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# Weak Formulations of Quasistatic Frictional Contact Problems

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## Abstract

We consider a general mathematical model which describes the quasistatic contact of a deformable body with an obstacle, the so-called foundation. The material's behaviour is modeled with a visco-elastic-type constitutive law and the contact is described with a general interface law associated to a version of Coulomb's law of dry friction. We list the **assumptions** on the data and provide relevant examples of constitutive laws and boundary conditions. Then, we derive two different variational formulations of the model in which the **unknowns are** the displacement and the strain field, respectively. We prove the equivalence of these formulations. Finally, we use recent arguments of sweeping process in order to obtain the existence of a unique weak solution to the contact model.

**AMS Subject Classification :** 74M15, 74M10, 74G25, 47J22, 49J40, 47J20, 34G25.

**Key words :** viscoelastic material, frictional contact problem, variational formulation, sweeping process, history-dependent operator.

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Contact phenomena arise in industry and everyday life. They are modeled by strongly nonlinear boundary value problems which, usually, do not have classical solutions. For this reason, in the last decades, a considerable effort has been done in the study of variational analysis of different contact models. The literature in the field is extensive. It includes the books [9, 10, 12, 25, 26] and, more recently [7, 27, 28]. There, various models of contact have been considered together with their variational formulations. Then, existence and uniqueness results have been obtained by using various functional arguments, including arguments of monotonicity, convexity, nonsmooth analysis, multivalued analysis and fixed point. The numerical analysis of various models of contact can be found in [12, 13, 15], for instance.

The notion of variational formulation for a contact problem varies from author to author and even from paper to paper. For contact models which have a convex structure, most of the formulations considered in the literature are in a form of a variational inequality in which the unknown is either the displacement or the velocity field. References in the field include [7, 12, 25, 27]. The contact models formulated in terms of locally Lipschitz functions lead to hemivariational inequalities. References in the field are [26, 28], for instance. Weak formulations of contact problems in which the unknown is the stress field are also called dual formulations. Usually, such formulations lead to variational inequalities or inclusions. Some examples have been considered in [27, 29] for the convex case and in [14, 28] for the nonconvex case. There, existence, uniqueness and equivalence results have been obtained.

Sweeping processes are differential inclusions governed by the normal cones of a family of convex moving sets. Introduced in early seventy's in the pioneering works of Moreau [18, 19, 20], sweeping processes have been intensively studied in the last decades, as illustrated in [8, 11, 16, 17] and, more recently, in [6, 21, 24, 30]. Arguments of sweeping process in the variational analysis of mathematical models of contact have been considered in [1, 2, 22, 23]. There, abstract existence and uniqueness results for various classes of sweeping processes have been obtained by using the properties of history-dependent operators. Then, these results have been used in the study of frictionless or frictional contact models with viscoelastic materials. The sweeping process considered in [1, 2] was formulated in terms of displacement while the sweeping process considered in [22, 23] was formulated in terms of the strain field.

The aim of this current paper is twofold. The first one is to establish two different variational formulations for quasistatic frictional contact problems with viscoelastic materials and to prove their equivalence. Thus, we consider a general class of contact problems with a convex structure for which we provide a first variational formulation in which the unknown is the displacement field and a sweeping process formulation in which the unknown is the strain field. Deriving these formulations and proving their equivalence show that the displacement field and the strain field play symmetric roles

in the structure of the mathematical models of contact, which represents the first trait of novelty of this paper. Our second aim is to deduce existence and uniqueness results for the corresponding contact problems and, to this end, we use a sweeping process argument. At the best of our knowledge this represents the second trait of novelty of this paper.

The rest of the paper is organized as follows. In Section 2 we present preliminary material needed in the rest of the paper. In Section 3 we introduce the general contact model considered and list the assumptions on the data. Then we provide examples of constitutive laws and boundary conditions which satisfy these assumptions. Section 4 is devoted to the weak formulations of the contact models while Section 5 provides their equivalence. The unique weak solvability of the models is presented in Section 6. We end this paper with Section 7 in which we present some concluding remarks.

## 2 Preliminaries

In this section we introduce some notation and preliminary material. The notation we introduce here will be used everywhere in the next sections, associated to particular choices of spaces and operators. All the function spaces we consider in this paper are real spaces, even if we do not mention it explicitly.

**Function spaces in Contact Mechanics.** Everywhere below  $d \in \{2, 3\}$  and  $\mathbb{S}^d$  stands for the space of second order symmetric tensors on  $\mathbb{R}^d$ . Moreover “ $\cdot$ ”,  $\|\cdot\|$  and  $\mathbf{0}$  represent the inner product, the Euclidean norm and the zero element of the spaces  $\mathbb{R}^d$  and  $\mathbb{S}^d$ , respectively. In addition,  $\Omega \subset \mathbb{R}^d$  is a bounded domain with a Lipschitz continuous boundary divided into three mutually disjoint measurable sets  $\Gamma_1, \Gamma_2$  and  $\Gamma_3$ , such that the measure of  $\Gamma_1$  is positive. A typical point in  $\Omega \cup \Gamma$  will be denoted by  $\mathbf{x}$  and the outward unit normal at  $\Gamma$  will be denoted by  $\boldsymbol{\nu}$ . Nevertheless, for simplicity, we sometimes skip the dependence of variables with respect to  $\mathbf{x}$ .

We use the standard notation for the Lebesgue and Sobolev spaces associated to  $\Omega$  and  $\Gamma$ . For an element  $\mathbf{v} \in H^1(\Omega)$  we still write  $\mathbf{v}$  for its trace  $\gamma\mathbf{v} \in L^2(\Gamma)$  and  $v_\nu, \mathbf{v}_\tau$  for the normal and tangential traces on the boundary, i.e.,  $v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$  and  $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$ . Moreover,  $\boldsymbol{\varepsilon}(\mathbf{v})$  will denote the symmetric part of the gradient of  $\mathbf{v}$ , i.e.,

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + \nabla^T\mathbf{v}).$$

In addition, we shall use the function spaces

$$V = \{ \mathbf{v} = (v_i) : v_i \in H^1(\Omega), v_i = 0 \text{ on } \Gamma_1 \quad \forall i = 1, \dots, d \}, \quad (2.1)$$

$$Q = \{ \boldsymbol{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \quad \forall i, j = 1, \dots, d \}. \quad (2.2)$$

These are real Hilbert spaces endowed with the inner products

$$(\mathbf{u}, \mathbf{v})_V = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, dx \quad (2.3)$$

and the associated norms  $\|\cdot\|_V$  and  $\|\cdot\|_Q$ , respectively. Note that, by definition, we have

$$\|\mathbf{v}\|_V = \|\varepsilon(\mathbf{v})\|_Q \quad \text{for all } \mathbf{v} \in V. \quad (2.4)$$

Finally, using the Sobolev trace theorem yields

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq c_{tr} \|\mathbf{v}\|_V \quad \text{for all } \mathbf{v} \in V, \quad (2.5)$$

where  $c_{tr}$  is a positive constant depending on  $\Omega$ ,  $\Gamma_1$  and  $\Gamma_3$ .

**History-dependent operators.** Let  $(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$  be normed spaces and let  $I$  be a time interval of the form  $I = [0, T]$  with  $T > 0$  or the unbounded interval  $\mathbb{R}_+ = [0, +\infty)$ . We denote by  $C(I; X)$  the vector space of continuous functions defined on  $I$  with values in  $X$ . For any differentiable function  $v : I \rightarrow X$  we use  $\dot{v}$  for the derivative of  $v$  with respect to the time variable  $t \in I$ . We also denote by  $C^1(I; X)$  the vector space of continuously differentiable functions on  $I$  with values in  $X$ . Obviously, the inclusion  $v \in C^1(I; X)$  holds if and only if  $v \in C(I; X)$  and  $\dot{v} \in C(I; X)$ . Moreover, it is well known that for any function  $v \in C^1(I; X)$ , the following equality holds:

$$v(t) = \int_0^t \dot{v}(s) ds + v(0) \quad \text{for all } t \in I. \quad (2.6)$$

We use similar notation for the vector spaces of continuous and continuously differentiable functions on  $I$  with values in  $Y$  which will be denoted by  $C(I; Y)$  and  $C^1(I; Y)$ , respectively.

Assume now that  $A : X \rightarrow Y$  and  $\Lambda : C(I; X) \rightarrow C(I; Y)$ . Then, for any function  $u \in C(I; X)$  we use the shorthand notation  $\Lambda u(t)$  to represent the value of the function  $\Lambda u$  at the point  $t \in I$ , that is,  $\Lambda u(t) := (\Lambda u)(t)$ . Moreover,  $A + \Lambda$  will represent a shorthand notation for the operator which maps any function  $u \in C(I; X)$  to the function  $t \mapsto Au(t) + \Lambda u(t) \in C(I; Y)$ .

