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Calculus for eight-dimensional hypercomplex algebras

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ABSTRACT

This paper introduces the basis of calculus with the formulation of partial derivatives over hypercomplex variables for two 8-dimensional hypercomplex algebras: octonions and split-biquaternions. The work can be seen as a direct extension of Wirtinger/ $\mathbb{C}\mathbb{R}$ calculus for complex-numbers algebra, and $\mathbb{H}\mathbb{R}$ calculus or generalized $\mathbb{H}\mathbb{R}$ calculus for quaternion algebra. We first show that the approach based on rotations over basis components used in the development of $\mathbb{H}\mathbb{R}$ calculus or generalized $\mathbb{H}\mathbb{R}$ calculus is not suitable for the two considered 8-dimensional hypercomplex algebras. Instead, we justify the use of transformations based on Hadamard matrices, which allows a common framework for the development of partial derivatives for complex-numbers, quaternion, octonion and split-biquaternion algebras. The results of this paper are an important step towards the development of optimization algorithms and adaptive filtering algorithms of hypercomplex variables for 8-dimensional hypercomplex algebras.

KEYWORDS: calculus for 8-dimensional hypercomplex algebras, octonion algebra, split-biquaternion algebra, normed and seminormed 8-dimensional composition algebras, normed and seminormed 8-dimensional division algebras.

I. INTRODUCTION

The use of Wirtinger/ $\mathbb{C}\mathbb{R}$ calculus for complex-numbers algebra has been well know in the last few decades. With partial derivatives over complex variables, it provides a simple and efficient representation for performing calculus and optimization with functions of complex-variables, whether or not the functions are analytic or holomorphic (unlike complex analysis)[1],[2],[3]. More recently, this concept has been extended to hypercomplex quaternion algebra, with the development of $\mathbb{H}\mathbb{R}$ calculus and generalized $\mathbb{H}\mathbb{R}$ calculus

[4],[5],[6]. Just like $\mathbb{C}\mathbb{R}$ calculus for complex-numbers algebra, $\mathbb{H}\mathbb{R}$ calculus provides a simple and efficient notation for calculus in quaternion algebra with partial derivatives over hypercomplex variables, and it can also provide some benefits over the alternative calculus formulations using only derivatives of the (real-valued) individual components of quaternions [7].

In this paper, we provide the basis of calculus with the formulation of partial derivatives over hypercomplex variables in two 8-dimensional hypercomplex algebras: the octonion algebra and the split-biquaternion algebra. Such partial derivatives are an important step towards the development of optimization algorithms and adaptive filtering algorithms of hypercomplex variables for 8-dimensional hypercomplex algebras.

While the 8-dimensional octonion algebra is a normed (composition) algebra over \mathbb{R} and a division algebra over \mathbb{R} , it is not an associative algebra, which can certainly be a limitation for some practical uses of the algebra and for the development of calculus in this algebra. In this paper, we nevertheless show some developments of calculus for octonions, despite the fact that the last step in these developments is in principle invalid because of the lack of associativity, for two reasons:

- It will provide some insights for the development of calculus for the associative octonion-like split-biquaternion algebra, and allow some insightful comparisons with the development of $\mathbb{C}\mathbb{R}$ calculus and $\mathbb{H}\mathbb{R}$ calculus.
- It may later lead to some heuristic approximative algorithms, since despite the lack of associativity in octonion algebra the product $\mathbf{o}_x \mathbf{o}_y$ of two octonion numbers can be significantly correlated with the product $\mathbf{o}_y \mathbf{o}_x$ [8] and also $(\mathbf{o}_x \mathbf{o}_y)^* = \mathbf{o}_y^* \mathbf{o}_x^*$, i.e., some stochastic estimations of the derivatives may become possible.

The octonion-like split-biquaternion algebra is an 8-dimensional associative algebra (unlike the octonion algebra). It is an even subalgebra of Clifford algebra $\mathcal{C}l_{4,0}(\mathbb{R})$, is isomorphic to $\mathcal{C}l_{0,3}(\mathbb{R})$, and can be obtained from a sum of two quaternion algebras [9],[10]. It has recently been explained [11] that this octonion-like algebra is a seminormed composition algebra over \mathbb{R} (i.e., preservation of two seminorms under multiplication) and is also a seminormed division algebra over \mathbb{R} (i.e., division by any number is possible as long as the two seminorms

of this number are non-zero). It was also pointed out that for sparse split-biquaternion numbers (i.e., when one of the components in the split-biquaternion number is dominant), the seminorm becomes similar to the Euclidian norm. The split biquaternion algebra was also found to be the same as the 1d-up approach to conformal geometric algebra, presented with some potential application in [12].

In $\mathbb{H}\mathbb{R}$ calculus, for the development of the partial derivatives over quaternion variables, the approach of performing rotations of variables over the different unit basis of quaternions was followed [4],[6]. These rotations become involutions and they lead to a form where each basis component in the resulting rotated quaternion variables is only a function of the same component in the original quaternion variable, which is a requirement for the formulation of simple partial derivatives over quaternion variables. The resulting rotations are referred to as simplified rotations in this paper. This approach was extended to a more general form in the generalized $\mathbb{H}\mathbb{R}$ calculus [5],[6], where the different basis used for the simplified rotations are the original quaternion unit basis rotated over an arbitrary quaternion number. This led to more general results, where instead of having partial derivatives over a quaternion variable rotated over the quaternion basis components, a partial derivative over a quaternion variable rotated over any arbitrary quaternion number is obtained.

Essentially, for the development of $\mathbb{H}\mathbb{R}$ calculus or calculus involving partial derivatives over complex or hypercomplex variables, the concept is to perform transformations (e.g. simplified norm-preserving rotations/involutions in [4],[6]) such that transformed variables are generated from an original variable, and such that the individual components in the original variable can be expressed as a linear combination of the transformed variables, using real-valued coefficients. The transformed variables, which also include the original variable (unit transformation), correspond to the complex or hypercomplex variables for which partial derivatives are to be formulated. While the choice in [4],[5],[6] of using simplified rotations over the basis quaternion components as transformations may not be unique, it leads to the same known simple forms as in $\mathbb{C}\mathbb{R}$ calculus for the resulting partial derivatives over quaternion variables in $\mathbb{H}\mathbb{R}$ calculus or generalized $\mathbb{H}\mathbb{R}$ calculus. However, this approach using simplified rotations over the basis components is not directly transferable to other complex or hypercomplex algebras. For complex-numbers algebra, a rotation over any basis of complex numbers leads to the same number as the original number and is thus not useful. For the octonion algebra, performing simplified rotations over the

octonion basis components leads to a non-intuitive formulation of the partial derivatives over octonion variables, with somewhat arbitrary coefficients that don't have the same intuitive form as in $\mathbb{C}\mathbb{R}$ calculus or $\mathbb{H}\mathbb{R}$ calculus. And more importantly, for the octonion-like split-biquaternion algebra, performing simplified rotations over basis components leads to only four different split-biquaternion variables, while eight linearly independent variables are required for the reconstruction of the individual basis components in the original split-biquaternion variable, which is an important step in the formulation of the partial derivatives.

It is possible to replace the approach of using simplified rotations over basis components (used in [4],[5],[6] for $\mathbb{H}\mathbb{R}$ calculus or generalized $\mathbb{H}\mathbb{R}$ calculus) by an approach which is applicable to all the considered algebras in this paper (complex numbers, quaternion, octonion and split-biquaternion algebras), and which leads to an intuitive formulation where the partial derivatives over complex or hypercomplex variables all have the same form for the different algebras. This is achieved by formulating the transformation as a real-valued transformation matrix with some constraints which are justified in the paper: the first column of the matrix has unit elements, all the elements of the matrix have unit absolute value, and all elements of the inverse matrix also have a constant absolute value. For all the complex and hypercomplex algebras considered in this paper, the orthogonal and symmetric Hadamard matrix provides a solution for the required transformation matrix, and the transformation matrix and resulting partial derivatives become unique if the first row of the matrix is also required to have positive unit values.

The structure of this paper is as follows. A summary of the development of $\mathbb{H}\mathbb{R}$ calculus and generalized $\mathbb{H}\mathbb{R}$ calculus using simplified rotations as in [4],[5],[6] will first be presented. Then the limitations of developing calculus for octonions and octonion-like split-biquaternions using simplified rotations over basis components will be illustrated. After this, a new calculus (partial derivatives over hypercomplex variables) will be derived for octonions and octonion-like split-biquaternions with a transformation of variables using the Hadamard matrix. For completeness, the development of the well-known Wirtinger/ $\mathbb{C}\mathbb{R}$ calculus using the proposed Hadamard matrix transformation will then also be shown.

II. SUMMARY OF $\mathbb{H}\mathbb{R}$ CALCULUS AND GENERALIZED $\mathbb{H}\mathbb{R}$ CALCULUS USING SIMPLIFIED ROTATIONS

The quaternion associative (but non-commutative) normed division algebra over \mathbb{R} has components $e_0 = 1$, $e_1 = i$, $e_2 = j$, $e_3 = k$ as an orthogonal basis, with $e_0^2 = 1$, $e_i^2 = -1$ $1 \leq i \leq 3$ and the multiplication rules of Table I.

Table I Multiplication rules in the quaternion algebra

$e_i e_j$		e_j			
		$e_0 = 1$	$e_1 = i$	$e_2 = j$	$e_3 = k$
e_i	$e_0 = 1$	1	$e_1 = i$	$e_2 = j$	$e_3 = k$
	$e_1 = i$	$e_1 = i$	-1	$e_3 = k$	$-e_2 = -j$
	$e_2 = j$	$e_2 = j$	$-e_3 = -k$	-1	$e_1 = i$
	$e_3 = k$	$e_3 = k$	$e_2 = j$	$-e_1 = -i$	-1

$$\mathbf{q} = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 = q_0 + q_1 i + q_2 j + q_3 k \quad (1)$$

$$\mathbf{q}^* = q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3 = q_0 - q_1 i - q_2 j - q_3 k \quad (2)$$

$$\|\mathbf{q}\|^2 = \mathbf{q}\mathbf{q}^* = q_0^2 + q_1^2 + q_2^2 + q_3^2 \quad (3)$$

$$(\mathbf{q}_x \mathbf{q}_y)^* = \mathbf{q}_y^* \mathbf{q}_x^* \quad (4).$$

In general, quaternions are non-commutative:

$$\mathbf{q}_x \mathbf{q}_y \neq \mathbf{q}_y \mathbf{q}_x \quad (5).$$

However, when the product $\mathbf{q}_x \mathbf{q}_y$ is a scalar (i.e., only the real-valued or scalar first component e_0 is non-zero), the product becomes commutative.

To formulate the partial derivatives over quaternion variables, three transformed versions of a variable \mathbf{q} are produced. \mathbf{q} and the three transformed variables are the quaternion variables for which partial derivatives formulations are to be obtained. \mathbf{q} and the three

transformed variables must be such that the coefficients q_0, q_1, q_2, q_3 in \mathbf{q} can reversely be expressed as a linear combination of \mathbf{q} and transformed variables, with real-valued coefficients. \mathbf{q} and the transformed variables must therefore be linearly independent. Then from the expression of the total differential of a function $df(\mathbf{q})$ and using the differentials dq_0, dq_1, dq_2, dq_3 , the partial derivatives over \mathbf{q} and transformed variables will be formulated.

One possible transformation to generate the three transformed versions of \mathbf{q} is to use a multiplication with form $\mathbf{q}^u = \mathbf{u}\mathbf{q}\mathbf{u}^{-1}$, which is a norm-preserving rotation. Moreover, the transformation can be further simplified if the norm-preserving rotation $\mathbf{q}^u = \mathbf{u}\mathbf{q}\mathbf{u}^{-1}$ is also involution. Some values of \mathbf{u} leading to an involution rotation are those for which $\mathbf{u}^2 = 1$ or $\mathbf{u}^2 = -1$, then leading to a simplified form $\mathbf{q}^u = \mathbf{u}\mathbf{q}\mathbf{u}$. For the simple formulation of the partial derivatives, it is important that after the rotation or involution each component in the transformed \mathbf{q}^u ($q_0^u, q_1^u, q_2^u, q_3^u$) is only a function of the corresponding coefficient in \mathbf{q} , i.e., q_0, q_1, q_2, q_3 . This is not necessarily achieved by all choices of rotations $\mathbf{q}^u = \mathbf{u}\mathbf{q}\mathbf{u}^{-1}$ or involutions $\mathbf{q}^u = \mathbf{u}\mathbf{q}\mathbf{u}$.

