Structure of polyzetas and monoid factorization
C.V. Bui, Vincel Hoang Ngoc Minh, Q. H. Ngo

To cite this version:
C.V. Bui, Vincel Hoang Ngoc Minh, Q. H. Ngo. Structure of polyzetas and monoid factorization. 2022. hal-03558930

HAL Id: hal-03558930
https://hal.science/hal-03558930
Preprint submitted on 5 Feb 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Structure of polyzetas and monoid factorization

C.V Bui, V. Hoang Ngoc Minh, and Q.H. Ngo

1 Hue University, 77, Nguyen Hue, Hue, Viet Nam, bvchien.vn@gmail.com
2 University of Lille, 1 Place Délion, 59024 Lille, France, LIPN - UMR 7030, CNRS, 93430 Villetaneuse, France, vincel.hoang-ngoc-minh@univ-lille.fr, minh@lipn.univ-paris13.fr
3 Hai Phong University, 171, Phan Dang Luu, Hai Phong, VietNam, hoannq@dhhp.edu.vn

Abstract. We review simultaneously the essential steps to establish the equations bridging the algebraic structures of converging polyzetas, via their noncommutative generating series put in factorized form MRS. These equations then allow us to describe polynomial relations, homogeneous in weight, among polyzetas, by local coordinates identification.

Keywords: Zeta function · monoid factorization · rewriting system.

1 Introduction

For any integral multiindex \((s_1, \ldots, s_r) \in \mathbb{N}_\geq 1, r \geq 1\), the polylogarithm \(\text{Li}_{s_1, \ldots, s_r}\) and the harmonic sum \(H_{s_1, \ldots, s_r}\) are defined, for \(z \in \mathbb{C}, |z| < 1\) and \(n \in \mathbb{N}\), as follows

\[
\text{Li}_{s_1, \ldots, s_r}(z) := \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \cdots n_r^{s_r}},
\]

\[
H_{s_1, \ldots, s_r}(n) := \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}.
\]

They satisfy

\[
(1 - z)^{-1} \text{Li}_{s_1, \ldots, s_r}(z) = \sum_{n \geq 1} H_{s_1, \ldots, s_r}(n) z^n.
\]

In particular, for \(s_1 > 1\), the following limits exist and, after a theorem by Abel, one has

\[
\lim_{z \to 1} \text{Li}_{s_1, \ldots, s_r}(z) = \lim_{n \to \infty} H_{s_1, \ldots, s_r}(n) =: \zeta_r(s_1, \ldots, s_r),
\]

so-called MZV \([20]\), or polyzeta \([8]\) for the contraction of polymorphism \(\zeta\) \([14,15]\), see Section 3.3.

4 after the names of G. Mélançon, C. Reutenauer, M.P. Schützenberger
These objects were appeared in the regularization of solutions of the following first order differential equation, with singularities in \( \{0, 1, +\infty\} \):

\[
(DE) \quad \frac{dG(z)}{dz} = \left( \frac{x_0}{z} + \frac{x_1}{1-z} \right) G(z)
\]

and noncommutative indeterminates in \( \Omega \).

Let \( \mathcal{H}(\Omega) \) denote the ring of holomorphic functions on the simply connected domain \( \Omega := \mathbb{C} - \{0, 1\} \) with \( 1_\Omega : \Omega \to \mathcal{H}(\Omega) \) (mapping \( z \) to 1) as the neutral element. Let us also introduce the differential forms

\[
\omega_0(z) := z^{-1}dz \quad \text{and} \quad \omega_1(z) := (1-z)^{-1}dz.
\]

The resolution of \( (DE) \), on the completion of \( \mathcal{H}(\Omega)(X) \), uses the so-called Chen series, of \( \omega_0 \) and \( \omega_1 \) and along a path \( z_0 \rightsquigarrow z \) on \( \Omega \), defined by \[7\]:

\[
C_{z_0 \rightsquigarrow z} := \sum_{w \in X^*} \alpha_{z_0}^z(w)w \in \mathcal{H}(\Omega)(X) = \mathcal{H}(\Omega)\langle X \rangle,
\]

where \( X^* \) is the free monoid generated by the alphabet \( X \) (equipping \( 1_{X^*} \) as neutral element) and, for a subdivision \( (z_0, z_1, \ldots, z_k, z) \) of \( z_0 \rightsquigarrow z \) and the coefficient \( \alpha_{z_0}^z(w) \in \mathcal{H}(\Omega) \) is defined, for any \( w = x_1 \cdots x_k \in X^* \), as follow\[6\]

\[
\alpha_{z_0}^z(w) = \int_{z_0}^{z} \omega_1(z_1) \int_{z_0}^{z_1} \omega_1(z_2) \cdots \int_{z_0}^{z_{k-1}} \omega_1(z_k)dz_k \quad \text{and} \quad \alpha_{z_0}^z(1_{X^*}) = 1_\Omega.
\]

The series \( C_{z_0 \rightsquigarrow z} \) is group-like \[18\], i.e. that there exists a primitive series \( L_{z_0 \rightsquigarrow z} \) such that

\[
e^{L_{z_0 \rightsquigarrow z}} = C_{z_0 \rightsquigarrow z}.
\]

In \[11\], essentially interested in solutions of \( (9) \) over the interval \( ]0, 1[ \) and, using the involution \( z \mapsto 1-z \), Drinfel’d stated that \( (9) \) admits a unique solution \( G_0 \) (resp. \( G_1 \)) satisfying

\[
G_0(z) \sim_0 z^{x_0} \quad \text{and} \quad G_1(z) \sim_1 (1-z)^{-x_1}.
\]

Since \( G_0 \) and \( G_1 \) are group-like series then there is a unique group-like series \( \Phi_{KZ} \in \mathbb{R}(\langle X \rangle) \), so-called Drinfel’d series (or Drinfel’d associator), such that

\[
G_0 = G_1 \Phi_{KZ}.
\]

After that, Lê and Murakami expressed, in particular, the divergent coefficients of \( \Phi_{KZ} \) as linear combinations of \( \{\zeta_{r}(s_1, \ldots, s_r)\}_{r \geq 1, (s_1, \ldots, s_r) \in \mathbb{N}^*_r, s_1 \geq 2} \), via a regularization based on representation of the chord diagram algebras \[16\].

---

5 For \( x_0 = A/2\pi \) and \( x_1 = -B/2\pi \), \( (DE) \) is nothing else \( (KZ_0) \) introduced in \[14\].

6 This only depends on the homotopy class of \( z_0 \rightsquigarrow z \), and the endpoints \( (z_0, z) \).
In all the sequel, for simplification, we will also adopt the notation $\zeta$ for $\zeta_r, r \in \mathbb{N}$. Moreover, any multiindex $(s_1, \ldots, s_r) \in \mathbb{N}_r^r, r \geq 1$ can be viewed as a sequence of the monoid generated by $\mathbb{N}_r^r$ then let us note that there are one-to-one correspondences among sequences of monoids:

$$(s_1, \ldots, s_r) \in \mathbb{N}_r^r \leftrightarrow y_{s_1} \cdots y_{s_r} \in Y^* \frac{\pi_Y}{\pi_X} x_0^{s_1-1} x_1 \cdots x_0^{s_r-1} x_1 \in X^* x_1, \quad (12)$$

where $Y := \{y_k\}_{k \geq 1}$ generating the free monoid $Y^*$ (its neutral element is $1_{Y^*}$) and $\pi_X$ is the conc-morphism of Cauchy algebras of polynomials with coefficients in a commutative ring $A\ [1]$, $A\langle Y \rangle \xrightarrow{\text{conc}} A\langle X \rangle, \quad \pi_Y x_k \xrightarrow{\text{adj}} x_0^{k-1} x_1. \quad (14)$

This morphism $\pi_X$ admits an adjoint $\pi_Y$ for the two standard scalar products which has a simple combinatorial description: the restriction of $\pi_Y$ to the subalgebra $(A1_{X^*} \oplus A\langle X \rangle x_1, \text{conc}, 1_{X^*})$, is an isomorphism given by

$$\pi_Y(x_0^{k-1} x_1) = y_k \quad (15)$$

(and the kernel of the non-restricted $\pi_Y$ is $A\langle X \rangle x_0$).

Once, the correspondences given in (12) are adopted, the notation $\zeta$ previously introduced, can be viewed then as a map from monoid to $\mathbb{R}[14,15]$ as we will precise in the sequel, using (4) and the fact that $\{L_{s_1,\ldots,s_r}\}_{(s_1,\ldots,s_r)\in \mathbb{N}_r^r}$ given in (1) (resp. $\{H_{s_1,\ldots,s_r}\}_{(s_1,\ldots,s_r)\in \mathbb{N}_r^r}$ given in (2)) is obtained as images by isomorphism from the shuffle (resp. stuffle) algebra over $X$ (resp. $Y$), to subalgebra of holomorphic (resp. arithmetic) functions [1 3] (resp. [10]).

In this work, we review simultaneously the essential steps to furnish $G_0$ and $\Phi_{KZ}$ and then the equations bridging the algebraic structures of converging polyzetas. This allows to obtain algebraic generating systems of polyzetas (see tables in Appendix). The organization of this paper is follows:

- In Section 2 algebraic combinatorial frameworks will be recalled. In particular, we will give an explicit isomorphism, $\varphi_{\pi_1}$, from shuffle bialgebra to stuffle bialgebra. Doing with $\varphi_{\pi_1}$, the construction by Mélançon-Reutenauer-Schützenberger (MRS, in short), initially elaborated in shuffle bialgebra and useful to factorize group-like series, will be extended in the stuffle bialgebra for the similar factorizations. It is done via explicit constructions of pairs of bases in duality and thanks to the Cartier-Quillen-Milnor-Moore (CQMM, in short) and Poincaré-Birkhoff-Witt (PBW, in short) theorems.

- In Section 3 to obtain a structure of polyzetas, the polylogarithms and harmonic sums will be encoded by words over various alphabets. In particular, their noncommutative generating series will be put in the MRS form. We

\[ \text{7 although some of the properties already hold for a general commutative semiring } A. \]

\[ \text{8 That is to say } (\forall p \in A\langle X \rangle) (\forall q \in A\langle Y \rangle) \left( (\pi_Y p|q)|y = \langle p|\pi_X q \rangle x \right). \]

\[ \text{9 for the contraction of sticky shuffle.} \]
will insist on the fact that the noncommutative generating series of polylogarithms is the actual solution of (4) and the generating series of the finite parts of their singular expansions corresponds to $\Phi_{KZ}$. The latest will be also put in MRS form without divergent zeta values as local coordinates and will provide an algorithm describing the image and the kernel of the polymorphism $\zeta$.

