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Analysis and Control of Stationary Inclusions in Contact Mechanics

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Abstract

We start with a mathematical model which describes the frictionless contact of an elastic body with an obstacle and prove that it leads to a stationary inclusion for the strain field. Then, inspired by this contact model, we consider a general stationary inclusion in a real Hilbert space, governed by three parameters. We prove the unique solvability of the inclusion as well as the continuous dependence of its solution with respect to the parameters. We use these results in the study of an associated optimal control problem for which we prove existence and convergence results. The proofs are based on arguments of monotonicity, compactness, convex analysis and lower semicontinuity. Then, we apply these abstract results to the mathematical model of contact and provide the corresponding mechanical interpretations.

Key words: elastic material, frictionless contact, stationary inclusion, Mosco convergence, optimal control, optimal pair.

AMS Subject Classification: 74M15, 74M10, 47J22, 49J40, 49J21, 34G25.

1 Introduction

The Mathematical Theory of Contact Mechanics is that part of Applied Mathematics which deals with the study of mathematical models that describe phenomena of contact between deformable bodies or between a deformable body and an obstacle, the so-called foundation. It provides the analysis of such models, including results of existence, uniqueness and continuous dependence of the solutions with respect to

the data and parameters. It also deals with the optimal control of the corresponding models as well as with their numerical approximation. Its main feature is the cross fertilization between models and applications, on one hand, and the nonlinear functional analysis, on the other hand. References in the field include the books [6, 7, 8, 9, 13, 18, 19] and, more recently, [5, 21, 22]. Optimal control results in the study of various mathematical models of contact can be found in [3, 4, 10, 11, 12, 20].

Stated as strongly nonlinear boundary value problems which usually do not have classical solutions, the mathematical models of contact lead to a large variety of weak formulations, expressed in terms of variational or hemivariational inequalities. These inequalities could be elliptic, time-dependent or evolutionary, in function of the mechanical process (static, quasistatic or dynamic) considered, the constitutive law (elastic, viscoelastic, viscoplastic,...) choosed to model the material's behavior and the interface law (frictionless, frictional,...) used in the construction of each model. Usually, the corresponding unknowns are the displacement or the velocity field and, on occasion, the stress field. Employing such kind of formulations allows us to analyze the corresponding models through arguments from the theory of variatioanal and hemivariational inequalities which can be found in various books, including [14, 17, 18, 19], for instance.

Currently, there is an interest in variational formulations of contact models in the form of a time-dependent inclusion or a sweeping process. There, the unknown could be either the displacement field or the strain field, as illustrated in [1, 2] and [15, 16], respectively. Using such kind of formulations in the study of contact models requires to adapt the arguments of abstract stationary or differential inclusions and, very often, to develop new arguments in their analysis and control.

The current paper signs up in this direction. Indeed, here we consider a stationary inclusion in an abstract Hilbert spaces, inspired by a mathematical model which describes the equilibrium of an elastic body in contact with an obstacle. The structure of the inclusion, including the operators and the parameters involved, is motivated by the contact model and, at the best of our knowledge, is new. Our aim is three fold. The first one is to provide existence, uniqueness and continuous dependence results for the abstract inclusion. The second one is to prove the existence and convergence of the optimal pairs of an associated optimal control problem. Finally, our third aim is to illustrate the use of these abstract results in the study of a static problem of contact with elastic materials. Achieving these goals contributes to develop the analysis and control theory of abstract inclusions and provides mathematical tools which can be used the study of nonstandard formulations of contact problems. At the best of our knowledge, this represents the main trait of novelty of this work.

The rest of the manuscript is organized as follows. In Section 2 we present a mathematical model of contact, list the assumption on the data and derive a variational formulation, in a form of an inclusion for the stain field. Motivated by this contact model, in Section 3 we introduce a stationary inclusion in a real Hilbert space and

prove its unique solvability as well as a first convergence result. In Section 4 we study the dependence of the solution with respect to the data and parameters and provide additional convergence results. We use these results in Section 5 where we consider an associated optimal control problem. In Section 6 we turn back to the contact model introduced Section 2 and illustrate the applicability of the abstract results in the study of this problem. We end this paper with Section 7 in which we present some concluding remarks.

2 An elastic contact model

The physical setting is as follows. An elastic body occupies, in its reference configuration, the domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$), with a smooth boundary Γ . The body is held fixed on a part $\Gamma_1 \subset \Gamma$, is acted upon by given surface tractions on a part $\Gamma_2 \subset \Gamma$, and is in contact on the part $\Gamma_3 \subset \Gamma$ with an obstacle. We assume that Γ_1 , Γ_2 and Γ_3 are measurable subsets, represent a partition Γ and, moreover, $\text{meas}(\Gamma_1) > 0$. In addition, we denote by \mathbb{S}^d the space of second order symmetric on \mathbb{R}^d and use the notation “ \cdot ”, $\|\cdot\|$, $\mathbf{0}$ for the inner product, the Euclidian norm and the zero element on the spaces \mathbb{R}^d and \mathbb{S}^d , respectively.

To describe the equilibrium of the elastic body in the physical setting above we denote by $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ the displacement field, by $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$ the strain field, and by $\boldsymbol{\varepsilon}(\mathbf{u}) : \Omega \rightarrow \mathbb{S}$ the linearized strain tensor, that is

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u}).$$

Note that here and below, for simplicity, we do not mention the dependence of various functions with respect to the spatial variable $\mathbf{x} \in \Omega \cup \Gamma$.

We model the material's behaviour with a constitutive law of the form

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega, \tag{2.1}$$

where \mathcal{A} represent the elasticity operator, assumed to satisfy the following conditions.

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad \|\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\ \quad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) } \mathcal{A}(\mathbf{x}, \mathbf{0}) = \mathbf{0} \text{ a.e. } \mathbf{x} \in \Omega. \end{array} \right. \tag{2.2}$$

Next, for simplicity, we neglect the body forces and, therefore, the stress field satisfies the equation of equilibrium

$$\operatorname{Div} \boldsymbol{\sigma} = \mathbf{0} \quad \text{in } \Omega. \quad (2.3)$$

The displacement and traction boundary conditions corresponding to the physical setting described above are

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad (2.4)$$

$$\boldsymbol{\sigma} \boldsymbol{\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2. \quad (2.5)$$

Here $\boldsymbol{\nu}$ represents the outward unit vector to Ω and \mathbf{f}_2 denotes the density of surface tractions, assumed to have the regularity

$$\mathbf{f}_2 \in L^2(\Gamma_2)^d. \quad (2.6)$$

We now turn to describe the contact conditions on Γ_3 and, to this end, we use the indices ν and τ for the normal and tangential components of vectors and tensors, i.e.,

$$u_\nu = \mathbf{u} \cdot \boldsymbol{\nu}, \quad \mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}, \quad \sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}.$$

We assume that the obstacle has a rigid-perfect plastic behaviour. We denote by F its yield limit and we assume that it depends on a parameter θ , the temperature of the contact surface, for instance. Then the contact condition on Γ_3 is the following:

$$-F(\theta) \leq \sigma_\nu \leq 0, \quad \sigma_\nu = \begin{cases} 0 & \text{if } u_\nu < 0, \\ -F(\theta) & \text{if } u_\nu > 0 \end{cases} \quad \text{on } \Gamma_3. \quad (2.7)$$

Note that (2.7) shows that penetration occur only when the magnitude of the normal stress has the critical value $F(\theta)$, which represents the behaviour of a rigid-plastic obstacle. Moreover, when there is separation (i.e., when $u_\nu < 0$) then the reaction of the foundation vanishes. Details on interface laws similar to (3.8) can be find in [22].

Our assumptions on the data F , θ and g are the following.

$$\left\{ \begin{array}{l} \text{(a) } F : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+. \\ \text{(b) There exists } L_F > 0 \text{ such that} \\ \quad |F(\mathbf{x}, \theta_1) - F(\mathbf{x}, \theta_2)| \leq L_F |\theta_1 - \theta_2| \\ \quad \text{for all } \theta_1, \theta_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c) The mapping } \mathbf{x} \mapsto F(\mathbf{x}, \theta) \text{ is measurable on } \Gamma_3, \\ \quad \text{for all } \theta \in \mathbb{R}. \\ \text{(d) } F(\mathbf{x}, 0) = 0 \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{array} \right. \quad (2.8)$$

$$\theta \in L^2(\Gamma_3). \quad (2.9)$$

Finally, we assume that the contact is frictionless and, therefore, the friction force vanishes on the contact surface, i.e.,

$$\boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_3. \quad (2.10)$$

We now gather the above equations and boundary condition to obtain the classical formulation of our contact model.