A natural choice for the three transformed variables obtained from \mathbf{q} are to use $\mathbf{u} = i, j, k$:

$$\mathbf{q}^i = i\mathbf{q}i^{-1} = -i\mathbf{q}i = q_0 + q_1i - q_2j - q_3k \quad (6)$$

$$\mathbf{q}^j = j\mathbf{q}j^{-1} = -j\mathbf{q}j = q_0 - q_1i + q_2j - q_3k \quad (7)$$

$$\mathbf{q}^k = k\mathbf{q}k^{-1} = -k\mathbf{q}k = q_0 - q_1i - q_2j + q_3k \quad (8)$$

To express each component $q_n e_n$ $0 \leq n \leq 3$ of \mathbf{q} as a linear combination of $\mathbf{q}, \mathbf{q}^i, \mathbf{q}^j, \mathbf{q}^k$ with real-valued coefficients, we have:

$$q_n e_n = c_{n1}\mathbf{q} + c_{ni}\mathbf{q}^i + c_{nj}\mathbf{q}^j + c_{nk}\mathbf{q}^k \quad 0 \leq n \leq 3. \quad (9)$$

Re-writing this in matrix form with real-valued elements:

$$\begin{bmatrix} 0 \\ q_n \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} q_0 & q_0 & q_0 & q_0 \\ q_1 & q_1 & -q_1 & -q_1 \\ q_2 & -q_2 & q_2 & -q_2 \\ q_3 & -q_3 & -q_3 & q_3 \end{bmatrix} \begin{bmatrix} c_{n1} \\ c_{ni} \\ c_{nj} \\ c_{nk} \end{bmatrix} \quad \vec{\mathbf{q}}_n = \vec{\mathbf{Q}}\vec{\mathbf{c}}_n \quad 0 \leq n \leq 3 \quad (10)$$

Since each component in $\mathbf{q}^i, \mathbf{q}^j, \mathbf{q}^k$ involves only the corresponding component q_n in \mathbf{q} , each row in the above matrix can be normalized by the coefficient q_n , to have a formulation which is independent of the coefficients in \mathbf{q} :

$$\begin{bmatrix} 0 \\ 1 \text{ (row } n) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_{n1} \\ c_{ni} \\ c_{nj} \\ c_{nk} \end{bmatrix} \quad \bar{\mathbf{q}}'_n = \bar{\bar{\mathbf{Q}}}' \bar{\mathbf{c}}_n \quad 0 \leq n \leq 3 \quad (11).$$

This leads to:

$$\begin{aligned} \bar{\mathbf{c}}_n &= \bar{\bar{\mathbf{Q}}}'^{-1} \bar{\mathbf{q}}'_n \quad 0 \leq n \leq 3 \\ \bar{\mathbf{c}}_n &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}^{-1} \times \bar{\mathbf{q}}'_n = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \text{ (row } n) \\ 0 \\ 0 \end{bmatrix} \quad 0 \leq n \leq 3 \quad (12). \end{aligned}$$

Therefore, each $\bar{\mathbf{c}}_n$ $0 \leq n \leq 3$ has the values from the corresponding column in $\bar{\bar{\mathbf{Q}}}'^{-1}$:

$$\begin{aligned} \bar{\mathbf{c}}_0 &= \frac{1}{4} [1 \ 1 \ 1 \ 1]^T & \bar{\mathbf{c}}_1 &= \frac{1}{4} [1 \ 1 \ -1 \ -1]^T \\ \bar{\mathbf{c}}_2 &= \frac{1}{4} [1 \ -1 \ 1 \ -1]^T & \bar{\mathbf{c}}_3 &= \frac{1}{4} [1 \ -1 \ -1 \ 1]^T \end{aligned} \quad (13),$$

leading to

$$q_n = e_n^{-1} (c_{n1} \mathbf{q} + c_{ni} \mathbf{q}^i + c_{nj} \mathbf{q}^j + c_{nk} \mathbf{q}^k) \quad (14)$$

with $e_n^{-1} = e_n$ $n = 0$, $e_n^{-1} = -e_n$ $1 \leq n \leq 3$, and

$$\begin{aligned} q_0 &= \frac{1}{4} (\mathbf{q} + \mathbf{q}^i + \mathbf{q}^j + \mathbf{q}^k) & q_1 &= \frac{-i}{4} (\mathbf{q} + \mathbf{q}^i - \mathbf{q}^j - \mathbf{q}^k) \\ q_2 &= \frac{-j}{4} (\mathbf{q} - \mathbf{q}^i + \mathbf{q}^j - \mathbf{q}^k) & q_3 &= \frac{-k}{4} (\mathbf{q} - \mathbf{q}^i - \mathbf{q}^j + \mathbf{q}^k) \end{aligned} \quad (15).$$

It should be noted that the results in (15) are quaternion variables where only the 1st component is non-zero, therefore they are real-valued scalars.

If we wish to express $q_n e_n$ $0 \leq n \leq 3$ in terms of conjugates $\mathbf{q}^*, (\mathbf{q}^i)^*, (\mathbf{q}^j)^*, (\mathbf{q}^k)^*$, then a conjugation operation can be applied to (15):

$$\begin{aligned} q_0 &= \frac{1}{4}(\mathbf{q}^* + (\mathbf{q}^i)^* + (\mathbf{q}^j)^* + (\mathbf{q}^k)^*) & q_1 &= \frac{i}{4}(\mathbf{q}^* + (\mathbf{q}^i)^* - (\mathbf{q}^j)^* - (\mathbf{q}^k)^*) \\ q_2 &= \frac{j}{4}(\mathbf{q}^* - (\mathbf{q}^i)^* + (\mathbf{q}^j)^* - (\mathbf{q}^k)^*) & q_3 &= \frac{k}{4}(\mathbf{q}^* - (\mathbf{q}^i)^* - (\mathbf{q}^j)^* + (\mathbf{q}^k)^*) \end{aligned} \quad (16)$$

The total differential of a function $f(\mathbf{q}) = f(q_0, q_1, q_2, q_3)$ $q_n \in \mathbb{R}$ $0 \leq n \leq 3$ can be expressed as:

$$df(\mathbf{q}) = \frac{\partial f(\mathbf{q})}{\partial q_0} dq_0 + \frac{\partial f(\mathbf{q})}{\partial q_1} dq_1 + \frac{\partial f(\mathbf{q})}{\partial q_2} dq_2 + \frac{\partial f(\mathbf{q})}{\partial q_3} dq_3 \quad (17)$$

or

$$df(\mathbf{q}) = dq_0 \frac{\partial f(\mathbf{q})}{\partial q_0} + dq_1 \frac{\partial f(\mathbf{q})}{\partial q_1} + dq_2 \frac{\partial f(\mathbf{q})}{\partial q_2} + dq_3 \frac{\partial f(\mathbf{q})}{\partial q_3} \quad (dq_n \in \mathbb{R}) \quad (18).$$

(17) will lead to left derivatives while (18) will lead to right derivatives, which may differ since quaternions are non-commutative. However, for real-valued functions of quaternions $f(\mathbf{q})$, the left derivatives and right derivatives have been shown to be the same [5].

The differentials dq_0, dq_1, dq_2, dq_3 are obtained from (15) or (16). To obtain the partial derivatives, we substitute dq_0, dq_1, dq_2, dq_3 into (17) or (18). There are therefore four different possibilities, depending on partial derivatives over the variables $\mathbf{q}, \mathbf{q}^i, \mathbf{q}^j, \mathbf{q}^k$ or $\mathbf{q}^*, (\mathbf{q}^i)^*, (\mathbf{q}^j)^*, (\mathbf{q}^k)^*$, and left derivatives or right derivatives. Note that for the right derivatives in (18), norm-preserving involutions $-iq_1i, -jq_2j, -kq_3k$ should first be applied to (15) or (16), to get equations where the i, j, k terms appear at the end of the equation.

The formulation of left side partial derivatives over the variables $\mathbf{q}, \mathbf{q}^i, \mathbf{q}^j, \mathbf{q}^k$ is shown in detail in (19). Grouping together the terms with $d\mathbf{q}, d\mathbf{q}^i, d\mathbf{q}^j, d\mathbf{q}^k$ and defining $\frac{\partial}{\partial \mathbf{q}^n}$ $n = 1, i, j, k$ as the factor multiplying $d\mathbf{q}^n$, we obtain:

$$\begin{aligned}
df(\mathbf{q}) &= \frac{\partial f(\mathbf{q})}{\partial q_0} dq_0 + \frac{\partial f(\mathbf{q})}{\partial q_1} dq_1 + \frac{\partial f(\mathbf{q})}{\partial q_2} dq_2 + \frac{\partial f(\mathbf{q})}{\partial q_3} dq_3 \\
&= \frac{1}{4} \frac{\partial f(\mathbf{q})}{\partial q_0} (d\mathbf{q} + d\mathbf{q}^i + d\mathbf{q}^j + d\mathbf{q}^k) - \frac{1}{4} \frac{\partial f(\mathbf{q})}{\partial q_1} i (d\mathbf{q} + d\mathbf{q}^i - d\mathbf{q}^j - d\mathbf{q}^k) \\
&\quad - \frac{1}{4} \frac{\partial f(\mathbf{q})}{\partial q_2} j (d\mathbf{q} - d\mathbf{q}^i + d\mathbf{q}^j - d\mathbf{q}^k) - \frac{1}{4} \frac{\partial f(\mathbf{q})}{\partial q_3} k (d\mathbf{q} - d\mathbf{q}^i - d\mathbf{q}^j + d\mathbf{q}^k) \\
&= \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} - \frac{\partial f(\mathbf{q})}{\partial q_1} i - \frac{\partial f(\mathbf{q})}{\partial q_2} j - \frac{\partial f(\mathbf{q})}{\partial q_3} k \right) d\mathbf{q} \\
&\quad + \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} - \frac{\partial f(\mathbf{q})}{\partial q_1} i + \frac{\partial f(\mathbf{q})}{\partial q_2} j + \frac{\partial f(\mathbf{q})}{\partial q_3} k \right) d\mathbf{q}^i \\
&\quad + \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} + \frac{\partial f(\mathbf{q})}{\partial q_1} i - \frac{\partial f(\mathbf{q})}{\partial q_2} j + \frac{\partial f(\mathbf{q})}{\partial q_3} k \right) d\mathbf{q}^j \\
&\quad + \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} + \frac{\partial f(\mathbf{q})}{\partial q_1} i + \frac{\partial f(\mathbf{q})}{\partial q_2} j - \frac{\partial f(\mathbf{q})}{\partial q_3} k \right) d\mathbf{q}^k \\
&\equiv \frac{\partial f(\mathbf{q})}{\partial \mathbf{q}} d\mathbf{q} + \frac{\partial f(\mathbf{q})}{\partial \mathbf{q}^i} d\mathbf{q}^i + \frac{\partial f(\mathbf{q})}{\partial \mathbf{q}^j} d\mathbf{q}^j + \frac{\partial f(\mathbf{q})}{\partial \mathbf{q}^k} d\mathbf{q}^k \tag{19}
\end{aligned}$$

From the above, we note in particular the result for the left partial derivative over \mathbf{q} :

$$\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}} \equiv \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} - \frac{\partial f(\mathbf{q})}{\partial q_1} i - \frac{\partial f(\mathbf{q})}{\partial q_2} j - \frac{\partial f(\mathbf{q})}{\partial q_3} k \right) \tag{20}$$

Using the same procedure but with (16) instead of (15), the left partial derivative over \mathbf{q}^* can also be found:

$$\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}^*} \equiv \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} + \frac{\partial f(\mathbf{q})}{\partial q_1} i + \frac{\partial f(\mathbf{q})}{\partial q_2} j + \frac{\partial f(\mathbf{q})}{\partial q_3} k \right) \tag{21}$$

For right side derivatives, i.e., when (18) is used instead of (17), the following results are obtained:

$$\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}} \equiv \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} - i \frac{\partial f(\mathbf{q})}{\partial q_1} - j \frac{\partial f(\mathbf{q})}{\partial q_2} - k \frac{\partial f(\mathbf{q})}{\partial q_3} \right) \tag{22}$$

$$\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}^*} \equiv \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} + i \frac{\partial f(\mathbf{q})}{\partial q_1} + j \frac{\partial f(\mathbf{q})}{\partial q_2} + k \frac{\partial f(\mathbf{q})}{\partial q_3} \right) \tag{23}$$

These partial derivatives over quaternion variables are the fundamental expressions for $\mathbb{H}\mathbb{R}$ calculus in quaternion algebra, for which several other properties such as chain rule, product rule, gradient vector, Hessian matrix, Taylor series expansion can also be developed [5][6].

Even though the choice of the transformed variables $\mathbf{q}, \mathbf{q}^i, \mathbf{q}^j, \mathbf{q}^k$ or $\mathbf{q}^*, (\mathbf{q}^i)^*, (\mathbf{q}^j)^*, (\mathbf{q}^k)^*$ to derive $\mathbb{H}\mathbb{R}$ calculus can be considered natural, it can also be seen as somewhat arbitrary. Some more general results for partial derivatives over $\mathbf{q}^{\mathbf{u}}$ or $\mathbf{q}^{\mathbf{u}*}$, where \mathbf{u} is an arbitrary quaternion, have been developed with the generalized $\mathbb{H}\mathbb{R}$ calculus [5]. In general, performing a norm-preserving rotation of \mathbf{q} around an arbitrary quaternion number \mathbf{u} gives:

$$\mathbf{q}^{\mathbf{u}} = \mathbf{u}\mathbf{q}\mathbf{u}^{-1} = \mathbf{u}(q_0 + iq_1 + jq_2 + kq_3)\mathbf{u}^{-1} = q_0 1^{\mathbf{u}} + q_1 i^{\mathbf{u}} + q_2 j^{\mathbf{u}} + q_3 k^{\mathbf{u}} \quad (24)$$

$$\text{with } 1^{\mathbf{u}} = \mathbf{u}\mathbf{1}\mathbf{u}^{-1} = 1 \quad i^{\mathbf{u}} = \mathbf{u}i\mathbf{u}^{-1} \quad j^{\mathbf{u}} = \mathbf{u}j\mathbf{u}^{-1} \quad k^{\mathbf{u}} = \mathbf{u}k\mathbf{u}^{-1}.$$

$i^{\mathbf{u}}, j^{\mathbf{u}}, k^{\mathbf{u}}$ aren't necessarily trivial to compute (except when $\mathbf{u} = i, j, k$), but $1, i^{\mathbf{u}}, j^{\mathbf{u}}, k^{\mathbf{u}}$ for any non-zero \mathbf{u} still form an orthonormal base [5], i.e., with $i^{\mathbf{u}}i^{\mathbf{u}} = -1$, $j^{\mathbf{u}}j^{\mathbf{u}} = -1$, $k^{\mathbf{u}}k^{\mathbf{u}} = -1$ and $i^{\mathbf{u}}j^{\mathbf{u}}k^{\mathbf{u}} = -1$ (just like $ii = -1$, $jj = -1$, $kk = -1$, and $ijk = -1$). In other words, the basic rules of the algebra (and the involutions and their properties) are preserved with the new basis.