2 Combinatorial frameworks

In this section, unless explicitly stated, all tensor products will be considered over the ambient ring (or field), i.e. $A$.

2.1 Noncommutative polynomials

In Section 1, the encoding alphabets $X$ and $Y$ were already introduced. For all matters concerning finite ($X$ and similar) or infinite ($Y$ and similar) alphabets, we will use a generic model noted $X$ (generating the free monoid $X^*$ equipping $1_X$ as its neural element) in order to state their common combinatorial features.

Note also that once $X$ has been totally ordered, the set of Lyndon words over $X$ will be denoted by $LynX$. A pair of Lyndon words $(l_1, l_2)$ is called the standard factorization of a Lyndon word (and will be noted $(l_1, l_2) = st(l)$) if $l = l_1l_2$ and $l_2$ is the longest nontrivial proper right factor of $l$ or, equivalently, its smallest such (for the lexicographic ordering, see [17] for proofs and details).

As an algebra the $A$-module $A(X)$ is also equipped with the associative commutative and unital shuffle product which is defined, for any $x, y \in X$ and $u, v \in X^*$, by the recursion

$$u \shuffle 1_{X^*} = 1_{X^*} \shuffle u = u \quad \text{and} \quad xu \shuffle yv = x(u \shuffle yv) + y(xu \shuffle v), \quad (16)$$

or equivalently, by its coproduct (which is a morphism for concatenations, defined, for each letter $x \in X$, as follows

$$\Delta \shuffle x = 1_{X^*} \otimes x + x \otimes 1_{X^*}.$$  \quad (17)

According to a theorem by Radford [19], $LynX$ form a pure transcendence basis of the $A$-shuffle algebras $(A(X), \shuffle, 1_X)$ (see also [19]).

Similarly, the $A$-module $A(Y)$ is also equipped with the associative commutative and unital stuffle product defined, for $u, v, w \in Y^*$ and $y_i, y_j \in Y$, by

$$w \stuffle 1_{Y^*} = 1_{Y^*} \stuffle w = w \quad \text{and} \quad y_iu \stuffle y_jv = y_i(u \stuffle y_jv) + y_j(y_iu \stuffle v) + y_{i+j}(u \stuffle v). \quad (18)$$

It can be dualized according to $(y_k \in Y)$

$$\Delta \stuffle y_k := y_k \otimes 1_{Y^*} + 1_{Y^*} \otimes y_k + \sum_{i+j=k} y_i \otimes y_j \quad (19)$$

\footnote{For technical reasons, the orders $x_0 < x_1$ (for $X$) and $y_1 > \ldots > y_n > y_{n+1} > \ldots$ (for $Y$) are usual.}
which is also a \textit{conc}-morphism and the $A$-stuffle algebra $(A(\langle X \rangle), \conc, 1_{\langle X \rangle})$ admits $\mathcal{L}ynY$ as a pure transcendence basis (see [14][15]).

Recall also that the associative unital concatenation, $\text{conc}$, admits the co-law, denoted by $\Delta_{\text{conc}}$, which is defined, for any $w \in X^*$, as follows

$$\Delta_{\text{conc}}w = \sum_{u,v \in X^*, uv = w} u \otimes v,$$

(20)

\section*{2.2 Noncommutative series}

In all the sequel, $\text{Lie}_A(\langle X \rangle)$ (resp. $\text{Lie}_A(\langle X \rangle)$) equipped the Lie bracket, $[\cdot, \cdot]$, and $A(\langle X \rangle)$ equipped the concatenation product, $\text{conc}$, denote, respectively, the Lie algebra of series (resp. Lie polynomials) and the Cauchy algebra of series over $X$ with coefficients in $A$ [19].

The product $\text{conc}$ is extended over $A(\langle X \rangle)$, as follows [1]

$$\forall S, R \in A(\langle X \rangle), \quad SR = \sum_{w \in X^*} \left( \sum_{u,v \in X^*, uv = w} \langle S|u\rangle\langle R|v\rangle \right) w.$$

(21)

In fact, as $A$-modules, $A(\langle X \rangle)$ is the dual of $A(\langle X \rangle)$, i.e.

$$A(\langle X \rangle)^\vee = A(\langle X \rangle),$$

(22)

via the pairing [1]

$$A(\langle X \rangle) \otimes_A A(\langle X \rangle) \rightarrow A, \quad T \otimes P \mapsto \langle T|w\rangle\langle P|v\rangle.$$

(23)

Now, let us extend $\conc$ and $\conc$, for any series $S, R \in A(\langle X \rangle)$ (resp. $A(\langle Y \rangle)$), as follows

$$S \conc R = \sum_{u,v \in X^*} \langle S|u\rangle\langle R|v\rangle u \conc v,$$

(25)

$$S \conc R = \sum_{u,v \in Y^*} \langle S|u\rangle\langle R|v\rangle u \conc v.$$

(26)

\textbf{Definition 1 (Extended Friedrichs criterion, [6]).} Let $S \in A(\langle Y \rangle)$ (resp. $A(\langle X \rangle)$). Then

1. $S$ is said to be a $\conc$ (resp. $\text{conc}$ and $\conc$) character if and only if, for any $w, v \in Y^*$ (resp. $X^*$), $\langle S|w\rangle\langle S|v\rangle = \langle S|w \conc v\rangle$ (resp. $\langle S|w\rangle\langle S|v\rangle = \langle S|uv\rangle$ and $\langle S|w\rangle\langle S|v\rangle = \langle S|w \conc v\rangle$).

2. $S$ is said to be an infinitesimal $\conc$ (resp. $\text{conc}$ and $\conc$) character if and only if, for any $w, v \in Y^*$ (resp. $X^*$), $\langle S|w \conc v\rangle = \langle S|w\rangle\langle v|1_{Y^*}\rangle + \langle w|1_{Y^*}\rangle\langle S|v\rangle$ (resp. $\langle S|uv\rangle = \langle S|w\rangle\langle v|1_{X^*}\rangle + \langle w|1_{X^*}\rangle\langle S|v\rangle$ and $\langle S|w \conc v\rangle = \langle S|w\rangle\langle v|1_{X^*}\rangle + \langle w|1_{X^*}\rangle\langle S|v\rangle$).
Let us also extend the coproduct $\Delta_{\mathfrak{u}}$ (resp. $\Delta_{\text{conc}}$ and $\Delta_{\mathfrak{u}}$), given in (19) (resp. (20) and (17)), over $A(\langle Y \rangle)$ (resp. $A(\langle X \rangle)$) by linearity and infinite sums as follows

\begin{equation}
\forall S \in A(\langle Y \rangle), \quad \Delta_{\mathfrak{u}} S = \sum_{w \in Y^*} \langle S|w\rangle \Delta_{\mathfrak{u}} w \in A(\langle Y^* \otimes Y^* \rangle), \tag{27}
\end{equation}

\begin{equation}
\forall S \in A(\langle X \rangle), \quad \Delta_{\mathfrak{u}} S = \sum_{w \in X^*} \langle S|w\rangle \Delta_{\mathfrak{u}} w \in A(\langle X^* \otimes X^* \rangle), \tag{28}
\end{equation}

\begin{equation}
\forall S \in A(\langle X \rangle), \quad \Delta_{\text{conc}} S = \sum_{w \in X^*} \langle S|w\rangle \Delta_{\text{conc}} w \in A(\langle X^* \otimes X^* \rangle). \tag{29}
\end{equation}

**Remark 1.** In (27)–(29), $A(\langle X \rangle) \otimes A(\langle X \rangle)$ contains the elements of the form $\sum_{i \in I} \text{finite } G_i \otimes D_i$ (with $(G_i, D_i) \in A(\langle X \rangle) \times A(\langle X \rangle))$ which can be interpreted as double series. But, a priori, the images of different dual laws cannot be, in general, reduced to such sums. Furthermore, the arrow tensor products of series $\rightarrow$ **double series** may not be into, when $A$ is only a ring.

**Definition 2.** Let $\Phi$ denote the injection

$A(\langle X \rangle)^\vee \otimes A(\langle X \rangle)^\vee \longrightarrow (A(\langle X \rangle) \otimes A(\langle X \rangle))^\vee$

and let $S \in A(\langle Y \rangle)$ (resp. $A(\langle X \rangle)$). Then

1. $S$ is said to be a group-like series if and only if $\langle S|1_{X^*}\rangle = 1$ and $\Delta_{\mathfrak{u}} S = \Phi(S \otimes S)$ (resp. $\Delta_{\text{conc}} S = \Phi(S \otimes S)$ and $\Delta_{\mathfrak{u}} S = \Phi(S \otimes S)$).
2. $S$ is said to be a primitive series if and only if $\Delta_{\mathfrak{u}} S = 1_{Y^*} \otimes S + S \otimes 1_{Y^*}$ (resp. $\Delta_{\text{conc}} S = 1_{X^*} \otimes S + S \otimes 1_{X^*}$ and $\Delta_{\mathfrak{u}} S = 1_{X^*} \otimes S + S \otimes 1_{X^*}$).

The following results are standard facts since works by Ree [18] (see also [0,14,19]).

**Proposition 1.** 1. Let $P := \{ P \in A(Y) \ (\text{resp. } A(X)) \mid \Delta_{\mathfrak{u}} P = P \otimes 1_{Y^*} + 1_{Y^*} \otimes P \ (\text{resp. } \Delta_{\text{conc}} P = P \otimes 1_{X^*} + 1_{X^*} \otimes P) \text{ and } \Delta_{\mathfrak{u}} P = P \otimes 1_{X^*} + 1_{X^*} \otimes P \}$. Then $P$ is closed by the Lie bracket and linear combinations.

2. Let $S \in A(\langle Y \rangle)$ (resp. $A(\langle X \rangle)$). Then $S$ is primitive, for $\Delta_{\mathfrak{u}}$ (resp. $\Delta_{\text{conc}}$ and $\Delta_{\mathfrak{u}}$), if and only if, for any $u, v \in Y^* Y$ (resp. $X^* X$), we get $\langle S|u \omega v \rangle = 0$ (resp. $\langle S|u v \rangle = 0$ and $\langle S|u \omega v \rangle = 0$).

