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Analysis and control of an electro-elastic contact problem

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Abstract

We consider a mathematical model that describes the frictional contact of an electro-elastic body with a semi-insulator foundation. The process is static; the contact is bilateral and associated to Tresca's friction law. We list the assumptions on the data and derive a variational formulation of the model, in the form of a system that couples two inclusions in which the unknowns are the strain field and the electric field. Then we prove the unique solvability of the system, as well as the continuous dependence of its solution with respect to the data. We use these results in the study of an associated optimal control problem, for which we prove an existence result. The proofs are based on arguments of monotonicity, compactness, convex analysis, and lower semicontinuity.

Keywords

Electro-elastic material, frictional contact, stationary inclusion, weak solution, optimal control

1. Introduction

The mathematical theory of contact mechanics is that part of applied mathematics that deals with the study of mathematical models that describe phenomena of contact between deformable bodies or between a deformable body and an obstacle, the so-called foundation. Stated as strongly nonlinear boundary value problems, which usually do not have classical solutions, such models lead to a large variety of weak formulations, expressed in terms of inequalities. Employing such kinds of formulation allows us to analyse the corresponding models through arguments from the theory of variational and hemivariational inequalities. References in the field include the books [1–7] and, more recently, [8–10]. Results for related optimal control problems can be found in [11–15], for instance.

The piezoelectric effect is characterized by the coupling between the mechanical and electrical properties of the materials. This coupling leads to the appearance of an electric potential when the material is submitted to

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mechanical stress; conversely, mechanical stress is generated when an electric potential is applied. A deformable material that exhibits such a behavior is called a piezoelectric material. Piezoelectric materials are used as switches and actuators in many engineering systems, in radio electronics, electroacoustics, and measuring equipment. Piezoelectric materials with elastic mechanical properties are also called electro-elastic materials. References in the field include [16–19]. The analysis of mathematical models of contact with piezoelectric materials can be found in [9, 20], for instance.

Currently, there is an interest in variational formulation of contact models in the form of inclusions or sweeping processes. There, the unknown is either the displacement field or the strain field, as illustrated in [21, 22] and [23, 24], respectively. Using such kinds of formulation in the study of contact models requires adaptation of the arguments of abstract stationary or differential inclusions and, very often, the development of new arguments in their analysis and control. This paper follows this direction. Indeed, here we consider a mathematical model describing the equilibrium of a piezoelectric body in contact with an obstacle that leads to a system of two inclusions.

The aim in this paper is twofold. The first aim is to provide a new and nonstandard model of contact for electro-elastic materials and to prove its unique weak solvability. To this end, we use a recent existence and uniqueness result for stationary inclusions in Hilbert spaces proved in [25]. The second aim is to prove the continuous dependence of the solution with respect to the data and the solvability of an associated optimal control problem. Using such nonstandard formulation for a static piezoelectric contact problem represents the main trait of novelty of this work.

The rest of the manuscript is organized as follows. In Section 2, we introduce the notation we use, together with some preliminary results. In Section 3, we describe the mathematical model of piezoelectric contact, list the assumptions on the data and derive a variational formulation, in the form of a system in which the unknowns are the strain and the electric field. Then, in Section 4, we prove the unique solvability of the model, as well as the continuous dependence of the solution with respect to the data. Finally, we use these results in Section 5, where we consider an associate optimal control problem, for which we prove the existence of optimal pairs.

2. Notation and preliminaries

In this section, we present the notation we use, as well as some preliminary results. They concern the function spaces we need to introduce the piezoelectric contact problem, an abstract existence and uniqueness result for inclusions in real Hilbert spaces, and a version of the Weierstrass theorem, which we use in the study of the optimal control of the electro-elastic contact problem.

2.1. Function spaces

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a domain with smooth boundary Γ . We denote by $\boldsymbol{\nu}$ the outward unit normal to Γ and we assume that $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_a$, and Γ_b are measurable parts of Γ , such that $\text{meas}(\Gamma_1) > 0$ and $\text{meas}(\Gamma_a) > 0$. Moreover, both $\Gamma_1, \Gamma_2, \Gamma_3$, and $\Gamma_a, \Gamma_b, \Gamma_3$ represent a partition Γ . We also denote by \mathbb{S}^d the space of second-order symmetric tensors on \mathbb{R}^d and use the notation “ \cdot ”, $\|\cdot\|$, $\mathbf{0}$ for the inner product, the Euclidean norm, and the zero element of the spaces \mathbb{R}^d and \mathbb{S}^d , respectively.

We employ standard notation for Lebesgue and Sobolev spaces of real-valued functions defined on Ω and Γ . For an element $\boldsymbol{\nu} \in H^1(\Omega)^d$, we still write $\boldsymbol{\nu}$, ν_ν , and $\boldsymbol{\nu}_\tau$ for the trace, the normal trace, and the tangential trace of $\boldsymbol{\nu}$ to Γ , respectively, i.e.,

$$\nu_\nu = \boldsymbol{\nu} \cdot \boldsymbol{\nu}, \quad \boldsymbol{\nu}_\tau = \boldsymbol{\nu} - \nu_\nu \boldsymbol{\nu}.$$

Moreover, for the displacement and the stress field we need the spaces

$$\begin{aligned} V &= \{ \boldsymbol{\nu} = (\nu_i) : \nu_i \in H^1(\Omega), \nu_i = 0 \text{ on } \Gamma_1, \nu_\nu = 0 \text{ on } \Gamma_3, 1 \leq i \leq d \}, \\ Q &= \{ \boldsymbol{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega), 1 \leq i, j \leq d \}. \end{aligned}$$

It is well known that the spaces V and Q are real Hilbert spaces endowed with the canonical inner products

$$(\mathbf{u}, \boldsymbol{\nu})_V = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\nu}) \, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, dx \quad (1)$$

and the associated norms denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$. Here and in the following, $\boldsymbol{\varepsilon} : V \rightarrow Q$ represents the deformation operator defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{ij} + u_{ji}).$$

It follows from equation (1) that

$$\|\mathbf{v}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_Q \quad \forall \mathbf{v} \in V. \quad (2)$$

This equality will be used in various places in this paper. We denote by $\mathbf{0}_V$ the zero element of V and we recall that, as a consequence of the Sobolev trace theorem, there exists $c_0 > 0$, which depends on Ω , Γ_1 , and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq c_0 \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V. \quad (3)$$

If $\boldsymbol{\sigma} \in Q$ is a regular function, the normal component and the tangential part of the stress field $\boldsymbol{\sigma}$ on the boundary are defined by

$$\sigma_\nu = (\boldsymbol{\sigma} \mathbf{v}) \cdot \mathbf{v}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \mathbf{v} - \sigma_\nu \mathbf{v}.$$

Moreover, the following Green-type formula holds:

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_{\Omega} \text{Div } \boldsymbol{\sigma} \cdot \mathbf{v} \, dx = \int_{\Gamma} \boldsymbol{\sigma} \mathbf{v} \cdot \mathbf{v} \, da \quad \forall \mathbf{v} \in V. \quad (4)$$

Besides the spaces presented here, for the electric potential field we need the space

$$W = \{ \psi \in H^1(\Omega) : \psi = 0 \text{ a.e. on } \Gamma_a \}. \quad (5)$$