Problem \mathcal{P} . *Find a displacement field $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$ such that (2.1), (2.3), (2.4), (2.5), (2.7) and (2.10) hold.*

To derive a variational formulation of Problem \mathcal{P} we use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ . For an element $\mathbf{v} \in H^1(\Omega)^d$ we still write \mathbf{v} , v_ν and \mathbf{v}_τ for the trace, the normal trace and the tangential trace of \mathbf{v} to Γ , respectively. Moreover, we consider the following spaces:

$$\begin{aligned} V &= \{ \mathbf{v} = (v_i) : v_i \in H^1(\Omega), v_i = 0 \text{ on } \Gamma_1 \ \forall i = \overline{1, d} \}, \\ Q &= \{ \boldsymbol{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \ \forall i, j = \overline{1, d} \}. \end{aligned}$$

It is well known that the spaces V and Q are real Hilbert spaces endowed with the canonical inner products

$$(\mathbf{u}, \mathbf{v})_V = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, dx. \quad (2.11)$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$. It follows from here that

$$\|\mathbf{v}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_Q \quad \forall \mathbf{v} \in V. \quad (2.12)$$

This equality will be used in various places below. We denote by r^+ the positive part of r , by $\mathbf{0}_V$ the zero element of V and we recall that, as a consequence of the Sobolev trace theorem, there exists $c_{tr} > 0$ which depends on Ω and Γ_1 such that

$$\|\mathbf{v}\|_{L^2(\Gamma)^d} \leq c_{tr} \|\mathbf{v}\|_V \quad \text{for all } \mathbf{v} \in V. \quad (2.13)$$

Under the previous assumptions, we introduce the operator $A : Q \rightarrow Q$, the function $j(\theta, \cdot) : V \rightarrow \mathbb{R}$, the element $\mathbf{f} \in V$ and the set, $\Sigma(\mathbf{f}, \theta, g) \subset Q$ defined by

$$(A\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_{\Omega} \mathcal{A}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, dx \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in Q, \quad (2.14)$$

$$j(\theta, \mathbf{v}) = \int_{\Gamma_3} F(\theta)v_\nu^+ \, dx \quad \forall \boldsymbol{\sigma}, \mathbf{v} \in Q, \quad (2.15)$$

$$(\mathbf{f}, \mathbf{v})_V = \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v} \, da \quad \forall \mathbf{v} \in V, \quad (2.16)$$

$$\Sigma(\theta, \mathbf{f}) = \{ \boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\theta, \mathbf{v}) \geq (\mathbf{f}, \mathbf{v})_V \ \forall \mathbf{v} \in V \}. \quad (2.17)$$

Assume now that $(\mathbf{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy Problem \mathcal{P} and let $\mathbf{v} \in V$. Then, using standard arguments we deduce that

$$\begin{aligned} \int_{\Omega} \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})) \, dx + \int_{\Gamma_3} F(\theta) v_{\nu}^{+} \, da - \int_{\Gamma_3} F(\theta) u_{\nu}^{+} \, da \\ \geq \int_{\Gamma_2} \mathbf{f}_2 \cdot (\mathbf{v} - \mathbf{u}) \, da. \end{aligned} \quad (2.18)$$

Next, we use the notation (2.11), (2.15) and (2.16) to deduce that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}))_Q + j(\theta, \mathbf{v}) - j(\theta, \mathbf{u}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u})_V. \quad (2.19)$$

We now test in (2.19) with $\mathbf{v} = 2\mathbf{u}$ and $\mathbf{v} = \mathbf{0}_V$ to see that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}))_Q + j(\theta, \mathbf{u}) = (\mathbf{f}, \mathbf{u})_V. \quad (2.20)$$

Therefore, using (2.19) and (2.20) we find that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\theta, \mathbf{v}) \geq (\mathbf{f}, \mathbf{v})_V$$

which implies that

$$\boldsymbol{\sigma} \in \Sigma(\theta, \mathbf{f}). \quad (2.21)$$

To proceed, we use (2.17) and (2.20) to see that

$$(\boldsymbol{\tau} - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}))_Q \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(\theta, \mathbf{f})$$

and, using the notation

$$\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\mathbf{u}), \quad (2.22)$$

we find that

$$(\boldsymbol{\tau} - \boldsymbol{\sigma}, \boldsymbol{\omega})_Q \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(\theta, \mathbf{f}). \quad (2.23)$$

We now combine (2.21) and (2.23) to see that

$$-\boldsymbol{\omega} \in N_{\Sigma(\theta, \mathbf{f})}(\boldsymbol{\sigma}), \quad (2.24)$$

where $N_{\Sigma(\theta, \mathbf{f})}$ represents the outward normal cone of K in the sense of convex analysis. We now use the constitutive law (2.1) and notation (2.14), (2.22) to see that

$$\boldsymbol{\sigma} = A\boldsymbol{\omega}. \quad (2.25)$$

Therefore, using (2.21), (2.24) and (2.25) we deduce the following variational formulation of Problem \mathcal{P} .

Problem \mathcal{P}^V . Find a strain field $\boldsymbol{\omega} \in Q$ such that

$$-\boldsymbol{\omega} \in N_{\Sigma(\theta, \mathbf{f})}(A\boldsymbol{\omega}). \quad (2.26)$$

Note that (2.26) represents an inclusion for the unknown function $\boldsymbol{\omega}$. Once this function is obtained, the displacement field \mathbf{u} and the stress field $\boldsymbol{\sigma}$ are obtained by using equalities (2.22) and (2.25), respectively.

3 An abstract inclusion

Inclusion (2.26) suggests us to study an abstract stationary inclusion in the framework that we present in this section. First, everywhere below X represents a Hilbert space endowed with an inner product $(\cdot, \cdot)_X$ and its associated norm $\|\cdot\|_X := \sqrt{(\cdot, \cdot)_X}$. The set of parts of X is denoted by 2^X . Assume that:

$$\Sigma \text{ is a nonempty closed convex subset of } X. \quad (3.1)$$

$$\left\{ \begin{array}{l} A : X \rightarrow X \text{ is a strongly monotone and Lipschitz continuous operator,} \\ \text{i.e., there exist } m_A > 0 \text{ and } L_A > 0 \text{ such that} \\ \text{(a) } (Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \forall u, v \in X. \\ \text{(b) } \|Au - Av\|_X \leq L \|u - v\|_X \quad \forall u, v \in X \end{array} \right. \quad (3.2)$$

$$\eta \in X. \quad (3.3)$$

We denote in what follows by $N_\Sigma : X \rightarrow 2^X$ the outward normal cone of Σ in the sense of convex analysis and by $P_\Sigma : X \rightarrow \Sigma$ the projection operator on Σ . We recall that the following equivalence hold, for all $u, \xi \in X$:

$$\xi \in N_\Sigma(u) \iff u \in \Sigma, \quad (\xi, v - u)_X \leq 0 \quad \forall v \in \Sigma, \quad (3.4)$$

$$u = P_\Sigma \xi \iff u \in \Sigma, \quad (\xi - u, v - u)_X \leq 0 \quad \forall v \in \Sigma. \quad (3.5)$$

The inclusion we consider in this section is the following.

Problem \mathcal{I} . *Find an element $u \in X$ such that*

$$-u \in N_\Sigma(Au + \eta). \quad (3.6)$$

Note that, for simplicity, in (3.6) we do not mention explicitly the dependence of the set Σ on some parameters, say θ and f . We shall do it in the next sections, where such a dependence will play a crucial role in our study. Also, note that in contrast with the inclusion (2.26), a new parameter η appears in (3.6). Our interest in this parameter arises from the fact that it lies the background for the study of time-dependent contact problems with viscoelastic materials, similar to those studied in [2, 15].

The unique solvability of Problem \mathcal{I} is provided by the following existence and uniqueness result.

Theorem 3.1. *Assume (3.1)–(3.3). Then there exists a unique element $u \in X$ such that (3.6) holds.*

The proof of Theorem 3.1 follows from standard arguments that we need in the rest of the paper and, therefore, we provide it below.

Proof. Assumption (3.2) implies that the operator $A : X \rightarrow X$ is invertible and its inverse A^{-1} is strongly monotone and Lipschitz continuous with constants $m' = \frac{m_A}{L_A^2}$ and $L' = \frac{1}{m_A}$. Then, using the notation

$$\sigma = Au + \eta \quad (3.7)$$

we have

$$u = A^{-1}(\sigma - \eta). \quad (3.8)$$

Moreover, (3.4)–(3.8) show that the following equivalences hold, for any $\rho > 0$:

$$\begin{aligned} -u \in N_\Sigma(Au + \eta) &\iff -u \in N_\Sigma(\sigma) \\ &\iff \sigma \in \Sigma, \quad (\tau - \sigma, A^{-1}(\sigma - \eta)) \geq 0 \\ &\iff \sigma \in \Sigma, \quad (\tau - \sigma, \rho A^{-1}(\sigma - \eta)) \geq 0 \\ &\iff \sigma \in \Sigma, \quad (\tau - \sigma, \sigma - \sigma + \rho A^{-1}(\sigma - \eta)) \geq 0 \\ &\iff \sigma = P_\Sigma(\sigma - \rho A^{-1}(\sigma - \eta)). \end{aligned}$$

Consider now the operator $\Lambda_\rho : X \rightarrow X$ defined by

$$\Lambda_\rho \sigma = P_\Sigma(\sigma - \rho A^{-1}(\sigma - \eta)) \quad \forall \sigma \in X. \quad (3.9)$$

