $g = 0$.

Proof of Theorem B.1. Let $\Sigma \subset \mathbb{R}^3$ be a bounded, oriented, analytic surface. Denote by

$$j_\Sigma : \Sigma \to \mathbb{R}^3$$

the analytic embedding of Σ into $\mathbb{R}^3$. Let W be an analytic vector field defined on Σ. We will refer to it as Cauchy datum. We write W as

$$W = W_T + (W \cdot N)N,$$

where N is the unit normal vector to Σ, while W_T is the component of W tangent to Σ. Let $W^\flat$ be the dual 1-form of W, while $W_T^\flat$ denotes the dual 1-form of W_T. Let $g := W \cdot N$. Recall that the vector field W satisfies Condition ($*$) if

$$dW_T^\flat = g\sigma,$$

where here d is the exterior derivative on Σ and σ is the induced area-form on Σ.

Take local analytic coordinates in a neighborhood Ω of Σ:

$$(\rho, \xi_1, \xi_2),$$

where ρ is the signed distance function from Σ: if Ω is narrow enough, then this function is analytic. In these coordinates, the Euclidean metric reads as:

$$ds^2 = d\rho^2 + h_{ij}(\rho, \xi_1, \xi_2) d\xi_i d\xi_j,$$

where the last expression has to be understood as sum for $i, j = 1, 2$. In the sequel, we will use such Einstein convention, i.e. repeated indices have to be understood as summed. Denote as h^{ij} the inverse matrix of h_{ij} and $|h| := \det(h_{ij})$ its determinant.

The Beltrami field X reads in these coordinates as

$$X = a(\rho, \xi_1, \xi_2)\partial_\rho + b^i(\rho, \xi_1, \xi_2)\partial_{\xi_i}$$

for some functions $a(\rho, \xi_1, \xi_2), b^1(\rho, \xi_1, \xi_2), b^2(\rho, \xi_1, \xi_2)$. The 1-form associated to X is so

$$\beta := a(\rho, \xi_1, \xi_2) d\rho + c_i(\rho, \xi_1, \xi_2) d\xi_i,$$

with $c_i := h_{ij} b^j$. In terms of the 1-form β, the equation $\operatorname{curl} X = X$ reads as

$$\star d\beta = \beta, \tag{B.9}$$

where $\star$ denotes the Hodge star operator. In the coordinates (ρ, ξ_1, ξ_2), the term $\star d\beta$ has the form

$$\star d\beta =$$

$$\frac{1}{|h|^{1/2}}\left(\frac{\partial c_2}{\partial \xi_1} - \frac{\partial c_1}{\partial \xi_2}\right) d\rho + |h|^{1/2} h^{2i}\left(\frac{\partial a}{\partial \xi_i} - \frac{\partial c_i}{\partial \rho}\right) d\xi_1 + |h|^{1/2} h^{1i}\left(\frac{\partial c_i}{\partial \rho} - \frac{\partial a}{\partial \xi_i}\right) d\xi_2.$$

The equation $\star d\beta = \beta$ is then equivalent, in (ρ, ξ_1, ξ_2)-coordinates, to:

$$\begin{cases} \frac{1}{|h|^{1/2}}\left(\frac{\partial c_2}{\partial \xi_1} - \frac{\partial c_1}{\partial \xi_2}\right) = a & \text{(B.10)} \\ |h|^{1/2} h^{2i}\left(\frac{\partial a}{\partial \xi_i} - \frac{\partial c_i}{\partial \rho}\right) = c_1 & \text{(B.11)} \\ |h|^{1/2} h^{1i}\left(\frac{\partial c_i}{\partial \rho} - \frac{\partial a}{\partial \xi_i}\right) = c_2 & \text{(B.12)} \end{cases}$$