Note that W is a real Hilbert space with the inner product

$$(\varphi, \psi)_W = (\nabla \varphi, \nabla \psi)_{L^2(\Omega)^d} \quad (6)$$

and the associated norm $\|\cdot\|_W$. Moreover, from the Sobolev trace theorem, it follows that there exists a positive constant \tilde{c}_0 , which depends only on Ω , Γ_a , and Γ_3 , such that

$$\|\psi\|_{L^2(\Gamma_3)} \leq \tilde{c}_0 \|\psi\|_W \quad \forall \psi \in W. \quad (7)$$

Also, we denote by ψ^+ the positive part of $\psi \in W$ and we recall that, for a sufficiently regular function $\mathbf{D} \in L^2(\Omega)^d$, the Green-type formula holds:

$$(\mathbf{D}, \nabla \psi)_{L^2(\Omega)^d} + (\text{div } \mathbf{D}, \psi)_{L^2(\Omega)} = \int_{\Gamma} \mathbf{D} \cdot \mathbf{v} \, \psi \, da \quad \forall \psi \in W. \quad (8)$$

2.2. A stationary inclusion

In this section, X represents a Hilbert space endowed with the inner product $(\cdot, \cdot)_X$ and its associated norm $\|\cdot\|_X$. The set of parts of X is denoted by 2^X . Moreover, for any nonempty closed convex set $K \subset X$, we denote by $N_K : X \rightarrow 2^X$ the outward normal cone of K in the sense of convex analysis and by $P_K : X \rightarrow K$ the projection operator on K . We recall that the following equivalences hold, for all $u, \xi \in X$:

$$\xi \in N_K(u) \iff u \in K, \quad (\xi, v - u)_X \leq 0 \quad \forall v \in K. \quad (9)$$

$$u = P_K \xi \iff u \in K, \quad (\xi - u, v - u)_X \leq 0 \quad \forall v \in K. \quad (10)$$

Let $(Y, \|\cdot\|_Y)$ be a normed space, $\Sigma : Y \rightarrow 2^X$ a multivalued mapping, and $A : X \rightarrow X$ a nonlinear operator, such that the following conditions are satisfied:

$$\left\{ \begin{array}{l} \text{(a) For each } \theta \in Y, \text{ the set } \Sigma(\theta) \subset X \text{ is nonempty closed and convex.} \\ \text{(b) There exists } d_0 > 0 \text{ such that for each } \theta_1, \theta_2 \in Y \text{ and } u \in X \text{ one has} \\ \quad \|P_{\Sigma(\theta_1)} u - P_{\Sigma(\theta_2)} u\|_X \leq d_0 \|\theta_1 - \theta_2\|_Y. \end{array} \right. \quad (11)$$

$$\left\{ \begin{array}{l} A \text{ is a strongly monotone and Lipschitz continuous operator, i.e., there exist } m_A > 0 \text{ and } L_A > 0 \text{ such that} \\ \text{(a) } (Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \forall u, v \in X. \\ \text{(b) } \|Au - Av\|_X \leq L_A \|u - v\|_X \quad \forall u, v \in X. \end{array} \right. \quad (12)$$

Then, the inclusion we consider in this section is the following.

Problem. \mathcal{I} . Given $\theta \in Y$, find an element $u \in X$ such that

$$-u \in N_{\Sigma(\theta)}(Au). \quad (13)$$

The unique solvability of Problem \mathcal{I} , as well as its continuous dependence with respect to θ , is provided by the following result, proved in [25].

Theorem 1. *Assume equations (11) and (12). Then, for each element $\theta \in Y$, there exists a unique element $u = u(\theta) \in X$, such that equation (13) holds. Moreover, the operator $\theta \mapsto u(\theta) : Y \rightarrow X$ is Lipschitz continuous.*

2.3. An optimization problem

We end this section by recalling the following version of the Weierstrass theorem.

Theorem 2. *Let W be a reflexive Banach space, J a nonempty weakly closed subset of W , and $\mathcal{S} : J \rightarrow \mathbb{R}$ a weakly lower semicontinuous function. In addition, assume that \mathcal{S} is coercive, i.e., $\mathcal{S}(w) \rightarrow \infty$ as $\|w\|_W \rightarrow \infty$. Then, there exists at least an element w^* , such that*

$$w^* \in J, \quad \mathcal{S}(w^*) \leq \mathcal{S}(w) \quad \forall w \in J. \quad (14)$$

Theorem 2 will be used in Section 5. Its proof can be found in [26], for instance.

3. The model

In this section, we introduce the contact model, list the assumptions on the data, and derive its variational formulation.

The physical setting is as follows. An electro-elastic body occupies, in its reference configuration, the domain $\Omega \subset \mathbb{R}^d$ described in Section 2. The body is fixed on Γ_1 , is acted on by given surface tractions on Γ_2 , and is in contact, on Γ_3 , with an obstacle. We also assume that the electric potential vanishes on Γ_a and a surface electric charge of density q_b is prescribed on Γ_b . Then, the classical formulation of the problem is the following.

Problem 1. Find a displacement field $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$, an electric potential $\varphi : \Omega \rightarrow \mathbb{R}$, and an electric displacement field $\mathbf{D} : \Omega \rightarrow \mathbb{R}^d$, such that

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathcal{P}^\top \nabla \varphi \quad \text{in } \Omega, \quad (15)$$

$$\mathbf{D} = \mathcal{P}\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\beta} \nabla \varphi \quad \text{in } \Omega, \quad (16)$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega, \quad (17)$$

$$\text{div } \mathbf{D} - q_0 = 0 \quad \text{in } \Omega, \quad (18)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad (19)$$

$$\boldsymbol{\sigma} \mathbf{v} = \mathbf{f}_2 \quad \text{on } \Gamma_2, \quad (20)$$

$$\varphi = 0 \quad \text{on } \Gamma_a, \quad (21)$$

$$\mathbf{D} \cdot \mathbf{v} = q_b \quad \text{on } \Gamma_b, \quad (22)$$

$$u_v = 0 \quad \text{on } \Gamma_3, \quad (23)$$

$$\|\boldsymbol{\sigma}_\tau\| \leq F, \quad \boldsymbol{\sigma}_\tau = -F \frac{\mathbf{u}_\tau}{\|\mathbf{u}_\tau\|} \quad \text{if } \mathbf{u}_\tau \neq \mathbf{0} \quad \text{on } \Gamma_3, \quad (24)$$

$$0 \leq \mathbf{D} \cdot \mathbf{v} \leq G, \quad \mathbf{D} \cdot \mathbf{v} = \begin{cases} 0 & \text{if } \varphi < 0, \\ G & \text{if } \varphi > 0 \end{cases} \quad \text{on } \Gamma_3. \quad (25)$$

A brief description of the equations and boundary conditions in Problem 1 is the following. First, equations (15) and (16) represent the electro-elastic constitutive law of the material, in which \mathcal{F} is a given nonlinear elasticity operator, $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the linearized strain tensor, $\nabla\varphi$ is the electric field, \mathcal{P} represents the third-order piezoelectric tensor, \mathcal{P}^\top is its transpose, and $\boldsymbol{\beta}$ denotes the electric permittivity tensor. Equations (17) and (18) represent the balance equations for the stress and electric displacement fields, respectively, in which, recall, “Div” and “div” denote the divergence operator for tensor- and vector-valued functions, respectively. Next, equations (19) and (20) are the displacement and traction boundary conditions, respectively, and equations (21) and (22) represent the electric boundary conditions. Equation (23) is the contact condition. It shows that the contact is bilateral, i.e., there is no separation between the electro-elastic body and the foundation. Equation (24) is the Tresca law of dry friction, in which F represents the friction bound. Finally, equation (25) is the electrical contact condition, in which G denotes a given bound. We use this condition because we assume that the foundation is a semi-insulator. Indeed, equation (25) shows that there are electric charges on the contact zone Γ_3 when the electric potential is positive, and the charges vanish when the electric potential is negative. Equation (25) also shows that the electric potential on the contact surface is positive only when the magnitude of the electric charges has the critical value G . Details of piezoelectric contact problems similar to equations (15) to (25) can be found in [20].