The next definition introduces three important classes of operators defined on the space of continuous functions.

**Definition 2.1.** *Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two normed spaces. An operator  $\Lambda : C(I; X) \rightarrow C(I; Y)$  is said to be a pseudo history-dependent operator if for any nonempty compact set  $\mathcal{J} \subset I$ , there exist  $l_{\mathcal{J}}^{\Lambda} \geq 0$  and  $L_{\mathcal{J}}^{\Lambda} \geq 0$  such that*

$$\|\Lambda u_1(t) - \Lambda u_2(t)\|_Y \leq l_{\mathcal{J}}^{\Lambda} \|u_1(t) - u_2(t)\|_X + L_{\mathcal{J}}^{\Lambda} \int_0^t \|u_1(s) - u_2(s)\|_X ds \quad (2.7)$$

for all  $u_1, u_2 \in C(I; X)$  and  $t \in \mathcal{J}$ . If, in particular,  $l_{\mathcal{J}}^{\Lambda} \in [0, 1)$  for any nonempty compact set  $\mathcal{J} \subset I$ , then  $\Lambda$  is said to be an almost history-dependent operator and, if  $l_{\mathcal{J}}^{\Lambda} = 0$ , then  $\Lambda$  is said to be a history-dependent operator.

History-dependent and almost history-dependent operators arise in Contact Mechanics and Functional Analysis. For their basic properties we refer the reader to [28, Ch 2.]. We now complete Definition 2.1 with some elementary properties. To this end we denote by  $X \times X$  the product space endowed with norm  $\|\cdot\|_{X \times X}$  given by

$$\|\xi\|_{X \times X} = (\|x\|_X^2 + \|y\|_X^2)^{\frac{1}{2}} \quad \text{for all } \xi = (x, y) \in X \times X \quad (2.8)$$

and we recall that the following inequalities holds:

$$\|\xi\|_{X \times X} \leq \|x\|_X + \|y\|_X \leq \sqrt{2} \|\xi\|_{X \times X} \quad \text{for all } \xi = (x, y) \in X \times X. \quad (2.9)$$

Moreover, if  $X$  is a Hilbert space, then  $X \times X$  is a Hilbert space endowed with the canonical inner product.

In the rest of this paper we shall use the following elementary results.

**Proposition 2.2.** *Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two normed spaces and let  $\Lambda_1, \Lambda_2: C(I; X) \rightarrow C(I; Y)$  be two pseudo history-dependent operators. Define  $\Lambda: C(I; X) \rightarrow C(I; Y)$  by equality*

$$\Lambda v(t) = \Lambda_1 v(t) + \Lambda_2 v(t) \quad (2.10)$$

for all  $v \in C(I; X)$ ,  $t \in I$ . Then,  $\Lambda$  is a pseudo history-dependent operator.

**Proposition 2.3.** *Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two normed spaces,  $v_0 \in X$  and let  $\Lambda: C(I; X \times X) \rightarrow C(I; Y)$  be a pseudo history-dependent operator. Define  $J_{v_0}: C(I; X) \rightarrow C(I; X)$  and  $\Lambda_{v_0}: C(I; X) \rightarrow C(I; Y)$  by equalities*

$$J_{v_0} v(t) = \int_0^t v(s) ds + v_0, \quad \Lambda_{v_0} v(t) = \Lambda(J_{v_0} v, v)(t) \quad (2.11)$$

for all  $v \in C(I; X)$ ,  $t \in I$ . Then  $J_{v_0}$  is a history-dependent operator and  $\Lambda_{v_0}$  is a pseudo history-dependent operator. Moreover, if  $\Lambda$  is an almost history-dependent operator then  $\Lambda_{v_0}$  is an almost history-dependent operator, too. Finally, if the operator  $\Lambda$  is history-dependent then the operator  $\Lambda_{v_0}$  is history-dependent operator.

The proofs of Propositions 2.2 and 2.3 follow directly from Definition 2.1 and inequality (2.9) and, for this reason, we skip them. Moreover, with the notation in Proposition 2.3, it is easy to see that, for any nonempty compact set  $\mathcal{J} \subset I$ , we may assume that

$$l_{\mathcal{J}}^{\Lambda_{v_0}} \leq l_{\mathcal{J}}^{\Lambda}. \quad (2.12)$$

This inequality could be strict as it follows from the examples we present in Section 6 of this paper.

**Elements of convex analysis.** Below in this section  $X$  represents a real Hilbert space endowed with the inner product  $(\cdot, \cdot)_X$  and its associated norm  $\|\cdot\|_X$ , and  $2^X$  denotes the set of parts of  $X$ . For any nonempty closed convex subset  $K$  of  $X$  we

denoted by  $P_K : X \rightarrow K$  the projection operator on  $K$  and by  $N_K : X \rightarrow 2$  the outward normal cone of  $K$ , i.e.,

$$\begin{cases} \text{For any } f \in X, P_K f \text{ is the unique element of } K \text{ which satisfies} \\ \text{the inequality } \|f - P_K f\|_X \leq \|f - v\|_X \text{ for all } v \in K. \end{cases} \quad (2.13)$$

$$N_K(u) := \begin{cases} \{ \xi \in X : (\xi, v - u)_X \leq 0 \quad \forall v \in K \} & \text{if } u \in K, \\ \emptyset & \text{if } u \notin K. \end{cases} \quad (2.14)$$

It is well known that

$$u = P_K f \iff u \in K, \quad (u, v - u)_X \geq (f, v - u)_X \quad \text{for all } v \in K \quad (2.15)$$

and, using (2.14), we see that the following equivalence holds for all  $u, \xi \in X$ :

$$\xi \in N_K(u) \iff u \in K, \quad (\xi, v - u)_X \leq 0 \quad \text{for all } v \in K. \quad (2.16)$$

Moreover, combining the equivalences (2.15) and (2.16) it follows that

$$f - P_K f \in N_K(P_K f) \quad \text{for all } f \in X. \quad (2.17)$$

Given a function  $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , its subdifferential (in the sense of convex analysis) is the multivalued operator  $\partial\varphi : X \rightarrow 2^X$  defined by

$$\partial\varphi(u) := \{ \xi \in X : \varphi(v) - \varphi(u) \geq (\xi, v - u)_X \quad \forall v \in X \} \quad \text{for all } u \in X. \quad (2.18)$$

It follows from above that  $N_K$  represents the subdifferential (in the sense of convex analysis) of the indicator function  $\psi_K : X \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by

$$\psi_K(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

**Sweeping process.** Assume now that  $(Y, (\cdot, \cdot)_Y)$  is a Hilbert space. Consider a set-valued mapping  $K : Y \times I \rightarrow 2^X$ , the operators  $A : X \rightarrow X$ ,  $\mathcal{S} : C(I; X) \rightarrow C(I; X)$ ,  $\mathcal{R} : C(I; X) \rightarrow C(I; Y)$  and an element  $u_0$  which satisfy the following conditions.

- (K)  $K : Y \times I \rightarrow 2^X$  has nonempty closed convex values and, moreover:
- (a) the mapping  $(\theta, t) \mapsto P_{K(\theta, t)} u : Y \times I \rightarrow X$  is continuous, for all  $u \in X$ ;
  - (b) there exists  $c_0 > 0$  such that, for each  $\theta_1, \theta_2 \in Y$ ,  $t \in I$  and  $u \in X$ , one has

$$\|P_{K(\theta_1, t)} u - P_{K(\theta_2, t)} u\|_X \leq c_0 \|\theta_1 - \theta_2\|_Y. \quad (2.19)$$

- (A)  $A : X \rightarrow X$  is a strongly monotone Lipschitz continuous operator with constants  $m_A$  and  $L_A > 0$ , i.e.,

$$(Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \text{for all } u, v \in X. \quad (2.20)$$

$$\|Au - Av\|_X \leq L_A \|u - v\|_X \quad \text{for all } u, v \in X. \quad (2.21)$$

- ( $\mathcal{R}$ )  $\mathcal{R} : C(I; X) \rightarrow C(I; Y)$  is a pseudo history-dependent operator.
- ( $\mathcal{S}$ )  $\mathcal{S} : C(I; X) \rightarrow C(I; X)$  is a pseudo history-dependent operator.
- ( $\mathcal{U}$ )  $u_0 \in X$ .