To obtain the generalized $\mathbb{H}\mathbb{R}$ calculus, it is possible to follow exactly the same steps as for $\mathbb{H}\mathbb{R}$ calculus, but replacing $1, i, j, k$ by $1, i^{\mathbf{u}}, j^{\mathbf{u}}, k^{\mathbf{u}}$ and $\mathbf{q}, \mathbf{q}^i, \mathbf{q}^j, \mathbf{q}^k$ by $\mathbf{q}^{\mathbf{u}}, \mathbf{q}^{\mathbf{u}i^{\mathbf{u}}}, \mathbf{q}^{\mathbf{u}j^{\mathbf{u}}}, \mathbf{q}^{\mathbf{u}k^{\mathbf{u}}}$:

$$\mathbf{q}^{\mathbf{u}} = q_0 + q_1 i^{\mathbf{u}} + q_2 j^{\mathbf{u}} + q_3 k^{\mathbf{u}} \quad (25)$$

$$\mathbf{q}^{\mathbf{u}i^{\mathbf{u}}} = i^{\mathbf{u}}\mathbf{q}^{\mathbf{u}}(i^{\mathbf{u}})^{-1} = -i^{\mathbf{u}}\mathbf{q}^{\mathbf{u}}i^{\mathbf{u}} = q_0 + q_1 i^{\mathbf{u}} - q_2 j^{\mathbf{u}} - q_3 k^{\mathbf{u}}$$

$$\mathbf{q}^{\mathbf{u}j^{\mathbf{u}}} = j^{\mathbf{u}}\mathbf{q}^{\mathbf{u}}(j^{\mathbf{u}})^{-1} = -j^{\mathbf{u}}\mathbf{q}^{\mathbf{u}}j^{\mathbf{u}} = q_0 - q_1 i^{\mathbf{u}} + q_2 j^{\mathbf{u}} - q_3 k^{\mathbf{u}}$$

$$\mathbf{q}^{\mathbf{u}k^{\mathbf{u}}} = k^{\mathbf{u}}\mathbf{q}^{\mathbf{u}}(k^{\mathbf{u}})^{-1} = -k^{\mathbf{u}}\mathbf{q}^{\mathbf{u}}k^{\mathbf{u}} = q_0 - q_1 i^{\mathbf{u}} - q_2 j^{\mathbf{u}} + q_3 k^{\mathbf{u}}.$$

The result for the left partial derivative over $\mathbf{q}^{\mathbf{u}}$ with arbitrary quaternion value \mathbf{u} is then:

$$\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}^{\mathbf{u}}} \equiv \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} - \frac{\partial f(\mathbf{q})}{\partial q_1} i^{\mathbf{u}} - \frac{\partial f(\mathbf{q})}{\partial q_2} j^{\mathbf{u}} - \frac{\partial f(\mathbf{q})}{\partial q_3} k^{\mathbf{u}} \right) \quad (26).$$

For the left partial derivative over $\mathbf{q}^{\mathbf{u}^*}$, the corresponding result is:

$$\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}^{\mathbf{u}^*}} \equiv \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} + \frac{\partial f(\mathbf{q})}{\partial q_1} i^{\mathbf{u}} + \frac{\partial f(\mathbf{q})}{\partial q_2} j^{\mathbf{u}} + \frac{\partial f(\mathbf{q})}{\partial q_3} k^{\mathbf{u}} \right) \quad (27)$$

and for the right side derivatives the corresponding results are:

$$\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}^{\mathbf{u}}} \equiv \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} - i^{\mathbf{u}} \frac{\partial f(\mathbf{q})}{\partial q_1} - j^{\mathbf{u}} \frac{\partial f(\mathbf{q})}{\partial q_2} - k^{\mathbf{u}} \frac{\partial f(\mathbf{q})}{\partial q_3} \right) \quad (28)$$

$$\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}^{\mathbf{u}^*}} \equiv \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} + i^{\mathbf{u}} \frac{\partial f(\mathbf{q})}{\partial q_1} + j^{\mathbf{u}} \frac{\partial f(\mathbf{q})}{\partial q_2} + k^{\mathbf{u}} \frac{\partial f(\mathbf{q})}{\partial q_3} \right) \quad (29).$$

For $\mathbf{u} = 1, i, j, k$, we obtain the specific case of the derivatives previously introduced with $\mathbb{H}\mathbb{R}$ calculus.

III. CALCULUS FOR OCTONIONS USING SIMPLIFIED ROTATIONS

The octonion non-associative and non-commutative normed division algebra over \mathbb{R} has components e_i $0 \leq i \leq 7$ as an orthogonal basis, with $e_0^2 = 1$, $e_i^2 = -1$ $1 \leq i \leq 7$ and the multiplication rules of Table II.

Table II Multiplication rules in the octonion algebra

$e_i e_j$		e_j							
		$e_0 = 1$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_i	$e_0 = 1$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
	e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
	e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
	e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
	e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
	e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
	e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

$$\mathbf{o} = o_0 e_0 + o_1 e_1 + o_2 e_2 + o_3 e_3 + o_4 e_4 + o_5 e_5 + o_6 e_6 + o_7 e_7 \quad (30)$$

$$\mathbf{o}^* = o_0 e_0 - o_1 e_1 - o_2 e_2 - o_3 e_3 - o_4 e_4 - o_5 e_5 - o_6 e_6 - o_7 e_7 \quad (31)$$

$$\|\mathbf{o}\|^2 = \mathbf{o} \mathbf{o}^* = o_0^2 + o_1^2 + o_2^2 + o_3^2 + o_4^2 + o_5^2 + o_6^2 + o_7^2 \quad (32)$$

$$(\mathbf{o}_x \mathbf{o}_y)^* = \mathbf{o}_y^* \mathbf{o}_x^* \quad (33).$$

In general, octonions are non-associative and non-commutative:

$$\mathbf{o}_x (\mathbf{o}_y \mathbf{o}_z) \neq (\mathbf{o}_x \mathbf{o}_y) \mathbf{o}_z \quad \mathbf{o}_x \mathbf{o}_y \neq \mathbf{o}_y \mathbf{o}_x \quad (34).$$

However, when the product $\mathbf{o}_x \mathbf{o}_y$ is a scalar (i.e., only the real-valued or scalar first component e_0 is non-zero), the product becomes commutative. Moreover, the product of octonions by real-valued numbers are associative, as well as some specific forms such as:

$$(\mathbf{o}_x \mathbf{o}_y) \mathbf{o}_x^* = \mathbf{o}_x (\mathbf{o}_y \mathbf{o}_x^*) \quad (35).$$

More general properties of the octonions can also be mentioned here but will not be required in this paper: alternative algebra, Moufang identities and power associative property [13].

The same steps as in the previous development of $\mathbb{H}\mathbb{R}$ calculus and generalized $\mathbb{H}\mathbb{R}$ calculus using norm-preserving simplified rotations can be used in this section to generate transformed octonion variables, computed by rotations over the basis components of octonion algebra. The transformed variables are also the set of octonion variables over which the partial derivatives are to be formulated. The approach will lead to some limitations to be discussed later in this section.

The rotations of an octonion variable over the eight orthogonal basis components of octonion algebra lead to the following expressions. Note that the octonion multiplications below correspond to special cases where they are associative as in (35).

$$\begin{aligned}
\mathbf{o}^{e_0} &= (e_0 \mathbf{o})e_0^{-1} = e_0(\mathbf{o}e_0^{-1}) = (1 \times \mathbf{o}) \times 1^{-1} = 1 \times (\mathbf{o} \times 1) = \mathbf{o} \\
&= o_0 + o_1 e_1 + o_2 e_2 + o_3 e_3 + o_4 e_4 + o_5 e_5 + o_6 e_6 + o_7 e_7 \\
\mathbf{o}^{e_1} &= (e_1 \mathbf{o})e_1^{-1} = e_1(\mathbf{o}e_1^{-1}) = -(e_1 \mathbf{o})e_1 = -e_1(\mathbf{o}e_1) \\
&= o_0 + o_1 e_1 - o_2 e_2 - o_3 e_3 - o_4 e_4 - o_5 e_5 - o_6 e_6 - o_7 e_7 \\
\mathbf{o}^{e_2} &= (e_2 \mathbf{o})e_2^{-1} = e_2(\mathbf{o}e_2^{-1}) = -(e_2 \mathbf{o})e_2 = -e_2(\mathbf{o}e_2) \\
&= o_0 - o_1 e_1 + o_2 e_2 - o_3 e_3 - o_4 e_4 - o_5 e_5 - o_6 e_6 - o_7 e_7 \\
\mathbf{o}^{e_3} &= (e_3 \mathbf{o})e_3^{-1} = e_3(\mathbf{o}e_3^{-1}) = -(e_3 \mathbf{o})e_3 = -e_3(\mathbf{o}e_3) \\
&= o_0 - o_1 e_1 - o_2 e_2 + o_3 e_3 - o_4 e_4 - o_5 e_5 - o_6 e_6 - o_7 e_7 \\
\mathbf{o}^{e_4} &= (e_4 \mathbf{o})e_4^{-1} = e_4(\mathbf{o}e_4^{-1}) = -(e_4 \mathbf{o})e_4 = -e_4(\mathbf{o}e_4) \\
&= o_0 - o_1 e_1 - o_2 e_2 - o_3 e_3 + o_4 e_4 - o_5 e_5 - o_6 e_6 - o_7 e_7 \\
\mathbf{o}^{e_5} &= (e_5 \mathbf{o})e_5^{-1} = e_5(\mathbf{o}e_5^{-1}) = -(e_5 \mathbf{o})e_5 = -e_5(\mathbf{o}e_5) \\
&= o_0 - o_1 e_1 - o_2 e_2 - o_3 e_3 - o_4 e_4 + o_5 e_5 - o_6 e_6 - o_7 e_7 \\
\mathbf{o}^{e_6} &= (e_6 \mathbf{o})e_6^{-1} = e_6(\mathbf{o}e_6^{-1}) = -(e_6 \mathbf{o})e_6 = -e_6(\mathbf{o}e_6) \\
&= o_0 - o_1 e_1 - o_2 e_2 - o_3 e_3 - o_4 e_4 - o_5 e_5 + o_6 e_6 - o_7 e_7 \\
\mathbf{o}^{e_7} &= (e_7 \mathbf{o})e_7^{-1} = e_7(\mathbf{o}e_7^{-1}) = -(e_7 \mathbf{o})e_7 = -e_7(\mathbf{o}e_7) \\
&= o_0 - o_1 e_1 - o_2 e_2 - o_3 e_3 - o_4 e_4 - o_5 e_5 - o_6 e_6 + o_7 e_7
\end{aligned} \tag{36}$$

To express each individual component $o_n e_n$ $0 \leq n \leq 7$ in the original octonion variable \mathbf{o} as a linear combination of transformed octonion variables $\mathbf{o}^{e_0} = \mathbf{o}, \mathbf{o}^{e_1}, \mathbf{o}^{e_2}, \mathbf{o}^{e_3}, \mathbf{o}^{e_4}, \mathbf{o}^{e_5}, \mathbf{o}^{e_6}, \mathbf{o}^{e_7}$ with real-valued coefficients, we have:

$$o_n e_n = c_{n0} \mathbf{o}^{e_0} + c_{n1} \mathbf{o}^{e_1} + c_{n2} \mathbf{o}^{e_2} + c_{n3} \mathbf{o}^{e_3} + c_{n4} \mathbf{o}^{e_4} + c_{n5} \mathbf{o}^{e_5} + c_{n6} \mathbf{o}^{e_6} + c_{n7} \mathbf{o}^{e_7} \quad 0 \leq n \leq 7 \quad (37).$$

Re-writing this in matrix form with real-valued elements, where $\vec{\mathbf{o}}^{e_m}$ $0 \leq m \leq 7$ is a column vector including the eight real-valued components of the corresponding octonion variable \mathbf{o}^{e_m} $0 \leq m \leq 7$:

$$\begin{bmatrix} 0 \\ 0 \\ o_n \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_{n0} \vec{\mathbf{o}}^{e_0} + c_{n1} \vec{\mathbf{o}}^{e_1} + c_{n2} \vec{\mathbf{o}}^{e_2} + c_{n3} \vec{\mathbf{o}}^{e_3} + c_{n4} \vec{\mathbf{o}}^{e_4} + c_{n5} \vec{\mathbf{o}}^{e_5} + c_{n6} \vec{\mathbf{o}}^{e_6} + c_{n7} \vec{\mathbf{o}}^{e_7} = \begin{bmatrix} \vec{\mathbf{o}}^{e_0} \vec{\mathbf{o}}^{e_1} \vec{\mathbf{o}}^{e_2} \vec{\mathbf{o}}^{e_3} \vec{\mathbf{o}}^{e_4} \vec{\mathbf{o}}^{e_5} \vec{\mathbf{o}}^{e_6} \vec{\mathbf{o}}^{e_7} \end{bmatrix} \begin{bmatrix} c_{n0} \\ c_{n1} \\ c_{n2} \\ c_{n3} \\ c_{n4} \\ c_{n5} \\ c_{n6} \\ c_{n7} \end{bmatrix} \quad (38).$$

$$\vec{\mathbf{o}}_n = \vec{\vec{\mathbf{O}}} \vec{\mathbf{c}}_n \quad 0 \leq n \leq 7$$