**Remark 2.** By (20) and (17), any letter $x \in X$ is primitive, for $\Delta_{\text{conc}}$ and $\Delta_{\mathfrak{u}}$. By (19), the polynomials $\{\pi_1(y_k)\}_{k \geq 2}$ and only the letter $y_1$ are primitive, for $\Delta_{\mathfrak{u}}$.

**Proposition 2.** Let $S \in A(\langle Y \rangle)$ (resp. $A(\langle X \rangle)$). Then the following assertions are equivalent

1. $S$ is a $\omega$ (resp. conc and $\omega$)-character.
2. $S$ is group-like, for $\Delta_{\mathfrak{u}}$ (resp. $\Delta_{\text{conc}}$ and $\Delta_{\mathfrak{u}}$).
Theorem 1 (Extended Ree’s theorem, [6]). Let \( S \in A(\langle Y \rangle) \) (resp. \( A(\langle X \rangle) \)). Then the following assertions are equivalent

1. \( S \) is group-like, for \( \Delta_{\shuffle} \) (resp. \( \Delta_{\text{conc}} \) and \( \Delta_{\shuffle} \)).
2. \( \log S \) is primitive, for \( \Delta_{\shuffle} \) (resp. \( \Delta_{\text{conc}} \) and \( \Delta_{\shuffle} \)).

Corollary 1. Let \( S \in A(\langle Y \rangle) \) (resp. \( A(\langle X \rangle) \)). Then the following assertions are equivalent

1. \( S \) an infinitesimal \( \shuffle \) (resp. \( \text{conc} \) and \( \shuffle \))-character.
2. \( S \) is primitive, for \( \Delta_{\shuffle} \) (resp. \( \Delta_{\text{conc}} \) and \( \Delta_{\shuffle} \)).

2.3 Factorization in bialgebras

It is well known that the enveloping algebra \( U(\text{Lie}_A(X)) \) is isomorphic to the (connected, graded and co-commutative) bialgebra\([7]\) \( \mathcal{H}_{\shuffle}(X) = (A(X), \text{conc}, 1_X, \Delta_{\shuffle}, e) \) (the counit being here \( e(P) = \langle P | 1_X \rangle \)) and, via the pairing given in \([2] \) we can, classically, endow the algebra \( A(X) \) with the graded linear basis \( \{P_w\}_{w \in X^*} \) (expanded after any homogeneous basis \( \{P_l\}_{l \in \text{Dyn}X} \) of \( \text{Lie}_A(X) \)) and its graded dual basis \( \{S_w\}_{w \in X^*} \) (containing the pure transcendence basis \( \{S_l\}_{l \in \text{Dyn}X} \) of the \( A \)-shuffle algebra).

In the case when \( A \) is a \( \mathbb{Q} \)-algebra, we also have the following factorization of the diagonal series, i.e. \([19] \) (here all tensor products are over \( A \))

\[
D_X := \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \text{Dyn}X} e^{S_l \otimes P_l} \tag{30}
\]

and (still in case \( A \) is a \( \mathbb{Q} \)-algebra) dual bases of homogenous polynomials (in length) \( \{P_w\}_{w \in X^*} \) and \( \{S_w\}_{w \in X^*} \) can be constructed recursively as follows

\[
\begin{aligned}
P_x &= x, \quad S_x = x, & \text{for } x \in X; \\
P_l = [P_{i_1}, P_{i_2}], \quad S_l = yS_{y'}, & \text{for } l = y'y \in \text{Dyn}X - X' \\
P_w = P_{i_1}^{i_2} \cdots P_{i_k}, \quad S_w = \frac{S_{i_1^{i_2} i_2^{i_3} \cdots i_k^{i_{k+1}}}}{i_1! \cdots i_k!}, & \text{for } w = i_1^{i_2} \cdots i_k^{i_{k+1}}, \text{ with } i_1 > \ldots > i_k, l_1, \ldots, l_k \in \text{Dyn}X.
\end{aligned} \tag{31}
\]

11 In case \( A \) is a \( \mathbb{Q} \)-algebra, the isomorphism \( U(\text{Lie}_A(X)) \simeq \mathcal{H}_{\shuffle}(X) \) can be seen as an easy application of the CQMM theorem.

12 For \( X = X \) or \( Y \) the corresponding monoids are equipped with length functions, for \( X \) we consider the length of words and for \( Y \) the weight which is given by \( \ell(y_1 \cdots y_m) = i_1 + \ldots + i_n \). This naturally induces a grading of \( A(X) \) and \( \text{Lie}_A(X) \) in free modules of finite dimensions. For general \( X \), we consider the fine grading \([19] \) i.e. the grading by all partial degrees which, as well, induces a grading of \( A(X) \) and \( \text{Lie}_A(X) \) in free modules of finite dimensions.
Example 1 ([12]). Let \( X = \{x_0, x_1\} \) with \( x_0 < x_1 \).

<table>
<thead>
<tr>
<th>( l )</th>
<th>( P_i )</th>
<th>( S_l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 )</td>
<td>( x_0 )</td>
<td>( x_0 )</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( x_1 )</td>
<td>( x_1 )</td>
</tr>
<tr>
<td>( x_0x_1 )</td>
<td>([x_0, x_1])</td>
<td>( x_0x_1 )</td>
</tr>
<tr>
<td>( x_0^2x_1 )</td>
<td>( [x_0, [x_0, x_1]] )</td>
<td>( x_0^3x_1 )</td>
</tr>
<tr>
<td>( x_0^3x_1 )</td>
<td>( [x_0, [x_0, x_1]], x_1 )</td>
<td>( x_0^4x_1 )</td>
</tr>
<tr>
<td>( x_0^5x_1 )</td>
<td>( [x_0, [x_0, x_1]], x_1, x_1 )</td>
<td>( x_0^5x_1 )</td>
</tr>
<tr>
<td>( x_0^3x_1 )</td>
<td>( [x_0, [x_0, x_0, x_1]], x_1 )</td>
<td>( x_0^4x_1 )</td>
</tr>
<tr>
<td>( x_0^2x_1x_0x_1 )</td>
<td>( [x_0, [x_0, x_1]], [x_0, x_1] )</td>
<td>( 2x_0^3x_1^2 + x_0^2x_1x_0x_1 )</td>
</tr>
<tr>
<td>( x_0x_1x_0x_1^2 )</td>
<td>( [x_0, [x_0, x_1]], ([x_0, x_1], x_1) )</td>
<td>( x_0^4x_1^2 )</td>
</tr>
<tr>
<td>( x_0^2 )</td>
<td>( [x_0, [x_0, x_1]], x_1, x_1 )</td>
<td>( x_0^4x_1 )</td>
</tr>
<tr>
<td>( x_0^4x_1 )</td>
<td>( [x_0, [x_0, x_0, x_0, x_1]], x_1 )</td>
<td>( x_0^5x_1 )</td>
</tr>
<tr>
<td>( x_0^3x_1 )</td>
<td>( [x_0, [x_0, x_0, x_0, x_1]], x_1, x_1 )</td>
<td>( x_0^5x_1 )</td>
</tr>
<tr>
<td>( x_0^5x_1x_0x_1 )</td>
<td>( [x_0, [x_0, x_0, x_0, x_1]], [x_0, x_1] )</td>
<td>( 2x_0^4x_1^2 + x_0^3x_1x_0x_1 )</td>
</tr>
<tr>
<td>( x_0^4x_1 )</td>
<td>( [x_0, [x_0, x_0, x_0, x_1]], x_1, x_1 )</td>
<td>( x_0^5x_1 )</td>
</tr>
<tr>
<td>( x_0^5 )</td>
<td>( [x_0, [x_0, x_0, x_0, x_0, x_1]], x_1 )</td>
<td>( x_0^5x_1 )</td>
</tr>
</tbody>
</table>

The graded dual of \( \mathcal{H}_{\bot_1}(X) \) is

\[
\mathcal{H}_{\bot_1}^\perp(X) = (A(X), \ldots, 1_X, e, \Delta_{\text{conc}}, \epsilon) .
\] (32)

We get another connected, graded and co-commutative bialgebra which, in case \( A \) is a \( \mathbb{Q} \)-algebra, is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements,

\[
\mathcal{H}(Y) = (A(Y), \text{conc}, 1_Y, e, \Delta_e, \epsilon) \cong \mathcal{U}(\text{Prim}(\mathcal{H}(Y))),
\] (33)

where

\[
\text{Prim}(\mathcal{H}(Y)) = \text{Im}(\pi_1) = \text{span}_A \{ \pi_1(w) | w \in Y^+ \}
\] (34)

and \( \pi_1 \) is defined, for any \( w \in Y^+ \), by (see [14, 15])

\[
\pi_1(w) = w + \sum_{k=2}^{(w)} (-1)^{k-1} \sum_{u_1, \ldots, u_k \in Y^+} \langle w \rangle u_1 u_2 \ldots w u_k u_1 \ldots u_k ,
\] (35)

and, for any \( w = y_1 \ldots y_k \in Y^+ \), \( (w) \) denotes the number \( i_1 + \ldots + i_k \).
Now, let \( \{ \Pi_w \}_{w \in Y} \) be the linear basis, expanded by decreasing Poincaré-Birkhoff-Witt (PBW for short) after any basis \( \{ H_l \}_{l \in \mathcal{L}Y} \) of \( \mathcal{H}_{\omega}(Y) \) homogeneous in weight, and let \( \{ \Sigma_w \}_{w \in Y} \) be its dual basis which contains the pure transcendence basis \( \{ \Sigma_l \}_{l \in \mathcal{L}Y} \) of the A-stuffle algebra. One also has the factorization of the diagonal series \( \mathcal{D}_Y \), on \( \mathcal{H}_{\omega}(Y) \), which reads

\[
\mathcal{D}_Y := \sum_{w \in Y} w \otimes w = \sum_{w \in Y} \Sigma_w \otimes \Pi_w = \prod_{l \in \mathcal{L}Y} e^{\Sigma_l \otimes \Pi_l}.
\]

We are now in the position to state the following

**Theorem 2** ([14][15]). Let \( A \) be a \( \mathbb{Q} \)-algebra, then the endomorphism of algebras \( \varphi_{\pi_1} : (A(Y), \text{conc}, 1_{Y^*}) \to (A(Y), \text{conc}, 1_{Y^*}) \) mapping \( y_k \) to \( \pi_1(y_k) \), is an automorphism of \( A(Y) \) realizing an isomorphism of bialgebras between \( \mathcal{H}_{\omega}(Y) \) and

\[
\mathcal{H}_{\omega}(Y) \cong \mathcal{U}(\text{Prim}(\mathcal{H}_{\omega}(Y))).
\]

