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# Well-Posedness of Minimization Problems in Contact Mechanics

Mircea Sofonea<sup>1,2</sup> · Yi-bin Xiao<sup>1</sup>

## Abstract

We consider an abstract minimization problem in reflexive Banach spaces together with a specific family of approximating sets, constructed by perturbing the cost functional and the set of constraints. For this problem, we state and prove various well-posedness results in the sense of Tykhonov, under different assumptions on the data. The proofs are based on arguments of lower semicontinuity, compactness and Mosco convergence of sets. Our results are useful in the study of various mathematical models in contact mechanics. To provide examples, we introduce 2 models, which describe the equilibrium of an elastic body in contact with a rigid body covered by a rigid-plastic and an elastic material, respectively. The weak formulation of each model is in the form of a minimization problem for the displacement field. We use our abstract well-posedness results in the study of these minimization problems. In this way, we obtain existence, uniqueness and convergence results, and moreover, we provide their mechanical interpretations.

**Keywords** Minimization problem · Tykhonov well-posedness · Contact problem · Elastic material · Convergence results

**Mathematics Subject Classification** 49J45 · 90C48 · 90J31

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## 1 Introduction

The equilibrium of elastic bodies in potential contact with an obstacle is described by mathematical models, which consist of a system of partial differential equations associated with unilateral constraints. The literature in the field includes the books [1–8], among others. In a variational formulation, these elastic contact models are expressed in terms of variational or hemivariational inequalities and, under additional assumptions, in terms of minimization problems. Usually, the minimization problem for an elastic model aims to find a displacement field, which minimizes an energy functional on a constraint set of admissible displacement fields, where the energy functional is related to the constitutive laws and the problem data, i.e. the density of body force and surface traction, which act on the elastic body, and the friction bound. The constraint set is constructed by using the nonpenetrability conditions and could depend on the initial gap between the elastic body and the obstacle. The weak solvability of the elastic contact model is provided by the existence of solutions to the minimization problem. On the other hand, from mechanical point of view, it is interesting to study the dependence of the solutions on the problem data, i.e. to compare the solutions of the minimization problem with the solutions of its perturbed problem with perturbed energy functional and perturbed constraint set.

A comparison of the solutions to the minimization problem and its perturbed problem, including strong and weak convergence results, can be made by using the concept of well-posedness in the sense of Tykhonov introduced in [9] for a minimization problem. It is based on 2 main ingredients: the existence and uniqueness of the minimizer and the convergence of any approximating sequence to the minimizer. Note that the notion of approximating sequence depends on the choice of a specific family of sets, the so-called approximating sets. Therefore, the concept of well-posedness in the sense of Tykhonov, well-posedness for short, depends on this choice.

Following the pioneering work of Tykhonov, various concepts of well-posedness have been generalized for different optimization problems, such as extended well-posedness [10], Levitin–Polyak well-posedness [11] and generic well-posedness [12]. It is worth mentioning that the first basic criteria for well-posedness of optimization problems in metric spaces were established by Furi and Vignoli in [13,14]. For more details on well-posedness for optimization problems, we refer the readers to [15–17]. Extension of the concept of well-posedness to variational inequalities, mixed variational inequalities and hemivariational inequalities can be found in [18–26], for instance. Recently, a general concept of well-posedness in the sense of Tykhonov for abstract problems formulated on metric spaces was introduced in [27].

The aim of this paper is twofold. The first one is to study the well-posedness in the sense of Tykhonov for the minimization problem in the abstract framework of reflexive Banach spaces. To this end, we use a specific choice of approximating sets, inspired by the perturbed minimization problem. We provide sufficient conditions on the functional and the constraint set, which guarantee the weak and strong well-posedness of the minimization problem, and its weak well-posedness in generalized sense, too. To the best of our knowledge, this represents the first trait of novelty of the current paper. The second aim is to illustrate how these abstract results can be used in the study of contact problems with elastic materials. Our approach, based on Tykhonov

well-posedness concept, provides mathematical tools to deduce convergence results for various contact problems, which represents the second trait of novelty of our current paper. In particular, the obtained results represent a nontrivial extension of some of our previous results in [24,28,29], where the analysis of some new models, which describe the equilibrium of an elastic body in contact with a foundation, has been carried out. The model considered in [24] was frictionless and its variational formulation was in a form of an elliptic variational inequality with unilateral constraints for the stress field. The model considered in [28] was frictional, involved nonsmooth contact boundary condition, and therefore, its variational formulation was in the form of an elliptic variational–hemivariational inequality. Besides the unique solvability of the models, we proved in [24,28] the continuous dependence of solutions with respect to the data and discussed related optimal control problems, where we used arguments of minimization and convergence for specific lower semicontinuous functionals defined on particular Hilbert spaces. The corresponding proofs can be easily simplified by using the results we present in Sect. 3 of this current paper, which concern general functionals defined on abstract reflexive Banach spaces. We also stress that the contact model considered in [28] was constructed with the Signorini condition, a regularization of the Hencky elastic constitutive law, and was formulated in terms of the stress. The convergence results obtained there represent continuous dependence of solutions with respect to the data of the problem. In contrast, the contact model, which we present in Sect. 4 of this paper, is based on a different constitutive law, different contact boundary condition and leads to a minimization problem for the displacement field. Moreover, the convergence results obtained here are different, since they involve the solution of a regularized problem and are obtained by using Tykhonov well-posedness arguments.

The rest of the paper is structured as follows. In Sect. 2, we introduce the problem statement together with some preliminary material. In Sect. 3, we state and prove our main abstract results, Theorems 3.1 and 3.2. In Sect. 4, we introduce 2 mathematical models, which describe the equilibrium of an elastic body in contact with a rigid foundation covered by a rigid-plastic and an elastic material, respectively. Finally, in Sect. 5, we use our abstract mathematical tools in the study of these contact models and obtain existence, uniqueness and convergence results. We also provide the mechanical interpretations of these results.

## 2 Problem Statement

Everywhere below  $X$  is assumed to be a reflexive Banach space, unless stated otherwise. We use  $\|\cdot\|_X$  and  $0_X$  for the norm and the zero element of  $X$ , respectively. All the limits, upper and lower limits, below are considered as  $n \rightarrow \infty$ , even if we do not mention it explicitly. The symbols “ $\rightharpoonup$ ” and “ $\rightarrow$ ” denote the weak and the strong convergence in the space  $X$ , respectively. Moreover,  $K$  is a nonempty subset of  $X$  and  $J : X \rightarrow \mathbb{R}$  is a given functional. In this framework, we focus on the following minimization problem, which is referred to as Problem  $\mathcal{P}$ .

**Problem  $\mathcal{P}$**  Find  $u$  such that

$$u \in K, \quad J(u) := \min_{v \in K} J(v). \quad (1)$$

We denote in what follows by  $\mathcal{S}$  the set of solutions to Problem  $\mathcal{P}$ , i.e.

$$\mathcal{S} =: \{ u \in K : J(u) \leq J(v) \quad \forall v \in K \}. \quad (2)$$

In order to introduce the concept of well-posedness for Problem  $\mathcal{P}$ , we consider a family  $\{\Omega(\varepsilon)\}_{\varepsilon>0}$  of nonempty subsets of  $X$  and we introduce the following definitions.

**Definition 2.1** A sequence  $\{u_n\} \subset X$  is called an approximating sequence for Problem  $\mathcal{P}$  if there exists a sequence  $\{\varepsilon_n\} \subset \mathbb{R}$  with  $\varepsilon_n > 0$  and  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  such that  $u_n \in \Omega(\varepsilon_n)$  for each  $n \in \mathbb{N}$ .