We start by showing the **necessity** of Condition ($*$). The tangential component of X on Σ is

$$X_T = b^i(0, \xi_1, \xi_2)\partial_{\xi_i}.$$

Its dual 1-form is then $X_T^\flat = c_i(0, \xi_1, \xi_2) d\xi_i$. Thus

$$d(c_i(0, \xi_1, \xi_2) d\xi_i) = \left(\frac{\partial c_2(0, \xi_1, \xi_2)}{\partial \xi_1} - \frac{\partial c_1(0, \xi_1, \xi_2)}{\partial \xi_2}\right) d\xi_1 \wedge d\xi_2.$$

On the other hand the area-form σ on Σ is, since $N = \partial_\rho$,

$$\sigma = |h|^{1/2}(0, \xi_1, \xi_2) d\xi_1 \wedge d\xi_2.$$

Evaluating then (B.10) at $\rho = 0$, i.e. at Σ, we obtain

$$\frac{\partial c_2(0, \xi_1, \xi_2)}{\partial \xi_1} - \frac{\partial c_1(0, \xi_1, \xi_2)}{\partial \xi_2} = a(0, \xi_1, \xi_2)|h|^{1/2}(0, \xi_1, \xi_2),$$

and, denoting $g := a(0, \xi_1, \xi_2)$, we obtain Condition ($*$):

$$d(X_T^\flat) = g\sigma.$$

We now show the **sufficiency** of Condition ($*$). Consider the auxiliary problem:

$$\text{(A)} \qquad \begin{cases} (d + \delta)\psi = \star\psi & \text{in } \Omega \\ \psi|_\Sigma = W^\flat, \end{cases}$$

where $W^\flat$ is the dual 1-form of W, δ is the codifferential (i.e. for $\omega \in \Omega^k$ it holds $\delta\omega = (-1)^k \star d \star \omega$), and, denoting as Ω^k the space of k-forms, $\psi \in \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \Omega^3$. In particular, ψ can be decomposed as $\psi^0 \oplus \psi^1 \oplus \psi^2 \oplus \psi^3$, with ψ^k a k-form. The operator $d + \delta$ is a first order elliptic operator (notice that $(d + \delta)^2 = d\delta + \delta d = \Delta$ is the Hodge Laplacian). So, by the standard Cauchy-Kovalevskaya's Theorem (see [Tay11, Page 502, Proposition 4.2]), the Cauchy problem (A) has a unique analytic solution in Ω, for some Ω narrow enough.

Lemma B.4. *If W satisfies Condition ($*$), then $\delta\psi = 0$ in Ω.*

Proof of Lemma B.4. In [EPS12, Section 4] the authors define a parity operator

$$Q : \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \Omega^3 \to \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \Omega^3$$

as

$$Q\psi := \psi^0 \oplus -\psi^1 \oplus \psi^2 \oplus -\psi^3,$$

where $\psi = \psi^0 \oplus \psi^1 \oplus \psi^2 \oplus \psi^3$. First, since $\star d \star = -Q\delta$, applying $\star d$ at the equation (A) and since $\star\star = \mathrm{Id}$, we notice that $\delta\psi$ satisfies the elliptic equation

$$(d+\delta)\delta\psi = -\star Q\delta\psi.$$

Therefore, if we show that $\delta\psi|_\Sigma = 0$, then $\delta\psi = 0$ in Ω by uniqueness of the solution by the Cauchy-Kovalevskaya's Theorem, since the operator $d+\delta$ is elliptic.

Our aim is then proving that $\delta\psi|_\Sigma = 0$. Write ψ as follows:

$$\psi = \Psi + d\rho \wedge \tilde{\Psi},$$

where $\Psi, \tilde{\Psi}$ are forms in $\Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \Omega^3$ such that $i_{\partial_\rho}\Psi = i_{\partial_\rho}\tilde{\Psi} = 0$. Since $\psi|_\Sigma = W^\flat$, then