In particular, it can be easily checked that the following diagram commutes

\[
\begin{array}{ccc}
A(Y) & \xrightarrow{\Delta_{\omega}} & A(Y) \otimes A(Y) \\
\varphi_{\pi_1} \downarrow & & \downarrow \varphi_{\pi_1} \otimes \varphi_{\pi_1} \\
A(Y) & \xrightarrow{\Delta} & A(Y) \otimes A(Y)
\end{array}
\]

Hence, the bases \( \{ \Pi_w \}_{w \in Y} \) and \( \{ \Sigma_w \}_{w \in Y} \) of \( \mathcal{U}(\text{Prim}(\mathcal{H}_{\omega}(Y))) \) are images by \( \varphi_{\pi_1} \) and by the adjoint mapping of its inverse, \( \tilde{\varphi}_{\pi_1} \), of \( \{ F_w \}_{w \in Y} \) and \( \{ S_w \}_{w \in Y} \), respectively. Algorithmically, by Remark 2, the dual bases of homogeneous polynomials (in weight) \( \{ \Pi_w \}_{w \in Y} \) and \( \{ \Sigma_w \}_{w \in Y} \) can be constructed directly and recursively by

\[
\begin{align*}
\Pi_{y_s} &= \pi_1(y_s), & \Sigma_{y_s} &= y_s, & \text{for } y_s \in Y, \\
\Pi_l &= [\Pi_{i_1}, \Pi_{i_2}], & \Sigma_l &= \sum_{(s)} \frac{y_{s_{i_1}} \ldots y_{s_{i_k}}}{i_1! \ldots i_k!} \Sigma_{i_1 \ldots i_k}, & \text{for } l \in \mathcal{L}Y - Y, & \text{st}(l) = (i_1, i_2), \\
\Pi_w &= \Pi_{i_1} \ldots \Pi_{i_k}, & \Sigma_w &= \sum_{i_1! \ldots i_k!} \frac{\Sigma_{i_1} \ldots \Sigma_{i_k}}{i_1! \ldots i_k!}, & \text{for } w = l_{i_1} \ldots l_{i_k}, & \text{with } l_k > \ldots > l, l_1, l_1, \ldots, l_k \in \mathcal{L}Y.
\end{align*}
\]

In (\(*\)), the sum is taken over all \( \{ k_1, \ldots, k_i \} \subset \{ 1, \ldots, k \} \) and \( l_1 \geq \ldots \geq l_n \) such that \( (y_{s_{k_1}}, \ldots, y_{s_{k_i}}) \propto (y_{s_{k_1}}, \ldots, y_{s_{k_i}}, l_1, \ldots, l_n) \), where \( \propto \) denotes the transitive closure of the relation on standard sequences, denoted by \( \preceq \) [19].

\[\text{Again all tensor products will be taken over } A. \text{ Note that this factorization holds for any enveloping algebra as announced in [19]. Of course, the diagonal series no longer exists and must be replaced by the identity } \text{Id}_A.\]
Example 2 (3).

<table>
<thead>
<tr>
<th>$l$</th>
<th>$H_l$</th>
<th>$\Sigma_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_2$</td>
<td>$y_2 - \frac{1}{2} y_1^2$</td>
<td>$y_2$</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$y_1^2$</td>
<td>$y_1$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$y_3 - \frac{1}{2} y_1 y_2 - \frac{1}{2} y_2 y_1 + \frac{1}{2} y_1^2$</td>
<td>$\frac{1}{2} y_2 + y_1^2$</td>
</tr>
<tr>
<td>$y_2 y_1$</td>
<td>$y_2 y_1 - y_2 y_1$</td>
<td>$y_3 + y_2 y_1$</td>
</tr>
<tr>
<td>$y_1 y_2$</td>
<td>$y_2 y_1 - \frac{1}{2} y_1^2$</td>
<td>$y_1 y_2$</td>
</tr>
<tr>
<td>$y_3^3$</td>
<td>$y_3^3$</td>
<td>$1 y_3 + \frac{1}{2} y_2 y_1 + \frac{1}{2} y_1 y_2 + y_1^3$</td>
</tr>
<tr>
<td>$y_4$</td>
<td>$y_4 - \frac{1}{2} y_1 y_3 - \frac{1}{2} y_2^2 - \frac{1}{2} y_3 y_1$</td>
<td>$y_4$</td>
</tr>
<tr>
<td>$y_2 y_4$</td>
<td>$y_2 y_4 + \frac{1}{2} y_1 y_2 y_1 + \frac{1}{2} y_2 y_2 y_1 - \frac{1}{2} y_1^2$</td>
<td>$\frac{1}{2} y_4 + \frac{1}{2} y_3 y_1 + y_4$</td>
</tr>
<tr>
<td>$y_2 y_3$</td>
<td>$y_2 y_3 - \frac{3}{2} y_2^2 y_1 - y_3 y_1 + \frac{1}{2} y_1^2$</td>
<td>$\frac{1}{2} y_4 + \frac{1}{2} y_3 y_1 + y_4$</td>
</tr>
<tr>
<td>$y_1 y_3$</td>
<td>$y_1 y_3 - \frac{1}{2} y_1^2 y_2 - \frac{1}{2} y_2 y_2 y_1 + \frac{1}{2} y_1^2$</td>
<td>$y_4 + y_1 y_3 + y_3 y_2$</td>
</tr>
<tr>
<td>$y_1 y_2 y_1$</td>
<td>$y_1 y_2 y_1 - y_2 y_2 y_2$</td>
<td>$\frac{1}{2} y_4 + \frac{1}{2} y_3 y_1 + y_4$</td>
</tr>
<tr>
<td>$y_1^2 y_2$</td>
<td>$y_1^2 y_2 - \frac{3}{2} y_4$</td>
<td>$\frac{1}{2} y_4 + \frac{1}{2} y_3 y_1 + y_4$</td>
</tr>
<tr>
<td>$y_1^3$</td>
<td>$y_1^3$</td>
<td>$\frac{1}{2} y_4 + \frac{1}{2} y_3 y_1 + y_4$</td>
</tr>
</tbody>
</table>

3 Polylogarithms, harmonic sums and polyzetas

3.1 Noncommutative generating series of polylogarithms and harmonic sums

The following (morphisms) are injective (see [14,15])

\[
\begin{align*}
\text{Li}_n : (\mathbb{Q}(X), \omega, 1 X^r) & \rightarrow (\mathbb{Q}\{\text{Li}_w\}_{w \in X^r}, 1), \\
\text{x}^n_{r} \cdot x_1 \cdots x_0 & \mapsto \prod_{n=1}^{\infty} \frac{\text{Li}_{x_0^{n-1} x_1 \cdots x_0^{r-1} x_1}}{n!}, \\
\text{H}_n : (\mathbb{Q}(Y), \omega, 1 Y^r) & \rightarrow (\mathbb{Q}\{\text{H}_w\}_{w \in Y^r}, 1),
\end{align*}
\]

with the integers $n \geq 0, r \geq 1, s_1, \ldots, s_r \geq 1$.

Hence, the families $\{\text{Li}_w\}_{w \in X^r}$ and $\{\text{H}_w\}_{w \in Y^r}$ are linearly independent, or equivalently, $\{\text{Li}_l\}_{l \in \mathbb{C} Y X}$ and $\{\text{H}_l\}_{l \in \mathbb{C} Y X}$ (resp. $\{\text{Li}_l\}_{l \in \mathbb{C} X Y}$ and $\{\text{H}_l\}_{l \in \mathbb{C} X Y}$) are algebraically independent. It follows then

\[
(\mathbb{Q}\{\text{Li}_w\}_{w \in X^r}, 1) \cong (\mathbb{Q}\{\text{Li}_l\}_{l \in \mathbb{C} X Y^r}, 1) \cong (\mathbb{Q}\{\text{H}_w\}_{w \in Y^r}, 1) \cong (\mathbb{Q}\{\text{H}_l\}_{l \in \mathbb{C} X Y}, 1).
\]

Now, using $D_X$ and $D_Y$ given, respectively, in [19] and [30], the graphs of $\text{Li}_n$ and $\text{H}_n$, viewed as noncommutative generating series, are given as follows
\[
\begin{align*}
L := \sum_{w \in X^*} L_i w (L_i \otimes \text{Id}) = \prod_{l \in L \cap X} e^{L_i l_i}, \\
H := \sum_{w \in Y^*} H_l w (H_l \otimes \text{Id}) = \prod_{l \in L \cap Y} e^{H_l l_i},
\end{align*}
\]
and
\[
Z_{\downarrow \downarrow} := L_{\text{reg}}(1) \text{ and } Z_{\downarrow} := H_{\text{reg}}(\pm \infty),
\]
where, for convenience, we set also
\[
L_{\text{reg}} := \prod_{l \in L \cap X, l \neq x_0, x_1} e^{L_i l_i} \text{ and } H_{\text{reg}} := \prod_{l \in L \cap Y, l \neq y_1} e^{H_l l_i}.
\]

As for \(C_{z_0 \to z_1}\), the generating series \(L, L_{\text{reg}}, \) and then \(Z_{\downarrow \downarrow}\) (resp. \(H, H_{\text{reg}}, \) and then \(Z_{\downarrow}\)) are group-like, for \(\Delta_{\downarrow \downarrow}\) (resp. \(\Delta_{\downarrow}\)). Moreover, \(L\) is also a solution of \([5]\) and then \([9,10,15]\)
\[
L(z) = C_{z_0 \to z_1} L(z_0), \lim_{z \to 0} L(z) e^{-x_0 \log(z)} = 1, \lim_{z \to 1} e^{x_1 \log(1-z)} L(z) = Z_{\downarrow \downarrow},
\]
i.e. \([13]\)
\[
L(z) \sim_0 e^{x_0 \log(z)} \text{ and } L(z) \sim_1 e^{x_1 \log(1-z)} Z_{\downarrow \downarrow}.
\]

Via Newton-Girard identity type, we also get \([9,10,15]\)
\[
\sum_{k \geq 0} H_{y_1^k}(n) y_1^k = e^{\sum_{k \geq 1} H_{y_1^k}(n) (-y_1)^k / k}
\]
and then \([15]\)
\[
H(n) \sim_{+\infty} \left( \sum_{k \geq 0} H_{y_1^k}(n) y_1^k \right) \pi_Y(Z_{\downarrow \downarrow}).
\]

By \([30]\) and \([51]\), it follows that

**Theorem 3 (First Abel like theorem, [9,10,15]).**
\[
\lim_{z \to 1} e^{y_1 \log(1-z)} \pi_Y(L(z)) = \lim_{n \to \infty} e^{\sum_{k \geq 1} H_{y_1^k}(n) (-y_1)^k / k} H(n) = \pi_Y(Z_{\downarrow \downarrow}).
\]

As in Note \([5]\) changing \(x_0 = A/2i\pi\) and \(x_1 = -B/2i\pi\), \(L\) corresponds to \(G_0\) expected by Drinfeld’s \(Z_{\downarrow \downarrow}\) corresponds to \(\Phi_{KZ}\).