With these data we consider the problem of finding a function  $u : I \rightarrow X$  such that

$$\begin{cases} -\dot{u}(t) \in N_{K(\mathcal{R}\dot{u}(t), t)}(A\dot{u} + \mathcal{S}\dot{u}(t)) & \text{for all } t \in I, \\ u(0) = u_0. \end{cases} \quad (2.22)$$

The unique solvability of this sweeping process problem is given by the following existence and uniqueness result.

**Theorem 2.4.** *Assume ( $\mathcal{K}$ ), ( $\mathcal{A}$ ), ( $\mathcal{R}$ ), ( $\mathcal{S}$ ) and ( $\mathcal{U}$ ). Then there exists a constant  $\tilde{c}$  which depends only on  $m_A$ ,  $L_A$  and  $c_0$  such that the sweeping process (2.22) has a unique solution with regularity  $u \in C^1(I; X)$  provided that, for any **nonempty compact** set  $\mathcal{J} \subset I$ , the following inequality holds:*

$$l_{\mathcal{J}}^{\mathcal{R}} + l_{\mathcal{J}}^{\mathcal{S}} \leq \tilde{c}. \quad (2.23)$$

Theorem 2.4 was proved in [23], based on arguments of convex analysis and a fixed point result for almost history-dependent operators. We shall use it in Section 5 of the current paper.

### 3 The contact model

In this section we introduce a general mathematical model describing the mechanical state of a deformable body that occupies the domain  $\Omega$ , in the time interval of interest  $I$ . The body is fixed on the part  $\Gamma_1$  of its boundary, is acted upon by traction forces on  $\Gamma_2$  and is in potential contact on  $\Gamma_3$  with an obstacle, the so-called foundation. The model we consider contains as particular cases several models studied in the literature. It is based on the following mechanical assumptions: the material's behavior is viscoelastic, the process is quasistatic, the contact is frictional and the foundation is deformable, i.e., it allows penetration. Then, the problem under consideration is stated as follows.

**Problem  $\mathcal{P}$ .** *Find a displacement field  $\mathbf{u} : \Omega \times I \rightarrow \mathbb{R}^d$  and a stress field  $\boldsymbol{\sigma} : \Omega \times I \rightarrow \mathbb{S}^d$  such that*



$$\boldsymbol{\sigma}(t) = A\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{S}(\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\dot{\mathbf{u}}))(t), \quad \text{in } \Omega, \quad (3.1)$$

$$\text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \text{in } \Omega, \quad (3.2)$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_1, \quad (3.3)$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \mathbf{f}_2(t) \quad \text{on } \Gamma_2, \quad (3.4)$$

$$-\sigma_\nu(t) = \mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t) \quad \text{on } \Gamma_3, \quad (3.5)$$

$$\left. \begin{aligned} \|\boldsymbol{\sigma}_\tau(t)\| &\leq F_b(\sigma_\nu(t)), \\ -\boldsymbol{\sigma}_\tau(t) &= F_b(\sigma_\nu(t)) \frac{\dot{\mathbf{u}}_\tau(t)}{\|\dot{\mathbf{u}}_\tau(t)\|} \quad \text{if } \dot{\mathbf{u}}_\tau(t) \neq \mathbf{0} \end{aligned} \right\} \quad \text{on } \Gamma_3 \quad (3.6)$$

for all  $t \in I$  and, moreover,

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega. \quad (3.7)$$

Note that in (3.1)–(3.7) as well as in various places below we skip the dependence of various function with respect to the spatial variable  $\mathbf{x} \in \Omega \cup \Gamma$ . A description of the equations and boundary conditions above is the following.

First, equation (3.1) is the constitutive law in which  $A$  and  $\mathcal{S}$  are given operators which will be described below. Equation (3.2) is the equation of equilibrium in which  $\text{Div}$  denotes the divergence operator and  $\mathbf{f}_0$  represents the density of body forces. Conditions (3.3) and (3.4) are the displacement and traction boundary conditions, respectively. Here  $\mathbf{f}_2$  represents the density of surface tractions acting on  $\Gamma_2$ . Condition (3.5) is the contact condition in which  $\mathcal{R}$  is a given operator and (3.6) represents a version of Coulomb's law of dry friction in which  $F_b$  denotes the friction bound and  $\mathbf{u}_\tau$  represents the tangential displacement. Moreover,  $\sigma_\nu$  and  $\boldsymbol{\sigma}_\tau$  denote the normal and tangential components of the stress vector  $\boldsymbol{\sigma}\boldsymbol{\nu}$  on  $\Gamma$ , i.e.,  $\sigma_\nu = \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\nu}$  and  $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$ . Finally, condition (3.7) is the initial condition in which  $\mathbf{u}_0$  represents the initial displacement.

In the study of Problem  $\mathcal{P}$  we consider the following assumptions on the operators  $A$ ,  $\mathcal{S}$ ,  $\mathcal{R}$  and function  $F_b$ .

$$A : Q \rightarrow Q \text{ is a strongly monotone Lipschitz continuous operator.} \quad (3.8)$$

$$\mathcal{S} : C(I; Q \times Q) \rightarrow C(I; Q) \text{ is a pseudo history-dependent operator.} \quad (3.9)$$

$$\mathcal{R} : C(I; V \times V) \rightarrow C(I; L^2(\Gamma_3)) \text{ is a pseudo history-dependent operator.} \quad (3.10)$$

$$\begin{cases}
F_b: \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}. \\
\text{(b) There exists } L_b > 0 \text{ such that} \\
\quad |F_b(\mathbf{x}, r_1) - F_b(\mathbf{x}, r_2)| \leq L_b |r_1 - r_2| \\
\quad \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\
\text{(c) } F_b(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R}. \\
\text{(d) For a.e. } \mathbf{x} \in \Gamma_3, F_b(\mathbf{x}, r) = 0 \text{ if } r < 0 \text{ while } F_b(\mathbf{x}, r) \geq 0 \text{ if } r \geq 0.
\end{cases} \quad (3.11)$$

In what follows we provide **examples** of constitutive laws and contact conditions which satisfy these assumptions.

**Example 1.** A typical example of constitutive law of the form (3.1) in which conditions (3.8) and (3.9) are satisfied is given by

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{C}(t-s)\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)) ds, \quad (3.12)$$

where  $\mathcal{A}$  is a viscosity operator,  $\mathcal{B}$  is an elasticity operator and  $\mathcal{C}$  is a relaxation tensor. Such kind of laws have been considered in [3, 4, 5, 28]. Assume that the viscosity and the elasticity operators satisfy the following conditions.

$$\begin{cases}
\text{(a) } \mathcal{A}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\
\quad \|\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\quad (\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\
\quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(d) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\quad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(e) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0}) \text{ belongs to } Q.
\end{cases} \quad (3.13)$$

$$\begin{cases}
\text{(a) } \mathcal{B}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\
\quad \|\mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
\text{(c) The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\quad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(d) The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{0}) \text{ belongs to } Q.
\end{cases} \quad (3.14)$$

$$\mathcal{C} \in C(I; \mathbf{Q}_{\infty}). \quad (3.15)$$

Here  $\mathbf{Q}_\infty$  is the space defined by

$$\mathbf{Q}_\infty = \{ \mathcal{C} = (c_{ijkl}) \mid c_{ijkl} = c_{jikl} = c_{klij} \in L^\infty(\Omega) \},$$

equipped with the canonical norm. Then, following the arguments in [28], it is easy to see that, under the assumptions (3.13)–(3.15), the constitutive law (3.12) is of the form (3.1) with operators  $A$  and  $\mathcal{S}$  defined as follows:

$$(A\boldsymbol{\omega}, \boldsymbol{\tau})_Q = \int_\Omega \mathcal{A}\boldsymbol{\omega} \cdot \boldsymbol{\tau} \, dx \quad \text{for all } \boldsymbol{\omega}, \boldsymbol{\tau} \in Q, \quad (3.16)$$

$$(\mathcal{S}(\boldsymbol{\omega}, \boldsymbol{\theta})(t), \boldsymbol{\tau})_Q = \int_\Omega \mathcal{B}\boldsymbol{\omega}(t) \cdot \boldsymbol{\tau} \, dx + \left( \int_0^t \mathcal{C}(t-s)\boldsymbol{\theta}(s) \, ds, \boldsymbol{\tau} \right)_Q \quad (3.17)$$

for all  $(\boldsymbol{\omega}, \boldsymbol{\theta}) \in C(I; Q \times Q)$ ,  $t \in I$ ,  $\boldsymbol{\tau} \in Q$ .