$$\Psi|_\Sigma = c_i d\xi_i \quad \text{and} \quad \tilde{\Psi}|_\Sigma = g.$$

From (A), it holds that

$$\delta\psi|_\Sigma = (\star\psi - d\psi)|_\Sigma. \tag{B.13}$$

By the Cauchy condition, it holds that $\star\psi|_\Sigma = \star W^\flat|_\Sigma$. Using that $\star d\rho = |h|^{1/2} d\xi_1 \wedge d\xi_2$, we deduce that

$$\star\psi|_\Sigma = \star W^\flat|_\Sigma = \star(g d\rho + c_1(0,\xi_1,\xi_2)d\xi_1 + c_2(0,\xi_1,\xi_2)d\xi_2) =$$

$$g|h|^{1/2} d\xi_1 \wedge d\xi_2 + d\rho \wedge \tilde{\alpha}, \tag{B.14}$$

for some 1-form $\tilde{\alpha}$. Denoting as $\bar{d}$ the exterior derivative only in the variables ξ_1, ξ_2, we have

$$d\psi = \bar{d}\Psi + d\rho \wedge \frac{\partial \Psi}{\partial \rho} - d\rho \wedge \bar{d}\tilde{\Psi} = \bar{d}\Psi + d\rho \wedge \left(\frac{\partial \Psi}{\partial \rho} - \bar{d}\tilde{\Psi} \right).$$

Since $\Psi|_\Sigma = c_1(0,\xi_1,\xi_2)d\xi_1 + c_2(0,\xi_1,\xi_2)d\xi_2$, it holds, using Condition ($*$),

$$\bar{d}\Psi|_\Sigma = g|h|^{1/2} d\xi_1 \wedge d\xi_2.$$

Consequently

$$d\psi|_\Sigma = g|h|^{1/2} d\xi_1 \wedge d\xi_2 + d\rho \wedge \hat{\alpha}, \tag{B.15}$$

for some $\hat{\alpha} = \hat{\alpha}^0 \oplus \hat{\alpha}^1 \oplus \hat{\alpha}^2$ on Σ. Putting together (B.13), (B.14) and (B.15), we deduce that

$$\delta\psi|_\Sigma = d\rho \wedge \check{\alpha},$$

for some $\check{\alpha} = \tilde{\alpha} - \hat{\alpha} = \check{\alpha}^0 \oplus \check{\alpha}^1 \oplus \check{\alpha}^2$ on Σ. Noticing that $\psi|_\Sigma = W^\flat$ is a 1-form, we obtain:

(i) since $\delta\psi|_\Sigma = d\rho \wedge \check{\alpha}$ does not contain 0-forms, the components $\delta\psi^0|_\Sigma$ and $\delta\psi^1|_\Sigma$ are both zero;

(*ii*) for $p = 2, 3$ we have

$$\delta\psi^p|_\Sigma = (-1)^p(\star d \star \psi^p)|_\Sigma = (-1)^p \star (d\rho \wedge \beta)$$

for some form β. This comes from the fact that

$$d(\star\psi^2)|_\Sigma = d\rho \wedge \partial_\rho(\star\psi^2)|_{\rho=0} + \bar{d}(\star\psi^2)|_{\rho=0},$$

but since $\psi^2|_{\rho=0} = 0$, then $\bar{d}(\star\psi^2)|_{\rho=0} = \bar{d}(\star\psi^2|_{\rho=0}) = 0$. Similarly, since $\psi^3|_\Sigma = 0$, it holds that

$$d(\star\psi^3)|_\Sigma = d\rho \wedge \partial_\rho(\star\psi^3)|_{\rho=0}.$$

Therefore, for $p = 2, 3$, $\delta\psi^p|_\Sigma$ only contains elements with $d\xi_1$ and $d\xi_2$, for $p = 2$, and only $d\xi_1 \wedge d\xi_2$ for $p = 3$.