Then, it follows from above that the following equivalence holds:

$$-u \in N_\Sigma(Au + \eta) \iff \exists \rho > 0 \text{ such that } \Lambda_\rho \sigma = \sigma. \quad (3.10)$$

Next, we claim that if

$$0 < \rho < \frac{2m'}{L'^2} \quad (3.11)$$

then the operator Λ_ρ is a contraction on X . To prove this claim we fix $\sigma_1, \sigma_2 \in X$ and denote $z_1 = \sigma_1 - \eta$, $z_2 = \sigma_2 - \eta$. Then, we use the definition of Λ_ρ , the nonexpansivity of the projection operator P_K and the properties (3.2) of the operator A to see that

$$\begin{aligned} \|\Lambda_\rho \sigma_1 - \Lambda_\rho \sigma_2\|_X^2 &\leq \|(\sigma_1 - \rho A^{-1}(\sigma_1 - \eta)) - (\sigma_2 - \rho A^{-1}(\sigma_2 - \eta))\|_X^2 \\ &= \|(z_1 - z_2) - \rho(A^{-1}z_1 - A^{-1}z_2)\|_X^2 \\ &= \|z_1 - z_2\|_X^2 - 2\rho(z_1 - z_2, A^{-1}z_1 - A^{-1}z_2) + \rho^2\|A^{-1}z_1 - A^{-1}z_2\|_X^2 \\ &\leq (1 - 2\rho m' + \rho^2 L'^2)\|z_1 - z_2\|_X^2 \end{aligned}$$

and, therefore,

$$\|\Lambda_\rho \sigma_1 - \Lambda_\rho \sigma_2\|_X \leq k_\rho \|\sigma_1 - \sigma_2\|_X, \quad (3.12)$$

where

$$k_\rho := \sqrt{1 - 2\rho m' + \rho^2 L'^2}.$$

Thanks to the smallness assumption (3.11), we obviously have

$$k_\rho \in (0, 1). \quad (3.13)$$

Combining (3.12) and (3.13), we see that Λ_ρ is a contraction on X , as claimed.

Thus, we are in a position to apply the Banach fixed point theorem to get a unique $\sigma \in X$ such that $\Lambda_\rho \sigma = \sigma$. Then, it follows from (3.10) that the element u defined by (3.8) is the unique solution of the inclusion (3.6), which concludes the proof. \square

The solution of inclusion (3.6) depends on the data Σ and η . For this reason, we proceed our analysis with a result which shows the continuous dependence of the solution with respect to these data. To present it, we consider two sequences $\{\Sigma_n\}$ and $\{\eta_n\}$ such that, for each $n \in \mathbb{N}$, the following hold.

$$\Sigma_n \text{ is a nonempty closed convex subset of } X. \quad (3.14)$$

$$\eta_n \in X. \quad (3.15)$$

Then, using Theorem 3.1 it follows that for each $n \in \mathbb{N}$ there exists a unique solution to the following inclusion problem.

Problem \mathcal{I}_n . Find an element $u_n \in X$ such that

$$-u_n \in N_{\Sigma_n}(Au_n + \eta_n). \quad (3.16)$$

We now consider the following assumptions.

$$\eta_n \rightarrow \eta \quad \text{in } X, \quad (3.17)$$

$$\Sigma_n \xrightarrow{M} \Sigma, \quad (3.18)$$

$$\Sigma_n \xrightarrow{W^i} \Sigma, \quad (3.19)$$

where the symbols “ \xrightarrow{M} ” and “ $\xrightarrow{W^i}$ ” denote the convergence in the sense of Mosco and Wijsman, respectively. For the convenience of the reader we recall that the sequence $\{\Sigma_n\}$ is said to converge to the set Σ in the sense of Mosco if the following conditions hold:

- (a) For each $u \in \Sigma$, there exists a sequence $\{u_n\}$ such that $u_n \in \Sigma_n$ for each $n \in \mathbb{N}$ and $u_n \rightarrow u$ in X .
- (b) For each sequence $\{u_n\}$ such that $u_n \in \Sigma_n$ for each $n \in \mathbb{N}$ and $u_n \rightarrow u$ weakly in X , we have $u \in \Sigma$.

The sequence $\{\Sigma_n\}$ is said to converge to the set Σ in the sense of Wijsman, if

$$\lim_{n \rightarrow \infty} \inf_{v \in \Sigma_n} \|u - v\|_X = \inf_{v \in \Sigma} \|u - v\|_X \quad \text{for any } u \in X. \quad (3.20)$$

We have the following convergence result.

Theorem 3.2. *Assume (3.1)–(3.3), (3.14), (3.15), (3.17) and either (3.18) and (3.19). Then, the solution of the inclusion (3.16) converge to the solution of the inclusion (3.6), i.e.,*

$$u_n \rightarrow u \quad \text{in } X. \quad (3.21)$$

Proof. Let $n \in \mathbb{N}$ be fixed and use the notation

$$\sigma_n = Au_n + \eta_n, \quad (3.22)$$

which implies that

$$u_n = A^{-1}(\sigma_n - \eta_n). \quad (3.23)$$

We chose $\rho > 0$ which satisfies the inequality (3.11), then we use (3.9) and (3.10) to see that

$$\sigma = P_\Sigma(\sigma - \rho A^{-1}(\sigma - \eta)), \quad \sigma_n = P_{\Sigma_n}(\sigma_n - \rho A^{-1}(\sigma_n - \eta_n))$$

where, here and below, P_{Σ_n} represents the projection operator on Σ_n . This implies that

$$\begin{aligned} \|\sigma_n - \sigma\|_X &\leq \|P_{\Sigma_n}(\sigma_n - \rho A^{-1}(\sigma_n - \eta_n)) - P_{\Sigma_n}(\sigma - \rho A^{-1}(\sigma - \eta))\|_X \\ &\quad + \|P_{\Sigma_n}(\sigma - \rho A^{-1}(\sigma - \eta)) - P_\Sigma(\sigma - \rho A^{-1}(\sigma - \eta))\|_X \end{aligned}$$

and, using the nonexpansivity of the operator P_{Σ_n} , we find that

$$\begin{aligned} \|\sigma_n - \sigma\|_X &\leq \|((\sigma_n - \eta_n) - \rho A^{-1}(\sigma_n - \eta_n)) - ((\sigma - \eta) - \rho A^{-1}(\sigma - \eta))\|_X \\ &\quad + \|\eta_n - \eta\|_X + \|P_{\Sigma_n}(\sigma - \rho A^{-1}(\sigma - \eta)) - P_\Sigma(\sigma - \rho A^{-1}(\sigma - \eta))\|_X. \end{aligned} \quad (3.24)$$

Note that arguments similar to those used to obtain (3.12) imply that

$$\begin{aligned} &\|((\sigma_n - \eta_n) - \rho A^{-1}(\sigma_n - \eta_n)) - ((\sigma - \eta) - \rho A^{-1}(\sigma - \eta))\|_X \\ &\leq k_\rho \|(\sigma_n - \eta_n) - (\sigma - \eta)\|_X \leq k_\rho \|\sigma_n - \sigma\|_X + k_\rho \|\eta_n - \eta\|_X \end{aligned}$$

and, therefore, (3.24) yields

$$\begin{aligned} \|\sigma_n - \sigma\|_X &\leq \frac{1 + k_\rho}{1 - k_\rho} \|\eta_n - \eta\|_X \\ &\quad + \frac{1}{1 - k_\rho} \|P_{\Sigma_n}(\sigma - \rho A^{-1}(\sigma - \eta)) - P_\Sigma(\sigma - \rho A^{-1}(\sigma - \eta))\|_X. \end{aligned} \quad (3.25)$$

We now use Theorem 3.3 in [23] which states that the convergences (3.18) and (3.19) are equivalent and, moreover, they are equivalent with the following condition

$$P_{\Sigma_n}\xi \rightarrow P_\Sigma\xi \quad \text{in } X, \quad \forall \xi \in X. \quad (3.26)$$

Therefore, using (3.25), (3.17) and (3.26) we deduce that

$$\sigma_n \rightarrow \sigma \quad \text{in } X. \quad (3.27)$$

Finally, equalities (3.8), (3.23) and convergences (3.27), (3.17) imply that (3.21) holds, which concludes the proof. \square

4 Additional continuous dependence results

In this section we assume that the set Σ depends on two parameters and study the dependence of the solution of the inclusion \mathcal{I} with respect to these parameters. To this end we consider in what follows two normed spaces $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ and a multivalued mapping $\Sigma : Y \times Z \rightarrow 2^X$ which satisfies the following condition.

$$\left\{ \begin{array}{l} \text{(a) For each } (\theta, f) \in Y \times Z \text{ the set } \Sigma(\theta, f) \subset X \text{ is} \\ \quad \text{nonempty closed and convex.} \\ \text{(b) There exists } c_0 > 0 \text{ such that for each} \\ \quad \theta_1, \theta_2 \in Y, f_1, f_2 \in Z \text{ and } u \in X \text{ one has} \\ \quad \|P_{\Sigma(\theta_1, f_1)}u - P_{\Sigma(\theta_2, f_2)}u\|_X \leq c_0(\|\theta_1 - \theta_2\|_Y + \|f_1 - f_2\|_Z). \end{array} \right. \quad (4.1)$$

Here and below $Y \times Z$ represent the product of the spaces Y and Z , endowed with one of the canonical equivalent norms. We also use $X \times Y \times Z$ for the product space of the spaces X, Y and Z .