Moreover, it is easy to see that conditions (3.8) and (3.9) are satisfied. Finally, note that, when  $\mathcal{C}$  vanishes, equation (3.12) reduces to the well-known Kelvin-Voigt constitutive law.  $\square$

**Example 2.** A second example of constitutive laws of the form (3.1) in which conditions (3.8) and (3.9) are satisfied can be obtained by using rheological arguments, as follows. Consider a rheological model obtained by connecting in parallel a purely viscous element with an elastic-visco-plastic element. Then, for any  $t \in I$ , we have

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^v(t) + \boldsymbol{\sigma}^{vp}(t), \quad (3.18)$$

where  $\boldsymbol{\sigma}^v$  and  $\boldsymbol{\sigma}^{vp}$  represent the stresses in the viscous element and the elastic-visco-plastic element, respectively. Assume that the constitutive law of the viscous element is given by

$$\boldsymbol{\sigma}^v(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \quad (3.19)$$

where the viscosity operator  $\mathcal{A}$  satisfies condition (3.13) and the constitutive law of the elastic-visco-plastic element is given by

$$\dot{\boldsymbol{\sigma}}^{vp}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}^{vp}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t))),$$

where

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{E} \in \mathbf{Q}_\infty. \\ \text{(b) There exists } m_\mathcal{E} > 0 \text{ such that} \\ \quad \mathcal{E}(\mathbf{x})\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_\mathcal{E}\|\boldsymbol{\tau}\|^2 \quad \text{for all } \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \end{array} \right. \quad (3.20)$$

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \rightarrow \mathbb{S}^d. \\ \text{(b) There exists } L_\mathcal{G} > 0 \text{ such that} \\ \quad \|\mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \\ \quad \leq L_\mathcal{G} (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|) \\ \quad \text{for all } \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \\ \quad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{G}(\mathbf{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } Q. \end{array} \right. \quad (3.21)$$

Then, using (3.20) and (3.21) and assuming that  $\boldsymbol{\sigma}^{vp}(0) = \boldsymbol{\varepsilon}(0) = \mathbf{0}$ , it follows from Proposition 46 in [28] that there exists a history-dependent operator  $\mathcal{T} : C(I, Q) \rightarrow C(I; Q)$  such that

$$\boldsymbol{\sigma}^{vp}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \mathcal{T}\boldsymbol{\varepsilon}(\mathbf{u}(t)). \quad (3.22)$$

We now use (3.19) and (3.22) to see that, under assumptions (3.13), (3.20) and (3.21), the constitutive law (3.18) is of the form (3.1) with operator  $A$  defined by (3.16) and  $\mathcal{S}$  defined as follows:

$$(\mathcal{S}(\boldsymbol{\omega}, \boldsymbol{\theta})(t), \boldsymbol{\tau})_Q = \int_{\Omega} \mathcal{E}\boldsymbol{\omega}(t) \cdot \boldsymbol{\tau} \, dx + \int_{\Omega} \mathcal{T}\boldsymbol{\omega}(t) \cdot \boldsymbol{\tau} \, dx \quad (3.23)$$

$$\text{for all } (\boldsymbol{\omega}, \boldsymbol{\theta}) \in C(I; Q \times Q), \quad t \in I, \boldsymbol{\tau} \in Q.$$

Moreover, it is easy to see that conditions (3.8) and (3.9) are satisfied.  $\square$

**Example 3.** A typical example of contact condition of the form (3.5) is given by

$$-\sigma_{\nu} = a p_1(u_{\nu} - g) + b p_2(\dot{u}_{\nu}) \quad (3.24)$$

in which  $a, b \in L^{\infty}(\Gamma_3)$  are positive functions,  $g \in L^2(\Gamma_3)$  is a positive function and  $p_e$  ( $e = 1, 2$ ) are given functions such that

$$\left\{ \begin{array}{l} \text{(a) } p_e : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}. \\ \text{(b) There exists } L_e > 0 \text{ such that} \\ \quad |p_e(\mathbf{x}, r_1) - p_e(\mathbf{x}, r_2)| \leq L_e |r_1 - r_2| \\ \quad \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c) } p_e(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R}. \\ \text{(d) For a.e. } \mathbf{x} \in \Gamma_3, p_e(\mathbf{x}, r) = 0 \text{ if } r < 0 \text{ while } p_e(\mathbf{x}, r) \geq 0 \text{ if } r \geq 0. \end{array} \right. \quad (3.25)$$

Note that in this case condition (3.24) is of the form (3.5) with operator  $\mathcal{R}$  defined by

$$(\mathcal{R}(\mathbf{u}, \mathbf{v})(t), \xi)_{L^2(\Gamma_3)} = \int_{\Gamma_3} a p_1(u_{\nu}(t) - g) \xi \, da + \int_{\Gamma_3} b p_2(v_{\nu}(t)) \xi \, da \quad (3.26)$$

$$\text{for all } (\mathbf{u}, \mathbf{v}) \in C(I; V \times V), \quad t \in I, \quad \xi \in L^2(\Gamma_3).$$

which, obviously, **satisfies** condition (3.10). Note also that, when  $b$  vanishes, the contact condition (3.24) reduces to the classical normal compliance condition and, when  $a$  vanishes, the contact condition (3.24) reduces to the classical normal damped response condition. Details can be found in [12, 28].  $\square$

**Example 4.** A typical example of function  $F_b$  is given by  $F_b(\mathbf{x}, r) = \mu(\mathbf{x})r_+$  for all  $r \in \mathbb{R}$ ,  $\mathbf{x} \in \Gamma_3$ , in which  $\mu \in L^{\infty}(\Gamma_3)$  is a given positive function, the coefficient of friction, and  $r_+$  denotes the positive part of  $r$ . Using this choice in (3.6) leads to the classical Coulomb's law of dry friction.  $\square$

**Example 5.** A second example of  $F_b$  which satisfies (3.11) is given by  $F_b(\mathbf{x}, r) = \mu(\mathbf{x})r_+(1 - \delta(\mathbf{x})r)_+$  for all  $r \in \mathbb{R}$ ,  $\mathbf{x} \in \Gamma_3$ , in which  $\mu, \delta \in L^\infty(\Gamma_3)$  are given positive functions and, again,  $r_+$  denotes the positive part of  $r$ . Using this choice in (3.6) leads to a version of Coulomb's used in [12] and the references therein.  $\square$

We conclude from above that our results in Sections 4 and 5 are valid for any contact model which combines one of the constitutive laws in Examples 1 or 2, the contact condition in Example 3 and one of the friction laws in Examples 4 or 5.

## 4 Two weak formulations

In this section we derive two variational formulation for Problem  $\mathcal{P}$ . To this end, besides assumptions (3.8), (3.9), (3.10) and (3.11) discussed in the previous section, we assume that the density of applied forces and the initial displacement have the regularity

$$\mathbf{f}_0 \in C(I; L^2(\Omega)^d). \quad (4.1)$$

$$\mathbf{f}_2 \in C(I; L^2(\Gamma_2)^d). \quad (4.2)$$

$$\mathbf{u}_0 \in V. \quad (4.3)$$

We shall keep these assumptions everywhere in this section, even if we do not mention it explicitly. Now, we consider the functions  $j: L^2(\Gamma_3) \times V \rightarrow \mathbb{R}$  and  $\mathbf{f}: I \rightarrow V$  defined by

$$j(\theta, \mathbf{v}) = \int_{\Gamma_3} \theta v_\nu da + \int_{\Gamma_3} F_b(\theta) \|\mathbf{v}_\tau\| da \quad \text{for all } \theta \in L^2(\Gamma_3), \mathbf{v} \in V, \quad (4.4)$$

$$(\mathbf{f}(t), \mathbf{v})_V = \int_{\Omega} \mathbf{f}_0(t) \cdot \mathbf{v} dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v} da \quad \text{for all } \mathbf{v} \in V, t \in I. \quad (4.5)$$

Assume that  $(\mathbf{u}, \boldsymbol{\sigma})$  represents a regular solution of Problem  $\mathcal{P}$  and let  $\mathbf{v} \in V$ ,  $t \in I$  be arbitrarily fixed. Then, using integration by parts and standard arguments we find that

$$\begin{aligned} & \int_{\Omega} \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t))) dx \\ & + \int_{\Gamma_3} \mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t)(v_\nu - \dot{u}_\nu(t)) + \int_{\Gamma_3} F_b(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t))(\|\mathbf{v}_\tau(s)\| - \|\dot{\mathbf{u}}_\tau(s)\|) da \\ & \geq \int_{\Omega} \mathbf{f}_0(t) \cdot (\mathbf{v} - \dot{\mathbf{u}}(t)) dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot (\mathbf{v} - \dot{\mathbf{u}}(t)) da. \end{aligned}$$

Therefore, using notation (4.4) and (4.5) we see that

$$\begin{aligned} & (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + j(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t), \mathbf{v}) \\ & - j(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V. \end{aligned} \quad (4.6)$$

We now use the constitutive law (3.1), inequality (4.6) and the initial condition (3.7) to obtain the following variational formulation of Problem  $\mathcal{P}$ .