In \([10]\), three algorithms for asymptotic expansion of \(\{H_w\}_{w \in Y^*}\), in the comparison scale \(\{n^a H_1^k(n)\}_{a \in \mathbb{Z}, k \in \mathbb{N}}\), are proposed.
Hence, the coefficients of $Z_{\omega}$ and $Z_{\omega^1}$, defined in (47), represent, respectively, the following finite parties \[9,10,15\]
\[
\text{f.p.}_{z-1} \text{Li}_w(z) = (Z_{\omega^1} \omega^1 w), \quad \{(1-z)^a \log^b((1-z)^{-1})\}_{a \in \mathbb{Z}, b \in \mathbb{N}}, \tag{53}
\]
\[
\text{f.p.}_{n \to \infty} \text{H}_w(n) = (Z_{\omega} \omega^1 w), \quad \{n^a \Pi(b)(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}. \tag{54}
\]

**Example 3 (910)**. With the correspondences given in (12), one has in convergence case,
\[
\text{Li}_{2,1}(z) = \zeta(3) + (1-z) \log(1-z) - (1-z)^{-1} - (1-z) \log^2(1-z)/2 + (1-z)^2(-\log^2(1-z) + \log(1-z))/4 + \ldots,
\]
\[
\text{H}_{2,1}(n) = \zeta(3) - (\log(n) + 1 + \gamma)/n + \log(n)/2n + \ldots.
\]
and then
\[
\text{f.p.}_{z \to 1} \text{Li}_{2,1}(z) = \text{f.p.}_{n \to \infty} \text{H}_{2,1}(n) = \zeta(2,1) = \zeta(3).
\]
In divergence case, one also has
\[
\text{Li}_{1,2}(z) = 2 - 2\zeta(3) - \zeta(2) \log(1-z) - 2(1-z) \log(1-z) + (1-z) \log^2(1-z)
\]
\[
+ (1-z)^2(-\log^2(1-z) + \log(1-z))/2 + \ldots,
\]
\[
\text{H}_{1,2}(n) = \zeta(2)\gamma - 2\zeta(3) + \zeta(2) \log(n) + \zeta(2)/2n + \ldots.
\]
Since numerically,
\[
\zeta(2)\gamma = 0.949481711111498152454556410223170493364000\ldots,
\]
then one deduces that
\[
\text{f.p.}_{z \to 1} \text{Li}_{1,2}(z) = 2 - 2\zeta(3) \neq \text{f.p.}_{n \to \infty} \text{H}_{1,2}(n) = \zeta(2)\gamma - 2\zeta(3).
\]

By (41) and (12), let us consider the following \(\mathbb{Q}\)-algebra of convergent polyzeetas, being algebraically generated by \(\{\zeta(l)\}_{l \in \mathcal{L}_\mathbb{N} \setminus X} \) (resp. \(\{\zeta(S\ell)\}_{l \in \mathcal{L}_\mathbb{N} \setminus X} \)), or equivalently, by \(\{\zeta(l)\}_{l \in \mathcal{L}_\mathbb{N} \setminus \{y_1\}} \) (resp. \(\{\zeta(S\ell)\}_{l \in \mathcal{L}_\mathbb{N} \setminus \{y_1\}} \)).
\[
\mathcal{Z} := \text{span}_\mathbb{Q}\{\zeta(w)\}_{w \in \mathcal{X} \setminus \{x_1\}} = \text{span}_\mathbb{Q}\{\zeta(w)\}_{w \in (\mathcal{Y} \setminus \{y_1\})^{Y^*}}. \tag{55}
\]
For any \(k \geq 1\) let
\[
\mathcal{Z}_k := \text{span}_\mathbb{Q}\{\zeta(w)\}_{w \in \mathcal{X} \setminus \{x_1\}} = \text{span}_\mathbb{Q}\{\zeta(w)\}_{w \in (\mathcal{Y} \setminus \{y_1\})^{Y^*}, (w) = k}. \tag{56}
\]
Now, considering the third and last generating series of polyzeetas \[9,10,15\]
\[
\mathcal{Z}_\gamma = \sum_{w \in \mathcal{Y}^*} \gamma_{w} w, \tag{57}
\]
\[16\] The weight of \(w = x_{i_1} \ldots x_{i_r} \in \mathcal{X}\) is defined, in cases \(\mathcal{X} = \mathcal{X}\), by \((w) = |w| = r\) and, in cases \(\mathcal{X} = \mathcal{Y}\), by \((w) = s_1 + \ldots + s_r\).
which is the graph of the following $\omega$-character

\[ \gamma_* : (\mathbb{Q}(Y), \omega, 1_Y) \to (\mathbb{R}, ., 1), \]

\[ w \mapsto \gamma_w = \text{f.p.}_{n \to +\infty} H_w(n), \quad \{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}, \]

yielding, for any $w \in (Y \setminus y_1)Y^*$,

\[ \gamma_w = \zeta(w) \text{ and } \gamma_{y_1} = \gamma. \]

Hence, using the diagonal series $D_Y$ and the series $Z_{\omega}$ introduced, respectively, in (68) and (67), it follows that (65)

\[ Z_\gamma = (\gamma_* \otimes \text{Id})D_Y = e^{\gamma_{y_1}} \prod_{i \in \mathcal{L}(\gamma_1 \setminus \{y_1\})} e^{\zeta(i) H_1} = e^{\gamma_{y_1}} Z_{\omega}, \]

meaning also that the series generating $Z_\gamma$ is group-like, for $\Delta_{\omega}$. Moreover,

Theorem 4 (64). Let \( y \in \mathbb{P} \)

\[ B(y_1) := \exp \left( \gamma_{y_1} - \sum_{k \geq 2} \zeta(k) \frac{(-y_1)^k}{k} \right) \text{ and } B'(y_1) := \exp \left( \sum_{k \geq 2} \zeta(k) \frac{(-y_1)^k}{k} \right). \]

Then

\[ Z_\gamma = B(y_1) \pi_Y Z_{\omega}, \text{ or equivalently by cancellation, } Z_{\omega} = B'(y_1) \pi_Y Z_{\omega}. \]

Identifying the coefficients in these identities, we get

Corollary 2 (64). \( \gamma_{y_1}^k = \sum_{s_1, \ldots, s_k > 0, s_1 + \ldots + s_k = k} \frac{(-1)^k}{s_1! \cdots s_k!} (-\gamma)^s_1 \left( -\zeta(2) \right)^{s_2} \cdots \left( -\zeta(k) \right)^{s_k}, \)

\[ \gamma_{y_1}^k \omega = \sum_{i=0}^k \frac{\zeta(x_0)((-x_1)^{k-i} \omega \pi_Y \omega)}{i!} \left( \sum_{j=1}^i b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots) \right), \]

where \( k \in \mathbb{N}_+, w \in Y^* \) and \( b_{n,k}(t_1, \ldots, t_k) \) are Bell polynomials.

Example 4 (Generalized Euler’s constant, 9). With the correspondences given in (12), we get

\[ \gamma_{1,1} = \frac{1}{2} (\gamma^2 - \zeta(2)), \]

\[ \gamma_{1,1,1} = \frac{1}{6} (\gamma^3 - 3 \zeta(2) \gamma + 2 \zeta(3)). \]

\[ \gamma_{1,7} = \zeta(7) \gamma + \zeta(3) \zeta(5) - \frac{54}{175} \zeta(2)^4, \]

\[ \gamma_{1,6} = \frac{4}{35} \zeta(2)^3 \gamma^2 + (\zeta(2) \zeta(5) + \frac{2}{5} \zeta(3) \zeta(2)^2 - 4 \zeta(7) \gamma) \]

\[ + \zeta(6, 2) + \frac{19}{35} \zeta(2)^4 + \frac{1}{2} \zeta(2) \zeta(3)^2 - 4 \zeta(3) \zeta(5). \]

\[ \text{For } l \geq 1, \{\log B(y_1)|y_1\} = \text{f.p.}_{n \to +\infty} \left( \sum_{i \geq 1} H_{y_1}(n)(-y_1)^i / l | y_1^i \right), \{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}. \]
3.2 Polymorphism zeta

By (1), (55) and (56), the following polymorphism is surjective (see [41, 15])

$$
\zeta : (Q(X) \ast x_0Q(Y)x_1, \omega, 1_X^*) \rightarrow (Z, 1), \quad \zeta(x_0, \ldots, x_r) = (s_0, \ldots, s_r),
$$

where $Z$ is the $Q$-algebra algebraically generated by the convergent polyzetas $\{n^a \log^b(1-a)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$.