In the study of the piezoelectric contact problem (equations (15) to (25)), we assume that the elasticity operator, the piezoelectric tensor, and the electric permittivity tensor satisfy the following conditions:

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{F}} > 0 \text{ such that} \\ \quad \|\mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that} \\ \quad (\mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \quad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) The mapping } \mathbf{x} \mapsto \mathcal{F}(\mathbf{x}, \mathbf{0}) \text{ belongs to } Q. \end{array} \right. \quad (26)$$

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{P} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{R}^d. \\ \text{(b) } \mathcal{P}(\mathbf{x}, \boldsymbol{\tau}) = (p_{ijk}(\mathbf{x})\tau_{jk}) \quad \forall \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) } p_{ijk} = p_{ikj} \in L^\infty(\Omega), \quad 1 \leq i, j, k \leq d. \end{array} \right. \quad (27)$$

$$\left\{ \begin{array}{l} \text{(a) } \boldsymbol{\beta} : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d. \\ \text{(b) } \boldsymbol{\beta}(\mathbf{x}, \mathbf{E}) = (\beta_{ij}(\mathbf{x})E_j) \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) } \beta_{ij} = \beta_{ji} \in L^\infty(\Omega), \quad 1 \leq i, j \leq d. \\ \text{(d) There exists } m_{\boldsymbol{\beta}} > 0 \text{ such that} \\ \quad \boldsymbol{\beta}(\mathbf{x}, \mathbf{E}) \cdot \mathbf{E} \geq m_{\boldsymbol{\beta}} \|\mathbf{E}\|^2 \quad \forall \mathbf{E} \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Omega. \end{array} \right. \quad (28)$$

The densities of body forces, surface tractions, volume, and surface electric charges satisfy

$$\mathbf{f}_0 \in L^2(\Omega)^d, \quad \mathbf{f}_2 \in L^2(\Gamma_2)^d, \quad (29)$$

$$q_0 \in L^2(\Omega), \quad q_b \in L^2(\Gamma_b). \quad (30)$$

Finally, the bounds F and G are such that

$$F \in L^2(\Gamma_3) \text{ and } F(\mathbf{x}) \geq 0 \text{ a.e. } \mathbf{x} \in \Gamma_3, \quad (31)$$

$$G \in L^2(\Gamma_3) \text{ and } G(\mathbf{x}) \geq 0 \text{ a.e. } \mathbf{x} \in \Gamma_3. \quad (32)$$

Under the previous assumptions, we introduce the functions $j : V \rightarrow \mathbb{R}$, $h : W \rightarrow \mathbb{R}$, the elements $\mathbf{f} \in V$, $g \in W$, and the sets $\Sigma_1(\mathbf{f}) \subset Q$, $\Sigma_2(q) \subset L^2(\Omega)^d$ defined by

$$j(\mathbf{v}) = \int_{\Gamma_3} F \|\mathbf{v}_\tau\| \, da \quad \forall \mathbf{v} \in V, \quad (33)$$

$$h(\psi) = \int_{\Gamma_3} G \psi^+ \, da \quad \forall \psi \in W, \quad (34)$$

$$(\mathbf{f}, \mathbf{v})_V = \int_{\Omega} \mathbf{f}_0 \cdot \mathbf{v} \, dx + \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v} \, da \quad \forall \mathbf{v} \in V, \quad (35)$$

$$(q, \psi)_W = \int_{\Omega} q_0 \psi \, dx - \int_{\Gamma_b} q_b \psi \, da \quad \forall \psi \in W. \quad (36)$$

$$\Sigma_1(\mathbf{f}) = \{ \boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\mathbf{v}) \geq (\mathbf{f}, \mathbf{v})_V \quad \forall \mathbf{v} \in V \}, \quad (37)$$

$$\Sigma_2(q) = \{ \mathbf{E} \in L^2(\Omega)^d : (\mathbf{E}, \nabla \psi)_{L^2(\Omega)^d} + h(\psi) \geq (q, \psi)_W \quad \forall \psi \in W \}. \quad (38)$$

Assume now that $(\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D})$ are sufficiently regular functions that satisfy Problem 1 and let $\mathbf{v} \in V$. Then, using standard arguments based on the Green formula (equation (4)), we deduce that

$$\int_{\Omega} \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})) \, dx + \int_{\Gamma_3} F \|\mathbf{v}_\tau\| \, da - \int_{\Gamma_3} F \|\mathbf{u}_\tau\| \, da \geq \int_{\Omega} \mathbf{f}_0 \cdot (\mathbf{v} - \mathbf{u}) \, dx + \int_{\Gamma_2} \mathbf{f}_2 \cdot (\mathbf{v} - \mathbf{u}) \, da. \quad (39)$$

Next, we use equations (1), (33) and (35) to deduce that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}))_Q + j(\mathbf{v}) - j(\mathbf{u}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u})_V. \quad (40)$$

We now test equation (40) with $\mathbf{v} = 2\mathbf{u}$ and $\mathbf{v} = \mathbf{0}_V$, to see that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}))_Q + j(\mathbf{u}) = (\mathbf{f}, \mathbf{u})_V. \quad (41)$$

Therefore, using equations (40) and (41), we find that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\mathbf{v}) \geq (\mathbf{f}, \mathbf{v})_V,$$

which implies that

$$\boldsymbol{\sigma} \in \Sigma_1(\mathbf{f}). \quad (42)$$

To proceed, we use equations (37) and (41) to see that

$$(\boldsymbol{\tau} - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}))_Q \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma_1(\mathbf{f})$$

and, using the notation

$$\boldsymbol{\omega}_1 = \boldsymbol{\varepsilon}(\mathbf{u}), \quad (43)$$

we find that

$$(\boldsymbol{\tau} - \boldsymbol{\sigma}, \boldsymbol{\omega}_1)_Q \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma_1(\mathbf{f}). \quad (44)$$

We now combine equations (42) and (44); then we use equation (9) to see that

$$-\boldsymbol{\omega}_1 \in N_{\Sigma_1(\mathbf{f})}(\boldsymbol{\sigma}) \quad (45)$$

where, recall, $N_{\Sigma_1(\mathbf{f})}$ represents the outward normal cone of $\Sigma_1(\mathbf{f})$ in the sense of convex analysis.