Given a triple $(\eta, \theta, f) \in X \times Y \times Z$ we now consider the following inclusion.

Problem \mathcal{J} . Find an element $u \in X$ such that

$$-u \in N_{\Sigma(\theta, f)}(Au + \eta). \quad (4.2)$$

The unique solvability of Problem \mathcal{J} as well as its dependence with respect to the data is provided by the following result.

Theorem 4.1. Assume (3.2) and (4.1). Then, for each element $(\eta, \theta, f) \in X \times Y \times Z$ there exists a unique element $u = u(\eta, \theta, f) \in X$ such that (4.2) holds. Moreover, the mapping $(\eta, \theta, f) \mapsto u(\eta, \theta, f) : X \times Y \times Z \rightarrow X$ is Lipschitz continuous.

Proof. The existence and uniqueness part is a direct consequence of Theorem 3.1. For the Lipschitz continuity part, consider two elements $(\eta_1, \theta_1, f_1), (\eta_2, \theta_2, f_2) \in X \times Y \times Z$ and denote by $u_1, u_2 \in X$ the corresponding solutions of Problem \mathcal{J} , that is $u_1 = u(\eta_1, \theta_1, f_1)$ and $u_2 = u(\eta_2, \theta_2, f_2)$. We have

$$-u_1 \in N_{\Sigma(\theta_1, f_1)}(Au_1 + \eta_1), \quad -u_2 \in N_{\Sigma(\theta_2, f_2)}(Au_2 + \eta_2).$$

We now use the notation,

$$\sigma_1 = Au_1 + \eta_1, \quad \sigma_2 = Au_2 + \eta_2 \quad (4.3)$$

or, equivalently,

$$u_1 = A^{-1}(\sigma_1 - \eta_1), \quad u_2 = A^{-1}(\sigma_2 - \eta_2). \quad (4.4)$$

Next, we chose $\rho > 0$ which satisfies the inequality (3.11) and recall that the arguments in the proof of Theorem 3.1, including (3.10) and (3.9), imply that

$$\sigma_1 = P_{\Sigma(\theta_1, f_1)}(\sigma_1 - \rho A^{-1}(\sigma_1 - \eta_1)), \quad \sigma_2 = P_{\Sigma(\theta_2, f_2)}(\sigma_2 - \rho A^{-1}(\sigma_2 - \eta_2)).$$

Therefore,

$$\begin{aligned} & \|\sigma_1 - \sigma_2\|_X \\ & \leq \|P_{\Sigma(\theta_1, f_1)}(\sigma_1 - \rho A^{-1}(\sigma_1 - \eta_1)) - P_{\Sigma(\theta_1, f_1)}(\sigma_2 - \rho A^{-1}(\sigma_2 - \eta_2))\|_X \\ & \quad + \|P_{\Sigma_1(\theta_1, f_1)}(\sigma_2 - \rho A^{-1}(\sigma_2 - \eta_2)) - P_{\Sigma(\theta_2, f_2)}(\sigma_2 - \rho A^{-1}(\sigma_2 - \eta_2))\|_X \end{aligned}$$

and, using the nonexpansivity of the projection together with assumption (4.1)(b) we find that

$$\begin{aligned} & \|\sigma_1 - \sigma_2\|_X \\ & \leq \|((\sigma_1 - \eta_1) - \rho A^{-1}(\sigma_1 - \eta_1)) - ((\sigma_2 - \eta_2) - \rho A^{-1}(\sigma_2 - \eta_2))\|_X \\ & \quad + \|\eta_1 - \eta_2\|_X + c_0(\|\theta_1 - \theta_2\|_X + \|f_1 - f_2\|_Z). \end{aligned} \quad (4.5)$$

Note that arguments similar to those used to obtain (3.12) imply that

$$\begin{aligned} & \|((\sigma_1 - \eta_1) - \rho A^{-1}(\sigma_1 - \eta_1)) - ((\sigma_2 - \eta_2) - \rho A^{-1}(\sigma_2 - \eta_2))\|_X \\ & \leq k_\rho \|(\sigma_1 - \eta_1) - (\sigma_2 - \eta_2)\|_X \leq k_\rho \|\sigma_1 - \sigma_2\|_X + k_\rho \|\eta_1 - \eta_2\|_X \end{aligned}$$

and, therefore, (4.5) yields

$$\|\sigma_1 - \sigma_2\|_X \leq \frac{1 + k_\rho}{1 - k_\rho} \|\eta_1 - \eta_2\|_X + \frac{c_0}{1 - k_\rho} (\|\theta_1 - \theta_2\|_X + \|f_1 - f_2\|_Z). \quad (4.6)$$

Finally, (4.4) combined with the Lipschitz continuity of the operator A^{-1} and inequality (4.6) show that there exists a constant $d_\rho > 0$ such that

$$\|u_1 - u_2\|_X \leq d_\rho (\|\eta_1 - \eta_2\|_X + \|\theta_1 - \theta_2\|_Y + \|f_1 - f_2\|_Z), \quad (4.7)$$

which concludes the proof. \square

Theorem 4.1 allows us to obtain existence, uniqueness and continuous results in the study of time-independent inclusions. To present them we consider $T > 0$ and, for any normed space $(W, \|\cdot\|_W)$, we use the notation $C([0, T]; W)$ for the space of continuous functions defined on $[0, T]$ with values in W , endowed with the norm

$$\|w\|_{C([0, T]; W)} = \max_{t \in [0, T]} \|w(t)\|_W. \quad (4.8)$$

Given a triple of functions $(\eta, \theta, f) : [0, T] \rightarrow X \times Y \times Z$ we consider the following inclusion.

Problem \mathcal{J}_T . Find an element $u : [0, T] \rightarrow X$ such that

$$-u(t) \in N_{\Sigma(\theta(t), f(t))}(Au(t) + \eta(t)) \quad \text{for } t \in [0, T]. \quad (4.9)$$

The unique solvability of Problem \mathcal{J}_T as well as its dependence with respect to the data is provided by the following result.

Corollary 4.2. *Assume (3.2) and (4.1). Then, for each triple of functions $(\eta, \theta, f) \in C([0, T]; X \times Y \times Z)$ there exists a unique function $u = u(\eta, \theta, f) \in C([0, T]; X)$ such that (4.9) holds. Moreover, the mapping $(\eta, \theta, f) \mapsto u(\eta, \theta, f) : C([0, T]; X \times Y \times Z) \mapsto C([0, T]; X)$ is Lipschitz continuous.*

Proof. First, we note that Theorem 3.1 guarantees the existence of a unique element $u(t)$ which solves the time-dependent inclusion (4.9), for any $t \in [0, T]$. Moreover, inequality (4.7) shows that, for any $t_1, t_2 \in [0, T]$, the following inequality holds:

$$\|u(t_1) - u(t_2)\|_X \leq d_\rho(\|\eta(t_1) - \eta(t_2)\|_X + \|\theta(t_1) - \theta(t_2)\|_Y + \|f(t_1) - f(t_2)\|_Z). \quad (4.10)$$

Therefore, since $t \mapsto \eta(t), \theta(t), f(t) : [0, T] \rightarrow X, Y, Z$ are continuous functions we deduce that $t \mapsto u(t) : [0, T] \rightarrow X$ is a continuous function. This proves the existence part of the Corollary 4.2. The uniqueness part follows from the unique solvability of the inclusion (4.9) at each $t \in [0, T]$. Finally, the Lipschitz continuity of the mapping $(\eta, \theta, f) : C([0, T]; X \times Y \times Z) \mapsto C([0, T]; X)$ is now a direct consequence of the inequality (4.7) combined with definition (4.8). \square

For the result below we assume that $k \in \{0, 1\}$, $1 \leq p \leq \infty$ and, for any normed space $(W, \|\cdot\|_W)$, use the standard notation for the Lebesgue spaces $L^p(0, T; W)$ and the Sobolev spaces $W^{k,p}(0, T; W)$. Therefore, $W^{0,p}(0, T; W) = L^p(0, T; W)$.

Corollary 4.3. *Assume (3.2) and (4.1) and, moreover, assume that Y, Z are reflexive Banach spaces. Then, for each element $(\eta, \theta, f) \in W^{k,p}(0, T; X \times Y \times Z)$ there exists a unique element $u = u(\eta, \theta, f) \in W^{k,p}(0, T; X)$ such that (4.9) holds. Moreover, the mapping $(\eta, \theta, f) \mapsto u(\eta, \theta, f) : W^{k,p}(0, T; X \times Y \times Z) \rightarrow W^{k,p}(0, T; X)$ is Lipschitz continuous.*

The proof of Corollary 4.3 is based on inequality (4.7) combined with arguments similar to those used in the proof of Corollary 4.2 and, therefore, we skip it.

5 An optimal control problem

In this section we study the existence and convergence of optimal pairs for an optimal control problem associated to a stationary inclusion of the form (4.2). To this end we recall the following version of the Weierstrass theorem.