Since each component in \mathbf{o}^{e_m} $0 \leq m \leq 7$ (i.e., coefficient for each basis component e_0 to e_7) involves only the corresponding coefficient o_n in \mathbf{o} , each row in the above matrix can be normalized by the coefficient o_n , to have a formulation which is independent of the coefficients in \mathbf{o} :

$$\begin{bmatrix} 0 \\ 0 \\ 1 \text{ (row } n) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_{n0} \\ c_{n1} \\ c_{n2} \\ c_{n3} \\ c_{n4} \\ c_{n5} \\ c_{n6} \\ c_{n7} \end{bmatrix} \quad \vec{\mathbf{o}}'_n = \vec{\vec{\mathbf{O}}}' \vec{\mathbf{c}}_n \quad 0 \leq n \leq 7 \quad (39)$$

leading to:

$$\vec{\mathbf{c}}_n = \vec{\vec{\mathbf{O}}}'^{-1} \vec{\mathbf{o}}'_n \quad 0 \leq n \leq 7 \quad (40)$$

$$\begin{bmatrix} c_{n0} \\ c_{n1} \\ c_{n2} \\ c_{n3} \\ c_{n4} \\ c_{n5} \\ c_{n6} \\ c_{n7} \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 5 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 5 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 5 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 5 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 5 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \text{ (row } n) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, each $\bar{\mathbf{c}}_n$ $0 \leq n \leq 7$ has the values from the corresponding column in $\bar{\mathbf{O}}'^{-1}$, leading to:

$$o_n = e_n^{-1} (c_{n0} \mathbf{o}^{e_0} + c_{n1} \mathbf{o}^{e_1} + c_{n2} \mathbf{o}^{e_2} + c_{n3} \mathbf{o}^{e_3} + c_{n4} \mathbf{o}^{e_4} + c_{n5} \mathbf{o}^{e_5} + c_{n6} \mathbf{o}^{e_6} + c_{n7} \mathbf{o}^{e_7}) \quad (41)$$

with $e_n^{-1} = -e_n$ $1 \leq n \leq 7$, $e_n^{-1} = e_n$ $n = 0$, and:

$$o_0 = \frac{1}{12} (5\mathbf{o}^{e_0} + \mathbf{o}^{e_1} + \mathbf{o}^{e_2} + \mathbf{o}^{e_3} + \mathbf{o}^{e_4} + \mathbf{o}^{e_5} + \mathbf{o}^{e_6} + \mathbf{o}^{e_7}) \quad (42)$$

$$o_1 = \frac{-e_1}{12} (\mathbf{o}^{e_0} + 5\mathbf{o}^{e_1} - \mathbf{o}^{e_2} - \mathbf{o}^{e_3} - \mathbf{o}^{e_4} - \mathbf{o}^{e_5} - \mathbf{o}^{e_6} - \mathbf{o}^{e_7})$$

$$o_2 = \frac{-e_2}{12} (\mathbf{o}^{e_0} - \mathbf{o}^{e_1} + 5\mathbf{o}^{e_2} - \mathbf{o}^{e_3} - \mathbf{o}^{e_4} - \mathbf{o}^{e_5} - \mathbf{o}^{e_6} - \mathbf{o}^{e_7})$$

$$o_3 = \frac{-e_3}{12} (\mathbf{o}^{e_0} - \mathbf{o}^{e_1} - \mathbf{o}^{e_2} + 5\mathbf{o}^{e_3} - \mathbf{o}^{e_4} - \mathbf{o}^{e_5} - \mathbf{o}^{e_6} - \mathbf{o}^{e_7})$$

$$o_4 = \frac{-e_4}{12} (\mathbf{o}^{e_0} - \mathbf{o}^{e_1} - \mathbf{o}^{e_2} - \mathbf{o}^{e_3} + 5\mathbf{o}^{e_4} - \mathbf{o}^{e_5} - \mathbf{o}^{e_6} - \mathbf{o}^{e_7})$$

$$o_5 = \frac{-e_5}{12} (\mathbf{o}^{e_0} - \mathbf{o}^{e_1} - \mathbf{o}^{e_2} - \mathbf{o}^{e_3} - \mathbf{o}^{e_4} + 5\mathbf{o}^{e_5} - \mathbf{o}^{e_6} - \mathbf{o}^{e_7})$$

$$o_6 = \frac{-e_6}{12} (\mathbf{o}^{e_0} - \mathbf{o}^{e_1} - \mathbf{o}^{e_2} - \mathbf{o}^{e_3} - \mathbf{o}^{e_4} - \mathbf{o}^{e_5} + 5\mathbf{o}^{e_6} - \mathbf{o}^{e_7})$$

$$o_7 = \frac{-e_7}{12} (\mathbf{o}^{e_0} - \mathbf{o}^{e_1} - \mathbf{o}^{e_2} - \mathbf{o}^{e_3} - \mathbf{o}^{e_4} - \mathbf{o}^{e_5} - \mathbf{o}^{e_6} + 5\mathbf{o}^{e_7})$$

It should be noted that the results in (42) are octonion variables where only the 1st component is non-zero, therefore they are real-valued scalars.

If we wish to express $o_n e_n$ $0 \leq n \leq 7$ in terms of conjugates $(\mathbf{o}^{e_m})^*$ $0 \leq m \leq 7$, then a conjugation operation can be applied to (42):

$$o_0 = \frac{1}{12} (5(\mathbf{o}^{e_0})^* + (\mathbf{o}^{e_1})^* + (\mathbf{o}^{e_2})^* + (\mathbf{o}^{e_3})^* + (\mathbf{o}^{e_4})^* + (\mathbf{o}^{e_5})^* + (\mathbf{o}^{e_6})^* + (\mathbf{o}^{e_7})^*) \quad (43)$$

$$o_1 = \frac{e_1}{12} ((\mathbf{o}^{e_0})^* + 5(\mathbf{o}^{e_1})^* - (\mathbf{o}^{e_2})^* - (\mathbf{o}^{e_3})^* - (\mathbf{o}^{e_4})^* - (\mathbf{o}^{e_5})^* - (\mathbf{o}^{e_6})^* - (\mathbf{o}^{e_7})^*)$$

$$o_2 = \frac{e_2}{12} ((\mathbf{o}^{e_0})^* - (\mathbf{o}^{e_1})^* + 5(\mathbf{o}^{e_2})^* - (\mathbf{o}^{e_3})^* - (\mathbf{o}^{e_4})^* - (\mathbf{o}^{e_5})^* - (\mathbf{o}^{e_6})^* - (\mathbf{o}^{e_7})^*)$$

$$o_3 = \frac{e_3}{12} ((\mathbf{o}^{e_0})^* - (\mathbf{o}^{e_1})^* - (\mathbf{o}^{e_2})^* + 5(\mathbf{o}^{e_3})^* - (\mathbf{o}^{e_4})^* - (\mathbf{o}^{e_5})^* - (\mathbf{o}^{e_6})^* - (\mathbf{o}^{e_7})^*)$$

$$o_4 = \frac{e_4}{12} ((\mathbf{o}^{e_0})^* - (\mathbf{o}^{e_1})^* - (\mathbf{o}^{e_2})^* - (\mathbf{o}^{e_3})^* + 5(\mathbf{o}^{e_4})^* - (\mathbf{o}^{e_5})^* - (\mathbf{o}^{e_6})^* - (\mathbf{o}^{e_7})^*)$$

$$o_5 = \frac{e_5}{12} ((\mathbf{o}^{e_0})^* - (\mathbf{o}^{e_1})^* - (\mathbf{o}^{e_2})^* - (\mathbf{o}^{e_3})^* - (\mathbf{o}^{e_4})^* + 5(\mathbf{o}^{e_5})^* - (\mathbf{o}^{e_6})^* - (\mathbf{o}^{e_7})^*)$$

$$o_6 = \frac{e_6}{12} ((\mathbf{o}^{e_0})^* - (\mathbf{o}^{e_1})^* - (\mathbf{o}^{e_2})^* - (\mathbf{o}^{e_3})^* - (\mathbf{o}^{e_4})^* - (\mathbf{o}^{e_5})^* + 5(\mathbf{o}^{e_6})^* - (\mathbf{o}^{e_7})^*)$$

$$o_7 = \frac{e_7}{12} ((\mathbf{o}^{e_0})^* - (\mathbf{o}^{e_1})^* - (\mathbf{o}^{e_2})^* - (\mathbf{o}^{e_3})^* - (\mathbf{o}^{e_4})^* - (\mathbf{o}^{e_5})^* - (\mathbf{o}^{e_6})^* + 5(\mathbf{o}^{e_7})^*)$$

The total differential of a function $f(\mathbf{o}) = f(o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ $o_n \in \mathbb{R}$ $0 \leq n \leq 7$ can be expressed as:

$$\begin{aligned} df(\mathbf{o}) &= \frac{\partial f(\mathbf{o})}{\partial o_0} do_0 + \frac{\partial f(\mathbf{o})}{\partial o_1} do_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} do_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} do_3 \\ &+ \frac{\partial f(\mathbf{o})}{\partial o_4} do_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} do_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} do_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} do_7 \end{aligned} \quad (44)$$

or

$$\begin{aligned}
df(\mathbf{o}) &= do_0 \frac{\partial f(\mathbf{o})}{\partial o_0} + do_1 \frac{\partial f(\mathbf{o})}{\partial o_1} + do_2 \frac{\partial f(\mathbf{o})}{\partial o_2} + do_3 \frac{\partial f(\mathbf{o})}{\partial o_3} \\
&+ do_4 \frac{\partial f(\mathbf{o})}{\partial o_4} + do_5 \frac{\partial f(\mathbf{o})}{\partial o_5} + do_6 \frac{\partial f(\mathbf{o})}{\partial o_6} + do_7 \frac{\partial f(\mathbf{o})}{\partial o_7}
\end{aligned} \tag{45}$$

$(do_n \in \mathbb{R})$

(44) will lead to left derivatives while (45) will lead to right derivatives, which may differ since octonions are non-commutative.

The differentials $do_0, do_1, do_2, do_3, do_4, do_5, do_6, do_7$ are obtained from (42) or (43). To obtain the partial derivatives, we substitute $do_0, do_1, do_2, do_3, do_4, do_5, do_6, do_7$ into (44) or (45). There are therefore four different possibilities, depending on partial derivatives over the variables $(\mathbf{o}^{e_0} = \mathbf{o}), \mathbf{o}^{e_1}, \mathbf{o}^{e_2}, \mathbf{o}^{e_3}, \mathbf{o}^{e_4}, \mathbf{o}^{e_5}, \mathbf{o}^{e_6}, \mathbf{o}^{e_7}$ or $(\mathbf{o}^{e_0} = \mathbf{o})^*, (\mathbf{o}^{e_1})^*, (\mathbf{o}^{e_2})^*, (\mathbf{o}^{e_3})^*, (\mathbf{o}^{e_4})^*, (\mathbf{o}^{e_5})^*, (\mathbf{o}^{e_6})^*, (\mathbf{o}^{e_7})^*$, and left derivatives or right derivatives. Note that for the right derivatives in (45), norm-preserving involutions $(e_n o_n) e_n^{-1} = e_n (o_n e_n^{-1}) = -(e_n o_n) e_n = -e_n (o_n e_n)$ $1 \leq n \leq 7$ would first be applied to (42) or (43), to get equations where the e_n term appears at the end of the equation.

The formulation of left side partial derivatives over the variables $(\mathbf{o}^{e_0} = \mathbf{o}), \mathbf{o}^{e_1}, \mathbf{o}^{e_2}, \mathbf{o}^{e_3}, \mathbf{o}^{e_4}, \mathbf{o}^{e_5}, \mathbf{o}^{e_6}, \mathbf{o}^{e_7}$ is shown in detail in (46) and (47). Grouping together the terms with $do, do^{e_1}, do^{e_2}, do^{e_3}, do^{e_4}, do^{e_5}, do^{e_6}, do^{e_7}$ and defining $\frac{\partial}{\partial \mathbf{o}^{e_m}}$ $0 \leq m \leq 7$ as the factor multiplying do^{e_m} $0 \leq m \leq 7$, we obtain the definition of the partial derivatives $\frac{\partial}{\partial \mathbf{o}^{e_m}}$ $0 \leq m \leq 7$. However, this last step and the results that it provides have some significant shortcomings:

- The last step in the development is invalid because of the lack of octonion associativity in general: $\frac{\partial f(\mathbf{o})}{\partial o_n} (e_n do^{e_m}) \neq \left(\frac{\partial f(\mathbf{o})}{\partial o_n} e_n \right) do^{e_m}$. However, for reasons mentioned in the introduction, this development is shown nevertheless.
- The form of the partial derivatives over octonion variables obtained is not the intuitive form expected from a generalization of $\mathbb{C}\mathbb{R}$ calculus and $\mathbb{H}\mathbb{R}$ calculus. This is a first motivation for the use of transformations with Hadamard matrices in later sections of this paper, which will not suffer from this issue.

$$\begin{aligned}
df(\mathbf{o}) = & \frac{\partial f(\mathbf{o})}{\partial o_0} \left(\frac{1}{12} (5d\mathbf{o} + d\mathbf{o}^{e_1} + d\mathbf{o}^{e_2} + d\mathbf{o}^{e_3} + d\mathbf{o}^{e_4} + d\mathbf{o}^{e_5} + d\mathbf{o}^{e_6} + d\mathbf{o}^{e_7}) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_1} \left(\frac{-e_1}{12} (d\mathbf{o} + 5d\mathbf{o}^{e_1} - d\mathbf{o}^{e_2} - d\mathbf{o}^{e_3} - d\mathbf{o}^{e_4} - d\mathbf{o}^{e_5} - d\mathbf{o}^{e_6} - d\mathbf{o}^{e_7}) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_2} \left(\frac{-e_2}{12} (d\mathbf{o} - d\mathbf{o}^{e_1} + 5d\mathbf{o}^{e_2} - d\mathbf{o}^{e_3} - d\mathbf{o}^{e_4} - d\mathbf{o}^{e_5} - d\mathbf{o}^{e_6} - d\mathbf{o}^{e_7}) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_3} \left(\frac{-e_3}{12} (d\mathbf{o} - d\mathbf{o}^{e_1} - d\mathbf{o}^{e_2} + 5d\mathbf{o}^{e_3} - d\mathbf{o}^{e_4} - d\mathbf{o}^{e_5} - d\mathbf{o}^{e_6} - d\mathbf{o}^{e_7}) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_4} \left(\frac{-e_4}{12} (d\mathbf{o} - d\mathbf{o}^{e_1} - d\mathbf{o}^{e_2} - d\mathbf{o}^{e_3} + 5d\mathbf{o}^{e_4} - d\mathbf{o}^{e_5} - d\mathbf{o}^{e_6} - d\mathbf{o}^{e_7}) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_5} \left(\frac{-e_5}{12} (d\mathbf{o} - d\mathbf{o}^{e_1} - d\mathbf{o}^{e_2} - d\mathbf{o}^{e_3} - d\mathbf{o}^{e_4} + 5d\mathbf{o}^{e_5} - d\mathbf{o}^{e_6} - d\mathbf{o}^{e_7}) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_6} \left(\frac{-e_6}{12} (d\mathbf{o} - d\mathbf{o}^{e_1} - d\mathbf{o}^{e_2} - d\mathbf{o}^{e_3} - d\mathbf{o}^{e_4} - d\mathbf{o}^{e_5} + 5d\mathbf{o}^{e_6} - d\mathbf{o}^{e_7}) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_7} \left(\frac{-e_7}{12} (d\mathbf{o} - d\mathbf{o}^{e_1} - d\mathbf{o}^{e_2} - d\mathbf{o}^{e_3} - d\mathbf{o}^{e_4} - d\mathbf{o}^{e_5} - d\mathbf{o}^{e_6} + 5d\mathbf{o}^{e_7}) \right)
\end{aligned} \tag{46}$$

$$\begin{aligned}
df(\mathbf{o}) &= \frac{1}{12} \left(5 \frac{\partial f(\mathbf{o})}{\partial o_0} - \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 - \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o} \\
&+ \frac{1}{12} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} - 5 \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^{e_1} \\
&+ \frac{1}{12} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - 5 \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^{e_2} \\
&+ \frac{1}{12} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - 5 \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^{e_3} \\
&+ \frac{1}{12} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - 5 \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^{e_4} \\
&+ \frac{1}{12} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - 5 \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^{e_5} \\
&+ \frac{1}{12} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - 5 \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^{e_6} \\
&+ \frac{1}{12} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 - 5 \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^{e_7} \\
&\equiv \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}} d\mathbf{o} + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^{e_1}} d\mathbf{o}^{e_1} + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^{e_2}} d\mathbf{o}^{e_2} + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^{e_3}} d\mathbf{o}^{e_3} + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^{e_4}} d\mathbf{o}^{e_4} + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^{e_5}} d\mathbf{o}^{e_5} + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^{e_6}} d\mathbf{o}^{e_6} + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^{e_7}} d\mathbf{o}^{e_7} \quad (47).
\end{aligned}$$