**Problem  $\mathcal{P}_u$ .** Find a displacement field  $\mathbf{u} : I \rightarrow V$  such that

$$(A\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{S}(\varepsilon(\mathbf{u}), \varepsilon(\dot{\mathbf{u}}))(t), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}(t)))_Q + j(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t), \mathbf{v}) \quad (4.7)$$

$$-j(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \text{for all } \mathbf{v} \in V, t \in I,$$

$$\mathbf{u}(0) = \mathbf{u}_0. \quad (4.8)$$

We now provide a different variational formulation for Problem  $\mathcal{P}$  and, to this end, we introduce the multivalued mapping  $\Sigma : L^2(\Gamma_3) \times I \rightarrow 2^Q$  and the element  $\boldsymbol{\omega}_0$  defined by

$$\Sigma(\theta, t) = \{ \boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \varepsilon(\mathbf{v}))_Q + j(\theta, \mathbf{v}) \geq (\mathbf{f}(t), \mathbf{v})_V \quad \forall \mathbf{v} \in V \} \quad (4.9)$$

for all  $\theta \in L^2(\Gamma_3), t \in I$ .

$$\boldsymbol{\omega}_0 = \varepsilon(\mathbf{u}_0). \quad (4.10)$$

To proceed, we need the following two results.

**Lemma 4.1.** Assume (3.11), (4.1) and (4.2). Then the multivalued mapping  $\Sigma : L^2(\Gamma_3) \times I \rightarrow 2^Q$  has nonempty closed convex values.

*Proof.* Let  $\theta \in L^2(\Gamma_3)$  and  $t \in I$  be fixed. Since the function  $\mathbf{v} \mapsto j(\theta, \mathbf{v}) : V \rightarrow \mathbb{R}$  is subdifferentiable and vanishes in  $\mathbf{0}_V$ , we deduce from (2.18) that there exists an element  $\mathbf{g} \in V$  such that  $j(\theta, \mathbf{v}) \geq (\mathbf{g}, \mathbf{v})_V$  for all  $\mathbf{v} \in V$ . Moreover, recall that  $(\mathbf{g}, \mathbf{v})_V = (\varepsilon(\mathbf{g}), \varepsilon(\mathbf{v}))_Q$  and  $(\mathbf{f}(t), \mathbf{v})_V = (\varepsilon(\mathbf{f}(t)), \varepsilon(\mathbf{v}))_Q$ . Therefore, using the notation  $\boldsymbol{\xi} = \varepsilon(\mathbf{f}(t)) - \varepsilon(\mathbf{g})$  we find that

$$(\boldsymbol{\xi}, \varepsilon(\mathbf{v}))_Q + j(\theta, \mathbf{v}) \geq (\mathbf{f}(t), \mathbf{v})_V \quad \text{for all } \mathbf{v} \in V. \quad (4.11)$$

We now combine (4.9) and (4.11) to see that  $\boldsymbol{\xi} \in \Sigma(\theta, t)$  and, therefore,  $\Sigma(\theta, t)$  is not empty. On the other hand, it is easy to see that  $\Sigma(\theta, t)$  is a closed convex subset of  $Q$ , which concludes the proof.  $\square$

**Lemma 4.2.** There exists a linear continuous operator  $G : Q \rightarrow V$  such that for all  $\boldsymbol{\omega} \in Q, \mathbf{u} \in V$  the following implication holds:

$$\boldsymbol{\omega} = \varepsilon(\mathbf{u}) \implies \mathbf{u} = G\boldsymbol{\omega}. \quad (4.12)$$

*Proof.* First we recall that, since the  $d - 1$  measure of  $\Gamma_1$  is positive, the range of the deformation operator  $\varepsilon : V \rightarrow Q$ , denoted by  $\varepsilon(V)$ , is a closed subspace of  $Q$ . A proof of this result can be found on [28, p.212]. Denote by  $P : Q \rightarrow \varepsilon(V)$  the orthogonal projection operator on  $\varepsilon(V) \subset Q$  and note that equality (2.4) shows that

$\varepsilon: V \rightarrow \varepsilon(V)$  is a linear invertible operator. This allows us to consider its inverse,  $\varepsilon^{-1}: \varepsilon(V) \rightarrow V$ . Define now the operator  $G: Q \rightarrow V$  by equalities

$$G\boldsymbol{\omega} = \varepsilon^{-1}P\boldsymbol{\omega} \quad \text{for all } \boldsymbol{\omega} \in Q. \quad (4.13)$$

Then, it is easy to see that  $G$  is linear, continuous and, moreover, it satisfies (4.12).  $\square$

Based on Lemmas 4.1 and 4.2 we are in a position to define the normal cone  $N_{\Sigma(\theta,t)} \subset Q$  for each  $\theta \in L^2(\Gamma_3)$  and  $t \in I$ , as well as the element  $\mathcal{R}(G\boldsymbol{\omega}, G\dot{\boldsymbol{\omega}})(t) \in L^2(\Gamma_3)$  for each  $\boldsymbol{\omega} \in C^1(I, Q)$  and  $t \in I$ .

This allows us to consider the following sweeping process problem.

**Problem  $\mathcal{P}_\omega$ .** Find a strain field  $\boldsymbol{\omega}: I \rightarrow Q$  such that

$$\begin{aligned} -\dot{\boldsymbol{\omega}}(t) &\in N_{\Sigma(\mathcal{R}(G\boldsymbol{\omega}, G\dot{\boldsymbol{\omega}})(t), t)}(A\dot{\boldsymbol{\omega}}(t) + \mathcal{S}(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}})(t)) \quad \text{for all } t \in I, \\ \boldsymbol{\omega}(0) &= \boldsymbol{\omega}_0. \end{aligned}$$

We end this section with the following comments. First, Problem  $\mathcal{P}_\omega$  can be obtained formally from Problem  $\mathcal{P}$  by using inequality (4.6), definitions (4.9), (4.13) and arguments that will be presented in the next section. For this reason we refer to this problem as variational formulation of the contact problem  $\mathcal{P}$ , too. Next, we underline that problems  $\mathcal{P}_u$  and  $\mathcal{P}_\omega$  have a different structure, since Problem  $\mathcal{P}_u$  is an evolutionary variational inequality for the displacement field while Problem  $\mathcal{P}_\omega$  is a sweeping process in which the unknown is the strain field. Nevertheless, in the next section we shall see that these problems are connected by an equivalence result, Theorem 5.2.

## 5 An equivalence result

We start this section with the following preliminary result.

**Lemma 5.1.** Assume (3.11), (4.1), (4.2) and let  $\boldsymbol{\omega}, \boldsymbol{\sigma} \in Q$ ,  $\theta \in L^2(\Gamma_3)$ ,  $t \in I$  such that

$$-\boldsymbol{\omega} \in N_{\Sigma(\theta,t)}(\boldsymbol{\sigma}). \quad (5.1)$$

Then, there exists a unique element  $\mathbf{u} \in V$  such that  $\boldsymbol{\omega} = \varepsilon(\mathbf{u})$  and, moreover,

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u})) + j(\theta, \mathbf{v}) - j(\theta, \mathbf{u}) \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{u})_V \quad \text{for all } \mathbf{v} \in V. \quad (5.2)$$

*Proof.* First, we note that inclusion (5.1) implies that

$$\boldsymbol{\sigma} \in \Sigma(\theta, t), \quad (\boldsymbol{\tau} - \boldsymbol{\sigma}, \boldsymbol{\omega})_Q \geq 0 \quad \text{for all } \boldsymbol{\tau} \in \Sigma(\theta, t). \quad (5.3)$$

Let  $\mathbf{z} \in \varepsilon(V)^\perp$  where, here and below,  $M^\perp$  represents the orthogonal of the set  $M$  in  $Q$ . Then  $(\mathbf{z}, \varepsilon(\mathbf{v}))_Q = 0$  for all  $\mathbf{v} \in V$ , which implies that  $\boldsymbol{\sigma} \pm \mathbf{z} \in \Sigma(\theta, t)$ .