Theorem 5.1. *Let W be a reflexive Banach space, \tilde{J} a nonempty weakly closed subset of W and $\mathcal{S} : \tilde{J} \rightarrow \mathbb{R}$ a weakly lower semicontinuous function. In addition, assume that either \tilde{J} is bounded or \mathcal{S} is coercive, i.e., $\mathcal{S}(v) \rightarrow \infty$ as $\|v\|_W \rightarrow \infty$. Then, there exists at least one element w^* such that*

$$w^* \in \tilde{J}, \quad \mathcal{S}(w^*) \leq \mathcal{S}(w) \quad \forall w \in \tilde{J}. \quad (5.1)$$

Next, we keep the functional framework in Section 4 and, in addition, we consider a reflexive Banach W equipped with the norm $\|\cdot\|_W$ as well as an operator $B : W \rightarrow Z$. Then, for each $(\eta, \theta, w) \in X \times Y \times W$, under the assumptions of Theorem 4.1, we denote by $u = u(\eta, \theta, Bw)$ the solution of Problem \mathcal{J} with $f = Bw$. Thus,

$$-u \in N_{\Sigma(\theta, Bw)}(Au + \eta). \quad (5.2)$$

Consider now two sets $I \subset X \times Y$ and $J \subset W$ and, for each $(\eta, \theta) \in I$, let $K(\eta, \theta)$ denote a subset of J which depends on (η, θ) . With these notation define the set of admissible pairs for inclusion (5.2) by equality

$$\mathcal{V}_{ad}(\eta, \theta) = \{ (u, w) : w \in K(\eta, \theta), u = u(\eta, \theta, Bw) \}. \quad (5.3)$$

In other words, a pair (u, w) belongs to $\mathcal{V}_{ad}(\eta, \theta)$ if and only if $w \in K(\eta, \theta)$ and, moreover, u is the solution of inclusion (5.2). Consider also a cost functional $\mathcal{L} : X \times J \rightarrow \mathbb{R}$. Then, given $(\eta, \theta) \in I$, the optimal control problem we are interested in is stated as follows.

Problem \mathcal{Q} . Find $(u^*, w^*) \in \mathcal{V}_{ad}(\eta, \theta)$ such that

$$\mathcal{L}(u^*, w^*) = \min_{(u, w) \in \mathcal{V}_{ad}(\eta, \theta)} \mathcal{L}(u, w). \quad (5.4)$$

Note that Problem \mathcal{Q} and its solutions depend on the couple (η, θ) but, for simplicity, since no confusion arises, we do not mention explicitly this dependence. In the study of this problem we consider the following assumptions.

$$\left\{ \begin{array}{l} \text{For each } (\eta, \theta) \in I \text{ the set } K(\eta, \theta) \\ \text{is a nonempty weakly closed subset of } W. \end{array} \right. \quad (5.5)$$

$$\left\{ \begin{array}{l} B : W \rightarrow Z \text{ is a compact operator, i.e.,} \\ w_k \rightharpoonup w \text{ in } W \implies Bw_k \rightarrow Bw \text{ in } Z. \end{array} \right. \quad (5.6)$$

$$\left\{ \begin{array}{l} \text{For all sequences } \{u_k\} \subset X \text{ and } \{w_k\} \subset J \text{ such that} \\ u_k \rightarrow u \text{ in } X, w_k \rightharpoonup w \text{ in } W, \text{ we have} \\ \liminf_{k \rightarrow \infty} \mathcal{L}(u_k, w_k) \geq \mathcal{L}(u, w). \end{array} \right. \quad (5.7)$$

$$\left\{ \begin{array}{l} \text{There exists } h : J \rightarrow \mathbb{R} \text{ such that} \\ \text{(a) } \mathcal{L}(u, w) \geq h(w) \quad \forall u \in X, w \in J, \\ \text{(b) } \|w_k\|_W \rightarrow +\infty \implies h(w_k) \rightarrow \infty. \end{array} \right. \quad (5.8)$$

$$J \text{ is a bounded subset of } W. \quad (5.9)$$

Example 1. A typical example of function \mathcal{L} which satisfies conditions (5.7) and (5.8) is obtained by taking

$$\mathcal{L}(u, w) = g(u) + h(w) \quad \forall u \in X, w \in J,$$

where $g : X \rightarrow \mathbb{R}$ is a continuous positive function and $h : J \rightarrow \mathbb{R}$ is a weakly lower semicontinuous coercive function, i.e., it satisfies condition (5.8)(b).

Our first result in this section is the following existence result.

Theorem 5.2. Assume (3.2), (4.1). Moreover, assume (5.5), (5.6), (5.7) and, in addition, assume that either (5.8) or (5.9) hold. Then Problem \mathcal{Q} has at least one solution (u^*, w^*) , for each $(\eta, \theta) \in I$.

Proof. Let $(\eta, \theta) \in I$ and let $\mathcal{S} : J \rightarrow \mathbb{R}$ be the function defined by

$$\mathcal{S}(w) = \mathcal{L}(u(\eta, \theta, Bw), w) \quad \forall w \in J. \quad (5.10)$$

Note that, again, \mathcal{S} depends on the couple (η, θ) but, for simplicity, we do not mention explicitly this dependence. We now consider the following auxiliary problem:

$$\text{find } w^* \in K(\eta, \theta) \text{ such that } \mathcal{S}(w^*) = \min_{w \in K(\eta, \theta)} \mathcal{S}(w). \quad (5.11)$$

We claim that this problem has at least one solution w^* and, to this end, we use Theorem 5.1 with $\tilde{J} = K(\eta, \theta)$.

Consider a sequence $\{w_k\} \subset K(\eta, \theta)$ such that $w_k \rightharpoonup w$ in W . Then, using assumption (5.6) it follows that $Bw_k \rightarrow Bw$ in Z and, therefore, Theorem 4.1 implies that

$$u(\eta, \theta, Bw_k) \rightarrow u(\eta, \theta, Bw) \quad \text{in } X.$$

We now use definition (5.10) and assumption (5.7) to see that

$$\liminf_{k \rightarrow \infty} \mathcal{S}(w_k) \geq \mathcal{S}(w).$$

We conclude from here that the function $\mathcal{S} : K(\eta, \theta) \rightarrow \mathbb{R}$ is lower semicontinuous.

Assume now that (5.8) holds. Then, for any sequence $\{w_k\} \subset K(\eta, \theta)$, we have

$$\mathcal{S}(w_k) = \mathcal{L}(u(\eta, \theta, w_k), w_k) \geq h(w_k).$$

Therefore, if $\|w_k\|_W \rightarrow \infty$ we deduce that $\mathcal{S}(w_k) \rightarrow \infty$ which shows that the function \mathcal{S} is coercive. Recall also the assumption (5.5) and the reflexivity of the space W . The existence of at least one solution to problem (5.11) is now a direct consequence of Theorem 5.1. On the other hand, if (5.9) holds it follows that $K(\eta, \theta)$ is a bounded subset on W and, therefore, we are still in a position to apply Theorem 5.1. We deduce from here that, if either (5.8) or (5.9) hold, then problem (5.11) has at least one solution.

Finally, using the definitions (5.10) and (5.3) it is easy to see that

$$\begin{cases} (u^*, w^*) \text{ is a solution of Problem } \mathcal{Q} \text{ if and only if} \\ w^* \text{ is a solution of problem (5.11) and } u^* = u(\eta, \theta, Bw^*). \end{cases} \quad (5.12)$$

Theorem 5.2 is a direct consequence of the equivalence (5.12) combined with the solvability of the optimization problem (5.11). \square

In what follows we assume that $(\eta, \theta) \in I$ is fixed and, for each $n \in \mathbb{N}$, we consider a perturbation $(\eta_n, \theta_n) \in I$ of (η, θ) , together with the set of admissible pairs defined by

$$\mathcal{V}_{ad}(\eta_n, \theta_n) = \{ (u, w) : w \in K(\eta_n, \theta_n), u = u(\eta_n, \theta_n, Bw) \}. \quad (5.13)$$

With these data, for each $n \in \mathbb{N}$ we consider the following perturbation of Problem \mathcal{Q} .