This polymorphism is partially defined and can algebraically be extended as characters according, respectively, to $\omega$ and $\omega$ as follows

$$
\zeta_{\omega, \omega} : (Q(X), \omega, 1_X^*) \rightarrow (\mathbb{R}, 1), \quad \zeta_{\omega, \omega} : (Q(Y), \omega, 1_Y^*) \rightarrow (\mathbb{R}, 1)
$$

such that, for any $w \in X^*$, one has (see [14, 15])

$$
\zeta_{\omega, \omega}(w) = \log(1) = \log(1) - \log(1 - z) = n^a \log^b(1-z)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}.
$$

From (64), (65) and (56), one obtains the following convergent polyzetas [18]

$$
\forall l \in \text{Lyn} X \setminus X, \quad \zeta_{\omega, \omega}(l) = \zeta_{\omega, \omega}((\pi_Y l)^* = \gamma_{\pi_Y l} = \zeta(l)
$$

and, for the algebraic generator $x_0$,

$$
\zeta_{\omega, \omega}(x_0) = 0 = \log(1)
$$

and, for the algebraic generators $x_1$ and $y_1$ (corresponding to divergent cases),

$$
\zeta_{\omega, \omega}(x_1) = 0 = \log(1 - z) = \log^b(1-z)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}.
$$

Of course, via (58), one also obtains

$$
\gamma_{y_1} = \gamma = \log(1) = \log^b(1-z)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}.
$$

Remark 3. Note that (65) – (71) mean also that

$$
x_0, x_1 \in \ker \zeta_{\omega, \omega} \text{ and ker } \zeta_{\omega, \omega} \ni y_1 \notin \ker \gamma.
$$

Recall that $S_l$ (resp. $\Sigma_l$) is polynomial on Lyndon words, for $l \in \text{Lyn} X$ (resp. $\text{Lyn} Y$). Recall also that $S_{x_0} = x_0$, $S_{x_1} = x_1$ (see [31]) and $\Sigma_{y_1} = y_1$ (see [37]).
In \([13]\), polynomial relations among \(\{\zeta(l)\}_{l \in \Lyn X \setminus X}\), homogenous in weight, are obtained using the double shuffle relations, \(i.e.
\)
\[
\mathcal{I}_X = \{ x_1 \shuffle l_2 - \pi_X (y_1 \shuffle \pi_Y (l_2)), \quad l_1 \shuffle l_2 - \pi_X (\pi_Y (l_1) \shuffle \pi_Y (l_2)) \}_{l_1, l_2 \in \Lyn X \setminus X},
\]
and it is conjectured that \(\ker \zeta\) equals the \(\shuffle\)-ideal generated by \(\mathcal{I}_X\) \([13]\).

In \([66]\) (resp. \([67]\)), once the decomposition of \(\pi_Y S_l\) (resp. \(\pi_X \Sigma_l\)) on the base \(\{\Sigma_l\}_{l \in \Lyn Y}\) (resp. \(\{S_l\}_{l \in \Lyn X}\)) is achieved as previously, the polymorphism \(\zeta\) yields, polynomial relations, homogenous in weight, among \(\{\zeta(S_l)\}_{l \in \Lyn X \setminus X}\) (resp. \(\{\zeta(\Sigma_l)\}_{l \in \Lyn Y \setminus \{y_1\}\})

### 3.3 Algorithm LocaleCordinateIdenfication

Now, let us use the following algorithm to identify locale coordinates \([21]\), via identities of Theorem \([3]\).

The identification of local coordinates in \(Z_\gamma = B(y_1) \pi_Y Z_{\shuffle}\), leads to

1. A family of algebraic generators \(\mathcal{L}_{\text{irr}}^\infty(\mathcal{X})\) of \(\mathcal{Z}\) (see Example \([7]\) in Appendix)

\[
\mathcal{Z}_{\text{irr}}^p(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{\text{irr}}^{p}(\mathcal{X}) \subset \cdots \subset \mathcal{Z}_{\text{irr}}^\infty(\mathcal{X}) = \bigcup_{p \geq 2} \mathcal{Z}_{\text{irr}}^{p}(\mathcal{X})
\]

and their inverse image by a section of \(\zeta\) (see Example \([9]\) in Appendix)

\[
\mathcal{L}_{\text{irr}}^p(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{\text{irr}}^{p}(\mathcal{X}) \subset \cdots \subset \mathcal{L}_{\text{irr}}^\infty(\mathcal{X}) = \bigcup_{p \geq 2} \mathcal{L}_{\text{irr}}^{p}(\mathcal{X})
\]

such that the following restriction is bijective

\[
\zeta : \mathbb{Q}[\mathcal{L}_{\text{irr}}^\infty(\mathcal{X})] \longrightarrow \mathcal{Z}.
\]

Moreover,

\[
\mathcal{Z} = \mathbb{Q}[\mathcal{Z}_{\text{irr}}^\infty(\mathcal{X})] = \mathbb{Q}[\{\zeta(p)\}_{p \in \mathcal{L}_{\text{irr}}^\infty(\mathcal{X})}].
\]

2. The following sub ideals of \(\ker \zeta\)

\[
\mathcal{R}_Y := (\text{span}_\mathbb{Q}\{Q_i\}_{i \in \Lyn Y \setminus \{y_1\}}; \shuffle, 1_Y^*),
\]

\[
\mathcal{R}_X := (\text{span}_\mathbb{Q}\{Q_i\}_{i \in \Lyn X \setminus X}; \shuffle, 1_X^*)
\]

are generated by the polynomials \(\{Q_i\}_{i \in \Lyn X \setminus X}\) (see Example \([8]\) in Appendix) homogenous in weight \((= (l))\) such that the following assertions are equivalent (see Examples \([8]\) in Appendix)

---

\(^{19}\) or identically, in virtue of the Perrin’s lemma (\(i.e.\) \(l \in \Lyn X \setminus \{x_0\}\) if and only if \(\pi_Y l \in \Lyn Y\), see \([13]\), the \(\shuffle\)-ideal generated by \(\mathcal{I}_Y = \pi_Y \mathcal{I}_X\) (\(i.e.\) polynomial relations among \(\{\zeta(l)\}_{l \in \Lyn Y \setminus \{y_1\}}\), also homogenous in weight).

Since \(\zeta\) is a polymorphism (see \([62]\)) then it is also equivalent to the fact that \(\ker \zeta\) is generated by the ideal \(I = \{x_1 \shuffle l_2 - x_1 \shuffle l_2, l_1 \shuffle l_2 - l_1 \shuffle l_2\}_{l_1, l_2 \in \Lyn X \setminus X}\).

\(^{20}\) Testing up to order 10 the Zagier’s dimension conjecture holds.

\(^{21}\) This algorithm is partially implemented in \([3]\). Testing up to order 12, the Zagier’s dimension conjecture holds.
(a) \( Q_l = 0 \), 
(b) \( \Sigma_l \to \Sigma_l \) (resp. \( S_l \to S_l \)), 
(c) \( \Sigma_l \in L_{irr}^{\infty}(Y) \) (resp. \( S_l \in L_{irr}^{\infty}(X) \)).

Any polynomial \( Q_l \), not equal 0, is led by \( \Sigma_l \) (resp. \( S_l \)), being transcendent over the sub algebra \( \mathbb{Q}[L_{irr}^{\infty}(X)] \). Moreover, being homogenous polynomial of weight \( p = (l) \), \( \Sigma_l \) (resp. \( S_l \)) belongs to \( \mathbb{Q}[L_{irr}^{\leq p}(X)] \) and
\[
\Sigma_l \to Y_l \text{ (resp. } S_l \to U_l). \tag{79}
\]

In other terms,
\[
\Sigma_l = Q_l + Y_l \text{ (resp. } S_l = Q_l + U_l), \tag{80}
\]
i.e.
\[
\text{span}_{\mathbb{Q}}\{S_l\}_{l \in L_{irr}^{\infty}(X \setminus \{x_l\})} = R_X \oplus \text{span}_{\mathbb{Q}}L_{irr}^{\infty}(X), \tag{81}
\]
(resp. \( \text{span}_{\mathbb{Q}}\{\Sigma_l\}_{l \in L_{irr}^{\infty}(Y \setminus \{y_l\})} = R_Y \oplus \text{span}_{\mathbb{Q}}L_{irr}^{\infty}(Y) \)). \tag{82}

LocaleCordinateIdentification

\( N \in \mathbb{N}_{\geq 1} \%
Z_{irr}^{\infty}(Y) = \emptyset \text{ and } Z_{irr}^{\infty}(X) = \emptyset; \)
\( L_{irr}^{\infty}(Y) = \emptyset \text{ and } L_{irr}^{\infty}(X) = \emptyset; \)
\( R_{irr}(Y) = \emptyset \) and \( R_{irr}(X) = \emptyset; \)
for \( p \) ranges in \( 2, \ldots, N \) do
for \( l \) ranges in the totally ordered \( \mathcal{L}_{yn}^{p}(X) \) do
identify the coefficients of \( H_l \) in \( Z_{\gamma} = B(y_l)^{\pi_Y}Z_{\omega}; \)
identify the coefficients of \( P_l \) in \( \pi_X Z_{\gamma} = B(x_l)^{\pi_X}Z_{\omega}; \)
end_for;
for \( l \) ranges in the totally ordered \( \mathcal{L}_{yn}^{p}(X) \) do
express the local coordinate \( \zeta(\Sigma_l) \) as rewriting rule;
if \( \zeta(\Sigma_l) \to \zeta(\Sigma_l) \)
then \( Z_{irr}^{\infty}(Y) = Z_{irr}^{\infty}(Y) \cup \{\zeta(\Sigma_l)\} \) and \( L_{irr}^{\infty}(Y) = L_{irr}^{\infty}(Y) \cup \{\Sigma_l\} \)
else \( R_{irr}(Y) = R_{irr}(Y) \cup \{\Sigma_l \to Y_l\}; \)
express the local coordinate \( \zeta(S_l) \) as rewriting rule;
if \( \zeta(S_l) \to \zeta(S_l) \)
then \( Z_{irr}^{\infty}(X) = Z_{irr}^{\infty}(X) \cup \{\zeta(S_l)\} \) and \( L_{irr}^{\infty}(X) = L_{irr}^{\infty}(X) \cup \{S_l\} \)
else \( R_{irr}(X) = R_{irr}(X) \cup \{S_l \to U_l\} \)
end_for
end_for

By Radford’s theorem \( \equiv \), \( \zeta(w) \in \mathbb{Q}[Z_{irr}^{\infty}(X)], \) for \( w \in x_0X^*x_1 \) (resp. \( Y \setminus \{y_1\}Y^* \)). It follows that, for any \( P \in \mathbb{Q}[\{S_l\}_{l \in L_{irr}^{\infty}(Y \setminus \{y_l\})}] \) such that \( P \notin \ker \zeta \geq \mathcal{R}_X \), one obtains, by linearity, \( \zeta(P) \in \mathbb{Q}[Z_{irr}^{\infty}(X)] \).