Next, we use the Green formula (equation(8)) and equations (34) and (36) to see that

$$(-\mathbf{D}, \nabla \psi - \nabla \varphi)_{L^2(\Omega)^d} + h(\psi) - h(\varphi) \geq (q, \psi - \varphi)_W \quad \forall \psi \in W. \quad (46)$$

A careful analysis reveals that equation (46) is similar to equation (40), the difference arising from the fact that the spaces Q , V , the functions $\boldsymbol{\sigma}$, \mathbf{v} , \mathbf{u} , \mathbf{f} , j , and the operator $\boldsymbol{\varepsilon}$ are now replaced by the spaces $L^2(\Omega)^d$, W ,

the functions $-D$, ψ , φ , q , h , and the operator ∇ , respectively. Therefore, using arguments similar to those used for equations (40) to (45), we deduce that

$$-\omega_2 \in N_{\Sigma_2(q)}(-D) \quad (47)$$

where, here and in the following, we employ the notation

$$\omega_2 = \nabla\varphi. \quad (48)$$

We now use the inclusions (equations (45) and (47)), the constitutive laws (equations (15) and (16)), and equations (43) and (48) to deduce the following variational formulation of Problem 1.

Problem 2. Find a strain field $\omega_1 \in Q$ and an electric field $\omega_2 \in L^2(\Omega)^d$, such that

$$-\omega_1 \in N_{\Sigma_1(f)}(\mathcal{F}\omega_1 + \mathcal{P}^\top\omega_2), \quad (49)$$

$$-\omega_2 \in N_{\Sigma_2(q)}(\beta\omega_2 - \mathcal{P}\omega_1). \quad (50)$$

Note that Problem 2 represents a system that couples two inclusions for the unknowns ω_1 and ω_2 . Once these functions are known, the displacement field \mathbf{u} and the electric potential field φ are obtained by using equations (43) and (48), respectively, as shown in the next section. This, in turn, allows us to obtain the stress field σ and the electric displacement field D , by using the constitutive law (equations (15) and (16)).

4. Existence, uniqueness, and continuous dependence results

We start this section with the following existence, uniqueness, and continuous dependence result concerning Problem 2.

Theorem 3. *Assume equations (26) to (32). Then Problem 2 has a unique solution $\omega = (\omega_1, \omega_2) \in Q \times L^2(\Omega)^d$. Moreover, the operators $(f, q) \mapsto \omega_1 = \omega_1(f, q) : V \times W \rightarrow Q$ and $(f, q) \mapsto \omega_2 = \omega_2(f, q) : V \times W \rightarrow L^2(\Omega)^d$ are Lipschitz continuous.*

The proof of Theorem 3 is based on some preliminary results that we present in the following, where we assume everywhere that equations (26) to (32) hold, even if, for simplicity, we do not mention it explicitly. We start with the following result.

Lemma 1. *Let $\omega_1, \sigma \in Q, f \in V$ be such that*

$$-\omega_1 \in N_{\Sigma_1(f)}(\sigma). \quad (51)$$

Then, there exists a unique element $\mathbf{u} \in V$ such that $\omega_1 = \varepsilon(\mathbf{u})$ and, moreover,

$$(\sigma, \varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}))_Q + j(\mathbf{v}) - j(\mathbf{u}) \geq (f, \mathbf{v} - \mathbf{u})_V \quad \forall \mathbf{v} \in V. \quad (52)$$

Proof. First, we note that the inclusion (equation (51)) implies that

$$\sigma \in \Sigma_1(f), \quad (\tau - \sigma, \omega_1)_Q \geq 0 \quad \forall \tau \in \Sigma_1(f). \quad (53)$$

Let $z \in \varepsilon(V)^\perp$, where, here and in the following, M^\perp represents the orthogonal of $M \subset Q$. Then $(z, \varepsilon(\mathbf{v}))_Q = 0$ for all $\mathbf{v} \in V$ and, using equation (37), we find that $\sigma \pm z \in \Sigma_1(f)$. Therefore, testing with $\tau = \sigma \pm z$ in equation (53), we deduce that $(z, \omega_1)_Q = 0$, which shows that $\omega_1 \in \varepsilon(V)^{\perp\perp} = \varepsilon(V)$. Note that the last equation follows since $\varepsilon(V)$ is a closed subspace of Q , as guaranteed by Theorem 4.1 in [9]. Next, inclusion $\omega_1 \in \varepsilon(V)$ implies that there exists an element $\mathbf{u} \in V$ such that

$$\omega_1 = \varepsilon(\mathbf{u}). \quad (54)$$

Moreover, equation (2) guarantees that \mathbf{u} is unique.

Next, by the subdifferentiability of the function j at \mathbf{u} , we know that there exists an element $\mathbf{g} \in V$ such that

$$j(\mathbf{v}) - j(\mathbf{u}) \geq (\mathbf{g}, \mathbf{v} - \mathbf{u})_V = (\varepsilon(\mathbf{g}), \varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}))_Q$$

and, taking $\boldsymbol{\tau}_0 := \boldsymbol{\varepsilon}(\boldsymbol{f}) - \boldsymbol{\varepsilon}(\boldsymbol{g})$, we deduce that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall \boldsymbol{v} \in V. \quad (55)$$

We now test with $\boldsymbol{v} = 2\boldsymbol{u}$ and $\boldsymbol{v} = \mathbf{0}_V$ in this equation to deduce that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u})_V. \quad (56)$$

Therefore, combining equations (55) and (56), we find that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\boldsymbol{v}) \geq (\boldsymbol{f}, \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V,$$

which shows that $\boldsymbol{\tau}_0 \in \Sigma_1(\boldsymbol{f})$. This regularity and equations (53) and (54) imply that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q \geq (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q$$

and using equation (56) yields

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\boldsymbol{u}) \leq (\boldsymbol{f}, \boldsymbol{u})_V. \quad (57)$$

Moreover, since $\boldsymbol{\sigma} \in \Sigma_1(\boldsymbol{f})$ and $\boldsymbol{u} \in V$, the converse inequality holds, i.e.,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{u})_V. \quad (58)$$

We now combine equations (57) and (58) to see that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u})_V. \quad (59)$$

Then, using equation (59) and inclusion $\boldsymbol{\sigma} \in \Sigma_1(\boldsymbol{f})$, it follows that equation (52) holds. \square

We now proceed with the following result.

Lemma 2. *Let $\boldsymbol{\omega}_2, \boldsymbol{D} \in L^2(\Omega)^d$, $q \in W$ be such that*

$$-\boldsymbol{\omega}_2 \in N_{\Sigma_2(q)}(-\boldsymbol{D}). \quad (60)$$

Then, there exists a unique element $\varphi \in W$ such that $\boldsymbol{\omega}_2 = \nabla\varphi$ and, moreover,

$$(-\boldsymbol{D}, \nabla\psi - \nabla\varphi)_{L^2(\Omega)^d} + h(\psi) - h(\varphi) \geq (q, \psi - \varphi)_W \quad \forall \psi \in W. \quad (61)$$

The proof of Lemma 2 is similar to that of Lemma 1, replacing the spaces Q, V , the functions $\boldsymbol{\omega}_1, \boldsymbol{\sigma}, \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{f}$, and j , the set $\Sigma_1(\boldsymbol{f})$, and the operator $\boldsymbol{\varepsilon}$ by the spaces $L^2(\Omega)^d, W$, the functions $\boldsymbol{\omega}_2, -\boldsymbol{D}, \psi, \varphi, q$, and h , the set $\Sigma_2(q)$, and the operator ∇ , respectively. Since the modifications are straightforward, we skip the details in the proof.