For simplicity, for any sequence  $\{\varepsilon_n\}$  satisfying the conditions of Definition 2.1, we shall write  $0 < \varepsilon_n \rightarrow 0$ .

**Definition 2.2** Problem  $\mathcal{P}$  is said to be:

(a) strongly (weakly) well posed if it has a unique solution and every approximating sequence for Problem  $\mathcal{P}$  converges strongly (weakly) in  $X$  to the solution;

(b) strongly (weakly) well posed in generalized sense if it has at least one solution and every approximating sequence for Problem  $\mathcal{P}$  contains a subsequence which converges strongly (weakly) to some point of its solution set.

It follows from the definition above that, if Problem  $\mathcal{P}$  is strongly well posed (strongly well posed in generalized sense), then it is weakly well posed (weakly well posed in generalized sense). Moreover, if it is strongly (weakly) well posed, then it is strongly (weakly) well posed in generalized sense.

Note that the concepts of approximating sequence and well-posedness above depend on the family  $\{\Omega(\varepsilon)\}_{\varepsilon>0}$ . For this reason, when necessary, we shall refer to them as “well-posedness with respect to the family  $\{\Omega(\varepsilon)\}_{\varepsilon>0}$ ” or “approximating sequence with respect to the family  $\{\Omega(\varepsilon)\}_{\varepsilon>0}$ ”. We now proceed with the following 2 examples.

**Example 2.1** Let  $X$  be a Hilbert space,  $K := \{ v \in X : \|v\|_X \leq k \}$  with  $k > 0$  and  $P_K : X \rightarrow K$  be the projection map on the set  $K$ . We define the functional  $J : X \rightarrow \mathbb{R}$  as follows

$$J(v) := \|v - P_K v\|_X \quad \forall v \in X, \quad (3)$$

and for each  $\varepsilon > 0$ , we let  $\Omega(\varepsilon) := \{ v \in X : \|v\|_X \leq k + \varepsilon \}$ . With these notation, we claim that Problem  $\mathcal{P}$  is weakly well posed in generalized sense. Indeed, if  $\{u_n\}$  is an approximating sequence, then Definition 2.1 implies that there exists a sequence  $\{\varepsilon_n\} \subset \mathbb{R}$  such that  $0 < \varepsilon_n \rightarrow 0$  and  $\|u_n\|_X \leq k + \varepsilon_n$  for each  $n \in \mathbb{N}$ . This implies that the sequence  $\{u_n\}$  is bounded, and therefore, there exist a subsequence, still denoted

by  $\{u_n\}$ , and a point  $u \in X$  such that  $u_n \rightarrow u$  in  $X$ . We now use the weak lower semicontinuity of the norm to see that

$$\|u\|_X \leq \liminf \|u_n\|_X \leq \liminf (k + \varepsilon_n) = k,$$

and therefore, we deduce that  $u \in K$ . On the other hand, it is easy to see that the set of solutions of problem  $\mathcal{P}$  with  $J$  defined by (3) is  $K$  and, therefore,  $u \in \mathcal{S}$ . We now use Definition 2.2(b) to deduce that Problem  $\mathcal{P}$  is weakly well posed in generalized sense. This problem is neither weakly well posed nor strongly well posed (in the sense of Definition 2.2(a)) since, in general, its set of solutions is not a singleton.

**Example 2.2** Let  $K := X$ ,  $a \in X$ ,  $J : X \rightarrow \mathbb{R}$  be the functional given by

$$J(v) := \|v - a\|_X \quad \forall v \in X, \quad (4)$$

and for each  $\varepsilon > 0$ , we let  $\Omega(\varepsilon) := \{v \in X : \|v - a\|_X \leq \varepsilon\}$ . With these notation, we claim that Problem  $\mathcal{P}$  is strongly well posed. Indeed, let us first note that this problem has a unique solution since  $\mathcal{S} = \{a\}$ . Moreover, if  $\{u_n\}$  is an approximating sequence, then Definition 2.1 shows that there exists a sequence  $\{\varepsilon_n\} \subset \mathbb{R}$  such that  $0 < \varepsilon_n \rightarrow 0$  and  $\|u_n - a\|_X \leq \varepsilon_n$  for each  $n \in \mathbb{N}$ . This implies that  $u_n \rightarrow a$ . We now use Definition 2.2(a) to deduce that Problem  $\mathcal{P}$  is strongly well posed.

Next, recall that it was assumed in [27] that the inclusion

$$\mathcal{S} \subset \Omega(\varepsilon) \quad (5)$$

holds for any  $\varepsilon > 0$  and note that this inclusion is satisfied in Examples 2.1 and 2.2, too. Based on this inclusion, a characterization for the well-posedness of Problem  $\mathcal{P}$  was obtained in [27], in terms of the metric properties of the sets  $\Omega(\varepsilon)$ . Nevertheless, in what follows, we shall use a choice of the approximating sets, which does not guarantee the inclusion (5). Therefore, the well-posedness of Problem  $\mathcal{P}$  will be obtained by using different arguments, which consists one of the traits of novelties of this paper. More precisely, we consider a specific family of sets  $\{\Omega(\varepsilon)\}_{\varepsilon>0}$  defined as follows: For each  $\varepsilon > 0$ , we assume that  $K_\varepsilon$  is a nonempty subset of  $X$  and  $J_\varepsilon : X \rightarrow \mathbb{R}$  is a functional, which represent the perturbations of the set  $K$  and the functional  $J$ , respectively. We denote by Problem  $\mathcal{P}_\varepsilon$  the following minimization problem.

**Problem  $\mathcal{P}_\varepsilon$**  Find  $u$  such that

$$u \in K_\varepsilon, \quad J_\varepsilon(u) := \min_{v \in K_\varepsilon} J_\varepsilon(v). \quad (6)$$

Then,  $\Omega(\varepsilon)$  represents the set of solutions of Problem  $\mathcal{P}_\varepsilon$ , i.e.

$$\Omega(\varepsilon) := \{u \in K_\varepsilon : J_\varepsilon(u) \leq J_\varepsilon(v) \quad \forall v \in K_\varepsilon\}. \quad (7)$$

Our aim in what follows is to provide necessary and sufficient conditions, which guarantee the well-posedness and the well-posedness in generalized sense of Problem

$\mathcal{P}$  with respect to the family of sets  $\{\Omega(\varepsilon)\}_{\varepsilon>0}$  defined by (7). To this end, we start by consider the following assumptions.

- ( $K_1$ )  $K$  is a nonempty weakly closed subset of  $X$ .
- ( $K_2$ )  $K$  is a nonempty convex closed subset of  $X$ .
- ( $J_1$ )  $\begin{cases} J \text{ is a weakly lower semicontinuous functional, i.e.} \\ \text{for any sequence } \{u_n\} \subset X \text{ such that } u_n \rightarrow u \text{ in } X, \text{ one has} \\ \liminf J(u_n) \geq J(u). \end{cases}$
- ( $J_2$ )  $\begin{cases} J \text{ is a coercive functional, i.e. for any sequence } \{u_n\} \subset X \\ \text{such that } \|u_n\|_X \rightarrow \infty, \text{ one has } J(u_n) \rightarrow \infty. \end{cases}$
- ( $J_3$ )  $\begin{cases} J \text{ is a strictly convex functional, i.e.} \\ (1-t)J(u) + tJ(v) - J((1-t)u + tv) > 0, \\ \text{for all } u, v \in X, u \neq v, t \in (0, 1). \end{cases}$

Note that assumption ( $K_2$ ) implies assumption ( $K_1$ ). We end this preliminary section with the following version of the Weierstrass theorem, which provides the solvability and the unique solvability of Problem  $\mathcal{P}$ .