Therefore, testing with  $\boldsymbol{\tau} = \boldsymbol{\sigma} \pm \boldsymbol{z}$  in (5.3) we deduce that  $(\boldsymbol{z}, \boldsymbol{\omega})_Q = 0$  which shows that  $\boldsymbol{\omega} \in \boldsymbol{\varepsilon}(V)^{\perp\perp} = \boldsymbol{\varepsilon}(V)$ . This implies that there exists an element  $\boldsymbol{u} \in V$  such that

$$\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\boldsymbol{u}). \quad (5.4)$$

Moreover, (2.4) guarantees that  $\boldsymbol{u}$  is unique.

Next, by the subdifferentiability of the function  $j(\theta, \cdot)$  in  $\boldsymbol{u}$  we know that there exists an element  $\boldsymbol{g} \in V$  such that

$$j(\theta, \boldsymbol{v}) - j(\theta, \boldsymbol{u}) \geq (\boldsymbol{g}, \boldsymbol{v} - \boldsymbol{u})_V = (\boldsymbol{\varepsilon}(\boldsymbol{g}), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q$$

and, taking  $\boldsymbol{\tau}_0 := \boldsymbol{\varepsilon}(\boldsymbol{f}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{g})$ , we deduce that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\theta, \boldsymbol{v}) - j(\theta, \boldsymbol{u}) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u})_V \quad \text{for all } \boldsymbol{v} \in V. \quad (5.5)$$

We now test with  $\boldsymbol{v} = 2\boldsymbol{u}$  and  $\boldsymbol{v} = \mathbf{0}_V$  in this inequality to deduce that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\theta, \boldsymbol{u}) = (\boldsymbol{f}(t), \boldsymbol{u})_V. \quad (5.6)$$

Therefore, combining (5.5) and (5.6) we find that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\theta, \boldsymbol{v}) \geq (\boldsymbol{f}(t), \boldsymbol{u})_V \quad \text{for all } \boldsymbol{v} \in V$$

and, hence,  $\boldsymbol{\tau}_0 \in \Sigma(\theta, t)$ . This regularity, (5.3) and (5.4) imply that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q \geq (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q$$

and, using (5.6) yields

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\theta, \boldsymbol{u}) \leq (\boldsymbol{f}(t), \boldsymbol{u})_V. \quad (5.7)$$

On the other hand, since  $\boldsymbol{\sigma} \in \Sigma(\theta, t)$  and  $\boldsymbol{u} \in V$  the converse inequality holds, i.e.,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\theta, \boldsymbol{u}) \geq (\boldsymbol{f}(t), \boldsymbol{u})_V. \quad (5.8)$$

We now combine (5.7) and (5.8) to see that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + j(\theta, \boldsymbol{u}) = (\boldsymbol{f}(t), \boldsymbol{u})_V. \quad (5.9)$$

Then, we use (5.9), the regularity  $\boldsymbol{\sigma} \in \Sigma(\theta, t)$  and definition (4.9) to deduce that (5.2) holds.  $\square$

Our main result in this section is the following.

**Theorem 5.2.** *Assume (3.8), (3.9), (3.10), (3.11), (4.1)–(4.3) and let  $\boldsymbol{u} : I \rightarrow V$ ,  $\boldsymbol{\omega} : I \rightarrow Q$ . Then, the following statements hold.*

a) *If  $\boldsymbol{u}$  is a solution to Problem  $\mathcal{P}_u$  with regularity  $\boldsymbol{u} \in C^1(I; V)$  and  $\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ , then  $\boldsymbol{\omega}$  is a solution to Problem  $\mathcal{P}_\omega$  with regularity  $\boldsymbol{\omega} \in C^1(I; Q)$ .*

b) *If  $\boldsymbol{\omega}$  is a solution to Problem  $\mathcal{P}_\omega$  with regularity  $\boldsymbol{\omega} \in C^1(I; Q)$ , there exists a unique function  $\boldsymbol{u} \in C^1(I; V)$  such that  $\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ . Moreover,  $\boldsymbol{u}$  is a solution to Problem  $\mathcal{P}_u$ .*



*Proof.* a) Assume that  $\mathbf{u}$  is a solution to Problem  $\mathcal{P}_u$  with regularity  $\mathbf{u} \in C^1(I; V)$  and  $\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\mathbf{u})$ . Then, it is obvious to see that  $\boldsymbol{\omega} \in C^1(I; Q)$ . Moreover, using (4.7), (4.8) and (4.10) we deduce that

$$(A\dot{\boldsymbol{\omega}}(t) + \mathcal{S}(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}})(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \dot{\boldsymbol{\omega}}(t))_Q + j(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t), \mathbf{v}) \quad (5.10)$$

$$-j(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \text{for all } t \in I,$$

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0. \quad (5.11)$$

Denote by  $\boldsymbol{\sigma} : I \rightarrow Q$  the function defined by

$$\boldsymbol{\sigma}(t) = A\dot{\boldsymbol{\omega}}(t) + \mathcal{S}(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}})(t) \quad \text{for all } t \in I \quad (5.12)$$

and note that (5.10) yields

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \dot{\boldsymbol{\omega}}(t))_Q + j(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t), \mathbf{v}) \quad (5.13)$$

$$-j(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \text{for all } t \in I.$$

Let  $t \in I$ . Then, taking successively  $\mathbf{v} = 2\dot{\mathbf{u}}(t)$  and  $\mathbf{v} = \mathbf{0}_V$  in (5.13) we obtain that

$$(\boldsymbol{\sigma}(t), \dot{\boldsymbol{\omega}}(t))_Q + j(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t), \dot{\mathbf{u}}(t)) = (\mathbf{f}(t), \dot{\mathbf{u}}(t))_V. \quad (5.14)$$

Therefore, using (5.13), (5.14) and (4.9) yields

$$\boldsymbol{\sigma}(t) \in \Sigma(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t), t), \quad (\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \dot{\boldsymbol{\omega}}(t))_Q \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t), t). \quad (5.15)$$

We now use (2.16) to see that

$$-\dot{\boldsymbol{\omega}}(t) \in N_{\Sigma(\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t), t)} \boldsymbol{\sigma}(t). \quad (5.16)$$

On the other hand, Lemma 4.2 implies that  $\mathbf{u} = G\boldsymbol{\omega}$  and  $\dot{\mathbf{u}} = G\dot{\boldsymbol{\omega}}$  which imply that

$$\mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t) = \mathcal{R}(G\boldsymbol{\omega}, G\dot{\boldsymbol{\omega}})(t). \quad (5.17)$$

We now substitute equalities (5.17) and (5.12), in (5.16) and use (5.11) to deduce that  $\boldsymbol{\omega}$  is a solution to Problem  $\mathcal{P}_\omega$ .

b) Conversely, assume that  $\boldsymbol{\omega}$  is a solution to Problem  $\mathcal{P}_\omega$  with regularity  $\boldsymbol{\omega} \in C^1(I; Q)$  and let  $\boldsymbol{\sigma}$  be the function defined by (5.12). Moreover, let  $\theta : I \rightarrow L^2(\Gamma_3)$  be the function defined by

$$\theta(t) = \mathcal{R}(G\boldsymbol{\omega}, G\dot{\boldsymbol{\omega}})(t) \quad \text{for all } t \in I. \quad (5.18)$$

We have

$$-\dot{\boldsymbol{\omega}}(t) \in N_{\Sigma(\theta(t), t)}(\boldsymbol{\sigma}(t)) \quad \text{for all } t \in I, \quad (5.19)$$

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0. \quad (5.20)$$

Let  $t \in I$ . We use (5.19) and Lemma 5.1 to see that there exists a unique element  $\mathbf{w}(t) \in V$  such that  $\dot{\boldsymbol{\omega}}(t) = \boldsymbol{\varepsilon}(\mathbf{w}(t))$  and, moreover,

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{w}(t)))_Q + j(\theta(t), \mathbf{v}) - j(\theta(t), \mathbf{w}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \mathbf{w}(t))_V \quad (5.21)$$

for all  $\mathbf{v} \in V$ . Equality  $\dot{\boldsymbol{\omega}}(t) = \boldsymbol{\varepsilon}(\mathbf{w}(t))$  and Lemma 4.2 imply that  $\mathbf{w} \in C(I; V)$  and, therefore, the function  $\mathbf{u} : I \rightarrow V$  defined by

$$\mathbf{u}(t) = \int_0^t \mathbf{w}(s) ds + \mathbf{u}_0 \quad \text{for all } t \in I \quad (5.22)$$

has the regularity  $\mathbf{u} \in C^1(I; V)$ . Moreover, using (4.10) and (5.20) it is easy to see that

$$\boldsymbol{\varepsilon}(\mathbf{u})(t) = \boldsymbol{\omega}(t) \quad (5.23)$$

and, obviously, it is the unique function in  $C^1(I; V)$  such that  $\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\omega}$ . We now combine (5.23), (4.12) and (5.18) to see that

$$\theta(t) = \mathcal{R}(\mathbf{u}, \dot{\mathbf{u}})(t). \quad (5.24)$$

In addition, (5.22) implies that  $\mathbf{u}(0) = \mathbf{u}_0$ . Therefore, using (5.21), (5.12) and (5.24) we deduce that  $\mathbf{u}$  is a solution to Problem  $\mathcal{P}_u$  which concludes the proof.  $\square$

## 6 Existence and uniqueness results

We start this section with the following preliminary result which completes the statement of Lemma 4.1.