From the above, we note in particular the result for the left partial derivative over octonion variable \mathbf{o} :

$$\frac{\partial f(\mathbf{o})}{\partial \mathbf{o}} = \frac{1}{12} \left(5 \frac{\partial f(\mathbf{o})}{\partial o_0} - \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 - \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) \quad (48).$$

This is not an intuitive result with the form expected from an 8-dimensional extension of 2-dimensional $\mathbb{C}\mathbb{R}$ calculus and 4-dimensional $\mathbb{H}\mathbb{R}$ calculus, where there would be no factor 5 and the factor $1/12$ would be replaced by $1/8$. This is a first indication that the approach of performing rotations with form $\mathbf{o}^{e_m} = -(e_m \mathbf{o})e_m = -e_m(\mathbf{o}e_m)$ over basis component is not appropriate for the development of a calculus for octonions or 8-dimensional algebras. A second and more severe indication will be provided in the next section for the split biquaternion algebra. After this, the subsequent sections will present the development of calculus using transformations from Hadamard matrices, which will be applicable to

$\mathbb{C}\mathbb{R}$ calculus, $\mathbb{H}\mathbb{R}$ calculus and calculus for octonions and octonion-like split-biquaternions, all leading the same intuitive form for the partial derivatives over complex or hypercomplex variables.

IV. CALCULUS FOR OCTONION-LIKE SPLIT-BIQUATERNIONS USING SIMPLIFIED ROTATIONS

The octonion-like split-biquaternion algebra is a non-commutative but associative algebra. It was illustrated in [11] that the split-biquaternion algebra is a seminormed composition algebra over \mathbb{R} (i.e., preservation of two seminorms under multiplication) and it is also a seminormed division algebra over \mathbb{R} (i.e., division by any number is possible as long as the two seminorms of this number are non-zero). The split-biquaternion algebra has components e_i $0 \leq i \leq 7$ as an orthogonal basis, with $e_0^2 = 1$, $e_7^2 = 1$, $e_i^2 = -1$ $1 \leq i \leq 6$ and the multiplication rules of Table III.

Table III Multiplication rules in the octonion-like split-biquaternion algebra

$e_i e_j$		e_j							
		$e_0 = 1$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_i	$e_0 = 1$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_1	e_1	-1	e_3	$-e_2$	$-e_5$	e_4	e_7	$-e_6$
	e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
	e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
	e_4	e_4	e_5	$-e_6$	e_7	-1	$-e_1$	e_2	$-e_3$
	e_5	e_5	$-e_4$	e_7	e_6	e_1	-1	$-e_3$	$-e_2$
	e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
	e_7	e_7	$-e_6$	$-e_5$	$-e_4$	$-e_3$	$-e_2$	$-e_1$	1

Note that the same notation \mathbf{o} is used in this section to represent numbers or variables in the octonion-like split-biquaternion algebra than for numbers or variables in the octonion algebra of the previous section:

$$\mathbf{o} = o_0 e_0 + o_1 e_1 + o_2 e_2 + o_3 e_3 + o_4 e_4 + o_5 e_5 + o_6 e_6 + o_7 e_7 \quad (49)$$

$$\mathbf{o}^* = o_0 e_0 - o_1 e_1 - o_2 e_2 - o_3 e_3 - o_4 e_4 - o_5 e_5 - o_6 e_6 + o_7 e_7 \quad (50)$$

$$(\mathbf{o}_x \mathbf{o}_y)^* = \mathbf{o}_y^* \mathbf{o}_x^* \quad (51)$$

$$\|\mathbf{o}\|^2 = \sum_{i=0}^7 o_i^2 - 2\lambda \sum_{i=1}^3 o_i o_{7-i} + 2\lambda o_0 o_7 \quad (52)$$

(52) leads to two possible seminorms for the octonion-like split-biquaternion algebra, for $\lambda = 1$ or $\lambda = -1$. Unlike the octonions of the previous section, numbers in the split-biquaternion algebra are associative, but in general they are not commutative:

$$\mathbf{o}_x (\mathbf{o}_y \mathbf{o}_z) = (\mathbf{o}_x \mathbf{o}_y) \mathbf{o}_z \quad \mathbf{o}_x \mathbf{o}_y \neq \mathbf{o}_y \mathbf{o}_x \quad (53).$$

However, when the product $\mathbf{o}_x \mathbf{o}_y$ is non-zero only for non-imaginary or scalar components

e_0, e_7 (such as in $\mathbf{o}_x \mathbf{o}_x^* = \left(\sum_{i=0}^7 o_i^2 \right) e_0 + 2 \left(-\sum_{i=1}^3 o_i o_{7-i} + o_0 o_7 \right) e_7$) the product becomes commutative:

$$(\mathbf{o}_x \mathbf{o}_x^*) \mathbf{o}_y = \mathbf{o}_y (\mathbf{o}_x \mathbf{o}_x^*) \quad (54).$$

This section will show the severe limitation of attempting to use the same steps as in the development of HIR calculus and generalized HIR calculus, with rotations of split-biquaternion variables over the basis components of the split-biquaternion algebra, to produce transformed split-biquaternion variables required as part of the procedure to formulate the partial derivatives over split-biquaternion variables.

The rotations of a split-biquaternion variable over the eight orthogonal basis components of the split-biquaternion algebra lead to the following expressions:

$$\begin{aligned} \mathbf{o}^{e_0} &= e_0 \mathbf{o} e_0^{-1} = 1 \times \mathbf{o} \times 1^{-1} = \mathbf{o} \\ &= o_0 + o_1 e_1 + o_2 e_2 + o_3 e_3 + o_4 e_4 + o_5 e_5 + o_6 e_6 + o_7 e_7 \\ \mathbf{o}^{e_1} &= e_1 \mathbf{o} e_1^{-1} = -e_1 \mathbf{o} e_1 \\ &= o_0 + o_1 e_1 - o_2 e_2 - o_3 e_3 - o_4 e_4 - o_5 e_5 + o_6 e_6 + o_7 e_7 \end{aligned} \quad (55)$$

$$\begin{aligned}\mathbf{o}^{e_2} &= e_2 \mathbf{o} e_2^{-1} = -e_2 \mathbf{o} e_2 \\ &= o_0 - o_1 e_1 + o_2 e_2 - o_3 e_3 - o_4 e_4 + o_5 e_5 - o_6 e_6 + o_7 e_7\end{aligned}$$

$$\begin{aligned}\mathbf{o}^{e_3} &= e_3 \mathbf{o} e_3^{-1} = -e_3 \mathbf{o} e_3 \\ &= o_0 - o_1 e_1 - o_2 e_2 + o_3 e_3 + o_4 e_4 - o_5 e_5 - o_6 e_6 + o_7 e_7\end{aligned}$$

$$\mathbf{o}^{e_4} = e_4 \mathbf{o} e_4^{-1} = -e_4 \mathbf{o} e_4 = \mathbf{o}^{e_3}$$

$$\mathbf{o}^{e_5} = e_5 \mathbf{o} e_5^{-1} = -e_5 \mathbf{o} e_5 = \mathbf{o}^{e_2}$$

$$\mathbf{o}^{e_6} = e_6 \mathbf{o} e_6^{-1} = -e_6 \mathbf{o} e_6 = \mathbf{o}^{e_1}$$

$$\mathbf{o}^{e_7} = e_7 \mathbf{o} e_7^{-1} = e_7 \mathbf{o} e_7 = \mathbf{o}^{e_0} = \mathbf{o}$$

Since $\mathbf{o}^{e_m} = \mathbf{o}^{e_{m-i}} \quad 0 \leq m \leq 3$, only four different transformed variables are generated by the eight involutions. This directly prevents to use of (37)-(42) to express the components $o_n e_n \quad 0 \leq n \leq 7$ of a split-biquaternion variable \mathbf{o} as a linear combination of transformed variables $\mathbf{o}^{e_0} = \mathbf{o}, \mathbf{o}^{e_1}, \mathbf{o}^{e_2}, \mathbf{o}^{e_3}, \mathbf{o}^{e_4}, \mathbf{o}^{e_5}, \mathbf{o}^{e_6}, \mathbf{o}^{e_7}$, because the resulting matrix $\overline{\mathbf{O}}$ in (38) has only rank four.

Therefore, the procedure of using simplified rotations over the basis components of an algebra to produced transformed variables required in the formulation of partial derivatives does not generalize from the 4-dimensional $\mathbb{H}\mathbb{R}$ calculus to the 8-dimensional octonion-like split-biquaternion algebra, and an alternative procedure needs to be developed to formulate the partial derivatives over split-biquaternion variables. This will be discussed in later sections.

V. CALCULUS FOR OCTONIONS USING HADAMARD MATRIX TRANSFORMATION

For calculus in the octonion algebra with the formulation of partial derivatives over octonion variables, we revisit the procedure of (37)-(42) for expressing the different components of an octonion variable $\mathbf{o} \quad (o_n e_n \quad 0 \leq n \leq 7)$ as a function a linear combination of \mathbf{o} and seven transformed variables $\mathbf{o}^m \quad 1 \leq m \leq 7$, with real-valued coefficients. Because \mathbf{o}^m will no longer be produced by rotations, the notation \mathbf{o}^m is used in this section and the following section, where m is simply a superscript index (and not a power), instead of the notation

$\mathbf{o}^{e_m} = e_m \mathbf{o} e_m^{-1}$ previously used for rotations. The original variable \mathbf{o} and the transformed variables \mathbf{o}^m $1 \leq m \leq 7$ are the octonion variables over which the partial derivatives are to be formulated. We have:

$$o_n e_n = c_{n0} \mathbf{o} + c_{n1} \mathbf{o}^1 + c_{n2} \mathbf{o}^2 + c_{n3} \mathbf{o}^3 + c_{n4} \mathbf{o}^4 + c_{n5} \mathbf{o}^5 + c_{n6} \mathbf{o}^6 + c_{n7} \mathbf{o}^7 \quad 0 \leq n \leq 7 \quad (56),$$

where the coefficients c_{nm} are real-valued.

Re-writing this in matrix form with real-valued elements, where $\vec{\mathbf{o}}^m$ $0 \leq m \leq 7$ is a column vector including the eight real-valued components of the corresponding octonion variable \mathbf{o}^m $0 \leq m \leq 7$:

$$\begin{bmatrix} 0 \\ 0 \\ o_n \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{matrix} c_{n0} \vec{\mathbf{o}} + c_{n1} \vec{\mathbf{o}}^1 + c_{n2} \vec{\mathbf{o}}^2 + c_{n3} \vec{\mathbf{o}}^3 \\ + c_{n4} \vec{\mathbf{o}}^4 + c_{n5} \vec{\mathbf{o}}^5 + c_{n6} \vec{\mathbf{o}}^6 + c_{n7} \vec{\mathbf{o}}^7 \end{matrix} = \begin{bmatrix} \vec{\mathbf{o}} \vec{\mathbf{o}}^1 \vec{\mathbf{o}}^2 \vec{\mathbf{o}}^3 \vec{\mathbf{o}}^4 \vec{\mathbf{o}}^5 \vec{\mathbf{o}}^6 \vec{\mathbf{o}}^7 \end{bmatrix} \begin{bmatrix} c_{n0} \\ c_{n1} \\ c_{n2} \\ c_{n3} \\ c_{n4} \\ c_{n5} \\ c_{n6} \\ c_{n7} \end{bmatrix} \quad (57)$$

$$\vec{\mathbf{o}}_n = \vec{\vec{\mathbf{O}}} \vec{\mathbf{c}}_n \quad 0 \leq n \leq 7$$

For a simple formulation of the partial derivatives, in the procedure to be developed (and as previously) it is important that in the transformed variables \mathbf{o}^m $1 \leq m \leq 7$ each component o_n^m $0 \leq n \leq 7$ (for basis e_0 to e_7) is only a function of the corresponding component o_n $0 \leq n \leq 7$ in \mathbf{o} . Then each row in matrix $\vec{\vec{\mathbf{O}}}$ in (57) can be normalized to $\vec{\vec{\mathbf{O}}}'$ in (58), with a formulation which is independent of the coefficients values in \mathbf{o} :

$$\begin{bmatrix} 0 \\ 0 \\ 1 \text{ (row } n) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{\mathbf{o}}'_n = \vec{\vec{\mathbf{O}}}' \vec{\mathbf{c}}_n \quad 0 \leq n \leq 7 \quad (58)$$

$$\vec{\mathbf{c}}_n = \vec{\vec{\mathbf{O}}}'^{-1} \vec{\mathbf{o}}'_n = \vec{\vec{\mathbf{O}}}'^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \text{ (row } n) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad 0 \leq n \leq 7 \quad (59)$$

Therefore, each $\vec{\mathbf{c}}_n$ $0 \leq n \leq 7$ will have the values from the corresponding column in $\vec{\vec{\mathbf{O}}}'^{-1}$. Once these coefficients $\vec{\mathbf{c}}_n$ are known, i.e., with proper choice of $\vec{\vec{\mathbf{O}}}'$ as discussed below, we will have:

$$o_n = e_n^{-1} (c_{n0} \mathbf{o} + c_{n1} \mathbf{o}^1 + c_{n2} \mathbf{o}^2 + c_{n3} \mathbf{o}^3 + c_{n4} \mathbf{o}^4 + c_{n5} \mathbf{o}^5 + c_{n6} \mathbf{o}^6 + c_{n7} \mathbf{o}^7) \quad (60).$$

$$\text{with } e_n^{-1} = -e_n \quad 1 \leq n \leq 7, \quad e_n^{-1} = e_n \quad n = 0$$