**Theorem 2.1** *Let  $X$  be a reflexive Banach space. Then, the following statements hold.*

- (i) *Under assumptions ( $K_1$ ), ( $J_1$ ) and ( $J_2$ ), there exists at least a solution to Problem  $\mathcal{P}$ , i.e.  $S \neq \emptyset$ .*
- (ii) *Under assumptions ( $K_2$ ), ( $J_1$ ), ( $J_2$ ) and ( $J_3$ ), there exists a unique solution to Problem  $\mathcal{P}$ , i.e.  $S$  is a singleton.*

Theorem 2.1 will be used in Sect. 3 to prove the solvability and the unique solvability of Problems  $\mathcal{P}$  and  $\mathcal{P}_\varepsilon$ . Its proof could be found in many books and survey, see, for instance, [7,30,31].

### 3 Well-Posedness Results

In order to study the well-posedness of Problem  $\mathcal{P}$ , we need the following additional assumptions on the perturbed sets  $K_\varepsilon$  and functionals  $J_\varepsilon$ .

- ( $K_{1\varepsilon}$ ) For each  $\varepsilon > 0$ ,  $K_\varepsilon$  is a nonempty weakly closed subset of  $X$ .
- ( $J_{1\varepsilon}$ )  $\begin{cases} J_\varepsilon \text{ is a weakly lower semicontinuous functional, i.e.} \\ \text{it satisfies condition } (J_1), \text{ for each } \varepsilon > 0. \end{cases}$
- ( $J_{2\varepsilon}$ )  $\begin{cases} J_\varepsilon \text{ is a coercive functional, i.e.} \\ \text{it satisfies condition } (J_2), \text{ for each } \varepsilon > 0. \end{cases}$
- ( $J_{3\varepsilon}$ )  $\begin{cases} \text{For any sequence } 0 < \varepsilon_n \rightarrow 0 \text{ and any sequence} \\ \{u_n\} \subset X \text{ such that } \|u_n\|_X \rightarrow \infty, \text{ one has } J_{\varepsilon_n}(u_n) \rightarrow \infty. \end{cases}$
- ( $J_{4\varepsilon}$ )  $\begin{cases} \text{For any sequence } 0 < \varepsilon_n \rightarrow 0 \text{ and any weakly convergent sequence} \\ \{u_n\} \subset X, \text{ one has } J_{\varepsilon_n}(u_n) - J(u_n) \rightarrow 0. \end{cases}$
- ( $J_{5\varepsilon}$ )  $\begin{cases} \text{For any sequence } 0 < \varepsilon_n \rightarrow 0 \text{ and any sequence} \\ \{u_n\} \subset X \text{ such that } u_n \rightarrow u \text{ in } X, \text{ one has } J_{\varepsilon_n}(u_n) - J_{\varepsilon_n}(u) \rightarrow 0. \end{cases}$

$$(M) \quad \left\{ \begin{array}{l} \text{The family } \{K_\varepsilon\} \text{ converges to } K \text{ in the sense of Mosco, i.e.} \\ \text{for any sequence } 0 < \varepsilon_n \rightarrow 0, \text{ the following two properties hold:} \\ (a) \text{ for each } v \in K, \text{ there exists a sequence } \{v_n\} \subset X \text{ such that} \\ \quad v_n \in K_{\varepsilon_n} \text{ for each } n \in \mathbb{N} \text{ and } v_n \rightarrow v \text{ in } X. \\ (b) \text{ for each sequence } \{v_n\} \subset X \text{ such that} \\ \quad v_n \in K_{\varepsilon_n} \text{ for each } n \in \mathbb{N} \text{ and } v_n \rightharpoonup v \text{ in } X, \text{ one has } v \in K. \end{array} \right.$$

Recall that the notion of Mosco convergence has been introduced in [32], and then, it was used in a large number of papers, including [28,33], for instance.

Our first result in this section is the following.

**Theorem 3.1** *Let  $X$  be a reflexive Banach space. Suppose that conditions  $(K_1)$ ,  $(J_1)$ ,  $(J_2)$ ,  $(K_{1\varepsilon})$ ,  $(J_{1\varepsilon})$ ,  $(J_{2\varepsilon})$ ,  $(J_{3\varepsilon})$ ,  $(J_{4\varepsilon})$ ,  $(J_{5\varepsilon})$  and  $(M)$  hold. Then, Problem  $\mathcal{P}$  is weakly well posed in generalized sense with respect to the family of approximating sets  $\{\Omega(\varepsilon)\}_{\varepsilon>0}$ .*

**Proof** First of all, since assumptions  $(K_1)$ ,  $(J_1)$  and  $(J_2)$  hold, it follows from Theorem 2.1 i) that there exists at least one solution to Problem  $\mathcal{P}$ , which implies that  $\mathcal{S} \neq \emptyset$ . Similarly, under assumptions  $(K_{1\varepsilon})$ ,  $(J_{1\varepsilon})$  and  $(J_{2\varepsilon})$ , there exists at least one solution to Problem  $\mathcal{P}_\varepsilon$ , which implies  $\Omega(\varepsilon) \neq \emptyset$  for each  $\varepsilon > 0$ .

Now, let  $\{u_n\}$  be an approximating sequence. Then, Definition 2.1 implies that there exists  $0 < \varepsilon_n \rightarrow 0$  such that  $u_n \in \Omega(\varepsilon_n)$  for all  $n \in \mathbb{N}$ . We claim that the sequence  $\{u_n\}$  is bounded in  $X$ . Indeed, if  $\{u_n\}$  is not bounded, then we can find a subsequence of the sequence  $\{u_n\}$ , again denoted by  $\{u_n\}$ , such that  $\|u_n\|_X \rightarrow \infty$ . Therefore, using assumptions  $(J_{3\varepsilon})$ , we deduce that

$$J_{\varepsilon_n}(u_n) \rightarrow \infty. \quad (8)$$

Let  $v$  be a given element in  $K$  and note that assumption  $(M)(a)$  implies that there exists a sequence  $\{v_n\}$  such that  $v_n \in K_{\varepsilon_n}$  for each  $n \in \mathbb{N}$  and

$$v_n \rightarrow v \quad \text{in } X. \quad (9)$$

Moreover, since  $u_n$  is a solution of Problem  $\mathcal{P}_{\varepsilon_n}$ , we obtain that  $J_{\varepsilon_n}(u_n) \leq J_{\varepsilon_n}(v_n)$ , and therefore,

$$J_{\varepsilon_n}(u_n) \leq [J_{\varepsilon_n}(v_n) - J_{\varepsilon_n}(v)] + [J_{\varepsilon_n}(v) - J(v)] + J(v) \quad \forall n \in \mathbb{N}. \quad (10)$$