**Lemma 6.1.** *Assume (3.11), (4.1), (4.2). Then, the multivalued mapping  $\Sigma : L^2(\Gamma_3) \times I \rightarrow 2^Q$  satisfies assumption (K) on the spaces  $X = Q$  and  $Y = L^2(\Gamma_3)$ .*

*Proof.* Assume that  $\theta_1, \theta_2 \in L^2(\Gamma_3)$ ,  $t_1, t_2 \in I$  and  $\mathbf{z} \in Q$  and denote

$$\boldsymbol{\sigma}_1 = P_{\Sigma(\theta_1, t_1)} \mathbf{z}, \quad \boldsymbol{\sigma}_2 = P_{\Sigma(\theta_2, t_2)} \mathbf{z}. \quad (6.1)$$

We use (2.17) to see that  $\mathbf{z} - P_{\Sigma(\theta_1, t_1)} \mathbf{z} \in N_{\Sigma(\theta_1, t_1)}(P_{\Sigma(\theta_1, t_1)} \mathbf{z})$  and, therefore, (6.1) implies that  $\mathbf{z} - \boldsymbol{\sigma}_1 \in N_{\Sigma(\theta_1, t_1)} \boldsymbol{\sigma}_1$ . Next, Lemma 5.1 implies that there exists a unique element  $\mathbf{u}_1 \in V$  such that

$$\boldsymbol{\sigma}_1 - \mathbf{z} = \boldsymbol{\varepsilon}(\mathbf{u}_1), \quad (6.2)$$

$$(\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}_1)) + j(\theta_1, \mathbf{v}) - j(\theta_1, \mathbf{u}_1) \geq (\mathbf{f}(t_1), \mathbf{v} - \mathbf{u}_1)_V \quad \text{for all } \mathbf{v} \in V. \quad (6.3)$$

Similar arguments show that there exists a unique element  $\mathbf{u}_2 \in V$  such that

$$\boldsymbol{\sigma}_2 - \mathbf{z} = \boldsymbol{\varepsilon}(\mathbf{u}_2), \quad (6.4)$$

$$(\boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}_2)) + j(\theta_2, \mathbf{v}) - j(\theta_2, \mathbf{u}_2) \geq (\mathbf{f}(t_2), \mathbf{v} - \mathbf{u}_2)_V \quad \text{for all } \mathbf{v} \in V. \quad (6.5)$$

We now take  $\mathbf{v} = \mathbf{u}_2$  in (6.3),  $\mathbf{v} = \mathbf{u}_1$  in (6.5) and add the resulting inequalities to obtain that

$$\begin{aligned} & (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}(\mathbf{u}_1) - \boldsymbol{\varepsilon}(\mathbf{u}_2))_Q \\ & \leq j(\theta_1, \mathbf{u}_2) - j(\theta_1, \mathbf{u}_1) + j(\theta_2, \mathbf{u}_1) - j(\theta_2, \mathbf{u}_2) + (\mathbf{f}(t_1) - \mathbf{f}(t_2), \mathbf{u}_1 - \mathbf{u}_2)_V. \end{aligned}$$

Then, using the identity  $\boldsymbol{\varepsilon}(\mathbf{u}_2) - \boldsymbol{\varepsilon}(\mathbf{u}_1) = \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1$ , guaranteed by (6.2) and (6.4), we find that

$$\begin{aligned} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_Q^2 & \leq j(\theta_1, \mathbf{u}_2) - j(\theta_1, \mathbf{u}_1) + j(\theta_2, \mathbf{u}_1) - j(\theta_2, \mathbf{u}_2) \\ & \quad + \|\mathbf{f}(t_1) - \mathbf{f}(t_2)\|_V \|\mathbf{u}_1 - \mathbf{u}_2\|_V. \end{aligned} \quad (6.6)$$

On the other hand, a standard calculation based on the definition (4.4), the properties of function  $F_b$  in (3.11) and the trace inequality (2.5) shows that

$$j(\theta_1, \mathbf{u}_2) - j(\theta_1, \mathbf{u}_1) + j(\theta_2, \mathbf{u}_1) - j(\theta_2, \mathbf{u}_2) \leq c_{tr}(L_b + 1) \|\theta_1 - \theta_2\|_{L^2(\Gamma_3)} \|\mathbf{u}_1 - \mathbf{u}_2\|_V. \quad (6.7)$$

We now combine (6.6) and (6.7), then we use equality  $\|\mathbf{u}_1 - \mathbf{u}_2\|_V = \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_Q$  to deduce that

$$\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_Q \leq c_{tr}(L_b + 1) \|\theta_1 - \theta_2\|_{L^2(\Gamma_3)} + \|\mathbf{f}(t_1) - \mathbf{f}(t_2)\|_V. \quad (6.8)$$

Finally, note that assumptions (4.1) and (4.2) imply that the element  $\mathbf{f}$  given by (4.5) has the regularity

$$\mathbf{f} \in C(I; V). \quad (6.9)$$

We now use (6.1), (6.8) and (6.9) to see that the set-valued mapping  $\Sigma(\cdot)$  satisfies assumption  $(\mathcal{K})$  with  $c_0 = c_{tr}(L_b + 1)$ .  $\square$

Next, we define the operators  $J_{\boldsymbol{\omega}_0} : C(I; Q) \rightarrow C(I; Q)$ ,  $\mathcal{S}_{\boldsymbol{\omega}_0} : C(I; Q) \rightarrow C(I; Q)$  and  $\mathcal{R}_{\boldsymbol{\omega}_0} : C(I; Q) \rightarrow C(I; L^2(\Gamma_3))$  by equalities

$$J_{\boldsymbol{\omega}_0} \boldsymbol{\omega}(t) = \int_0^t \boldsymbol{\omega}(s) ds + \boldsymbol{\omega}_0, \quad (6.10)$$

$$\mathcal{S}_{\boldsymbol{\omega}_0} \boldsymbol{\omega}(t) = \mathcal{S}(J_{\boldsymbol{\omega}_0} \boldsymbol{\omega}, \boldsymbol{\omega})(t), \quad (6.11)$$

$$\mathcal{R}_{\boldsymbol{\omega}_0} \boldsymbol{\omega}(t) = \mathcal{R}(J_{\boldsymbol{\omega}_0} G \boldsymbol{\omega}, G \boldsymbol{\omega})(t) \quad (6.12)$$

for all  $\boldsymbol{\omega} \in C(I; Q)$  and  $t \in I$ . Then, using assumptions (3.9), (3.10) and Proposition 2.3 it is easy to see that  $\mathcal{S}_{\boldsymbol{\omega}_0}$  and  $\mathcal{R}_{\boldsymbol{\omega}_0}$  are pseudo history-dependent operators and, therefore, for any nonempty compact set  $\mathcal{J} \subset I$  we are in a position to consider the constants  $l_{\mathcal{J}}^{\mathcal{S}_{\boldsymbol{\omega}_0}}$  and  $l_{\mathcal{J}}^{\mathcal{R}_{\boldsymbol{\omega}_0}}$ .

The unique solvability of Problem  $\mathcal{P}_{\boldsymbol{\omega}}$  is provided by the following existence and uniqueness result.