$\vec{\vec{\mathbf{O}}}'$ must be a full rank matrix with a first column of unit values (from the use of $\vec{\mathbf{o}}$ in (56) and the normalization used in $\vec{\vec{\mathbf{O}}}'$). Also, if conjugation-like operations are used (i.e., where only the sign of some coefficients o_n in $\vec{\mathbf{o}}$ can be changed) to produce the transformed variables $\vec{\mathbf{o}}^m$ $1 \leq m \leq 7$ (and columns of $\vec{\vec{\mathbf{O}}}'$), the absolute value of the resulting components in $\vec{\mathbf{o}}^m$ $1 \leq m \leq 7$ is the same as originally in $\vec{\mathbf{o}}$, and the Euclidian norms of all the transformed $\vec{\mathbf{o}}^m$ $1 \leq m \leq 7$ remain the same as the Euclidian norm of $\vec{\mathbf{o}}$, which is a desirable feature numerically (no scaling issue). In terms of the matrix $\vec{\vec{\mathbf{O}}}'$, this means having all real-valued elements having an absolute unit value. Another important feature is for the inverse

matrix $\vec{\vec{\mathbf{O}}}^{-1}$ to also have all elements having the same absolute value, because for the synthesis of each coefficient o_n in (60), the real-valued coefficients c_{nm} to be used in the linear combination are directly obtained from columns of $\vec{\vec{\mathbf{O}}}^{-1}$, so having constant absolute values for elements of $\vec{\vec{\mathbf{O}}}^{-1}$ leads to giving the same weight to all the terms $\vec{\mathbf{o}}$ and $\vec{\mathbf{o}}^m$ $1 \leq m \leq 7$ linearly combined in (60) (also a desirable feature). Considering those constraints or desirable features, the orthogonal and symmetric real-valued Hadamard matrix provides a solution for the required transformation matrix $\vec{\vec{\mathbf{O}}}'$, defining the conjugation-like operations to produce the transformed variables \mathbf{o}^m $1 \leq m \leq 7$ for which (together with variable \mathbf{o}) partial derivatives formulations are to be developed. Except for permutations not affecting the final formulations, the transformation matrix $\vec{\vec{\mathbf{O}}}'$ and the resulting partial derivatives become unique if the first row of the matrix $\vec{\vec{\mathbf{O}}}'$ is also required to have unit values, as in standard Hadamard matrices formulations. The inverse Hadamard matrix $\vec{\vec{\mathbf{O}}}^{-1} = \vec{\vec{\mathbf{O}}}' / 8$ will have elements all having the same absolute value.

It should be noted that in the case of $\mathbb{H}\mathbb{R}$ calculus for quaternions, the corresponding operation producing the \mathbf{q}^m $m = i, j, k$ transformed variables can also be seen as a conjugation-like transformation (changing the sign of some real-valued coefficients in \mathbf{q}), and the $\vec{\vec{\mathbf{Q}}}$ matrix in (11) has exactly the form of a 4-dimensional Hadamard matrix. The corresponding matrix in the case of Wirtinger/ $\mathbb{C}\mathbb{R}$ calculus for complex numbers is also a size-2 Hadamard matrix, as will be illustrated in a later section.

With a size-8 Hadamard matrix used as $\vec{\vec{\mathbf{O}}}'$, (58) and (59) become:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \text{ (row } n) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_{n0} \\ c_{n1} \\ c_{n2} \\ c_{n3} \\ c_{n4} \\ c_{n5} \\ c_{n6} \\ c_{n7} \end{bmatrix} \quad (61)$$

From the last seven columns of the matrix in (61) and using (57),(58), the conjugation-like operations (i.e., changing the sign of some real-valued coefficients in \mathbf{o}) producing the \mathbf{o}^m $1 \leq m \leq 7$ transformed variables are defined. Expressions for partial derivatives over octonion variable \mathbf{o} and these resulting transformed octonion variables \mathbf{o}^m $1 \leq m \leq 7$ will be developed. The c_{nm} coefficients are then obtained by:

$$\begin{bmatrix} c_{n0} \\ c_{n1} \\ c_{n2} \\ c_{n3} \\ c_{n4} \\ c_{n5} \\ c_{n6} \\ c_{n7} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \text{ (row } n) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (62)$$

and (60) becomes:

$$o_0 = \frac{1}{8}(\mathbf{o} + \mathbf{o}^1 + \mathbf{o}^2 + \mathbf{o}^3 + \mathbf{o}^4 + \mathbf{o}^5 + \mathbf{o}^6 + \mathbf{o}^7) \quad (63)$$

$$o_1 = \frac{-e_1}{8}(\mathbf{o} - \mathbf{o}^1 + \mathbf{o}^2 - \mathbf{o}^3 + \mathbf{o}^4 - \mathbf{o}^5 + \mathbf{o}^6 - \mathbf{o}^7)$$

$$o_2 = \frac{-e_2}{8}(\mathbf{o} + \mathbf{o}^1 - \mathbf{o}^2 - \mathbf{o}^3 + \mathbf{o}^4 + \mathbf{o}^5 - \mathbf{o}^6 - \mathbf{o}^7)$$

$$o_3 = \frac{-e_3}{8}(\mathbf{o} - \mathbf{o}^1 - \mathbf{o}^2 + \mathbf{o}^3 + \mathbf{o}^4 - \mathbf{o}^5 - \mathbf{o}^6 + \mathbf{o}^7)$$

$$o_4 = \frac{-e_4}{8}(\mathbf{o} + \mathbf{o}^1 + \mathbf{o}^2 + \mathbf{o}^3 - \mathbf{o}^4 - \mathbf{o}^5 - \mathbf{o}^6 - \mathbf{o}^7)$$

$$o_5 = \frac{-e_5}{8}(\mathbf{o} - \mathbf{o}^1 + \mathbf{o}^2 - \mathbf{o}^3 - \mathbf{o}^4 + \mathbf{o}^5 - \mathbf{o}^6 + \mathbf{o}^7)$$

$$o_6 = \frac{-e_6}{8}(\mathbf{o} + \mathbf{o}^1 - \mathbf{o}^2 - \mathbf{o}^3 - \mathbf{o}^4 - \mathbf{o}^5 + \mathbf{o}^6 + \mathbf{o}^7)$$

$$o_7 = \frac{-e_7}{8}(\mathbf{o} - \mathbf{o}^1 - \mathbf{o}^2 + \mathbf{o}^3 - \mathbf{o}^4 + \mathbf{o}^5 + \mathbf{o}^6 - \mathbf{o}^7)$$

It should be noted that the results in (63) are octonion variables where only the 1st component is non-zero, therefore they are real-valued scalars.

If we wish to express $o_n e_n$ $0 \leq n \leq 7$ in terms of conjugates $(\mathbf{o}^m)^*$ $0 \leq m \leq 7$, then a conjugation operation can be applied to (63):

$$o_0 = \frac{1}{8}((\mathbf{o})^* + (\mathbf{o}^1)^* + (\mathbf{o}^2)^* + (\mathbf{o}^3)^* + (\mathbf{o}^4)^* + (\mathbf{o}^5)^* + (\mathbf{o}^6)^* + (\mathbf{o}^7)^*) \quad (64)$$

$$o_1 = \frac{e_1}{8}((\mathbf{o})^* - (\mathbf{o}^1)^* + (\mathbf{o}^2)^* - (\mathbf{o}^3)^* + (\mathbf{o}^4)^* - (\mathbf{o}^5)^* + (\mathbf{o}^6)^* - (\mathbf{o}^7)^*)$$

$$o_2 = \frac{e_2}{8}((\mathbf{o})^* + (\mathbf{o}^1)^* - (\mathbf{o}^2)^* - (\mathbf{o}^3)^* + (\mathbf{o}^4)^* + (\mathbf{o}^5)^* - (\mathbf{o}^6)^* - (\mathbf{o}^7)^*)$$

$$o_3 = \frac{e_3}{8}((\mathbf{o})^* - (\mathbf{o}^1)^* - (\mathbf{o}^2)^* + (\mathbf{o}^3)^* + (\mathbf{o}^4)^* - (\mathbf{o}^5)^* - (\mathbf{o}^6)^* + (\mathbf{o}^7)^*)$$

$$o_4 = \frac{e_4}{8}((\mathbf{o})^* + (\mathbf{o}^1)^* + (\mathbf{o}^2)^* + (\mathbf{o}^3)^* - (\mathbf{o}^4)^* - (\mathbf{o}^5)^* - (\mathbf{o}^6)^* - (\mathbf{o}^7)^*)$$

$$o_5 = \frac{e_5}{8}((\mathbf{o})^* - (\mathbf{o}^1)^* + (\mathbf{o}^2)^* - (\mathbf{o}^3)^* - (\mathbf{o}^4)^* + (\mathbf{o}^5)^* - (\mathbf{o}^6)^* + (\mathbf{o}^7)^*)$$

$$o_6 = \frac{e_6}{8}((\mathbf{o})^* + (\mathbf{o}^1)^* - (\mathbf{o}^2)^* - (\mathbf{o}^3)^* - (\mathbf{o}^4)^* - (\mathbf{o}^5)^* + (\mathbf{o}^6)^* + (\mathbf{o}^7)^*)$$

$$o_7 = \frac{e_7}{8}((\mathbf{o})^* - (\mathbf{o}^1)^* - (\mathbf{o}^2)^* + (\mathbf{o}^3)^* - (\mathbf{o}^4)^* + (\mathbf{o}^5)^* + (\mathbf{o}^6)^* - (\mathbf{o}^7)^*)$$

The total differential of a function $f(\mathbf{o}) = f(o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ $o_n \in \mathbb{R}$ $0 \leq n \leq 7$ can be expressed as in (44) (left derivatives) or (45) (right derivatives), which may differ since octonions are non-commutative.

The differentials $do_0, do_1, do_2, do_3, do_4, do_5, do_6, do_7$ are obtained from (63) or (64). To obtain the partial derivatives, we substitute $do_0, do_1, do_2, do_3, do_4, do_5, do_6, do_7$ into (44) or (45). There are therefore four different possibilities, depending on partial derivatives over the variables $\mathbf{o}, \mathbf{o}^1, \mathbf{o}^2, \mathbf{o}^3, \mathbf{o}^4, \mathbf{o}^5, \mathbf{o}^6, \mathbf{o}^7$ or $(\mathbf{o})^*, (\mathbf{o}^1)^*, (\mathbf{o}^2)^*, (\mathbf{o}^3)^*, (\mathbf{o}^4)^*, (\mathbf{o}^5)^*, (\mathbf{o}^6)^*, (\mathbf{o}^7)^*$, and left derivatives or right derivatives. Note that for the right derivatives in (45), norm-preserving involutions

$(e_n o_n) e_n^{-1} = e_n (o_n e_n^{-1}) = -(e_n o_n) e_n = -e_n (o_n e_n) \quad 1 \leq n \leq 7$ would first be applied to (63) or (64), to get equations where the e_n term appears at the end of the equation.

The formulation of the left side partial derivative over the variables $\mathbf{o}, \mathbf{o}^1, \mathbf{o}^2, \mathbf{o}^3, \mathbf{o}^4, \mathbf{o}^5, \mathbf{o}^6, \mathbf{o}^7$ is shown in detail in (65) and (66). Grouping together the terms with $d\mathbf{o}, d\mathbf{o}^1, d\mathbf{o}^2, d\mathbf{o}^3, d\mathbf{o}^4, d\mathbf{o}^5, d\mathbf{o}^6, d\mathbf{o}^7$ and defining $\frac{\partial}{\partial \mathbf{o}^m} \quad 0 \leq m \leq 7$ as the factor multiplying

$d\mathbf{o}^m \quad 0 \leq m \leq 7$, we obtain the definition of the partial derivatives $\frac{\partial}{\partial \mathbf{o}^m} \quad 0 \leq m \leq 7$.