On the other hand, by the convergences (9) and  $\varepsilon_n \rightarrow 0$ , assumption  $(J_{5\varepsilon})$  implies that  $J_{\varepsilon_n}(v_n) - J_{\varepsilon_n}(v) \rightarrow 0$ , and in addition, assumption  $(J_{4\varepsilon})$  shows that  $J_{\varepsilon_n}(v) - J(v) \rightarrow 0$ . Thus, inequality (10) implies that the sequence  $\{J_{\varepsilon_n}(u_n)\}$  is bounded, which contradicts (8). We conclude from above that the sequence  $\{u_n\}$  is bounded in  $X$ , and therefore, there exists a subsequence of the sequence  $\{u_n\}$ , again denoted by  $\{u_n\}$ , and an element  $u \in X$  such that

$$u_n \rightharpoonup u \quad \text{in } X. \quad (11)$$



We now prove that  $u$  is a solution of Problem  $\mathcal{P}$ , i.e.  $u \in \mathcal{S}$ . To this end, we use assumption  $(M)(b)$  and the convergence (11) to deduce that

$$u \in K. \quad (12)$$

Next, we consider an arbitrary element  $v \in K$ , and using condition  $(M)(a)$ , we know that there exists a sequence  $\{v_n\}$  such that  $v_n \in K_{\varepsilon_n}$  for each  $n \in \mathbb{N}$  and (9) holds. Since  $u_n$  is the solution to Problem  $\mathcal{P}_{\varepsilon_n}$ , we have  $J_{\varepsilon_n}(u_n) \leq J_{\varepsilon_n}(v_n)$ , which implies that

$$\begin{aligned} 0 \leq & [J_{\varepsilon_n}(v_n) - J_{\varepsilon_n}(v)] + [J_{\varepsilon_n}(v) - J(v)] + [J(v) - J(u)] \\ & + [J(u) - J(u_n)] + [J(u_n) - J_{\varepsilon_n}(u_n)]. \end{aligned} \quad (13)$$

We now use the convergences  $\varepsilon_n \rightarrow 0$ , (9) and (11) combined with assumptions  $(J_{5\varepsilon})$ ,  $(J_{4\varepsilon})$ ,  $(J_1)$  to deduce that

$$J_{\varepsilon_n}(v_n) - J_{\varepsilon_n}(v) \rightarrow 0, \quad (14)$$

$$J_{\varepsilon_n}(v) - J(v) \rightarrow 0, \quad (15)$$

$$\limsup [J(u) - J(u_n)] \leq 0, \quad (16)$$

$$J(u_n) - J_{\varepsilon_n}(u_n) \rightarrow 0. \quad (17)$$

Therefore, passing to the upper limit in inequality (13) and using (14, 15, 16, 17), we find that

$$0 \leq J(v) - J(u). \quad (18)$$

We now combine (12) and (18) to deduce that  $u$  is a solution of Problem  $\mathcal{P}$ . Finally, we use the convergence (11) and Definition 2.2 b) to conclude the proof.  $\square$

We now proceed with some relevant particular cases, in which Theorem 3.1 works. The first one is when  $K_\varepsilon = K$ , for each  $\varepsilon > 0$ . Note that, in this case, Problem  $\mathcal{P}_\varepsilon$  is formulated as follows.

**Problem  $\mathcal{P}_\varepsilon^*$**  Find  $u$  such that

$$u \in K, \quad J_\varepsilon(u) := \min_{v \in K} J_\varepsilon(v).$$

Moreover, for each  $\varepsilon > 0$ , the corresponding set  $\Omega(\varepsilon)$  is given by

$$\Omega^*(\varepsilon) := \{u \in K : J_\varepsilon(u) \leq J_\varepsilon(v) \quad \forall v \in K\}.$$

Then, we have the following consequence of Theorem 3.1.

**Corollary 3.1** *Let  $X$  be a reflexive Banach space. Suppose that conditions  $(K_1)$ ,  $(J_1)$ ,  $(J_2)$ ,  $(J_{1\varepsilon})$ ,  $(J_{2\varepsilon})$ ,  $(J_{3\varepsilon})$  and  $(J_{4\varepsilon})$  hold. Then, Problem  $\mathcal{P}$  is weakly well posed in generalized sense with respect to the family of approximating sets  $\{\Omega^*(\varepsilon)\}_{\varepsilon>0}$ .*

**Proof** Since  $K_\varepsilon = K$ , it follows that assumption  $(K_1)$  guarantees condition  $(K_{1\varepsilon})$ . Moreover, condition  $(M)$  is satisfied. In addition, a careful analysis of the proof for Theorem 3.1 shows that condition  $(J_{5\varepsilon})$  is used only to prove the convergence (14). Or, in the particular case  $K_\varepsilon = K$ , we can avoid condition  $(J_{5\varepsilon})$  since we can take  $v_n = v$  for each  $n \in \mathbb{N}$ , and in this case, the convergence (14) obviously holds. With these remarks, we conclude the proof of Corollary 3.1 since it represents a simplified version of the proof of Theorem 3.1.  $\square$

The second particular case is when  $J_\varepsilon = J$  for each  $\varepsilon > 0$ . In this case, Problem  $\mathcal{P}_\varepsilon$  can be formulated as follows.

**Problem  $\mathcal{P}_\varepsilon^{**}$**  Find  $u$  such that

$$u \in K_\varepsilon, \quad J(u) := \min_{v \in K_\varepsilon} J(v).$$

Moreover, for each  $\varepsilon > 0$ , the corresponding set  $\Omega(\varepsilon)$  is given by

$$\Omega^{**}(\varepsilon) := \{u \in K_\varepsilon : J(u) \leq J(v) \quad \forall v \in K_\varepsilon\}.$$

Then, we can get the following consequence of Theorem 3.1.

**Corollary 3.2** *Let  $X$  be a reflexive Banach space. Suppose that assumptions  $(K_1)$ ,  $(K_{1\varepsilon})$ ,  $(J_1)$ ,  $(J_2)$ ,  $(M)$  hold and  $J : X \rightarrow \mathbb{R}$  is a continuous functional. Then, Problem  $\mathcal{P}$  is weakly well posed in generalized sense with respect to the family of approximating sets  $\{\Omega^{**}(\varepsilon)\}_{\varepsilon>0}$ .*

**Proof** Since  $J_\varepsilon = J$ , it follows that assumption  $(J_1)$  guarantees condition  $(J_{1\varepsilon})$ . Moreover, assumption  $(J_2)$  guarantees conditions  $(J_{2\varepsilon})$  and  $(J_{3\varepsilon})$ . In addition, condition  $(J_{4\varepsilon})$  is obviously satisfied and condition  $(J_{5\varepsilon})$  follows from the continuity of the functional  $J$ . Corollary 3.2 is now a direct consequence of Theorem 3.1.  $\square$

The third particular case is when  $K_\varepsilon = K$  and  $J_\varepsilon = J$  for each  $\varepsilon > 0$ . Then, Problem  $\mathcal{P}_\varepsilon$  reduces to Problem  $\mathcal{P}$  and  $\Omega(\varepsilon) = \mathcal{S}$  for each  $\varepsilon > 0$ . In this case, we have the following result.

**Corollary 3.3** *Let  $X$  be a reflexive Banach space and assume that conditions  $(K_1)$ ,  $(J_1)$  and  $(J_2)$  hold. Then, there exists at least a solution to Problem  $\mathcal{P}$ , and moreover, the set of solution is weakly sequentially compact in  $X$ .*

**Proof** The existence part is a direct consequence of Theorem 2.1. We now use Theorem 3.1 to see that Problem  $\mathcal{P}$  is weakly well posed in generalized sense with respect to the family of approximating sets  $\{\mathcal{S}\}_{\varepsilon>0}$ . Corollary 3.3 is now a direct consequence of Definition 2.2 b).  $\square$

Next, we consider the following assumption, which clearly reinforces assumption  $(J_3)$ .

$$(J_4) \quad \begin{cases} J \text{ is a strongly uniformly convex functional, i.e.} \\ \text{there exists } m > 0 \text{ such that} \\ (1-t)J(u) + tJ(v) - J((1-t)u + tv) \geq m t(1-t)\|u - v\|_X^2 \\ \text{for all } u, v \in X, t \in [0, 1]. \end{cases}$$

Our second result in this section is the following.