**Theorem 6.2.** *Assume (3.8)–(3.11) and (4.1)–(4.3) hold. Then there exists a constant  $\tilde{c}$  such that Problem  $\mathcal{P}_\omega$  has a unique solution with regularity  $\omega \in C^1(I; Q)$ , provided that, for any nonempty compact set  $\mathcal{J} \subset I$ , the following inequality holds:*

$$l_{\mathcal{J}}^{\mathcal{R}\omega_0} + l_{\mathcal{J}}^{\mathcal{S}\omega_0} \leq \tilde{c}. \quad (6.13)$$

*Proof.* We use Lemma 6.1, assumptions (3.8), (4.3), notation (4.10) and the properties of the operators  $\mathcal{S}\omega_0$  and  $\mathcal{R}\omega_0$  to see that we are in a position to employ Theorem 2.4 on the spaces  $X = Q$ ,  $Y = L^2(\Gamma_3)$ . In this way we deduce that there exists a constant  $\tilde{c}$  with the following property: if the smallness assumption (6.13) holds for any nonempty compact subset  $\mathcal{J} \subset I$ , then there exists a unique function  $\omega \in C^1(I; Q)$  such that

$$-\dot{\omega}(t) \in N_{\Sigma(\mathcal{R}\omega_0\dot{\omega}(t),t)}(A\dot{\omega}(t) + \mathcal{S}\omega_0\dot{\omega}(t)) \quad \text{for all } t \in I, \quad (6.14)$$

$$\omega(0) = \omega_0. \quad (6.15)$$

On the other hand, using (6.10)–(6.12) it is easy to see that

$$\mathcal{S}\omega_0\dot{\omega}(t) = \mathcal{S}(\omega, \dot{\omega})(t), \quad \mathcal{R}\omega_0\dot{\omega}(t) = \mathcal{R}(G\omega, G\dot{\omega})(t)$$

for all  $\omega \in C^1(I; Q)$ ,  $t \in I$ . These equalities show that a function  $\omega \in C^1(I; X)$  is a solution to problem  $\mathcal{P}_\omega$  if and only if  $\omega$  is a solution of the sweeping process (6.14)–(6.15).

Theorem 6.2 is now a consequence of the previous equivalence and the unique solvability of the sweeping process (6.14)–(6.15), guaranteed by Theorem 2.4.  $\square$

The following result concerns the unique solvability of Problem  $\mathcal{P}_u$ , which represents a direct consequence of Theorems 5.2 and 6.2.

**Corollary 6.3.** *Assume (3.8)–(3.11) and (4.1)–(4.3) hold. Then there exists a constant  $\tilde{c}$  such that Problem  $\mathcal{P}_u$  has a unique solution with regularity  $\mathbf{u} \in C^1(I; X)$ , provided that the smallness assumption (6.13) holds for any nonempty compact set  $\mathcal{J} \subset I$ .*

A couple of functions  $\mathbf{u} : I \rightarrow V$  and  $\boldsymbol{\sigma} : I \rightarrow Q$  such that  $\mathbf{u}$  is a solution to Problem  $\mathcal{P}_u$  and (3.1) holds for any  $t \in I$  is called a weak solution to the viscoelastic contact problem  $\mathcal{P}$ . We conclude from Corollary 6.3 that Problem  $\mathcal{P}$  has a unique weak solution, provided that the smallness assumption (6.13) holds.

We end this section with some comments and mechanical interpretation on the smallness assumption (6.13). To this end, everywhere below we assume that (3.8)–(3.11) and (4.1)–(4.3) hold and, for any nonempty compact set  $\mathcal{J} \subset I$ , we denote in what follows by  $l_{\mathcal{J}}^{\mathcal{R}}$  and  $l_{\mathcal{J}}^{\mathcal{S}}$  the constants in Definition 2.1 with  $\Lambda = \mathcal{R}$  and  $\Lambda = \mathcal{S}$ , respectively. Our comments are the following.

- 1) First, the statement of Theorem 2.4 shows that the constant  $\tilde{c}$  which appears in (6.13) depends on the constants  $m_A$ ,  $L_A$  and  $c_0$ . On the other hand, Lemma 6.1 shows that  $c_0 = c_{tr}(L_b + 1)$  where  $c_{tr}$  and  $L_b$  are the constants which appear in (2.5) and (3.11), respectively. We conclude from here that the constant  $\tilde{c}$  depends on the constitutive operator  $A$  (i.e., on the viscosity operator  $\mathcal{A}$ , see Examples 1 and 2), on the friction bound  $F_b$  and the geometry of the problem since, recall, the constant  $c_{tr}$  depends on  $\Omega$ ,  $\Gamma_1$  and  $\Gamma_3$ .
- 2) The statements of Theorem 6.2 and Corollary 6.3 still hold if the smallness assumption (6.13) is replaced by the inequality

$$l_{\mathcal{J}}^{\mathcal{R}} + l_{\mathcal{J}}^{\mathcal{S}} \leq \tilde{c}. \quad (6.16)$$

This remark is a direct consequence of inequality (2.12) and (2.4).

- 3) If the operator  $\mathcal{S}$  is such that  $\mathcal{S}(\boldsymbol{\omega}, \cdot) : C(I; Q) \rightarrow C(I, Q)$  is a history-dependent operator for each  $\boldsymbol{\omega} \in C(I; Q)$ , then the statements of Theorem 6.2 and Corollary 6.3 still hold if the smallness assumption (6.13) is replaced by one of the inequalities

$$l_{\mathcal{J}}^{\mathcal{R}\boldsymbol{\omega}_0} \leq \tilde{c}, \quad l_{\mathcal{J}}^{\mathcal{R}} \leq \tilde{c}. \quad (6.17)$$

This remark is based on the fact that, in this case, the operator  $\mathcal{S}\boldsymbol{\omega}_0$  is a history-dependent operator and, therefore,  $l_{\mathcal{J}}^{\mathcal{S}\boldsymbol{\omega}_0} = 0$ . Thus, whichever of the conditions (6.17) guarantees that inequality (6.13) holds. Note that this is the case of the constitutive laws presented in Examples 1 and 2, see the operators (3.17) and (3.23).

- 4) If the operator  $\mathcal{R}$  is such that  $\mathcal{R}(\mathbf{u}, \cdot) : C(I; V) \rightarrow C(I, L^2(\Gamma_3))$  is a history-dependent operator for each  $\mathbf{u} \in C(I; V)$ , then the statements of Theorem 6.2 and Corollary 6.3 still hold if the smallness assumption (6.13) is replaced by one of the inequalities

$$l_{\mathcal{J}}^{\mathcal{S}\boldsymbol{\omega}_0} \leq \tilde{c}, \quad l_{\mathcal{J}}^{\mathcal{S}} \leq \tilde{c}. \quad (6.18)$$

This remark is based on arguments similar to those presented in 3). Note that this is the case of the normal compliance contact condition laws presented in Example 4. Indeed, in this case we use the operator (3.26) with  $b \equiv 0$ .

- 5) Finally, if the additional conditions on the operators  $\mathcal{S}$  and  $\mathcal{R}$  described in 3) and 4) are satisfied, then the statements of Theorem 6.2 and Corollary 6.3 still hold without any smallness assumption. This is the case of the contact models based on the constitutive laws in Examples 1 and 2, associated to the normal compliance contact condition.

In this paper we considered a general frictional contact problem for viscoelastic materials. The constitutive law we used includes as particular cases various constitutive laws used in the literature, as the Kelvin-Voigt constitutive law, for instance. The contact condition presented here is very general, too, and includes as particular cases the normal compliance condition in a form with a gap function and the normal damped response condition. Friction was described with a version of Coulomb's law of dry friction. Under appropriate assumptions on the data we derived two different variational formulations of the problem: an evolutionary variational inequality with the unknown being the displacement field and a sweeping process problem with the unknown being the strain field. We proved that these two formulations are equivalent. Then, for each formulation, we proved an existence and uniqueness result.

Our results in this paper can be easily extended to models which take into account additional memory effects. For instance, in the contact condition is possible to consider a stiffness coefficient which depends on the accumulated penetration and, in the friction law, is possible to consider a slip-dependent or a total slip-dependent coefficient of friction. The variational analysis of the corresponding models can be carried out by using similar arguments, the difference arising in the choice of operator  $\mathcal{R}$ . Note also that a weak formulation of Problem  $\mathcal{P}$  in terms of the stress can also be considered, by using the arguments in [27, 28]. This formulation, the so-called dual formulation of Problem  $\mathcal{P}$ , leads to a special history-dependent variational inequality with constraints.

It follows from this paper that the displacement field and the strain tensor play symmetric role in the analysis of quasistatic viscoelastic contact problems. It also results that the variational formulation of such problems is not unique. Moreover, solving different variational formulations associated to quasistatic viscoelastic contact problems requires different functional methods and arguments. This illustrates the cross fertilization between the Mathematical Theory of Contact Mechanics, on one hand, and the Nonlinear Functional Analysis, on the other hand.

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