Next, let \( Q \in \mathcal{R}_X \cap \mathbb{Q}[L_{irr}^{\infty}(X)]. \) Since \( \mathcal{R}_X \subseteq \ker \zeta \) then \( \zeta(Q) = 0. \) Moreover, restricted on \( \mathbb{Q}[L_{irr}^{\infty}(X)], \) the polymorphism \( \zeta \) is bijective and then \( Q = 0. \) It follows that

\( \mathcal{L}_{yn}^{p}(X) \) denotes the set of Lyndon words over \( X \) of weight \( p \) (and \( X \) still denotes \( X \) or \( Y \)).
Proposition 3 ([14][15]).

\[ Q[[S]]_{i \in \mathcal{L}_{\text{sym}}(X \setminus X)} = R_X \oplus Q[\mathcal{L}_{\text{irr}}^\infty(X)], \]
\[ Q[[\Sigma]]_{i \in \mathcal{L}_{\text{sym}}(Y \setminus \{y_1\})} = R_Y \oplus Q[\mathcal{L}_{\text{irr}}^\infty(Y)]. \]

Hence, by duality (and via CQMM), one also has

\[ U(\mathcal{L}_{\text{Lie}}(X) \setminus X) = J_X \oplus U(\mathcal{L}_{\text{Lie}}(\{P\})_{i \in \mathcal{L}_{\text{sym}}(X \setminus X)}), \]
\[ U(\mathcal{L}_{\text{Lie}}(Y) \setminus \{y_1\}) = J_Y \oplus U(\mathcal{L}_{\text{Lie}}(\{\Pi\})_{i \in \mathcal{L}_{\text{sym}}(X \setminus \{y_1\})}), \]

where \( J_X \) (resp. \( J_Y \)) is a Lie ideal generated by \( \{P\}_{i \in \mathcal{L}_{\text{sym}}(X \setminus X)} \) (resp. \( \{\Pi\}_{i \in \mathcal{L}_{\text{sym}}(X \setminus X)} \)).

Now, let \( Q \in \ker \zeta \) such that \( \langle Q \rangle_{1_X} = 0 \). Then \( Q = Q_1 + Q_2 \), with \( Q_1 \in R_X \) and \( Q_2 \in Q[\mathcal{L}_{\text{irr}}^\infty(X)] \). Thus, \( Q \equiv_{R_X} Q_1 \in R_X \) and then

Corollary 3 ([14][15]).

\[ Q[[\zeta(p)]]_{p \in \mathcal{L}_{\text{irr}}^\infty(X)} = Z = \text{Im} \zeta \text{ and } R_X = \ker \zeta. \]

On the other hand, one also has

\[ Z \cong Q_1 1_X \oplus (Y \setminus \{y_1\})Q(Y) / \ker \zeta \cong Q_1 1_X \oplus x_0 Q(X)x_1 / \ker \zeta. \]  

(83)

Hence, as an ideal generated by homogenous in weight polynomials, \( \ker \zeta \) is graded and so is \( Z \):

Corollary 4 ([14][15]).

\[ Z = Q_1 \oplus \bigoplus_{k \geq 2} Z_k. \]

Now, let \( \xi = \zeta(P) \), where \( P \in Q(\mathcal{X}) \) and \( P \not\in \ker \zeta \), homogenous in weight. Since, for any \( p \) and \( n \geq 1 \), one has

\[ Z_p Z_n \subset Z_{p+n} \]  

(84)

then each monomial \( \xi^n \), for \( n \geq 1 \), is of different weight. Thus \( \xi \) could not satisfy

\[ \xi^n + a_{n-1}\xi^{n-1} + \ldots = 0, \text{ with } a_{n-1}, \ldots \in Q. \]  

(85)

In particular,

Corollary 5 ([14][15]). Any \( P \in \mathcal{L}_{\text{irr}}^\infty(X) \) is homogenous in weight then \( \zeta(P) \) is transcendent over \( Q \).

Example 5 (irreducible polyzetas up to weight 12, [17]).

\[ \mathcal{Z}_{\text{irr}}^{12}(X) = \{ \zeta(S_{x_0 x_1}), \zeta(S_{x_0^2 x_1}), \zeta(S_{x_0 x_1^2}), \zeta(S_{x_0 x_1^2 x_0 x_1}), \zeta(S_{x_0^2 x_1}), \zeta(S_{x_0 x_1^2 x_0}), \zeta(S_{x_0^2 x_1^2}), \zeta(S_{x_0 x_1^2 x_0}), \zeta(S_{x_0 x_1^2 x_0}) \}. \]

\[ \mathcal{L}_{\text{irr}}^{12}(X) = \{ S_{x_0 x_1}, S_{x_0^2 x_1}, S_{x_0 x_1^2}, S_{x_0 x_1^2 x_0 x_1}, S_{x_0^2 x_1}, S_{x_0 x_1 x_0 x_1}, S_{x_0^2 x_1}, S_{x_0 x_1^2 x_0 x_1}, S_{x_0^2 x_1}, S_{x_0 x_1^2 x_0} \}. \]

\[ Z_{\text{irr}}^{12}(Y) = \{ \zeta(\Sigma_{y_2}), \zeta(\Sigma_{y_3}), \zeta(\Sigma_{y_4}), \zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_6}), \zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_8}), \zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_{10}}) \}. \]

\[ \mathcal{L}_{\text{irr}}^{12}(Y) = \{ \Sigma_{y_2}, \Sigma_{y_3}, \Sigma_{y_4}, \Sigma_{y_5}, \Sigma_{y_6}, \Sigma_{y_7}, \Sigma_{y_8}, \Sigma_{y_9}, \Sigma_{y_{10}}, \Sigma_{y_{11}}, \Sigma_{y_{12}}, \Sigma_{y_{13}}, \Sigma_{y_{14}}, \Sigma_{y_{15}} \}. \]
4 Concluding remarks

In this work, we reviewed the essential steps, using noncommutative polynomials and series (introduced in Sections 2.1 and 2.2, respectively) to furnish

1. the actual solution of the noncommutative differential equation (DE) introduced in (3), satisfying the asymptotic conditions given in (10), i.e. the series \( L \) defined in (43) (the noncommutative generating series of polylogarithms).

2. the Drinfel’d associator, i.e. the series \( Z \), defined in (47) or in (64) (the noncommutative generating series of polyzetas).

To \( L \) and \( Z \) correspond also the noncommutative generating series of harmonic sums, \( H \) given in (46), and of polyzetas, \( Z_\gamma \) (given in (47) or in (65)). All these series were factorized using MRS factorizations, described in Section 2.3, with appropriate pair of bases in duality.

After that, in Section 3.2, thanks to Abel like results (see Theorem 3), we obtained equations bridging the algebraic structures of converging polyzetas \( \{ \zeta(S_l) \}_{l \in \text{LynX}} \) and \( \{ \zeta(S_l) \}_{l \in \text{LynY}\setminus\{y_1\}} \) (see Theorem 4) and, by local coordinates identification, leading to

1. the polynomial relations, homogenous in weight, among converging polyzetas (see Example 6 in Appendix),

2. the algebraic generators for polyzetas (see Example 5), \( \mathcal{L}^\infty_{\text{irr}}(X) \) and \( \mathcal{L}^\infty_{\text{irr}}(Y) \).

Then, using the algorithm LocaleCordonateIdentification (described in Section 3.3), the Zagier’s dimension conjecture [20] holds, up to weight 12, meaning that the irreducible polyzetas in \( Z^\leq_{\text{irr}}(X) \) (see Example 5) is algebraically independent over \( \mathbb{Q} \) (see [14], for example, for a short discussion).

To end, let \( l \in \text{LynX} \) such that \( l \neq y_1 \) and \( l \neq x_0, x_1 \). Then one has

\[ l \geq y_n \text{ and } l \geq x_0^{n-1}x_1 \]

and (see (51) and (37))

\[ \Sigma_{y_n} = y_n \in \text{LynY} \text{ and } S_{x_0^{n-1}x_1} = x_0^{n-1}x_1 \in \text{LynX}. \]

But, unfortunately,

\[ \{ \Sigma_{y_n} \}_{n>1} \not\in \mathcal{L}^\infty_{\text{irr}}(Y) \text{ and } \{ S_{x_0^{n-1}x_1} \}_{n>1} \not\in \mathcal{L}^\infty_{\text{irr}}(X). \]

Indeed,

1. \( \zeta(2) = \zeta(S_{y_2}) = \zeta(S_{x_2}) \) is then irreducible polyzeta (see Example 5),

2. By a Euler’s identity about the ratio \( \zeta(2k)/\pi^{2k}, k > 1 \), one deduces that

\[ \Sigma_{y_{2k}} = y_{2k} \not\in \mathcal{L}^\infty_{\text{irr}}(Y) \text{ and } S_{x_{2k-1}x_1} = x_0^{2k-1}x_1 \not\in \mathcal{L}^\infty_{\text{irr}}(X). \]

3. By Example 5, one obtains \( \zeta(3), \zeta(5), \zeta(7), \zeta(9), \zeta(11) \in Z^\leq_{\text{irr}}(X) \).

4. It could remains that, for \( n \geq 1 \),

\[ \Sigma_{y_{2n+1}} = y_{2n+1} \in \mathcal{L}^\infty_{\text{irr}}(Y) \text{ and } S_{x_{2n}x_1} = x_0^{2n}x_1 \in \mathcal{L}^\infty_{\text{irr}}(X). \]