**Theorem 3.2** *Let  $X$  be a reflexive Banach space. Then, the following 2 statements hold.*

- (i) *Under assumptions  $(K_2)$ ,  $(J_1)$ ,  $(J_2)$ ,  $(J_3)$ ,  $(K_{1\varepsilon})$ ,  $(J_{1\varepsilon})$ ,  $(J_{2\varepsilon})$ ,  $(J_{3\varepsilon})$ ,  $(J_{4\varepsilon})$ ,  $(J_{5\varepsilon})$  and  $(M)$ , Problem  $\mathcal{P}$  is weakly well posed with respect to the family of approximating sets  $\{\Omega(\varepsilon)\}_{\varepsilon>0}$ .*
- (ii) *Under assumptions  $(K_2)$ ,  $(J_1)$ ,  $(J_4)$ ,  $(K_{1\varepsilon})$ ,  $(J_{1\varepsilon})$ ,  $(J_{2\varepsilon})$ ,  $(J_{3\varepsilon})$ ,  $(J_{4\varepsilon})$ ,  $(J_{5\varepsilon})$  and  $(M)$ , Problem  $\mathcal{P}$  is strongly well posed with respect to the family of approximating sets  $\{\Omega(\varepsilon)\}_{\varepsilon>0}$ .*

**Proof** (i) With assumptions  $(K_2)$ ,  $(J_1)$ ,  $(J_2)$  and  $(J_3)$ , it follows from Theorem 2.1 ii) that there exists a unique solution to Problem  $\mathcal{P}$ , i.e.  $\mathcal{S}$  is a singleton. Now, let  $\{u_n\}$  be an approximating sequence. It follows from the proof of Theorem 3.1 that  $\{u_n\}$  is bounded and any weakly convergent subsequence of  $\{u_n\}$  converges to the solution of Problem  $\mathcal{P}$ , which is unique. We now use a standard result to deduce that the whole sequence  $\{u_n\}$  converges weakly to the unique solution of Problem  $\mathcal{P}$ . Therefore, using Definition 2.2 a), we conclude the proof of first part of the theorem.

(ii) Assume now that  $(K_1)$ ,  $(J_1)$ ,  $(J_4)$ ,  $(K_{1\varepsilon})$ ,  $(J_{1\varepsilon})$ ,  $(J_{2\varepsilon})$ ,  $(J_{3\varepsilon})$ ,  $(J_{4\varepsilon})$ ,  $(J_{5\varepsilon})$  and  $(M)$  hold. We recall that assumption  $(J_4)$  implies condition  $(J_3)$ . Moreover, using the fact that any convex lower semicontinuous function is bounded below by an affine function, it is easy to see that assumptions  $(J_1)$  and  $(J_4)$  imply condition  $(J_2)$ . Therefore, we are in a position to use the part i) of the theorem to get the unique solvability of Problem  $\mathcal{P}$  and the weak convergence of any approximating sequence to its unique solution. Denote by  $u$  the unique solution of Problem  $\mathcal{P}$  and let  $\{u_n\}$  be an approximating sequence. Then, it follows from above that

$$u_n \rightharpoonup u \quad \text{in } X. \quad (19)$$

Let  $\{\tilde{u}_n\}$  be a sequence such that  $\tilde{u}_n \in K_{\varepsilon_n}$  for each  $n \in \mathbb{N}$  and

$$\tilde{u}_n \rightarrow u \quad \text{in } X. \quad (20)$$

Recall that the existence of such sequence follows from assumption  $(M)$ (a). Then, using assumption  $(J_4)$  with  $t = \frac{1}{2}$ , we find that

$$\frac{m}{4} \|\tilde{u}_n - u_n\|_X^2 \leq \frac{1}{2} \left[ J(\tilde{u}_n) - J\left(\frac{\tilde{u}_n + u_n}{2}\right) \right] + \frac{1}{2} \left[ J(u_n) - J\left(\frac{\tilde{u}_n + u_n}{2}\right) \right]. \quad (21)$$

We write

$$J(\tilde{u}_n) - J\left(\frac{\tilde{u}_n + u_n}{2}\right) = [J(\tilde{u}_n) - J_{\varepsilon_n}(\tilde{u}_n)]$$

$$+[J_{\varepsilon_n}(\tilde{u}_n) - J(u)] + \left[ J(u) - J\left(\frac{\tilde{u}_n + u_n}{2}\right) \right], \quad (22)$$

and using the convergences (19), (20) and conditions  $(J_{4\varepsilon})$ ,  $(J_{5\varepsilon})$ ,  $(J_1)$ , we get that

$$\begin{aligned} J(\tilde{u}_n) - J_{\varepsilon_n}(\tilde{u}_n) &\rightarrow 0, \\ J_{\varepsilon_n}(\tilde{u}_n) - J(u) &\rightarrow 0, \\ \limsup \left[ J(u) - J\left(\frac{\tilde{u}_n + u_n}{2}\right) \right] &\leq 0. \end{aligned}$$

Therefore, it follows that

$$\limsup \left[ J(\tilde{u}_n) - J\left(\frac{\tilde{u}_n + u_n}{2}\right) \right] \leq 0. \quad (23)$$

On the other hand, we write

$$\begin{aligned} J(u_n) - J\left(\frac{\tilde{u}_n + u_n}{2}\right) &= [J(u_n) - J_{\varepsilon_n}(u_n)] \\ &\quad + \left[ J_{\varepsilon_n}(u_n) - J_{\varepsilon_n}\left(\frac{\tilde{u}_n + u_n}{2}\right) \right] \\ &\quad + \left[ J_{\varepsilon_n}\left(\frac{\tilde{u}_n + u_n}{2}\right) - J\left(\frac{\tilde{u}_n + u_n}{2}\right) \right]. \end{aligned} \quad (24)$$

Using the convergences (19), (20) and condition  $(J_{4\varepsilon})$ , we get that

$$\begin{aligned} J(u_n) - J_{\varepsilon_n}(u_n) &\rightarrow 0, \\ J_{\varepsilon_n}\left(\frac{\tilde{u}_n + u_n}{2}\right) - J\left(\frac{\tilde{u}_n + u_n}{2}\right) &\rightarrow 0, \end{aligned}$$

and moreover, since  $u_n$  is a solution to Problem  $\mathcal{P}_{\varepsilon_n}$ , we have

$$J_{\varepsilon_n}(u_n) - J_{\varepsilon_n}\left(\frac{\tilde{u}_n + u_n}{2}\right) \leq 0.$$

Therefore, with these three ingredients, equality (24) yields

$$\limsup \left[ J(u_n) - J\left(\frac{\tilde{u}_n + u_n}{2}\right) \right] \leq 0. \quad (25)$$

We now combine inequalities (21), (23) and (25) to deduce that

$$\limsup \| \tilde{u}_n - u_n \|_X^2 = 0,$$

which implies that

$$u_n - \tilde{u}_n \rightarrow 0_X \quad \text{in } X. \quad (26)$$

Finally, using the convergences (20) and (26), we get  $u_n \rightarrow u$  in  $X$ , which concludes the proof.  $\square$

We end this section with the remark that relevant version of Theorem 3.2 can be obtained for the particular cases where  $K_\varepsilon = K$  and  $J_\varepsilon = J$  for each  $\varepsilon > 0$ . Since the modifications are similar to those presented in Corollaries 3.1 and 3.2, we skip the details. Nevertheless, if the set  $\Omega(\varepsilon) = \{u_\varepsilon\}$  is a singleton for each  $\varepsilon > 0$ , then Theorem 3.2 provides a weak and strong convergence result, i.e.  $u_\varepsilon \rightarrow u$  in  $X$  and  $u_\varepsilon \rightarrow u$  in  $X$ , as  $\varepsilon \rightarrow 0$ , under corresponding assumptions on the data.