The fact that $\delta\psi|_\Sigma$ does not contain the element $d\rho$ contradicts the expression $\delta\psi|_\Sigma = d\rho \wedge \check{\alpha}$ previously found, unless

$$\delta\psi|_\Sigma = 0,$$

as we wanted to prove. This complete the proof of Lemma B.4. □

Going back to the problem (A), since $\delta\psi = 0$ in Ω, we have that

$$\begin{cases} d\psi = \star\psi & \text{in } \Omega \\ \psi|_\Sigma = W^\flat \end{cases}$$

and, using the identity $\star(\star\psi) = \psi$, it holds

$$\begin{cases} \star d\psi = \psi & \text{in } \Omega \\ \psi|_\Sigma = W^\flat. \end{cases}$$

Since $\star d$ maps 1-forms into 1-forms and $W^\flat$ is a 1-form, it holds $\psi^0 = \psi^2 = \psi^3 = 0$. Hence

$$\begin{cases} \star d\psi^1 = \psi^1 & \text{in } \Omega \\ \psi^1|_\Sigma = W^\flat. \end{cases}$$

Taking the vector field X dual to ψ^1, it satisfies then $\operatorname{curl} X = X$ and the sufficiency follows. This concludes the proof of the sufficiency and so of Theorem B.1. □

C Disjoint union of compact subsets with connected complement

The following result is used in the proof of Corollary 3.2. It is probably standard, but we include a proof for the sake of completeness.

Proposition C.1. *Let K_+ and K_- be two disjoint compact subsets of $\mathbb{R}^n$ with connected complement. Then $K = K_+ \sqcup K_-$ has connected complement.*

Proof. Since K is compact, it is clear that $\mathbb{R}^n \backslash K$ is connected if and only if $\mathbb{S}^n \backslash K$ is connected, where the n-sphere $\mathbb{S}^n$ is understood as the one-point compactification of $\mathbb{R}^n$. Alexander's duality then implies [Mas78] that

$$\tilde{H}_0(\mathbb{S}^n \backslash K; \mathbb{Z}) = \tilde{H}^{n-1}_{CE}(K; \mathbb{Z}),$$

where $\tilde{H}_0(\mathbb{S}^n\backslash K;\mathbb{Z})$ stands for the reduced singular homology of $\mathbb{S}^n\backslash K$ with integer coefficients (well defined because it is a manifold) and $\tilde{H}^{n-1}_{CE}(K;\mathbb{Z})$ denotes the reduced Čech cohomology, which is defined for any compact subset of $\mathbb{R}^n$. Now, since the reduced k-th cohomology group coincides with the cohomology group for $k \geq 1$ (this also holds for Čech cohomology, for which one can also write a long exact sequence [ES52]) and K_+ and K_- are disjoint, we infer that

$$\tilde{H}^{n-1}_{CE}(K;\mathbb{Z}) = H^{n-1}_{CE}(K;\mathbb{Z}) = H^{n-1}_{CE}(K_+;\mathbb{Z}) \oplus H^{n-1}_{CE}(K_-;\mathbb{Z})\,,$$

where to write the last isomorphism we have used the standard Mayer-Vietoris sequence for Čech cohomology [Mil16]. Then, using again Alexander's duality and that $H^{n-1}_{CE}(K_\pm;\mathbb{Z}) = \tilde{H}^{n-1}_{CE}(K_\pm;\mathbb{Z})$, we obtain that

$$H^{n-1}_{CE}(K_+;\mathbb{Z}) \oplus H^{n-1}_{CE}(K_-;\mathbb{Z}) = \tilde{H}_0(\mathbb{S}^n\backslash K_+;\mathbb{Z}) \oplus \tilde{H}_0(\mathbb{S}^n\backslash K_-;\mathbb{Z}) = 0$$

because $\mathbb{S}^n\backslash K_\pm$ are connected by assumption, and the 0-th reduced singular homology group of a manifold is trivial if and only if the manifold is connected. Putting all these computations together we finally conclude that

$$\tilde{H}_0(\mathbb{S}^n\backslash K;\mathbb{Z}) = 0\,,$$

so $\mathbb{S}^n\backslash K$ is connected, as we wanted to prove. □

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