As previously in this paper for calculus in octonion algebra, this last step is invalid because of the lack of octonion associativity in general: $\frac{\partial f(\mathbf{o})}{\partial o_n} (e_n d\mathbf{o}^m) \neq \left(\frac{\partial f(\mathbf{o})}{\partial o_n} e_n \right) d\mathbf{o}^m$. However, as previously mentioned in the introduction, this development is shown nevertheless for two reasons:

- It provides some insights for the development of calculus for the associative octonion-like split-biquaternion algebra (next section), and allow some insightful comparisons with the development of $\mathbb{C}\mathbb{R}$ calculus and $\mathbb{H}\mathbb{R}$ calculus;
- It may later lead to some heuristic approximative algorithms, since despite the lack of associativity in octonion algebra the product $\mathbf{o}_x \mathbf{o}_y$ of two octonion numbers can be significantly correlated with the product $\mathbf{o}_y \mathbf{o}_x$ [8] and also $(\mathbf{o}_x \mathbf{o}_y)^* = \mathbf{o}_y^* \mathbf{o}_x^*$, i.e., some stochastic estimations of the derivatives may become possible.

$$\begin{aligned}
df(\mathbf{o}) = & \frac{\partial f(\mathbf{o})}{\partial o_0} \left(\frac{1}{8} (d\mathbf{o} + d\mathbf{o}^1 + d\mathbf{o}^2 + d\mathbf{o}^3 + d\mathbf{o}^4 + d\mathbf{o}^5 + d\mathbf{o}^6 + d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_1} \left(\frac{-e_1}{8} (d\mathbf{o} - d\mathbf{o}^1 + d\mathbf{o}^2 - d\mathbf{o}^3 + d\mathbf{o}^4 - d\mathbf{o}^5 + d\mathbf{o}^6 - d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_2} \left(\frac{-e_2}{8} (d\mathbf{o} + d\mathbf{o}^1 - d\mathbf{o}^2 - d\mathbf{o}^3 + d\mathbf{o}^4 + d\mathbf{o}^5 - d\mathbf{o}^6 - d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_3} \left(\frac{-e_3}{8} (d\mathbf{o} - d\mathbf{o}^1 - d\mathbf{o}^2 + d\mathbf{o}^3 + d\mathbf{o}^4 - d\mathbf{o}^5 - d\mathbf{o}^6 + d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_4} \left(\frac{-e_4}{8} (d\mathbf{o} + d\mathbf{o}^1 + d\mathbf{o}^2 + d\mathbf{o}^3 - d\mathbf{o}^4 - d\mathbf{o}^5 - d\mathbf{o}^6 - d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_5} \left(\frac{-e_5}{8} (d\mathbf{o} - d\mathbf{o}^1 + d\mathbf{o}^2 - d\mathbf{o}^3 - d\mathbf{o}^4 + d\mathbf{o}^5 - d\mathbf{o}^6 + d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_6} \left(\frac{-e_6}{8} (d\mathbf{o} + d\mathbf{o}^1 - d\mathbf{o}^2 - d\mathbf{o}^3 - d\mathbf{o}^4 - d\mathbf{o}^5 + d\mathbf{o}^6 + d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_7} \left(\frac{-e_7}{8} (d\mathbf{o} - d\mathbf{o}^1 - d\mathbf{o}^2 + d\mathbf{o}^3 - d\mathbf{o}^4 + d\mathbf{o}^5 + d\mathbf{o}^6 - d\mathbf{o}^7) \right)
\end{aligned} \tag{65}$$

$$\begin{aligned}
df(\mathbf{o}) &= \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} - \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 - \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o} \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^1 \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} - \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^2 \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 - \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^3 \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} - \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^4 \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 - \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^5 \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} - \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 - \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^6 \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^7 \quad (66) \\
&\equiv \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}} d\mathbf{o} + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^1} d\mathbf{o}^1 + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^2} d\mathbf{o}^2 + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^3} d\mathbf{o}^3 + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^4} d\mathbf{o}^4 + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^5} d\mathbf{o}^5 + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^6} d\mathbf{o}^6 + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^7} d\mathbf{o}^7
\end{aligned}$$

From the above, we note in particular the result for the left partial derivative over octonion variable \mathbf{o} :

$$\frac{\partial f(\mathbf{o})}{\partial \mathbf{o}} = \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} - \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 - \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) \quad (67)$$

This corresponds to the intuitive expected extension of the partial derivatives for quaternion variables (4-dimensional $\mathbb{H}\mathbb{R}$ calculus):

$$\frac{\partial f(\mathbf{q})}{\partial \mathbf{q}} = \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} - \frac{\partial f(\mathbf{q})}{\partial q_1} e_1 - \frac{\partial f(\mathbf{q})}{\partial q_2} e_2 - \frac{\partial f(\mathbf{q})}{\partial q_3} e_3 \right) = \frac{1}{4} \left(\frac{\partial f(\mathbf{q})}{\partial q_0} - \frac{\partial f(\mathbf{q})}{\partial q_1} i - \frac{\partial f(\mathbf{q})}{\partial q_2} j - \frac{\partial f(\mathbf{q})}{\partial q_3} k \right) \quad (68)$$

and of the partial derivatives for complex numbers (2-dimensional Wirtinger/ $\mathbb{C}\mathbb{R}$ calculus):

$$\frac{\partial f(\mathbf{z})}{\partial \mathbf{z}} = \frac{1}{2} \left(\frac{\partial f(\mathbf{z})}{\partial z_0} - i \frac{\partial f(\mathbf{z})}{\partial z_1} \right) \quad (69).$$

Therefore, we observe that the development of calculus and partial derivatives over complex or hypercomplex variables using transformations from Hadamard matrices leads to consistent forms for different algebras. In the next section, we will see that this also applies to calculus for the associative octonion-like split-biquaternion algebra.

VI. CALCULUS FOR OCTONION-LIKE SPLIT-BIQUATERNIONS USING HADAMARD MATRIX TRANSFORMATION

For calculus in the split-biquaternion algebra with the formulation of partial derivatives over split-biquaternion variables, we apply the same approach using a transformation from a Hadamard matrix as in the previous section (for octonion algebra). However, since the split-biquaternion algebra is associative, the last step in the development of the partial derivatives will be strictly valid, unlike in the previous section.

Using the definitions and properties in (49)-(54) for the split-biquaternion algebra and (56)-(59), (61)-(62) for the transformation using conjugation-like operations (i.e., changing the sign of some real-valued coefficients in \mathbf{o}) to produce the \mathbf{o}^m $1 \leq m \leq 7$ transformed variables using the 8-dimensional Hadamard matrix, the different components of a split-biquaternion variable \mathbf{o} ($o_n e_n$ $0 \leq n \leq 7$) can be expressed as a linear combination of \mathbf{o} and seven split-biquaternion transformed variables \mathbf{o}^m $1 \leq m \leq 7$:

$$o_n = e_n^{-1} (c_{n0} \mathbf{o} + c_{n1} \mathbf{o}^1 + c_{n2} \mathbf{o}^2 + c_{n3} \mathbf{o}^3 + c_{n4} \mathbf{o}^4 + c_{n5} \mathbf{o}^5 + c_{n6} \mathbf{o}^6 + c_{n7} \mathbf{o}^7) \quad (70)$$

$$\text{with } e_n^{-1} = -e_n \quad 1 \leq n \leq 6, \quad e_n^{-1} = e_n \quad n = 0, 7.$$

$$o_0 = \frac{1}{8} (\mathbf{o} + \mathbf{o}^1 + \mathbf{o}^2 + \mathbf{o}^3 + \mathbf{o}^4 + \mathbf{o}^5 + \mathbf{o}^6 + \mathbf{o}^7) \quad (71)$$

$$o_1 = \frac{-e_1}{8} (\mathbf{o} - \mathbf{o}^1 + \mathbf{o}^2 - \mathbf{o}^3 + \mathbf{o}^4 - \mathbf{o}^5 + \mathbf{o}^6 - \mathbf{o}^7)$$

$$o_2 = \frac{-e_2}{8} (\mathbf{o} + \mathbf{o}^1 - \mathbf{o}^2 - \mathbf{o}^3 + \mathbf{o}^4 + \mathbf{o}^5 - \mathbf{o}^6 - \mathbf{o}^7)$$

$$o_3 = \frac{-e_3}{8} (\mathbf{o} - \mathbf{o}^1 - \mathbf{o}^2 + \mathbf{o}^3 + \mathbf{o}^4 - \mathbf{o}^5 - \mathbf{o}^6 + \mathbf{o}^7)$$

$$o_4 = \frac{-e_4}{8} (\mathbf{o} + \mathbf{o}^1 + \mathbf{o}^2 + \mathbf{o}^3 - \mathbf{o}^4 - \mathbf{o}^5 - \mathbf{o}^6 - \mathbf{o}^7)$$

$$o_5 = \frac{-e_5}{8} (\mathbf{o} - \mathbf{o}^1 + \mathbf{o}^2 - \mathbf{o}^3 - \mathbf{o}^4 + \mathbf{o}^5 - \mathbf{o}^6 + \mathbf{o}^7)$$

$$o_6 = \frac{-e_6}{8} (\mathbf{o} + \mathbf{o}^1 - \mathbf{o}^2 - \mathbf{o}^3 - \mathbf{o}^4 - \mathbf{o}^5 + \mathbf{o}^6 + \mathbf{o}^7)$$

$$o_7 = \frac{e_7}{8} (\mathbf{o} - \mathbf{o}^1 - \mathbf{o}^2 + \mathbf{o}^3 - \mathbf{o}^4 + \mathbf{o}^5 + \mathbf{o}^6 - \mathbf{o}^7)$$

It should be noted that the results in (71) are split-biquaternion variables where only the 1st component is non-zero, therefore they are real-valued scalars. (70) and (71) only differ from (60) and (63) (octonion algebra) for the component o_7 .

If we wish to express $o_n e_n$ $0 \leq n \leq 7$ in terms of conjugates $(\mathbf{o}^m)^*$ $0 \leq m \leq 7$, then a conjugation operation can be applied to (71), as per the definition in (50) for split-biquaternion numbers:

$$o_0 = \frac{1}{8} ((\mathbf{o})^* + (\mathbf{o}^1)^* + (\mathbf{o}^2)^* + (\mathbf{o}^3)^* + (\mathbf{o}^4)^* + (\mathbf{o}^5)^* + (\mathbf{o}^6)^* + (\mathbf{o}^7)^*) \quad (72)$$

$$o_1 = \frac{e_1}{8} ((\mathbf{o})^* - (\mathbf{o}^1)^* + (\mathbf{o}^2)^* - (\mathbf{o}^3)^* + (\mathbf{o}^4)^* - (\mathbf{o}^5)^* + (\mathbf{o}^6)^* - (\mathbf{o}^7)^*)$$

$$o_2 = \frac{e_2}{8} ((\mathbf{o})^* + (\mathbf{o}^1)^* - (\mathbf{o}^2)^* - (\mathbf{o}^3)^* + (\mathbf{o}^4)^* + (\mathbf{o}^5)^* - (\mathbf{o}^6)^* - (\mathbf{o}^7)^*)$$

$$o_3 = \frac{e_3}{8} ((\mathbf{o})^* - (\mathbf{o}^1)^* - (\mathbf{o}^2)^* + (\mathbf{o}^3)^* + (\mathbf{o}^4)^* - (\mathbf{o}^5)^* - (\mathbf{o}^6)^* + (\mathbf{o}^7)^*)$$

$$o_4 = \frac{e_4}{8} ((\mathbf{o})^* + (\mathbf{o}^1)^* + (\mathbf{o}^2)^* + (\mathbf{o}^3)^* - (\mathbf{o}^4)^* - (\mathbf{o}^5)^* - (\mathbf{o}^6)^* - (\mathbf{o}^7)^*)$$

$$o_5 = \frac{e_5}{8} ((\mathbf{o})^* - (\mathbf{o}^1)^* + (\mathbf{o}^2)^* - (\mathbf{o}^3)^* - (\mathbf{o}^4)^* + (\mathbf{o}^5)^* - (\mathbf{o}^6)^* + (\mathbf{o}^7)^*)$$

$$o_6 = \frac{e_6}{8} ((\mathbf{o})^* + (\mathbf{o}^1)^* - (\mathbf{o}^2)^* - (\mathbf{o}^3)^* - (\mathbf{o}^4)^* - (\mathbf{o}^5)^* + (\mathbf{o}^6)^* + (\mathbf{o}^7)^*)$$

$$o_7 = \frac{e_7}{8} ((\mathbf{o})^* - (\mathbf{o}^1)^* - (\mathbf{o}^2)^* + (\mathbf{o}^3)^* - (\mathbf{o}^4)^* + (\mathbf{o}^5)^* + (\mathbf{o}^6)^* - (\mathbf{o}^7)^*)$$

(73) has the same form as (64) for octonions, however the definition of the conjugation for split-biquaternions in (50) is different.

The total differential of a function $f(\mathbf{o}) = f(o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ $o_n \in \mathbb{R}$ $0 \leq n \leq 7$ can be expressed as in (44) (left derivatives) or (45) (right derivatives), which may differ since split-biquaternions are non-commutative.

The differentials $do_0, do_1, do_2, do_3, do_4, do_5, do_6, do_7$ are obtained (71) or (72). To obtain the partial derivatives, we substitute $do_0, do_1, do_2, do_3, do_4, do_5, do_6, do_7$ into (44) or (45). There are therefore four different possibilities, depending on partial derivatives over the variables $\mathbf{o}, \mathbf{o}^1, \mathbf{o}^2, \mathbf{o}^3, \mathbf{o}^4, \mathbf{o}^5, \mathbf{o}^6, \mathbf{o}^7$ or $(\mathbf{o})^*, (\mathbf{o}^1)^*, (\mathbf{o}^2)^*, (\mathbf{o}^3)^*, (\mathbf{o}^4)^*, (\mathbf{o}^5)^*, (\mathbf{o}^6)^*, (\mathbf{o}^7)^*$, and left derivatives or right derivatives. Note that for the right derivatives in (45), norm-preserving involutions $e_n o_n e_n^{-1} = -e_n o_n e_n^{-1}$ $1 \leq n \leq 6$ and $e_n o_n e_n^{-1} = e_n o_n e_n^{-1}$ $n = 7$ would first be applied to (71) or (72), to get equations where the e_n term appears at the end of the equation.