## 4 The Contact Models

We now turn to the application of our abstract results in the study of 2 elastic contact problems. The physical setting is the following. An deformable body occupies a bounded domain  $\Omega \subset \mathbb{R}^d$  ( $d = 1, 2, 3$ ) with a Lipschitz continuous boundary  $\Gamma$ , divided into three measurable disjoint parts  $\Gamma_1$ ,  $\Gamma_2$  and  $\Gamma_3$  such that  $meas(\Gamma_1) > 0$ . The body is fixed on  $\Gamma_1$ , and therefore, the displacement field vanishes there. It is acted upon by a given body force and is submitted to the action of a given surface traction on  $\Gamma_2$ . Moreover, the body is in contact with an obstacle on  $\Gamma_3$ , the so-called foundation. The mechanical process is static, and the contact is frictionless. We assume that the body is elastic and the foundation is made of a rigid obstacle covered by a deformable layer of rigid-plastic material of thickness  $g$ , say asperities. A two-dimensional version of the physical setting, corresponding to a rectangular body, is depicted in Fig. 1.

To construct the mathematical models, which correspond to this physical setting, we denote by  $\mathbf{u} := (u_i)$  and  $\boldsymbol{\sigma} := (\sigma_{ij})$  the displacement field and the stress field, respectively. Here and below, the indices  $i, j, k, l$  run between 1 and  $d$ , and unless stated otherwise, the summation convention over repeated indices is used. Moreover, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable  $\mathbf{x} \in \Omega \cup \Gamma$ , e.g.  $u_{i,j} = \partial u_i / \partial x_j$ . For simplicity, we do not indicate explicitly the dependence of various functions on the  $\mathbf{x}$ . Also,  $\boldsymbol{\epsilon}$  and  $\text{Div}$  will represent the deformation and the divergence operators, respectively, i.e.

$$\boldsymbol{\epsilon}(\mathbf{u}) := (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) := \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div } \boldsymbol{\sigma} := (\sigma_{ij,j}).$$

We denote by  $\mathbf{v} := (v_i)$  the outward unit normal at  $\Gamma$ , which exists almost everywhere, and  $u_\nu$ ,  $\mathbf{u}_\tau$  will represent the normal and tangential components of  $\mathbf{u}$  on  $\Gamma$  given by  $u_\nu := \mathbf{u} \cdot \mathbf{v}$  and  $\mathbf{u}_\tau := \mathbf{u} - u_\nu \mathbf{v}$ , respectively. Moreover,  $\sigma_\nu$  and  $\boldsymbol{\sigma}_\tau$  will represent the normal and tangential tractions on  $\Gamma$ , i.e.  $\sigma_\nu := (\boldsymbol{\sigma} \mathbf{v}) \cdot \mathbf{v}$  and  $\boldsymbol{\sigma}_\tau := \boldsymbol{\sigma} \mathbf{v} - \sigma_\nu \mathbf{v}$ . Here and below,  $A\boldsymbol{\omega}$  represents a short hand notation for the value of the tensor  $A$  in the element  $\boldsymbol{\omega}$ , and therefore,  $\boldsymbol{\sigma} \mathbf{v}$  represents the Cauchy stress vector. We use  $\mathbb{S}^d$  for the space of second-order symmetric tensors on  $\mathbb{R}^d$  and recall that the inner product and norm on  $\mathbb{R}^d$  and  $\mathbb{S}^d$  are defined by

$$\mathbf{u} \cdot \mathbf{v} := u_i v_i, \quad \|\mathbf{v}\| := (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}} \quad \forall \mathbf{u} = (u_i), \quad \mathbf{v} = (v_i) \in \mathbb{R}^d,$$

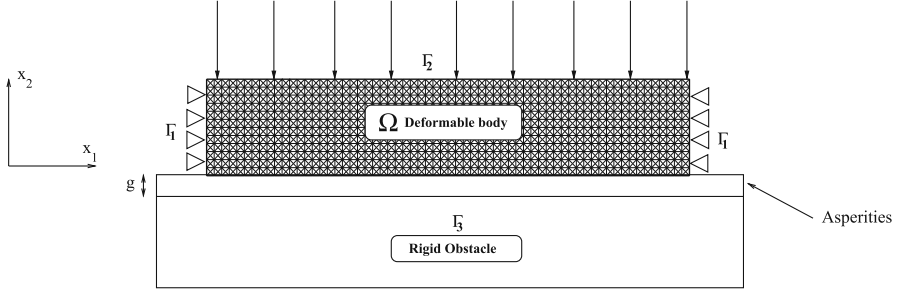


Fig. 1 Physical setting

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} := \sigma_{ij} \tau_{ij}, \quad \|\boldsymbol{\tau}\| := (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

Finally, the zero element of the spaces  $\mathbb{S}^d$  and  $\mathbb{R}^d$  will be denoted by  $\mathbf{0}$ .

The first contact model we consider in this section is as follows.

**Problem Q** Find a displacement field  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$  and a stress field  $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$  such that

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\epsilon}(\mathbf{u}) + \alpha(\boldsymbol{\epsilon}(\mathbf{u}) - P_B\boldsymbol{\epsilon}(\mathbf{u})) \quad \text{in } \Omega, \quad (27)$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega, \quad (28)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad (29)$$

$$\boldsymbol{\sigma} \mathbf{v} = \mathbf{f}_2 \quad \text{on } \Gamma_2, \quad (30)$$

$$u_\nu \leq g, \quad \left. \begin{array}{l} \sigma_\nu = 0, \quad \text{if } u_\nu < 0 \\ -F \leq \sigma_\nu \leq 0, \quad \text{if } u_\nu = 0 \\ \sigma_\nu = -F, \quad \text{if } 0 < u_\nu < g \\ \sigma_\nu \leq -F, \quad \text{if } u_\nu = g \end{array} \right\} \quad \text{on } \Gamma_3, \quad (31)$$

$$\boldsymbol{\sigma} \boldsymbol{\tau} = \mathbf{0} \quad \text{on } \Gamma_3. \quad (32)$$

We now provide a short description of the equations and boundary conditions in Problem Q.

First, Equ. (27) represents the elastic constitutive law of the material, in which  $\mathcal{E}$  is a fourth-order elasticity tensor,  $\alpha$  is a positive elastic coefficient,  $B$  is a nonempty closed convex set in the space  $\mathbb{S}^d$  and  $P_B : \mathbb{S}^d \rightarrow B$  denotes the projection operator. Such kind of constitutive laws could model the behaviour of some real materials like metals and have been used in [6,7,34,35], for instance. Usually, the set  $B$  is defined by

$$K := \{ \boldsymbol{\tau} \in \mathbb{S}^d : \mathcal{F}(\boldsymbol{\tau}) \leq 0 \}, \quad (33)$$

where  $\mathcal{F} : \mathbb{S}^d \rightarrow \mathbb{R}$  is a convex continuous function such that  $\mathcal{F}(\mathbf{0}) < 0$ . Recall that  $\boldsymbol{\tau} = P_B \boldsymbol{\tau}$  iff  $\boldsymbol{\tau} \in B$ , and therefore, we see from (27) that  $\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\epsilon}(\mathbf{u})$  iff  $\boldsymbol{\epsilon}(\mathbf{u}) \in B$ . It follows from here that the material behaves linearly as far as the strain tensor  $\boldsymbol{\epsilon}(\mathbf{u})$  belongs to  $B$ . The behaviour of the material is nonlinear only for strain tensors  $\boldsymbol{\epsilon}(\mathbf{u})$

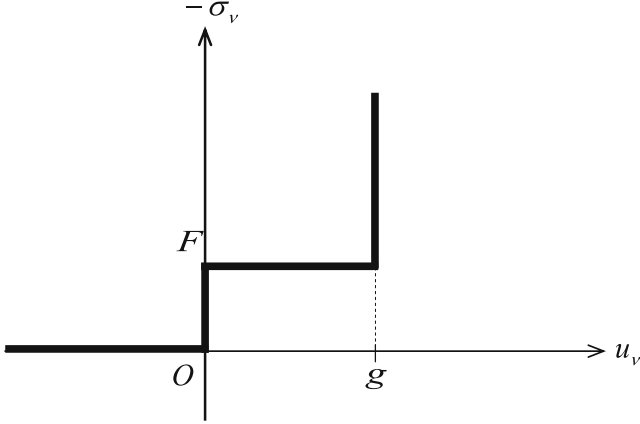


Fig.2 Contact condition (31)

such that  $\epsilon(\mathbf{u}) \notin B$ . We conclude from above that the set  $B$  represents the domain of linearly elastic behaviour of the material (27).