The next step is provided by the following statement.

Lemma 3. *For each $\boldsymbol{f} \in V$, the set $\Sigma_1(\boldsymbol{f}) \subset Q$ is nonempty closed and convex. Moreover, for each $\boldsymbol{f}, \boldsymbol{f}' \in V$, and $\boldsymbol{\omega}_1 \in Q$, one has*

$$\|P_{\Sigma_1(\boldsymbol{f})}\boldsymbol{\omega}_1 - P_{\Sigma_1(\boldsymbol{f}')}\boldsymbol{\omega}_1\|_Q \leq \|\boldsymbol{f} - \boldsymbol{f}'\|_V. \quad (62)$$

Proof. Let $\boldsymbol{f} \in V$ be fixed. Since the function $\boldsymbol{v} \mapsto j(\boldsymbol{v}) : V \rightarrow \mathbb{R}$ is subdifferentiable and vanishes in $\mathbf{0}_V$, we deduce that there exists an element $\boldsymbol{g} \in V$ such that $j(\boldsymbol{v}) \geq (\boldsymbol{g}, \boldsymbol{v})_V$ for all $\boldsymbol{v} \in V$. Moreover, recall that $(\boldsymbol{g}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{g}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q$ and $(\boldsymbol{f}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{f}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q$. Therefore, using the notation $\boldsymbol{\xi} = \boldsymbol{\varepsilon}(\boldsymbol{f}) - \boldsymbol{\varepsilon}(\boldsymbol{g})$, we find that

$$(\boldsymbol{\xi}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\boldsymbol{v}) \geq (\boldsymbol{f}, \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V. \quad (63)$$

We now combine equations (37) and (63) to see that $\boldsymbol{\xi} \in \Sigma_1(\boldsymbol{f})$ and, therefore, $\Sigma_1(\boldsymbol{f})$ is not empty. Conversely, it is easy to see that it is a closed convex subset of Q , which concludes the proof of the first part of the lemma.

Assume now that $\boldsymbol{f}, \boldsymbol{f}' \in V$, and $\boldsymbol{\omega}_1 \in Q$, and denote

$$\boldsymbol{\sigma} = P_{\Sigma_1(\boldsymbol{f})}\boldsymbol{\omega}_1, \quad \boldsymbol{\sigma}' = P_{\Sigma_1(\boldsymbol{f}')}\boldsymbol{\omega}_1. \quad (64)$$

We use equation (10) to see that

$$\sigma \in \Sigma_1(\mathbf{f}), \quad (\omega_1 - \sigma, \tau - \sigma)_Q \leq 0 \quad \forall \tau \in \Sigma_1(\mathbf{f})$$

and, therefore, equation (9) implies that $\omega_1 - \sigma \in N_{\Sigma_1(\mathbf{f})}(\sigma)$. Next, Lemma 1 implies that there exists a unique element $\mathbf{u} \in V$ such that

$$\sigma - \omega_1 = \boldsymbol{\varepsilon}(\mathbf{u}), \quad (65)$$

$$(\sigma, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}))_Q + j(\mathbf{v}) - j(\mathbf{u}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u})_V \quad \forall \mathbf{v} \in V. \quad (66)$$

Similar arguments show that there exists a unique element $\mathbf{u}' \in V$ such that

$$\sigma' - \omega_1 = \boldsymbol{\varepsilon}(\mathbf{u}'), \quad (67)$$

$$(\sigma', \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}'))_Q + j(\mathbf{v}) - j(\mathbf{u}') \geq (\mathbf{f}', \mathbf{v} - \mathbf{u}')_V \quad \forall \mathbf{v} \in V. \quad (68)$$

We now take $\mathbf{v} = \mathbf{u}'$ in equation (66); then we take $\mathbf{v} = \mathbf{u}$ in equation (68) and add the resulting inequalities to obtain

$$(\sigma - \sigma', \boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\mathbf{u}'))_Q \leq (\mathbf{f} - \mathbf{f}', \mathbf{u} - \mathbf{u}')_V.$$

Then, using the identity $\boldsymbol{\varepsilon}(\mathbf{u}') - \boldsymbol{\varepsilon}(\mathbf{u}) = \sigma' - \sigma$, guaranteed by equations (65) and (67), we find that

$$\|\sigma - \sigma'\|_Q^2 \leq \|\mathbf{f} - \mathbf{f}'\|_V \|\mathbf{u} - \mathbf{u}'\|_V.$$

Next, since $\|\mathbf{u} - \mathbf{u}'\|_V = \|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\varepsilon}(\mathbf{u}')\|_Q = \|\sigma - \sigma'\|_Q$, we deduce that

$$\|\sigma - \sigma'\|_Q \leq \|\mathbf{f} - \mathbf{f}'\|_V$$

and, therefore, equation (62) holds. \square

We now proceed with the following result.

Lemma 4. *For each $q \in W$ the set $\Sigma_2(q) \subset L^2(\Omega)^d$ is nonempty, closed, and convex. Moreover, for each $q, q' \in W$ and $\omega_2 \in L^2(\Omega)^d$, one has*

$$\|P_{\Sigma_2(q)}\omega_2 - P_{\Sigma_2(q')}\omega_2\|_{L^2(\Omega)^d} \leq \|q - q'\|_W. \quad (69)$$

The proof of Lemma 4 is similar to that of Lemma 3; therefore, we skip it.

We now have all the ingredients to provide the proof of Theorem 3.

Proof. We consider the product spaces $X = Q \times L^2(\Omega)^d$ and $Y = V \times W$, equipped with their canonical inner products, given by

$$(x, x')_X = (\sigma, \sigma')_Q + (\mathbf{D}, \mathbf{D}')_{L^2(\Omega)^d} \quad \forall x = (\sigma, \mathbf{D}), x' = (\sigma', \mathbf{D}') \in X, \quad (70)$$

$$(y, y')_Y = (\mathbf{u}, \mathbf{u}')_V + (\varphi, \varphi')_W \quad \forall y = (\mathbf{u}, \varphi), y' = (\mathbf{u}', \varphi') \in Y \quad (71)$$

and the associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively.

Next, we introduce the mapping $\Sigma : Y \rightarrow 2^X$ defined by

$$\Sigma(\theta) = \Sigma_1(\mathbf{f}) \times \Sigma_2(q) \quad \forall \theta = (\mathbf{f}, q) \in Y. \quad (72)$$

We use Lemmas 3 and 4 to see that, for each $\theta = (\mathbf{f}, q) \in Y$, the set $\Sigma(\theta)$ is nonempty, closed, and convex. Moreover, since

$$P_{\Sigma(\theta)}(\omega) = \left(P_{\Sigma_1(\mathbf{f})}(\omega_1), P_{\Sigma_2(q)}(\omega_2) \right) \quad \forall \omega = (\omega_1, \omega_2) \in Y,$$

from equations (62) and (69), we deduce that the multivalued mapping (equation (72)) satisfies equation (11).