23 See also [5,15], for the unicity of \( L \) and then of \( Z_\gamma \).

24 See also [4], for analogous study of the ratios \( \zeta(s_1, \ldots, s_r)/\pi^{s_1+\ldots+s_r} \).
References

5 Appendix : computational examples

**Example 6 (Polynomials relations on local coordinates, $\mathcal{X}$). By Theorem 4 we obtain**

<table>
<thead>
<tr>
<th>Relations on ${\zeta(\Sigma_l)}<em>{l \in \mathcal{L}</em>{\mathcal{Y}}} \setminus {y_l}$</th>
<th>Relations on ${\zeta(S_l)}<em>{l \in \mathcal{L}</em>{\mathcal{X}}} \setminus {x_l}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3$</td>
<td>$\zeta(\Sigma_{y_1}) = \frac{1}{2} \zeta(\Sigma_{y_2})$</td>
</tr>
<tr>
<td>$4$</td>
<td>$\zeta(\Sigma_{y_1}) = \frac{11}{10} \zeta(\Sigma_{y_2})$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(\Sigma_{y_2}) = \frac{3}{10} \zeta(\Sigma_{y_2})^{3}$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(\Sigma_{y_2})^{3} = \frac{3}{4} \zeta(\Sigma_{y_2})^{2}$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(S_{x_0}) = \frac{1}{10} \zeta(S_{x_0})^{2}$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(S_{x_0}^{2}) = \frac{1}{10} \zeta(S_{x_0})^{2}$</td>
</tr>
<tr>
<td>$5$</td>
<td>$\zeta(\Sigma_{y_1}) = 3\zeta(\Sigma_{y_2}) \zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_2})$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(\Sigma_{y_2}) = -\zeta(\Sigma_{y_2}) \zeta(\Sigma_{y_2}) + \frac{5}{2} \zeta(\Sigma_{y_2})$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(\Sigma_{y_2})^{2} = \frac{3}{2} \zeta(\Sigma_{y_2}) \zeta(\Sigma_{y_2}) - \frac{5}{3} \zeta(\Sigma_{y_2})$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(\Sigma_{y_2})^{3} = \frac{5}{4} \zeta(\Sigma_{y_2}) \zeta(\Sigma_{y_2}) + \frac{5}{2} \zeta(\Sigma_{y_2})$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(S_{x_0,x_1})^{2} = \frac{1}{2} \zeta(S_{x_0,x_1})^{2}$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(S_{x_0,x_1})^{3} = \zeta(S_{x_0,x_1})^{3}$</td>
</tr>
<tr>
<td>$6$</td>
<td>$\zeta(\Sigma_{y_1}) = \frac{11}{10} \zeta(\Sigma_{y_2})$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(\Sigma_{y_2}) = \zeta(\Sigma_{y_2})^{2} - \frac{3}{2} \zeta(\Sigma_{y_2})^{3}$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(\Sigma_{y_2})^{3} = \frac{9}{10} \zeta(\Sigma_{y_2})^{2} - \frac{3}{2} \zeta(\Sigma_{y_2})^{3} + \frac{5}{4} \zeta(\Sigma_{y_2})^{2}$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(\Sigma_{y_2})^{4} = \frac{1}{10} \zeta(\Sigma_{y_2})^{3} - \frac{3}{2} \zeta(\Sigma_{y_2})^{2}$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(\Sigma_{y_2})^{5} = \frac{3}{10} \zeta(\Sigma_{y_2})^{3} - \frac{3}{2} \zeta(\Sigma_{y_2})^{2}$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(\Sigma_{y_2})^{6} = \frac{3}{10} \zeta(\Sigma_{y_2})^{3} - \frac{3}{2} \zeta(\Sigma_{y_2})^{2}$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(\Sigma_{y_2})^{7} = \frac{3}{10} \zeta(\Sigma_{y_2})^{3} - \frac{3}{2} \zeta(\Sigma_{y_2})^{2}$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(\Sigma_{y_2})^{8} = \frac{3}{10} \zeta(\Sigma_{y_2})^{3} - \frac{3}{2} \zeta(\Sigma_{y_2})^{2}$</td>
</tr>
</tbody>
</table>
Example 7 (Rewriting system on irreducible coordinates, [2]). Replace “=” by “→”, we get

<table>
<thead>
<tr>
<th>Rewriting on ${\zeta(S_t)}<em>{t \in \mathcal{L}</em>{\gamma_{\infty}\setminus {y_1}}}$</th>
<th>Rewriting on ${\zeta(S_t)}<em>{t \in \mathcal{L}</em>{\gamma_{\infty}\setminus {x}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_2}) \rightarrow \frac{3}{2} \zeta(S_{y_2})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{1}{10} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_3}) \rightarrow \frac{3}{2} \zeta(S_{y_3})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{1}{5} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_3y_2}) \rightarrow \frac{5}{2} \zeta(S_{y_3y_2})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_4}) \rightarrow \frac{5}{2} \zeta(S_{y_4})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_4}) \rightarrow \frac{3}{2} \zeta(S_{y_4})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_4}) \rightarrow \frac{3}{2} \zeta(S_{y_4})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_5}) \rightarrow \frac{3}{2} \zeta(S_{y_5})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_5}) \rightarrow \frac{3}{2} \zeta(S_{y_5})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_6}) \rightarrow \frac{3}{2} \zeta(S_{y_6})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_6}) \rightarrow \frac{3}{2} \zeta(S_{y_6})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_6}) \rightarrow \frac{3}{2} \zeta(S_{y_6})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_6}) \rightarrow \frac{3}{2} \zeta(S_{y_6})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_6}) \rightarrow \frac{3}{2} \zeta(S_{y_6})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_6}) \rightarrow \frac{3}{2} \zeta(S_{y_6})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_6}) \rightarrow \frac{3}{2} \zeta(S_{y_6})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_6}) \rightarrow \frac{3}{2} \zeta(S_{y_6})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_6}) \rightarrow \frac{3}{2} \zeta(S_{y_6})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_6}) \rightarrow \frac{3}{2} \zeta(S_{y_6})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
<tr>
<td>$\zeta(S_{y_6}) \rightarrow \frac{3}{2} \zeta(S_{y_6})^2$</td>
<td>$\zeta(S_{x_0x}) \rightarrow \frac{5}{6} \zeta(S_{x_0x})^2$</td>
</tr>
</tbody>
</table>
Example 8 (Homogenous polynomials generating inside \( \ker \zeta \)). Since \( \zeta \) is surjective then

<table>
<thead>
<tr>
<th>( (Q_1)_{i \in \zeta \mathbb{Z}^n \setminus {1}} )</th>
<th>( (Q_1)_{i \in \zeta \mathbb{Z}^n \setminus {X}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \zeta(\Sigma_{y_3y_1}) - \frac{3}{2} \Sigma_{y_2} = 0 )</td>
<td>( \zeta(S_{x_0x_1} - S_{x_0x_1}) = 0 )</td>
</tr>
<tr>
<td>( \zeta(\Sigma_{y_4} - \frac{3}{2} \Sigma_{y_2}) = 0 )</td>
<td>( \zeta(S_{x_0x_1} - \frac{5}{5} S_{x_0x_1}) = 0 )</td>
</tr>
<tr>
<td>( \zeta(\Sigma_{y_4y_1} - \frac{3}{10} \Sigma_{y_2}) = 0 )</td>
<td>( \zeta(S_{x_0x_1} - \frac{10}{10} S_{x_0x_1}) = 0 )</td>
</tr>
<tr>
<td>( \zeta(\Sigma_{y_3y_1} - \frac{3}{2} \Sigma_{y_2}) = 0 )</td>
<td>( \zeta(S_{x_0x_1} - \frac{5}{5} S_{x_0x_1}) = 0 )</td>
</tr>
<tr>
<td>( \zeta(\Sigma_{y_3y_2} - \frac{3}{2} \Sigma_{y_2}) = 0 )</td>
<td>( \zeta(S_{x_0x_1} - \frac{5}{5} S_{x_0x_1}) = 0 )</td>
</tr>
<tr>
<td>( \zeta(\Sigma_{y_3y_2} - \frac{3}{10} \Sigma_{y_2}) = 0 )</td>
<td>( \zeta(S_{x_0x_1} - \frac{10}{10} S_{x_0x_1}) = 0 )</td>
</tr>
<tr>
<td>( \zeta(\Sigma_{y_3y_2} - \frac{3}{2} \Sigma_{y_2}) = 0 )</td>
<td>( \zeta(S_{x_0x_1} - \frac{5}{5} S_{x_0x_1}) = 0 )</td>
</tr>
<tr>
<td>( \zeta(\Sigma_{y_3y_2} - \frac{3}{10} \Sigma_{y_2}) = 0 )</td>
<td>( \zeta(S_{x_0x_1} - \frac{10}{10} S_{x_0x_1}) = 0 )</td>
</tr>
<tr>
<td>( \zeta(\Sigma_{y_3y_2} - \frac{3}{2} \Sigma_{y_2}) = 0 )</td>
<td>( \zeta(S_{x_0x_1} - \frac{5}{5} S_{x_0x_1}) = 0 )</td>
</tr>
</tbody>
</table>
Example 9 (Rewriting system on algebraic generators, [3]). Since $\zeta$ is surjective then

<table>
<thead>
<tr>
<th>Rewriting on ${\Sigma_j}_{j \in \mathcal{L}(\text{yn}) \setminus {y_1}}$</th>
<th>Rewriting on ${S_j}_{\mathcal{L}(\text{yn}) \setminus {x}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma y_2 y_3 \rightarrow \frac{1}{2} \Sigma y_3$</td>
<td>$S_{x_0 y_1} \rightarrow S_{\frac{1}{2} x_0 y_1}$</td>
</tr>
<tr>
<td>$\Sigma y_4 y_2 \rightarrow \frac{3}{2} \Sigma y_2 y_4$</td>
<td>$S_{x_2 x_1} \rightarrow \frac{1}{12} S_{2 x_2 x_1}$</td>
</tr>
<tr>
<td>$\Sigma y_3 y_1 y_2 \rightarrow \frac{1}{10} y_2 y_3$</td>
<td>$S_{x_0 x_1} \rightarrow \frac{3}{5} S_{2 x_0 x_1}$</td>
</tr>
<tr>
<td>$\Sigma y_2 y_7 \rightarrow \frac{3}{4} \Sigma y_2$</td>
<td>$S_{x_0 y_1} \rightarrow \frac{1}{8} S_{2 x_0 y_1}$</td>
</tr>
<tr>
<td>$\Sigma y_3 y_2 \rightarrow 3 \Sigma y_2 y_3 - 5 \Sigma y_3$</td>
<td>$S_{3 x_0 x_1} \rightarrow \frac{1}{2} S_{2 x_0 x_1}$</td>
</tr>
<tr>
<td>$\Sigma y_4 y_3 \rightarrow -5 \Sigma y_3 y_4 + 5 \Sigma y_4 y_3$</td>
<td>$S_{2 x_0 x_1} \rightarrow \frac{1}{10} S_{2 x_0 x_1} + S_{x_2 x_1} S_{x_0 x_1}$</td>
</tr>
<tr>
<td>$\Sigma y_2 y_2 y_7 \rightarrow \frac{2}{3} \Sigma y_2$</td>
<td>$S_{x_2 x_1} \rightarrow 2 S_{x_0 x_1} + 2 S_{x_0 x_1}$</td>
</tr>
<tr>
<td>$\Sigma y_2 y_7 \rightarrow \frac{1}{3} \Sigma y_2 y_3 y_2 + \frac{5}{3} \Sigma y_2$</td>
<td>$S_{x_2 x_1} \rightarrow 2 S_{x_0 x_1} + 2 S_{x_0 x_1}$</td>
</tr>
<tr>
<td>$\Sigma y_2 y_7 \rightarrow \frac{4}{3} \Sigma y_2 y_3 y_2 + \frac{5}{3} \Sigma y_2$</td>
<td>$S_{x_2 x_1} \rightarrow 2 S_{x_0 x_1} + 2 S_{x_0 x_1}$</td>
</tr>
</tbody>
</table>