Equation (28) is the equation of equilibrium, and conditions (29) and (30) represent the displacement and the traction boundary conditions, respectively. There,  $\mathbf{f}_0$  denotes the density of body force and  $\mathbf{f}_2$  represents the density of surface traction. Finally, condition (32) represents the frictionless contact condition.

We now turn to the contact condition (31), which is described by the maximal monotone multivalued relation between the normal displacement and the opposite of the normal stress represented in Fig. 2 and which was used in a large number of papers, including [8,36]. This condition can be derived in the following way. First, we assume that the normal stress has an additive decomposition of the form

$$\sigma_v = \sigma_v^P + \sigma_v^R \quad \text{on } \Gamma_3, \quad (34)$$

in which  $\sigma_v^P$  describes the reaction of the deformable layer and  $\sigma_v^R$  describes the reaction of the rigid obstacle.

We assume that  $\sigma_v^P$  satisfies the condition

$$-F \leq \sigma_v^P \leq 0, \quad \sigma_v^P = \begin{cases} 0, & \text{if } u_v < 0 \\ -F, & \text{if } u_v > 0, \end{cases} \quad (35)$$

where  $F$  is a given positive coefficient, which could depend on the spatial variable  $x$ . Using (35), we have

$$\begin{aligned} -F < \sigma_v^P \leq 0 &\implies u_v \leq 0, \\ \sigma_v^P = -F &\implies u_v \geq 0. \end{aligned}$$

This shows that the layer does not allow penetration (and, therefore, it behaves like a rigid body) as far as the inequality  $-F < \sigma_v^P \leq 0$  holds. It could allow penetration only

when  $\sigma_v^P = -F$ , and in this case, it offers no additional resistance. We conclude from here that the deformable layer has a rigid-plastic behaviour, as claimed. Moreover, the function  $F$  could be interpreted as its yield limit.

On the other hand, the rigid body does not allow penetration, and therefore, we assume that the normal displacement satisfies condition

$$u_v \leq g \quad \text{on } \Gamma_3. \quad (36)$$

Here, recall,  $g > 0$  is a given bound, which represents the thickness of the rigid-plastic layer. In addition, the part  $\sigma_v^R$  of the normal stress satisfies the Signorini condition in the form with the gap  $g$ , i.e.

$$\sigma_v^R \leq 0, \quad \sigma_v^R(u_v - g) = 0 \quad \text{on } \Gamma_3. \quad (37)$$

We now gather conditions (34, 35, 36, 37) and claim that, in this way, we obtain the contact condition (31). Indeed:

- (a) If  $u_v < 0$ , then (35) implies that  $\sigma_v^P = 0$ , (37) implies that  $\sigma_v^R = 0$ , and therefore, equality (34) shows that  $\sigma_v = 0$ .
- (b) If  $u_v = 0$ , then (35) implies that  $-F \leq \sigma_v^P \leq 0$ , (37) implies that  $\sigma_v^R = 0$ , and therefore, equality (34) shows that  $-F \leq \sigma_v \leq 0$ .
- (c) If  $0 < u_v < g$ , then (35) implies that  $\sigma_v^P = -F$ , (37) implies that  $\sigma_v^R = 0$ , and therefore, equality (34) shows that  $\sigma_v = -F$ .
- (d) If  $u_v = g$ , then (35) implies that  $\sigma_v^P = -F$ , (37) implies that  $\sigma_v^R \leq 0$ , and therefore, equality (34) shows that  $\sigma_v \leq -F$ .

We conclude from above that, in any case, condition (31) is satisfied, which proves the claim. Moreover, we have the following comments on these contact conditions. First, the foundation has a geometry conformal with the contact surface  $\Gamma_3$ , which could not be planar. In the deformed configuration, the points of  $\Gamma_3$  are in one of the following four status: separation with the foundation, contact without penetration of the deformable layer, contact with partial penetration of the deformable layer, contact with the rigid obstacle (since the deformable layer is completely squeezing). These four status correspond to the situations a)-d) described above and divide the surface  $\Gamma_3$  into four regions, which are not known a priori, are part of the problem and represent free boundaries.

The second contact model we consider in this section is governed by a parameter  $\varepsilon > 0$  and is stated as follows.

**Problem  $\mathcal{Q}_\varepsilon$**  Find a displacement field  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$  and a stress field  $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$  such that

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\epsilon}(\mathbf{u}) + \alpha_\varepsilon(\boldsymbol{\epsilon}(\mathbf{u}) - P_B\boldsymbol{\epsilon}(\mathbf{u})) \quad \text{in } \Omega, \quad (38)$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_{0\varepsilon} = \mathbf{0} \quad \text{in } \Omega, \quad (39)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad (40)$$

$$\boldsymbol{\sigma} \mathbf{v} = \mathbf{f}_{2\varepsilon} \quad \text{on } \Gamma_2, \quad (41)$$



$$u_\nu \leq g_\varepsilon, \quad \left. \begin{array}{l} \sigma_\nu = 0, \quad \text{if } u_\nu < 0 \\ -\sigma_\nu = \frac{F u_\nu}{\sqrt{u_\nu^2 + k_\varepsilon^2}}, \quad \text{if } 0 < u_\nu < g_\varepsilon \\ \sigma_\nu \leq -\frac{F g_\varepsilon}{\sqrt{g_\varepsilon^2 + k_\varepsilon^2}}, \quad \text{if } u_\nu = g_\varepsilon \end{array} \right\} \quad \text{on } \Gamma_3, \quad (42)$$

$$\sigma_\tau = \mathbf{0} \quad \text{on } \Gamma_3. \quad (43)$$

The equations and boundary conditions (38, 39, 40, 41, 42, 43) have a similar meaning as the corresponding ones in Problem  $\mathcal{Q}$ . A brief comparison between the 2 contact models show that in Problem  $\mathcal{Q}_\varepsilon$  we replace the data  $\alpha$ ,  $f_0$ ,  $f_2$  and  $g$  with their perturbation  $\alpha_\varepsilon$ ,  $f_{0\varepsilon}$ ,  $f_{2\varepsilon}$  and  $g_\varepsilon$ , respectively. The second difference arises from the contact condition (42), which represents a regularization of the contact condition (31). Here,  $k_\varepsilon$  represents a deformability coefficient and  $r^+$  denotes the positive part of  $r$ , i.e.  $r^+ = \max\{0, r\}$ . This condition models the contact with a rigid body covered by a layer of elastic material of thickness  $g_\varepsilon$ . It can be deduced using arguments similar to those used to deduce the contact condition (31). Note that (42) shows that, if there is penetration on the elastic material, i.e.  $0 < u_\nu < g_\varepsilon$ , then the foundation exerts a pressure on the body, which depends on the penetration.