Furthermore, we consider the operator $A : X \rightarrow X$ defined by

$$(A\mathbf{w}, \mathbf{w}')_X = (\mathcal{F}\mathbf{w}_1 + \mathcal{P}^\top \mathbf{w}_2, \mathbf{w}'_1)_Q + (\boldsymbol{\beta}\mathbf{w}_2 - \mathcal{P}\mathbf{w}_1, \mathbf{w}'_2)_{L^2(\Omega)^d} \quad \forall \mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2), \mathbf{w}' = (\mathbf{w}'_1, \mathbf{w}'_2) \in X. \quad (73)$$

We now investigate the strong monotonicity and Lipschitz continuity of this operator. To this end, we consider two elements: $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$ and $\mathbf{w}' = (\mathbf{w}'_1, \mathbf{w}'_2) \in X$. We have

$$\begin{aligned} (A\mathbf{w} - A\mathbf{w}', \mathbf{w} - \mathbf{w}')_X &= (\mathcal{F}\mathbf{w}_1 - \mathcal{F}\mathbf{w}'_1, \mathbf{w}_1 - \mathbf{w}'_1)_Q + (\boldsymbol{\beta}\mathbf{w}_2 - \boldsymbol{\beta}\mathbf{w}'_2, \mathbf{w}_2 - \mathbf{w}'_2)_{L^2(\Omega)^d} \\ &\quad + (\mathcal{P}^\top \mathbf{w}_2 - \mathcal{P}^\top \mathbf{w}'_2, \mathbf{w}_1 - \mathbf{w}'_1)_Q - (\mathcal{P}\mathbf{w}_1 - \mathcal{P}\mathbf{w}'_1, \mathbf{w}_2 - \mathbf{w}'_2)_{L^2(\Omega)^d} \end{aligned}$$

and, since $(\mathcal{P}^\top \tilde{\mathbf{w}}_2, \tilde{\mathbf{w}}_1)_Q = (\mathcal{P}\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2)_{L^2(\Omega)^d}$ for all $\tilde{\mathbf{w}} = (\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2) \in X$, we deduce that

$$(A\mathbf{w} - A\mathbf{w}', \mathbf{w} - \mathbf{w}')_X = (\mathcal{F}\mathbf{w}_1 - \mathcal{F}\mathbf{w}'_1, \mathbf{w}_1 - \mathbf{w}'_1)_Q + (\boldsymbol{\beta}\mathbf{w}_2 - \boldsymbol{\beta}\mathbf{w}'_2, \mathbf{w}_2 - \mathbf{w}'_2)_{L^2(\Omega)^d}.$$

We now use equations (26)(c) and (28)(d) to see that there exists $c_1 > 0$, which depends only on \mathcal{F} , $\boldsymbol{\beta}$, Ω such that

$$(A\mathbf{w} - A\mathbf{w}', \mathbf{w} - \mathbf{w}')_X \geq c_1 (\|\mathbf{w}_1 - \mathbf{w}'_1\|_Q^2 + \|\mathbf{w}_2 - \mathbf{w}'_2\|_{L^2(\Omega)^d}^2)$$

and, keeping in mind equation (70), we obtain

$$(A\mathbf{w} - A\mathbf{w}', \mathbf{w} - \mathbf{w}')_X \geq c_1 \|\mathbf{w} - \mathbf{w}'\|_X^2. \quad (74)$$

In the same way, using equations (26) to (28), after some algebra, it follows that there exists $c_2 > 0$, which depends only on \mathcal{F} , $\boldsymbol{\beta}$, and \mathcal{P} such that

$$\begin{aligned} (A\mathbf{w} - A\mathbf{w}', \tilde{\mathbf{w}})_X &\leq c_2 (\|\mathbf{w}_1 - \mathbf{w}'_1\|_Q \|\tilde{\mathbf{w}}_1\|_Q + \|\mathbf{w}_2 - \mathbf{w}'_2\|_{L^2(\Omega)^d} \|\tilde{\mathbf{w}}_2\|_{L^2(\Omega)^d} \\ &\quad + \|\mathbf{w}_2 - \mathbf{w}'_2\|_{L^2(\Omega)^d} \|\tilde{\mathbf{w}}_1\|_Q + \|\mathbf{w}_1 - \mathbf{w}'_1\|_Q \|\tilde{\mathbf{w}}_2\|_{L^2(\Omega)^d}) \end{aligned}$$

for all $\tilde{\mathbf{w}} = (\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2) \in X$. Therefore, using equation (70), we see that

$$(A\mathbf{w} - A\mathbf{w}', \tilde{\mathbf{w}})_X \leq 4c_2 \|\mathbf{w} - \mathbf{w}'\|_X \|\tilde{\mathbf{w}}\|_X \quad \forall \tilde{\mathbf{w}} \in X$$

and, taking $\tilde{\mathbf{w}} = A\mathbf{w} - A\mathbf{w}' \in X$, we find

$$\|A\mathbf{w} - A\mathbf{w}'\|_X \leq 4c_2 \|\mathbf{w} - \mathbf{w}'\|_X. \quad (75)$$

Equations (74) and (75) show that the operator A defined by equation (73) satisfies equation (12) on the space X .

It follows from this that we are in a position to use Theorem 1 to conclude the proof of Theorem 3. \square

Next, inspired by the inclusions of equations (45) and (47) and the notation of equations (43) and (48), we introduce the following definition.

Definition 1. A quadruple of functions $(\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D}) \in V \times Q \times W \times L^2(\Omega)^d$, which satisfies equations (15) and (16) and the inclusions

$$-\boldsymbol{\varepsilon}(\mathbf{u}) \in N_{\Sigma_1(\mathcal{F})}(\boldsymbol{\sigma}), \quad (76)$$

$$-\nabla\varphi \in N_{\Sigma_2(\mathcal{Q})}(-\mathbf{D}), \quad (77)$$

is called a weak solution to the electro-elastic contact problem (equations (15) to (25)).

Using this definition, we deduce the following consequence of Theorem 3.

Corollary 1. Assume equations (26)–(32). Then Problem 1 has a unique weak solution $(\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D})$. Moreover, the operator $(\mathbf{f}_0, \mathbf{f}_2, q_0, q_2) \mapsto (\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D}) : L^2(\Omega)^d \times L^2(\Gamma_2)^d \times L^2(\Omega) \times L^2(\Gamma_b) \rightarrow V \times Q \times W \times L^2(\Omega)^d$ is Lipschitz continuous.