Problem \mathcal{Q}_n . Find $(u_n^*, w_n^*) \in \mathcal{V}_{ad}(\eta_n, \theta_n)$ such that

$$\mathcal{L}(u_n^*, w_n^*) = \min_{(u, w) \in \mathcal{V}_{ad}(\eta_n, \theta_n)} \mathcal{L}(u, w). \quad (5.14)$$

We also consider the function $\mathcal{S}_n : J \rightarrow \mathbb{R}$ defined by

$$\mathcal{S}_n(w) = \mathcal{L}(u(\eta_n, \theta_n, Bw), w) \quad \forall w \in J, \quad (5.15)$$

together with the optimization problem

$$\text{find } w_n^* \in K(\eta_n, \theta_n) \text{ such that } \mathcal{S}_n(w_n^*) = \min_{w \in K(\eta_n, \theta_n)} \mathcal{S}_n(w). \quad (5.16)$$

Then, if (3.2), (4.1), (5.5), (5.6), (5.7) and either (5.8) or (5.9) hold, it follows from Theorem 5.2 that Problem \mathcal{Q}_n has at least one solution (u_n^*, w_n^*) , for each $n \in \mathbb{N}$. Moreover, the proof of Theorem 5.2 shows that (5.16) has at least one solution w_n^* and, in addition,

$$\begin{cases} (u_n^*, w_n^*) \text{ is a solution of Problem } \mathcal{Q}_n \text{ if and only if} \\ w_n^* \text{ is a solution of problem (5.16) and } u_n^* = u(\eta_n, \theta_n, Bw_n^*). \end{cases} \quad (5.17)$$

In order to study the link between the solutions to Problems \mathcal{Q}_n and Problem \mathcal{Q} we consider the following assumptions.

$$\eta_n \rightarrow \eta \quad \text{in } X. \quad (5.18)$$

$$\theta_n \rightarrow \theta \quad \text{in } Y. \quad (5.19)$$

$$K(\eta_n, \theta_n) \xrightarrow{M} K(\eta, \theta) \quad \text{in } W. \quad (5.20)$$

$$\begin{cases} \mathcal{L} \text{ is continuous on } X \times J, \text{ i.e., for all sequences } \{u_k\} \subset X \\ \text{and } \{w_k\} \subset J \text{ such that } u_k \rightarrow u \text{ in } X, w_k \rightarrow w \text{ in } W, \\ \text{we have } \lim_{k \rightarrow \infty} \mathcal{L}(u_k, w_k) = \mathcal{L}(u, w). \end{cases} \quad (5.21)$$

Then, we have the following convergence result.

Theorem 5.3. *Assume (3.2), (4.1). Moreover, assume (5.5), (5.6), (5.7) and, in addition, assume that either (5.8) or (5.9) hold. For each $n \in \mathbb{N}$, let (u_n^*, w_n^*) be a solution of Problem \mathcal{Q}_n . Then, if (5.18)–(5.21) hold, there exists a subsequence of the sequence $\{(u_n^*, w_n^*)\}$, again denoted by $\{(u_n^*, w_n^*)\}$, and an element $(u^*, w^*) \in X \times W$, such that*

$$u_n^* \rightarrow u^* \quad \text{in } X, \quad (5.22)$$

$$w_n^* \rightharpoonup w^* \quad \text{in } W. \quad (5.23)$$

Moreover, (u^*, w^*) is a solution of Problem \mathcal{Q} .

Proof. The proof of Theorem 5.3 will be carried out in several steps that we present in what follows.

i) *In the first step we prove that there exists a subsequence of the sequence $\{w_n^*\}$, again denoted by $\{w_n^*\}$, and an element $w^* \in W$, such that (5.23) holds.*

To this end, we start by claiming that the sequence $\{w_n^*\}$ is bounded in W . This claim is obviously satisfied if we assume that (5.9) holds. Assume in what follows that (5.8) holds. If $\{w_n^*\}$ is not bounded in W , then we can find a subsequence of the sequence $\{w_n^*\}$, again denoted by $\{w_n^*\}$, such that $\|w_n^*\|_W \rightarrow \infty$. Therefore, using definition (5.15) and condition (5.8) we deduce that

$$\mathcal{S}_n(w_n^*) = \mathcal{L}(u(\eta_n, \theta_n, Bw_n^*), w_n^*) \geq h(w_n^*) \rightarrow \infty,$$

which implies that

$$\mathcal{S}_n(w_n^*) \rightarrow \infty. \quad (5.24)$$

Let s be a given element in $K(\eta, \theta)$ and note that assumption (5.20) implies that there exists a sequence $\{s_n\}$ such that $s_n \in K(\eta_n, \theta_n)$ for each $n \in \mathbb{N}$ and

$$s_n \rightarrow s \quad \text{in } W. \quad (5.25)$$

Moreover, since w_n^* is a solution of problem (5.16) we have

$$\mathcal{S}_n(w_n^*) \leq \mathcal{S}_n(s_n). \quad (5.26)$$

On the other hand, the convergences (5.18), (5.19), (5.25), assumption (5.6) and Theorem 4.1 show that

$$u(\eta_n, \theta_n, Bs_n) \rightarrow u(\eta, \theta, Bs) \quad \text{in } X \quad (5.27)$$

and, using assumption (5.21) we find that

$$\mathcal{L}(u(\eta_n, \theta_n, Bs_n), s_n) \rightarrow \mathcal{L}(u(\eta, \theta, Bs), s) \quad \text{as } n \rightarrow \infty,$$

which shows that

$$\lim_{n \rightarrow \infty} \mathcal{S}_n(s_n) = \mathcal{S}(s). \quad (5.28)$$

Thus, (5.26) implies that the sequence $\{\mathcal{S}_n(w_n^*)\}$ is bounded, which contradicts (5.24). We conclude from above that the sequence $\{w_n^*\}$ is bounded in W and, therefore, there exists a subsequence of the sequence $\{w_n^*\}$, again denoted by $\{w_n^*\}$, and an element $w^* \in W$, such that (5.23) holds.

ii) *We now prove that w^* is a solution of problem (5.11).*

To this end we recall that $w_n^* \in K(\eta_n, \theta_n)$, for all $n \in \mathbb{N}$. Therefore, using (5.23) and assumption (5.20), we deduce that $w^* \in K(\eta, \theta)$. Next, we consider an arbitrary element $s \in K(\eta, \theta)$ and, using assumption (5.20), again, we know that there exists a sequence $\{s_n\}$ such that $s_n \in K(\eta_n, \theta_n)$ for each $n \in \mathbb{N}$ and (5.25) holds. Moreover, it follows from the proof of the previous step that (5.27) and (5.28) hold, too.

Next, since w_n^* is a solution of problem (5.16) we have $\mathcal{S}_n(w_n^*) \leq \mathcal{S}_n(s_n)$ and, using (5.28) we find that

$$\liminf_{n \rightarrow \infty} \mathcal{S}_n(w_n^*) \leq \mathcal{S}(s). \quad (5.29)$$

On the other hand, using (5.27), (5.23) and (5.7), we have

$$\liminf_{n \rightarrow \infty} \mathcal{S}_n(w_n^*) = \liminf_{n \rightarrow \infty} \mathcal{L}(u(\eta_n, \theta_n, Bw_n^*), w_n^*) \geq \mathcal{L}(u(\eta, \theta, Bw^*), w^*) = \mathcal{S}(w^*)$$

and, using (5.29) yields $\mathcal{S}(w^*) \leq \mathcal{S}(s)$. Now, since $w^* \in K(\eta, \theta)$, we deduce that w^* is a solution of problem (5.11), which concludes the proof of this step.

iii) *End of proof.*

We now have all the ingredients to provide the proof of Theorem 5.3. First, the steps i) and ii) guarantee that there exists a subsequence of the sequence $\{w_n^*\}$, again denoted by $\{w_n^*\}$, and an element $w^* \in W$, such that (5.23) holds and, moreover, w^* is a solution of problem (5.11). Let $u^* = u(\eta, \theta, Bw^*)$. Then, the equivalence (5.12) shows that (u^*, w^*) is a solution of Problem \mathcal{Q} . On the other hand, since (u_n^*, w_n^*) is a solution of Problem \mathcal{Q}_n , (5.17) implies that $u_n^* = u(\eta_n, \theta_n, Bw_n^*)$ for all $n \in \mathbb{N}$. Then, the convergences (5.18), (5.19), (5.23) combined with assumption (5.6) show that $(\eta_n, \theta_n, Bw_n^*) \rightarrow (\eta, \theta, Bw^*)$ in $X \times Y \times Z$. It follows now from Theorem 4.1 that (5.22) holds, which concludes the proof. \square

6 Analysis and control of the contact model

In this section we apply our abstract results in the study of the contact problem in Section 2. Recall that this problem leads to the inclusion (2.26) with notation (2.14)–(2.17). Our first result in the study of this inclusion is the following

Theorem 6.1. *Assume (2.2) and (2.8). Then, for each element $(\theta, \mathbf{f}) \in L^2(\Gamma_3) \times V$ there exists a unique element $\boldsymbol{\omega} = \boldsymbol{\omega}(\theta, \mathbf{f}) \in Q$ such that (2.26) holds. Moreover, the mapping $(\theta, \mathbf{f}) \mapsto \boldsymbol{\omega}(\theta, \mathbf{f}) : L^2(\Gamma_3) \times V \rightarrow Q$ is Lipschitz continuous.*

The proof of Teorem 6.1 is based on two preliminary results that we present in what follows.