The formulation of the left side partial derivative over the variables $\mathbf{o}, \mathbf{o}^1, \mathbf{o}^2, \mathbf{o}^3, \mathbf{o}^4, \mathbf{o}^5, \mathbf{o}^6, \mathbf{o}^7$ is shown in detail in (74) and (75). Grouping together the terms with $d\mathbf{o}, d\mathbf{o}^1, d\mathbf{o}^2, d\mathbf{o}^3, d\mathbf{o}^4, d\mathbf{o}^5, d\mathbf{o}^6, d\mathbf{o}^7$ and defining $\frac{\partial}{\partial \mathbf{o}^m}$ $0 \leq m \leq 7$ as the factor multiplying

$d\mathbf{o}^m$ $0 \leq m \leq 7$, we obtain the definition of the partial derivatives $\frac{\partial}{\partial \mathbf{o}^m}$ $0 \leq m \leq 7$.

$$\begin{aligned}
df(\mathbf{o}) = & \frac{\partial f(\mathbf{o})}{\partial o_0} \left(\frac{1}{8} (d\mathbf{o} + d\mathbf{o}^1 + d\mathbf{o}^2 + d\mathbf{o}^3 + d\mathbf{o}^4 + d\mathbf{o}^5 + d\mathbf{o}^6 + d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_1} \left(\frac{-e_1}{8} (d\mathbf{o} - d\mathbf{o}^1 + d\mathbf{o}^2 - d\mathbf{o}^3 + d\mathbf{o}^4 - d\mathbf{o}^5 + d\mathbf{o}^6 - d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_2} \left(\frac{-e_2}{8} (d\mathbf{o} + d\mathbf{o}^1 - d\mathbf{o}^2 - d\mathbf{o}^3 + d\mathbf{o}^4 + d\mathbf{o}^5 - d\mathbf{o}^6 - d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_3} \left(\frac{-e_3}{8} (d\mathbf{o} - d\mathbf{o}^1 - d\mathbf{o}^2 + d\mathbf{o}^3 + d\mathbf{o}^4 - d\mathbf{o}^5 - d\mathbf{o}^6 + d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_4} \left(\frac{-e_4}{8} (d\mathbf{o} + d\mathbf{o}^1 + d\mathbf{o}^2 + d\mathbf{o}^3 - d\mathbf{o}^4 - d\mathbf{o}^5 - d\mathbf{o}^6 - d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_5} \left(\frac{-e_5}{8} (d\mathbf{o} - d\mathbf{o}^1 + d\mathbf{o}^2 - d\mathbf{o}^3 - d\mathbf{o}^4 + d\mathbf{o}^5 - d\mathbf{o}^6 + d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_6} \left(\frac{-e_6}{8} (d\mathbf{o} + d\mathbf{o}^1 - d\mathbf{o}^2 - d\mathbf{o}^3 - d\mathbf{o}^4 - d\mathbf{o}^5 + d\mathbf{o}^6 + d\mathbf{o}^7) \right) \\
& + \frac{\partial f(\mathbf{o})}{\partial o_7} \left(\frac{e_7}{8} (d\mathbf{o} - d\mathbf{o}^1 - d\mathbf{o}^2 + d\mathbf{o}^3 - d\mathbf{o}^4 + d\mathbf{o}^5 + d\mathbf{o}^6 - d\mathbf{o}^7) \right)
\end{aligned} \tag{74}$$

$$\begin{aligned}
df(\mathbf{o}) &= \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} - \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o} \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 - \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^1 \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} - \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 - \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^2 \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^3 \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} - \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 - \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^4 \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 + \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^5 \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} - \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 + \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 + \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^6 \\
&+ \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} + \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 + \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 + \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 - \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) d\mathbf{o}^7 \\
&\equiv \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}} d\mathbf{o} + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^1} d\mathbf{o}^1 + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^2} d\mathbf{o}^2 + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^3} d\mathbf{o}^3 + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^4} d\mathbf{o}^4 + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^5} d\mathbf{o}^5 + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^6} d\mathbf{o}^6 + \frac{\partial f(\mathbf{o})}{\partial \mathbf{o}^7} d\mathbf{o}^7 \tag{75}
\end{aligned}$$

From the above, we note in particular the result for the left partial derivative over the split-biquaternion variable \mathbf{o} :

$$\frac{\partial f(\mathbf{o})}{\partial \mathbf{o}} = \frac{1}{8} \left(\frac{\partial f(\mathbf{o})}{\partial o_0} - \frac{\partial f(\mathbf{o})}{\partial o_1} e_1 - \frac{\partial f(\mathbf{o})}{\partial o_2} e_2 - \frac{\partial f(\mathbf{o})}{\partial o_3} e_3 - \frac{\partial f(\mathbf{o})}{\partial o_4} e_4 - \frac{\partial f(\mathbf{o})}{\partial o_5} e_5 - \frac{\partial f(\mathbf{o})}{\partial o_6} e_6 + \frac{\partial f(\mathbf{o})}{\partial o_7} e_7 \right) \tag{76}$$

Considering the definition of $\frac{\partial f(\mathbf{o})}{\partial \mathbf{o}}$ for octonions in (67) (this is only possible if lack of octonion associativity is neglected in the last step of the development) and considering the different definitions of conjugation for octonions and split-biquaternions (i.e., the sign of $o_7 e_7$ is not changed under the conjugation operation in the split-biquaternion algebra), (76)

is the expected intuitive form of the partial derivative $\frac{\partial f(\mathbf{o})}{\partial \mathbf{o}}$ for split-biquaternions.

Therefore, we observe that the development of partial derivatives over hypercomplex variables using transformations from Hadamard matrices leads to consistent forms for

$\mathbb{H}\mathbb{R}$ calculus, for calculus in the octonion algebra and calculus in the octonion-like split-biquaternion algebra.

VII. $\mathbb{C}\mathbb{R}$ CALCULUS DEVELOPMENT USING HADAMARD MATRIX TRANSFORMATION

For completeness, and to illustrate that the general approach of performing a transformation using an Hadamard matrix can also be used for the development of classic Wirtinger/ $\mathbb{C}\mathbb{R}$ calculus, in this section the partial derivatives over complex variables in $\mathbb{C}\mathbb{R}$ calculus are derived using the same procedure as in the previous two sections. The well-known complex-numbers algebra is an associative and commutative normed division algebra over \mathbb{R} , with components $e_0 = 1$, $e_1 = i$ as an orthogonal basis, $e_0^2 = 1$, $e_1^2 = i^2 = -1$ and the well-known multiplication rules of Table IV.

Table IV Multiplication rules in the complex-numbers algebra

		e_j	
		$e_0 = 1$	$e_1 = i$
e_i	$e_0 = 1$	1	$e_1 = i$
	$e_1 = i$	$e_1 = i$	-1

$$\mathbf{z} = z_0 e_0 + z_1 e_1 = z_0 + z_1 i \quad (77)$$

$$\mathbf{z}^* = z_0 e_0 - z_1 e_1 = z_0 - z_1 i \quad (78)$$

$$\|\mathbf{z}\|^2 = \mathbf{z}\mathbf{z}^* = z_0^2 + z_1^2 \quad (79)$$

For complex numbers algebra, the concept of performing a rotation or involution of a variable \mathbf{z} over the basis component $e_1 = i$ is not useful to generate a transformed variable, because it leads to the same variable:

$$\mathbf{z}^i = i \mathbf{z} i^{-1} = -i \mathbf{z} i = -i i \mathbf{z} = \mathbf{z} \quad (80).$$

Instead, the size-2 Hadamard matrix can be applied as a transformation matrix as in previous sections, to generate two variables (the original complex variable \mathbf{z} and a transformed variable \mathbf{z}^1), which in turn can be linearly combined to obtain the individual components in \mathbf{z} ($z_0 e_0, z_1 e_1$):

$$z_n e_n = c_{n1} \mathbf{z} + c_{ni} \mathbf{z}^1 \quad n = 0,1 \quad e_0 = 1, e_1 = i \quad (81).$$

In matrix form with real-valued elements and the different components of the original variable \mathbf{z} and transformed variable \mathbf{z}^1 :

$$\begin{bmatrix} z_0 \\ 0 \end{bmatrix} = c_{00} \bar{\mathbf{z}} + c_{01} \bar{\mathbf{z}}^1 = \begin{bmatrix} \bar{\mathbf{z}} & \bar{\mathbf{z}}^1 \end{bmatrix} \begin{bmatrix} c_{00} \\ c_{01} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ z_1 \end{bmatrix} = c_{10} \bar{\mathbf{z}} + c_{11} \bar{\mathbf{z}}^1 = \begin{bmatrix} \bar{\mathbf{z}} & \bar{\mathbf{z}}^1 \end{bmatrix} \begin{bmatrix} c_{10} \\ c_{11} \end{bmatrix} \quad (82).$$

$$\bar{\mathbf{z}}_n = \vec{\vec{\mathbf{Z}}} \vec{\mathbf{c}}_n \quad n = 0,1$$

If each component in \mathbf{z}^1 (i.e., e_0, e_1 or $1, i$) involves only the coefficient z_n of the corresponding component in \mathbf{z} , each row in the above matrix can be normalized by the coefficient z_n , to have a formulation which is independent of the coefficients in \mathbf{z} :

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \bar{\mathbf{z}}'_0 = \vec{\vec{\mathbf{Z}}}' \vec{\mathbf{c}}_0 \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \bar{\mathbf{z}}'_1 = \vec{\vec{\mathbf{Z}}}' \vec{\mathbf{c}}_1 \quad \bar{\mathbf{z}}'_n = \vec{\vec{\mathbf{Z}}}' \vec{\mathbf{c}}_n \quad n = 0,1 \quad (83)$$

$$\vec{\mathbf{c}}_n = \vec{\vec{\mathbf{Z}}}'^{-1} \bar{\mathbf{z}}'_n \quad n = 0,1 \quad \vec{\mathbf{c}}_0 = \vec{\vec{\mathbf{Z}}}'^{-1} \bar{\mathbf{z}}'_0 = \vec{\vec{\mathbf{Z}}}'^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{\mathbf{c}}_1 = \vec{\vec{\mathbf{Z}}}'^{-1} \bar{\mathbf{z}}'_1 = \vec{\vec{\mathbf{Z}}}'^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (84)$$

Therefore, each $\vec{\mathbf{c}}_n$ $0 \leq n \leq 1$ will have the values from the corresponding column in $\vec{\vec{\mathbf{Z}}}'^{-1}$.

Once these coefficients $\vec{\mathbf{c}}_n$ are known, i.e., with proper choice of $\vec{\vec{\mathbf{Z}}}'$ below as the size-2 Hadamard matrix, we will have:

$$z_0 = e_0^{-1} (c_{00} \mathbf{z} + c_{01} \mathbf{z}^1) = c_{00} \mathbf{z} + c_{01} \mathbf{z}^1 \quad z_1 = e_1^{-1} (c_{10} \mathbf{z} + c_{11} \mathbf{z}^1) = -i (c_{10} \mathbf{z} + c_{11} \mathbf{z}^1) \quad (85).$$

With the size-2 Hadamard matrix as the matrix $\vec{\vec{\mathbf{Z}}}'$:

$$\vec{\vec{\mathbf{Z}}}' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (86)$$

$$\vec{\vec{\mathbf{Z}}}'^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (87)$$

$$\vec{\mathbf{c}}_0 = \begin{bmatrix} c_{00} \\ c_{01} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{\mathbf{c}}_1 = \begin{bmatrix} c_{10} \\ c_{11} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (88)$$

$$z_0 = \frac{1}{2}(\mathbf{z} + \mathbf{z}^1) \quad z_1 = \frac{-i}{2}(\mathbf{z} - \mathbf{z}^1) \quad (89).$$

It should be noted that the results in (89) are complex variables with only the 1st component being non-zero, therefore they are real-valued scalars.

From the definition of $\vec{\vec{\mathbf{Z}}}'$ and comparing the two columns, this defines the transformed variable \mathbf{z}^1 and we see that there is only a sign change for the component $e_1 = i$, therefore $\mathbf{z}^1 = \mathbf{z}^*$ and we have:

$$z_0 = \frac{1}{2}(\mathbf{z} + \mathbf{z}^*) \quad z_1 = \frac{-i}{2}(\mathbf{z} - \mathbf{z}^*) \quad (90).$$

The total differential of a function $f(\mathbf{z}) = f(z_0, z_1) \quad z_n \in \mathbb{R} \quad n = 0, 1$ can be expressed as:

$$df(\mathbf{z}) = \frac{\partial f(\mathbf{z})}{\partial z_0} dz_0 + \frac{\partial f(\mathbf{z})}{\partial z_1} dz_1 \quad (dz_n \in \mathbb{R}) \quad (91).$$

The differentials dz_0, dz_1 are obtained from (90). To obtain the partial derivatives, we substitute dz_0, dz_1 into (91). Grouping together the terms with $d\mathbf{z}, d\mathbf{z}^*$ and defining $\frac{\partial}{\partial \mathbf{z}}$ as

the factor multiplying $d\mathbf{z}$ and $\frac{\partial}{\partial \mathbf{z}^*}$ as the factor multiplying $d\mathbf{z}^*$, we obtain the definition

of the partial derivatives over the complex variables:

$$\begin{aligned} df(\mathbf{z}) &= \frac{\partial f(\mathbf{z})}{\partial z_0} \frac{1}{2}(d\mathbf{z} + d\mathbf{z}^*) + \frac{\partial f(\mathbf{z})}{\partial z_1} \frac{-i}{2}(d\mathbf{z} - d\mathbf{z}^*) \\ &= \frac{1}{2} \left(\frac{\partial f(\mathbf{z})}{\partial z_0} - i \frac{\partial f(\mathbf{z})}{\partial z_1} \right) d\mathbf{z} + \frac{1}{2} \left(\frac{\partial f(\mathbf{z})}{\partial z_0} + i \frac{\partial f(\mathbf{z})}{\partial z_1} \right) d\mathbf{z}^* \\ &\equiv \frac{\partial f(\mathbf{z})}{\partial \mathbf{z}} d\mathbf{z} + \frac{\partial f(\mathbf{z})}{\partial \mathbf{z}^*} d\mathbf{z}^* \end{aligned} \quad (92)$$

From the above, we note in particular that:

$$\frac{\partial f(\mathbf{z})}{\partial \mathbf{z}} = \frac{1}{2} \left(\frac{\partial f(\mathbf{z})}{\partial z_0} - i \frac{\partial f(\mathbf{z})}{\partial z_1} \right) \quad (93)$$

$$\frac{\partial f(\mathbf{z})}{\partial \mathbf{z}^*} = \frac{1}{2} \left(\frac{\partial f(\mathbf{z})}{\partial z_0} + i \frac{\partial f(\mathbf{z})}{\partial z_1} \right) \quad (94),$$

which are the well-known classic formulations of the partial derivatives over complex numbers using Wirtinger/ $\mathbb{C}\mathbb{R}$ calculus. Therefore, we observe that the use of the Hadamard matrix transformation in the development of the partial derivatives again leads to the correct result, as previously for $\mathbb{H}\mathbb{R}$ calculus, for calculus in octonion algebra and for calculus in octonion-like split-biquaternion algebra.

VIII. CONCLUSION

This paper introduced the basis of calculus with the formulation of partial derivatives over hypercomplex variables for two 8-dimensional hypercomplex algebras: octonions and octonion-like split-biquaternions. We showed that the approach based on rotations over basis components used in the development of $\mathbb{H}\mathbb{R}$ calculus or generalized $\mathbb{H}\mathbb{R}$ calculus is not suitable for the two considered 8-dimensional hypercomplex algebras. Instead, the use of transformations based on Hadamard matrices was found to lead to a common framework for the development of partial derivatives for complex-numbers, quaternion, octonion and split-biquaternion algebras. Some next steps to further develop the calculus of hypercomplex variables in the 8-dimensional hypercomplex algebras will include developing the chain rule / product rule, evaluating conditions under which left side partial derivatives and right side partial derivatives are equivalent, and applying the partial derivatives over hypercomplex variables for gradient vector based and Hessian matrix based optimization algorithms and adaptive filtering algorithms.

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