Proof. Let $\boldsymbol{\omega} = (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) \in Q \times L^2(\Omega)^d$ be the solution of Problem 2 obtained in Theorem 3. From Lemma 1 and Lemma 2, we know that there exists a unique couple of functions $(\mathbf{u}, \varphi) \in V \times W$ such that $\boldsymbol{\omega}_1 = \boldsymbol{\varepsilon}(\mathbf{u})$ and $\boldsymbol{\omega}_2 = \nabla\varphi$. Now define the functions $\boldsymbol{\sigma}$ and \mathbf{D} by equations (15) and (16). Then, it is easy to see that the quadruple $(\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D})$ is the unique weak solution of Problem 1. Moreover, recall that

$$\|\mathbf{u}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{u})\|_Q = \|\boldsymbol{\omega}_1\|_Q, \quad \|\varphi\|_W = \|\nabla\varphi\|_{L^2(\Omega)^d} = \|\boldsymbol{\omega}_2\|_{L^2(\Omega)^d}$$

and, in addition, Theorem 3 guarantees that the operators

$$(\mathbf{f}, q) \mapsto \boldsymbol{\omega}_1 = \boldsymbol{\omega}_1(\mathbf{f}, q) : V \times W \rightarrow Q, \quad (\mathbf{f}, q) \mapsto \boldsymbol{\omega}_2 = \boldsymbol{\omega}_2(\mathbf{f}, q) : V \times W \rightarrow L^2(\Omega)^d$$

are Lipschitz continuous. Therefore, we deduce that

$$(\mathbf{f}, q) \mapsto \mathbf{u} = \mathbf{u}(\mathbf{f}, q) : V \times W \rightarrow V, \quad (\mathbf{f}, q) \mapsto \varphi = \varphi(\mathbf{f}, q) : V \times W \rightarrow W$$

are Lipschitz continuous operators, too. Next, we use equations (15) and (16) and the properties of \mathcal{F} , \mathcal{P} , and $\boldsymbol{\beta}$ to see that the operators

$$(\mathbf{f}, q) \mapsto \boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{f}, q) : V \times W \rightarrow Q, \quad (\mathbf{f}, q) \mapsto \mathbf{D} = \mathbf{D}(\mathbf{f}, q) : V \times W \rightarrow L^2(\Omega)^d$$

are Lipschitz continuous. Finally, equations (35) and (36) imply that the operators

$$(\mathbf{f}_0, \mathbf{f}_2) \mapsto \mathbf{f} : L^2(\Omega)^d \times L^2(\Gamma_2)^d \rightarrow V, \quad (q_0, q_b) \mapsto q : L^2(\Omega) \times L^2(\Gamma_b) \rightarrow W$$

are Lipschitz continuous. It follows from here that the operator $(\mathbf{f}_0, \mathbf{f}_2, q_0, q_b) \mapsto (\mathbf{u}, \boldsymbol{\sigma}, \varphi, \mathbf{D})$, which maps the data $(\mathbf{f}_0, \mathbf{f}_2, q_0, q_b)$ into the weak solution of Problem 1 is a composition of Lipschitz continuous operators, which concludes the proof. \square

5. An optimal control problem

We now associate to Problem 2 an optimal control problem. To this end, we keep the functional framework in Section 4 and use notation “ \rightarrow ”, “ \rightharpoonup ” for the strong and weak convergence in various spaces that will be specified. Moreover, all the following limits are considered as $n \rightarrow \infty$, even if we do not mention it explicitly.

Next, we denote by Z the product space $Z = L^2(\Gamma_2)^d \times L^2(\Gamma_b)$ and we assume that \mathbf{f}_0 and q_0 are given functions, which have the regularity

$$\mathbf{f}_0 \in L^2(\Omega)^d, \quad q_0 \in L^2(\Omega). \quad (78)$$

In addition, we consider the operator $B : Z \rightarrow Y$ defined by

$$\theta = Bz \iff \theta = (\mathbf{f}, q), \quad z = (\mathbf{f}_2, q_b) \quad \text{and equations (35) and (36) hold.} \quad (79)$$

Then, under equations (26) to (28), (31), (32), and (78), it follows from Theorem 3 that for each $z = (\mathbf{f}_2, q_b) \in Z$, Problem 2 has a unique solution $\boldsymbol{\omega} = \boldsymbol{\omega}(Bz)$.

We now define the set of admissible pairs for Problem 2 by

$$\mathcal{V}_{\text{ad}} = \{(\boldsymbol{\omega}, z) : z \in Z, \boldsymbol{\omega} = \boldsymbol{\omega}(Bz)\}. \quad (80)$$

In other words, a pair $(\boldsymbol{\omega}, z)$ belongs to \mathcal{V}_{ad} if and only if $z \in Z$ and, moreover, $\boldsymbol{\omega}$ is the solution of Problem 2 with $(\mathbf{f}, q) = Bz$. Consider also a cost functional $\mathcal{L} : X \times Z \rightarrow \mathbb{R}$. Then, the optimal control problem we are interested in is stated as follows.

Problem (Q). Find $(\boldsymbol{\omega}^*, z^*) \in \mathcal{V}_{\text{ad}}$ such that

$$\mathcal{L}(\boldsymbol{\omega}^*, z^*) = \min_{(\boldsymbol{\omega}, z) \in \mathcal{V}_{\text{ad}}} \mathcal{L}(\boldsymbol{\omega}, z). \quad (81)$$

In the study of this problem, we consider the following assumptions:

$$\left\{ \begin{array}{l} \text{For all sequences } \{\boldsymbol{\omega}_n\} \subset X \text{ and } \{z_n\} \subset Z \text{ such that } \boldsymbol{\omega}_n \rightarrow \boldsymbol{\omega} \text{ in } X, \ z_n \rightharpoonup z \text{ in } Z, \text{ we have} \\ \liminf_{n \rightarrow \infty} \mathcal{L}(\boldsymbol{\omega}_n, z_n) \geq \mathcal{L}(\boldsymbol{\omega}, z). \end{array} \right. \quad (82)$$

$$\left\{ \begin{array}{l} \text{There exists } \mathcal{H} : Z \rightarrow \mathbb{R} \text{ such that} \\ \text{(a) } \mathcal{L}(\boldsymbol{\omega}, z) \geq \mathcal{H}(z) \quad \forall \boldsymbol{\omega} \in X, \ z \in Z. \\ \text{(b) } \|z_n\|_Z \rightarrow +\infty \implies \mathcal{H}(z_n) \rightarrow \infty. \end{array} \right. \quad (83)$$

Example 1. A typical example of function \mathcal{L} that satisfies equations (82) and (83) is obtained by taking

$$\mathcal{L}(\omega, z) = \mathcal{G}(\omega) + \mathcal{H}(z) \quad \forall \omega \in X, z \in Z,$$

where $\mathcal{G} : X \rightarrow \mathbb{R}$ is a continuous positive function and $\mathcal{H} : Z \rightarrow \mathbb{R}$ is a weakly lower semicontinuous coercive function, i.e., it satisfies equation (83)(b).

Our main result in this section is the following existence result.

Theorem 4. *Assume equations (26)–(28), (31), (32), (78), (82), and (83). Then Problem \mathcal{Q} has at least one solution (ω^*, z^*) .*

The proof of Theorem 4 is based on a preliminary result that we present in what follows.