Lemma 6.2. *Assume (2.2), (2.8) and let $\boldsymbol{\omega}, \boldsymbol{\sigma} \in Q, \theta \in L^2(\Gamma_3), \mathbf{f} \in V$ be such that*

$$-\boldsymbol{\omega} \in N_{\Sigma(\theta, \mathbf{f})} \boldsymbol{\sigma}. \quad (6.1)$$

Then, there exists a unique element $\mathbf{u} \in V$ such that $\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\mathbf{u})$ and, moreover,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}))_Q + j(\theta, \mathbf{v}) - j(\theta, \mathbf{u}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u})_V \quad \forall \mathbf{v} \in V. \quad (6.2)$$

Proof. First, we note that inclusion (6.1) implies that

$$\boldsymbol{\sigma} \in \Sigma(\theta, \mathbf{f}), \quad (\boldsymbol{\tau} - \boldsymbol{\sigma}, \boldsymbol{\omega})_Q \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(\theta, \mathbf{f}). \quad (6.3)$$

Let $\mathbf{z} \in \boldsymbol{\varepsilon}(V)^\perp$ where, here and below, M^\perp represents the orthogonal of $M \subset Q$. Then $(\mathbf{z}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q = 0$ for all $\mathbf{v} \in V$ and, using (2.17), we find that $\boldsymbol{\sigma} \pm \mathbf{z} \in \Sigma(\theta, \mathbf{f})$. Therefore, testing with $\boldsymbol{\tau} = \boldsymbol{\sigma} \pm \mathbf{z}$ in (6.3) we deduce that $(\mathbf{z}, \boldsymbol{\omega})_Q = 0$ which shows that $\boldsymbol{\omega} \in \boldsymbol{\varepsilon}(V)^{\perp\perp} = \boldsymbol{\varepsilon}(V)$. This implies that there exists a element $\mathbf{u} \in V$ such that

$$\boldsymbol{\omega} = \boldsymbol{\varepsilon}(\mathbf{u}). \quad (6.4)$$

Moreover, (2.12) guarantees that \mathbf{u} is unique.

Next, by the subdifferentiability of the function $j(\theta, \cdot)$ in \mathbf{u} we know that there exists an element $\mathbf{g} \in V$ such that

$$j(\theta, \mathbf{v}) - j(\theta, \mathbf{u}) \geq (\mathbf{g}, \mathbf{v} - \mathbf{u})_V = (\boldsymbol{\varepsilon}(\mathbf{g}), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}))_Q$$

and, taking $\boldsymbol{\tau}_0 := \boldsymbol{\varepsilon}(\mathbf{f}) - \boldsymbol{\varepsilon}(\mathbf{g})$ we deduce that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}))_Q + j(\theta, \mathbf{v}) - j(\theta, \mathbf{u}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u})_V \quad \forall \mathbf{v} \in V. \quad (6.5)$$

We now test with $\mathbf{v} = 2\mathbf{u}$ and $\mathbf{v} = \mathbf{0}_V$ in this inequality to deduce that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\mathbf{u}))_Q + j(\theta, \mathbf{u}) = (\mathbf{f}, \mathbf{u})_V. \quad (6.6)$$

Therefore, combining (6.5) and (6.6) we find that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\theta, \mathbf{v}) \geq (\mathbf{f}, \mathbf{v})_V \quad \forall \mathbf{v} \in V$$

which shows that $\boldsymbol{\tau}_0 \in \Sigma(\theta, \mathbf{f})$. This regularity, (6.3) and (6.4) imply that

$$(\boldsymbol{\tau}_0, \boldsymbol{\varepsilon}(\mathbf{u}))_Q \geq (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}))_Q$$

and, using (6.6) yields

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}))_Q + j(\theta, \mathbf{u}) \leq (\mathbf{f}, \mathbf{u})_V. \quad (6.7)$$

On the other hand, since $\boldsymbol{\sigma} \in \Sigma(\theta, \mathbf{f})$ and $\mathbf{u} \in V$ the converse inequality holds, i.e.

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}))_Q + j(\theta, \mathbf{u}) \geq (\mathbf{f}, \mathbf{u})_V. \quad (6.8)$$

We now combine (6.7) and (6.8) to see that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}))_Q + j(\theta, \mathbf{u}) = (\mathbf{f}, \mathbf{u})_V. \quad (6.9)$$

Then, using (6.9) and inclusion $\boldsymbol{\sigma} \in \Sigma(\theta, \mathbf{f})$ it follows that (6.2) holds. \square

Lemma 6.3. *Assume (2.8). Then, for each $(\theta, \mathbf{f}) \in L^2(\Gamma_3) \times V$ the set $\Sigma(\theta, \mathbf{f}) \subset Q$ is nonempty closed and convex. Moreover, there exists $c_0 > 0$ such that for each $\theta_1, \theta_2 \in L^2(\Gamma_3)$, $\mathbf{f}_1, \mathbf{f}_2 \in V$ and $\boldsymbol{\omega} \in Q$ one has*

$$\|P_{\Sigma(\theta_1, \mathbf{f}_1)}\boldsymbol{\omega} - P_{K(\theta_2, \mathbf{f}_2)}\boldsymbol{\omega}\|_X \leq c_0(\|\theta_1 - \theta_2\|_{L^2(\Gamma_3)} + \|\mathbf{f}_1 - \mathbf{f}_2\|_V). \quad (6.10)$$

Proof. Let $\theta \in L^2(\Gamma_3)$ and $\mathbf{f} \in V$ be fixed. Since the function $\mathbf{v} \mapsto j(\theta, \mathbf{v}) : V \rightarrow \mathbb{R}$ is subdifferentiable and vanishes in $\mathbf{0}_V$, we deduce that there exists an element $\mathbf{g} \in V$ such that $j(\theta, \mathbf{v}) \geq (\mathbf{g}, \mathbf{v})_V$ for all $\mathbf{v} \in V$. Moreover, recall that $(\mathbf{g}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{g}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q$ and $(\mathbf{f}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{f}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q$. Therefore, using notation $\boldsymbol{\xi} = \boldsymbol{\varepsilon}(\mathbf{f}) - \boldsymbol{\varepsilon}(\mathbf{g})$ we find that

$$(\boldsymbol{\xi}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + j(\theta, \mathbf{v}) \geq (\mathbf{f}, \mathbf{v})_V \quad \text{for all } \mathbf{v} \in V. \quad (6.11)$$

We now combine (2.17) and (6.11) to see that $\boldsymbol{\xi} \in \Sigma(\theta, \mathbf{f})$ and, therefore, $\Sigma(\theta, \mathbf{f})$ is not empty. On the other hand, it is easy to see that it is a closed convex subset of Q , which concludes the proof of the first part of the lemma.

Assume now that $\theta_1, \theta_2 \in L^2(\Gamma_3)$, $\mathbf{f}_1, \mathbf{f}_2 \in V$, $\boldsymbol{\omega} \in Q$ and denote

$$\boldsymbol{\sigma}_1 = P_{K(\theta_1, \mathbf{f}_1)}\boldsymbol{\omega}, \quad \boldsymbol{\sigma}_2 = P_{K(\theta_2, \mathbf{f}_2)}\boldsymbol{\omega}. \quad (6.12)$$

We use (3.5) to see that

$$\boldsymbol{\sigma}_1 \in \Sigma(\theta_1, \mathbf{f}_1), \quad (\boldsymbol{\omega} - \boldsymbol{\sigma}_1, \boldsymbol{\tau} - \boldsymbol{\sigma}_1) \leq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(\theta_1, \mathbf{f}_1)$$

and, therefore, (3.4) implies that $\boldsymbol{\omega} - \boldsymbol{\sigma}_1 \in N_{\Sigma(\theta_1, \mathbf{f}_1)}(\boldsymbol{\sigma}_1)$. Next, Lemma 6.2 guarantees that there exists a unique element $\mathbf{u}_1 \in V$ such that

$$\boldsymbol{\sigma}_1 - \boldsymbol{\omega} = \boldsymbol{\varepsilon}(\mathbf{u}_1), \quad (6.13)$$

$$(\boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}_1)) + j(\theta_1, \mathbf{v}) - j(\theta_1, \mathbf{u}_1) \geq (\mathbf{f}_1, \mathbf{v} - \mathbf{u}_1)_V \quad \forall \mathbf{v} \in V. \quad (6.14)$$

Similar arguments show that there exists a unique element $\mathbf{u}_2 \in V$ such that

$$\boldsymbol{\sigma}_2 - \boldsymbol{\omega} = \boldsymbol{\varepsilon}(\mathbf{u}_2), \quad (6.15)$$

$$(\boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}_2)) + j(\theta_2, \mathbf{v}) - j(\theta_2, \mathbf{u}_2) \geq (\mathbf{f}_2, \mathbf{v} - \mathbf{u}_2)_V \quad \forall \mathbf{v} \in V. \quad (6.16)$$

We now take $\mathbf{v} = \mathbf{u}_2$ in (6.14), then we take $\mathbf{v} = \mathbf{u}_1$ in (6.16) and add the resulting inequalities to obtain that

$$\begin{aligned} & (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}(\mathbf{u}_1) - \boldsymbol{\varepsilon}(\mathbf{u}_2))_Q \\ & \leq j(\theta_1, \mathbf{u}_2) - j(\theta_1, \mathbf{u}_1) + j(\theta_2, \mathbf{u}_1) - j(\theta_2, \mathbf{u}_2) + (\mathbf{f}_1 - \mathbf{f}_2, \mathbf{u}_1 - \mathbf{u}_2)_V. \end{aligned}$$