Lemma 5. *The operator $B : Z \rightarrow Y$ defined by equation (79) is a compact operator, i.e.,*

$$z_n \rightharpoonup z \text{ in } Z \implies Bz_n \rightarrow Bz \text{ in } Y. \quad (84)$$

Proof. Consider a sequence $\{z_n\} \subset Z$ such that $z_n = (f_{2n}, q_{bn})$ for each $n \in \mathbb{N}$ and $z_n \rightharpoonup z$ in Z , where $z = (f_2, q_b) \in Z$. Then,

$$f_{2n} \rightharpoonup f_2 \text{ in } L^2(\Gamma_2)^d \text{ and } q_{bn} \rightharpoonup q_b \text{ in } L^2(\Gamma_b). \quad (85)$$

Next, for each $n \in \mathbb{N}$, we define the elements $f_n \in V$ and q_n as

$$(f_n, v)_V = \int_{\Omega} f_0 \cdot v dx + \int_{\Gamma_2} f_{2n} \cdot v da \quad \forall v \in V, \quad (86)$$

$$(q_n, \psi)_W = \int_{\Omega} q_0 \psi dx - \int_{\Gamma_b} q_{bn} \psi da \quad \forall \psi \in W. \quad (87)$$

Then, using equations (35), (86), and (85), we find that

$$f_n \rightharpoonup f \text{ in } V \quad (88)$$

and, by a classical compactness result, we have

$$f_n \rightarrow f \text{ in } L^2(\Gamma_2)^d. \quad (89)$$

Moreover, a simple calculation based on equations (35) and (86) shows that

$$\begin{aligned} \|f_n - f\|_V^2 &= (f_n, f_n - f)_V - (f, f_n - f)_V \\ &= \int_{\Omega} f_0 (f_n - f) dx + \int_{\Gamma_2} f_{2n} (f_n - f) da - \int_{\Omega} f_0 (f_n - f) dx - \int_{\Gamma_2} f_2 (f_n - f) da \\ &= \int_{\Gamma_2} (f_{2n} - f_2) (f_n - f) da. \end{aligned}$$

We now use equations (85) and (89) to see that

$$\|f_n - f\|_V^2 = \int_{\Gamma_2} (f_{2n} - f_2) (f_n - f) da \rightarrow 0,$$

which implies that $f_n \rightarrow f$ in V . A similar argument, based on equations (36) and (87), shows that $q_n \rightarrow q$ in W . We conclude from this that $(f_n, q_n) \rightarrow (f, q)$ in $Y = V \times W$ and, therefore, the definition (equation (79)) of the operator B shows that $Bz_n \rightarrow Bz$ in Y . This proves the implication (equation (84)) and concludes the proof. \square

We are now in a position to provide the proof of Theorem 4.

Proof. Let $\mathcal{S} : Z \rightarrow \mathbb{R}$ be the function defined by

$$\mathcal{S}(z) = \mathcal{L}(\boldsymbol{\omega}(Bz), z) \quad \forall z \in Z. \quad (90)$$

Then, we consider the following auxiliary problem:

$$\text{find } z^* \in Z \text{ such that } \mathcal{S}(z^*) = \min_{z \in Z} \mathcal{S}(z). \quad (91)$$

We claim that this problem has at least one solution z^* and, to this end, we use Theorem 2.

Indeed, consider a sequence $\{z_n\} \subset Z$ such that $z_n \rightarrow z$ in Z . Then, Lemma 5 shows that $Bz_n \rightarrow Bz$ in Y and, therefore, Theorem 3 implies that $\boldsymbol{\omega}(Bz_n) \rightarrow \boldsymbol{\omega}(Bz)$ in X . We now use equations (90) and (82) to see that

$$\liminf_{n \rightarrow \infty} \mathcal{S}(z_n) \geq \mathcal{S}(z).$$

We conclude from here that the function $\mathcal{S} : Z \rightarrow \mathbb{R}$ is weakly lower semicontinuous. In addition, using equation (83)(a), for any sequence $\{z_n\} \subset Z$, we have

$$\mathcal{S}(z_n) = \mathcal{L}(\boldsymbol{\omega}(Bz_n), z_n) \geq \mathcal{H}(z_n) \quad \forall n \in \mathbb{N}.$$

Therefore, if $\|z_n\|_Z \rightarrow \infty$, by equation (83)(b) we deduce that $\mathcal{S}(z_n) \rightarrow \infty$, which shows that the function \mathcal{S} is coercive. Recall also the reflexivity of the space Z . The existence of at least one solution to equation (91) is now a direct consequence of Theorem 2.

Finally, using equations (80), (81), and (90), it is easy to see that

$$\begin{cases} (\boldsymbol{\omega}^*, z^*) \text{ is a solution of Problem } \mathcal{Q} \text{ if and only if} \\ z^* \text{ is a solution of equation (91) and } \boldsymbol{\omega}^* = \boldsymbol{\omega}(Bz^*). \end{cases} \quad (92)$$

Theorem 4 is a direct consequence of equation (92) combined with the solvability of the optimization problem (equation (91)). \square

Theorem 4 combined with Corollary 1 allows us to prove the existence of optimal pairs for various optimal control problems associated to the electro-elastic contact problem (equations (15) to (25)). Although some general results can be obtained in this matter, we restrict ourselves to providing only a representative example.

Example 2. Assume equations (26) to (28), (31), (32), (78), and, for each $z = (f_2, q_b) \in Z$, denote by $(\boldsymbol{u}_z, \boldsymbol{\sigma}_z, \varphi_z, \boldsymbol{D}_z)$ the weak solution of Problem 1. Let $\phi \in L^2(\Gamma_2)$ and let a_3, a_2 , and a_b be given positive constants. Then, we consider the following optimal control problem: find $z^* = (f_2^*, q_b^*) \in Z$ such that

$$\begin{cases} a_3 \int_{\Gamma_3} (\varphi_{z^*} - \phi)^2 da + a_2 \int_{\Gamma_2} (f_2^*)^2 da + a_b \int_{\Gamma_b} (q_b^*)^2 da \\ \leq a_3 \int_{\Gamma_3} (\varphi_z - \phi)^2 da + a_2 \int_{\Gamma_2} f_2^2 da + a_b \int_{\Gamma_b} q_b^2 da \end{cases} \quad \forall z = (f_2, q_b) \in Z. \quad (93)$$

Let $T : X \rightarrow W$ be the operator which associates to each element $\boldsymbol{\omega} = (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) \in X$ the element $\varphi \in W$ such that $\nabla \varphi = \boldsymbol{\omega}_2$ and consider the cost function $\mathcal{L} : X \times Z \rightarrow \mathbb{R}$ given by

$$\mathcal{L}(\boldsymbol{\omega}, z) = a_3 \int_{\Gamma_3} (T\boldsymbol{\omega} - \phi)^2 da + a_2 \int_{\Gamma_2} f_2^2 da + a_b \int_{\Gamma_b} q_b^2 da \quad \forall \boldsymbol{\omega} \in X, z = (f_2, q_b) \in Z \quad (94)$$

Then it is easy to see that equation (93) is of the form of equation (81), with the choice of equation (94) for the cost functional. Note that the arguments in the proof of Corollary 1 show that the operator $T : X \rightarrow W$ is continuous. Using this property, it is easy to see that the cost functional (equation (94)) satisfies equations (82) and (83). Therefore, Theorem 4 guarantees the existence of the solutions of the optimal control problem (equation (93)).


The mechanical interpretation of equation (93) is the following: given a contact process of the form of equations (15) to (25), we are looking for a density of surface tractions $f_2 \in L^2(\Gamma_2)^d$ and a density of electric

charges $q_b \in L^2(\Gamma_b)$ such that, on the contact surface Γ_3 , the corresponding electric potential φ is as close as possible to the “desired” electric potential ϕ . Furthermore, this choice must fulfill a minimum expenditure condition, which is taken into account by the last two terms in the cost functional (equation (94)). In fact, a compromise policy between the three aims (“ φ close to ϕ on Γ_3 ”, “minimal density of surface tractions f_2 ,” and “minimal density of electric charges q_b ”) must be found. The relative importance of each criterion with respect to the others is expressed by the choice of the weighting coefficients a_3 , a_2 , and a_b .

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