Then, using the identity $\boldsymbol{\varepsilon}(\mathbf{u}_2) - \boldsymbol{\varepsilon}(\mathbf{u}_1) = \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1$, guaranteed by (6.13) and (6.15), we find that

$$\begin{aligned} \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_Q^2 & \leq j(\theta_1, \mathbf{u}_2) - j(\theta_1, \mathbf{u}_1) + j(\theta_2, \mathbf{u}_1) - j(\theta_2, \mathbf{u}_2) \\ & \quad + \|\mathbf{f}_1 - \mathbf{f}_2\|_V \|\mathbf{u}_1 - \mathbf{u}_2\|_V. \end{aligned} \quad (6.17)$$

On the other hand, a standard calculation based on the definition (2.15), the properties (2.8) of the function F and the trace inequality (2.13) shows that

$$\begin{aligned} & j(\theta_1, \mathbf{u}_2) - j(\theta_1, \mathbf{u}_1) + j(\theta_2, \mathbf{u}_1) - j(\theta_2, \mathbf{u}_2) \\ & \leq c_{tr} L_F \|\theta_1 - \theta_2\|_{L^2(\Gamma_3)} \|\mathbf{u}_1 - \mathbf{u}_2\|_V. \end{aligned} \quad (6.18)$$

We now combine inequalities (6.17) and (6.18) and, since $\|\mathbf{u}_1 - \mathbf{u}_2\|_V = \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_Q$, we deduce that

$$\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_Q \leq c_{tr} L_F \|\theta_1 - \theta_2\|_{L^2(\Gamma_3)} + \|\mathbf{f}_1 - \mathbf{f}_2\|_V. \quad (6.19)$$

Then, using (6.12) and (6.19) we see that (6.10) holds with $c_0 = \max\{c_{tr} L_F, 1\}$. \square

We now have all the ingredients to provide the proof of Theorem 6.1.

Proof. We use assumption (2.2) on the constitutive function \mathcal{A} to see the operator A defined by (2.2) satisfies condition (3.2) on the space $X = Q$. Moreover, Lemme 6.3 guarantee that condition (4.1) is satisfied on the spaces $Y = L^2(\Gamma_3)$ and $Z = V$. So, we use Theorem 4.1 with $\boldsymbol{\eta} = \mathbf{0}_Q$ to conclude the proof. \square

Remark 1. A couple of functions $(\mathbf{u}, \boldsymbol{\sigma})$ such that $\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\omega}$ and $\boldsymbol{\sigma} = A\boldsymbol{\omega}$ where $\boldsymbol{\omega}$ is a solution to Problem \mathcal{P}^V is called a weak solution to the contact problem \mathcal{P} . It follows from Theorem 6.1 and Lemma 6.2 that, under assumptions (2.2), (2.6), (2.8) and (2.9), the weak solution of Problem \mathcal{P}^V exists, is unique, and depends continuously on the data \mathbf{f}_2 and θ .

We now associate to Problem \mathcal{P}^V an optimal control problem, in the framework described in Section 4. To this end we consider the sets $I \subset Q \times L^2(\Gamma_3)$, $J \subset W$ given by

$$I = \{\mathbf{0}_Q\} \times L^2(\Gamma_3), \quad J = \{\mathbf{f}_2 \in L^2(\Gamma_2)^d : \|\mathbf{f}_2\|_{L^2(\Gamma_2)^d} \leq f_0\}, \quad (6.20)$$

where $f_0 > 0$ is given. Moreover, for each $\theta \in L^2(\Gamma_3)$, let $\tilde{K}(\theta) \subset J$ be the subset given by

$$\tilde{K}(\theta) = \{ \mathbf{f}_2 \in J : \|\mathbf{f}_2\|_{L^2(\Gamma_2)^d} \leq \tilde{h}(\theta) \}, \quad (6.21)$$

where

$$\begin{cases} \tilde{h} : L^2(\Gamma_3) \rightarrow \mathbb{R} \text{ is a continuous function such that} \\ 0 \leq \tilde{h}(\theta) \leq f_0 \quad \forall \theta \in L^2(\Gamma_3). \end{cases} \quad (6.22)$$

In addition, denote by $B : L^2(\Gamma_3)^d \rightarrow V$ the operator defined by

$$\mathbf{f} = B\mathbf{f}_2 \iff (2.16) \text{ holds.} \quad (6.23)$$

Then, under the assumptions (2.2) and (2.8), it follows from Theorem 6.1 that for each $(\theta, \mathbf{f}_2) \in L^2(\Gamma_3) \times L^2(\Gamma_2)^d$, Problem \mathcal{P}^V has a unique solution $\boldsymbol{\omega} = \boldsymbol{\omega}(\theta, B\mathbf{f}_2)$. For each $\theta \in L^2(\Gamma_3)$ we define the set of admissible pairs for Problem \mathcal{P}^V by equality

$$\tilde{\mathcal{V}}_{ad}(\theta) = \{ (\boldsymbol{\omega}, \mathbf{f}_2) : \mathbf{f}_2 \in \tilde{K}(\theta), \boldsymbol{\omega} = \boldsymbol{\omega}(\theta, B\mathbf{f}_2) \}.$$

Consider also the cost functional $\mathcal{L} : Q \times J \rightarrow \mathbb{R}$ given by

$$\mathcal{L}(\boldsymbol{\omega}, \mathbf{f}_2) = \int_{\Omega} (A\boldsymbol{\omega} - \boldsymbol{\varphi})^2 dx \quad \forall \boldsymbol{\omega} \in Q, \quad \mathbf{f}_2 \in J, \quad (6.24)$$

where $\boldsymbol{\varphi} \in Q$ is given. Fix $\theta \in L^2(\Gamma_3)$. Then, the optimal control problem we are interested in is the following.

Problem \mathcal{T} . Find $(\boldsymbol{\omega}^*, \mathbf{f}_2^*) \in \tilde{\mathcal{V}}_{ad}(\theta)$ such that

$$\mathcal{L}(\boldsymbol{\omega}^*, \mathbf{f}_2^*) = \min_{(u, q) \in \tilde{\mathcal{V}}_{ad}(\theta)} \mathcal{L}(\boldsymbol{\omega}, \mathbf{f}_2).$$

With this choice, using equalities (6.24) and (2.25), it follows that the mechanical interpretation of Problem \mathcal{T} is the following: given a contact process described by the inclusion (2.26) with the data θ and $\mathbf{f} = B\mathbf{f}_2$, we are looking for a density of surface tractions $\mathbf{f}_2 \in L^2(\Gamma_2)^d$ with $\|\mathbf{f}_2\|_{L^2(\Gamma_2)^d} \leq \tilde{h}(\theta)$ such that the resulting stress field in the elastic body is as close as possible to the “desired” stress field $\boldsymbol{\varphi}$.

Note that, in this case assumptions (5.5), (5.6), (5.7) and (5.9) are satisfied with $X = Q$, $Y = L^2(\Gamma_3)$, $W = V$, I and J given by (6.20), $K(\boldsymbol{\eta}, \theta) = \tilde{K}(\theta)$ for all $(\boldsymbol{\eta}, \theta) = (\mathbf{0}_Q, \theta) \in I$ and B , \mathcal{L} given by (6.23) and (6.24), respectively. Therefore Theorem 5.2 guarantees the existence of the solutions of the optimal control problem \mathcal{T} . Moreover, note that, with the notation above, assumptions (5.19) and (6.22) imply the convergence $K(\theta_n) \xrightarrow{M} K(\theta)$ in $L^2(\Gamma_2)^d$ which shows that (5.20) holds with $\boldsymbol{\eta}_n = \boldsymbol{\eta} = \mathbf{0}_Q$. Finally, note that (5.21) holds, too. Therefore, the convergence result stated in Theorem 5.3 can be applied in the study of Problem \mathcal{T} .

7 Conclusions

We considered a boundary value problem which describes the frictionless contact of an elastic body with a foundation. Under appropriate assumption on the data we provided that the problem leads to a stationary inclusion in which the unknown is the strain field. Motivated by this example we studied a class of abstract inclusions in the framework of Hilbert spaces for which we obtained existence, uniqueness and continuous dependence results. In addition, we proved the existence of optimal pairs for an associated optimal control problem. Then we illustrate the use of the abstract results in the study of the contact model.

The present work shows that, besides the classical tools provided by the theory of variational inequalities, stationary inclusions arguments can be used in the study of elastic contact problems. Moreover, the results we present here give rise to several open problems and subjects for further research that we describe in what follows.

First, it would be interesting to derive necessary optimality conditions in the study of Problem \mathcal{Q} introduced in Section 5 . Due to the nonsmooth and nonconvex feature of the functional \mathcal{L} , the treatment of this problem requires the use of its approximation by smooth optimization problems. Next, it would be interesting to extend the results presented here in study of frictional contact models with elastic or viscoelastic materials. Some progress in this directions can be made by using Corollary 4.2 which lies the background to the study of time-dependent and history-dependent inclusions. Finally, another question is to study dynamic contact problems by using arguments of inclusions. It is likely that such kind of problems lead to differential evolutionary inclusions or sweeping process problems. A general theory of such inclusions with emphasis to applications would be a nice contribution to the Mathematical Theory of Contact Mechanics.

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