In the study of Problems  $\mathcal{Q}$  and  $\mathcal{Q}_\varepsilon$ , we need to introduce further notation and preliminary material. Everywhere below, we use the standard notation for Sobolev and Lebesgue spaces associated with  $\Omega$  and  $\Gamma$ . In particular, we use the spaces  $L^2(\Omega)^d$ ,  $L^2(\Gamma_2)^d$ ,  $L^2(\Gamma_3)$  and  $H^1(\Omega)^d$ , endowed with their canonical inner products and the associated norms. For an element  $\mathbf{v} \in H^1(\Omega)^d$ , we still write  $\mathbf{v}$  for the trace  $\gamma \mathbf{v}$  of  $\mathbf{v}$  to  $\Gamma$ . Moreover, we consider the space

$$X := \{ \mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1 \},$$

which is a real Hilbert space endowed with the canonical inner product

$$(\mathbf{u}, \mathbf{v})_X := \int_{\Omega} \boldsymbol{\epsilon}(\mathbf{u}) \cdot \boldsymbol{\epsilon}(\mathbf{v}) \, dx$$

and the associated norm  $\|\cdot\|_X$ . Recall that the completeness of the space  $X$  follows from the assumption  $\text{meas}(\Gamma_1) > 0$ , which allows the use of Korn's inequality. Also, there exists  $c_0 > 0$  depending on  $\Omega$ ,  $\Gamma_1$  and  $\Gamma_3$  such that

$$\|\mathbf{v}\|_{L^2(\Gamma)^d} \leq c_0 \|\mathbf{v}\|_X \quad \text{for all } \mathbf{v} \in X, \quad (44)$$

which represents a consequence of the Sobolev trace theorem.

In the study of the contact problem (27, 28, 29, 30, 31, 32), we assume that the elasticity tensor  $\mathcal{E}$  and the set  $B$  satisfy the following conditions.

- (a)  $\mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ ,
- (b)  $\mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega)$ ,  $1 \leq i, j, k, l \leq d$ ,
- (c) There exists  $m_\mathcal{E} > 0$  such that
 
$$\mathcal{E} \boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_\mathcal{E} \|\boldsymbol{\tau}\|^2 \quad \forall \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega.$$

$$B \text{ is a closed convex subset of } \mathbb{S}^d \text{ such that } \mathbf{0} \in B. \quad (46)$$

For the rest of the data, we assume that

$$\alpha > 0, \quad (47)$$

$$\mathbf{f}_0 \in L^2(\Omega)^d, \quad \mathbf{f}_2 \in L^2(\Gamma_2)^d, \quad (48)$$

$$F \in L^2(\Gamma_3), \quad F(\mathbf{x}) \geq 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3, \quad (49)$$

$$g > 0. \quad (50)$$

Under these assumptions, we introduce the set  $K \subset X$  and the functional  $J : X \rightarrow \mathbb{R}$  defined by

$$K := \{ \mathbf{v} \in X : v_\nu \leq g \quad \text{a.e. on } \Gamma_3 \}, \quad (51)$$

$$\begin{aligned} J(\mathbf{v}) := & \frac{1}{2} \int_{\Omega} \mathcal{E}\boldsymbol{\epsilon}(\mathbf{v}) \cdot \boldsymbol{\epsilon}(\mathbf{v}) \, dx + \frac{\alpha}{2} \int_{\Omega} \|\boldsymbol{\epsilon}(\mathbf{v}) - P_K \boldsymbol{\epsilon}(\mathbf{v})\|^2 \, dx \\ & - \int_{\Omega} \mathbf{f}_0 \cdot \mathbf{v} \, dx - \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v} \, da + \int_{\Gamma_3} F v_\nu^+ \, da \quad \forall \mathbf{v} \in X. \end{aligned} \quad (52)$$

Assume now that  $(\mathbf{u}, \boldsymbol{\sigma})$  represents a regular solution of Problem  $\mathcal{Q}$ . Then, using standard arguments, it can be shown that, for any  $\mathbf{v} \in K$ , the following inequalities hold.

$$\begin{aligned} \mathcal{E}\boldsymbol{\epsilon}(\mathbf{u}) \cdot (\boldsymbol{\epsilon}(\mathbf{v}) - \boldsymbol{\epsilon}(\mathbf{u})) & \leq \frac{1}{2} \mathcal{E}\boldsymbol{\epsilon}(\mathbf{v}) \cdot \boldsymbol{\epsilon}(\mathbf{v}) - \frac{1}{2} \mathcal{E}\boldsymbol{\epsilon}(\mathbf{u}) \cdot \boldsymbol{\epsilon}(\mathbf{u}) \quad \text{a.e. in } \Omega, \\ (\boldsymbol{\epsilon}(\mathbf{u}) - P_B \boldsymbol{\epsilon}(\mathbf{u})) \cdot (\boldsymbol{\epsilon}(\mathbf{v}) - \boldsymbol{\epsilon}(\mathbf{u})) \\ & \leq \frac{1}{2} \|\boldsymbol{\epsilon}(\mathbf{v}) - P_B \boldsymbol{\epsilon}(\mathbf{v})\|^2 - \frac{1}{2} \|\boldsymbol{\epsilon}(\mathbf{u}) - P_B \boldsymbol{\epsilon}(\mathbf{u})\|^2 \quad \text{a.e. in } \Omega, \\ -\boldsymbol{\sigma} \cdot (\mathbf{v}^* - \mathbf{u}^*) & \leq F v_\nu^+ - F u_\nu^+ \quad \text{a.e. on } \Gamma_3. \end{aligned}$$

Using these inequalities, notation (51), (52) and the Green formula, we deduce the following variational formulation of Problem  $\mathcal{Q}$ .

**Problem  $\mathcal{Q}^V$**  Find a displacement field  $\mathbf{u}$  such that

$$\mathbf{u} \in K, \quad J(\mathbf{u}) \leq J(\mathbf{v}) \quad \forall \mathbf{v} \in K. \quad (53)$$

Next, in the study of Problem  $\mathcal{Q}_\varepsilon$ , we assume the following additional assumptions for each  $\varepsilon > 0$ .

$$\alpha_\varepsilon > 0, \quad (54)$$

$$k_\varepsilon > 0, \quad (55)$$

$$\mathbf{f}_{0\varepsilon} \in L^2(\Omega)^d, \quad \mathbf{f}_{2\varepsilon} \in L^2(\Gamma_2)^d, \quad (56)$$

$$g_\varepsilon > 0. \quad (57)$$

Under these assumptions, we introduce the set  $K_\varepsilon \subset X$  and the functional  $J_\varepsilon : X \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} K_\varepsilon &:= \{ \mathbf{v} \in X : v_\nu \leq g_\varepsilon \text{ a.e. on } \Gamma_3 \}, \\ J_\varepsilon(\mathbf{v}) &:= \frac{1}{2} \int_\Omega \mathcal{E}\boldsymbol{\epsilon}(\mathbf{v}) \cdot \boldsymbol{\epsilon}(\mathbf{v}) \, dx + \frac{\alpha_\varepsilon}{2} \int_\Omega \|\boldsymbol{\epsilon}(\mathbf{v}) - P_B\boldsymbol{\epsilon}(\mathbf{v})\|^2 \, dx \\ &\quad - \int_\Omega \mathbf{f}_{0\varepsilon} \cdot \mathbf{v} \, dx - \int_{\Gamma_2} \mathbf{f}_{2\varepsilon} \cdot \mathbf{v} \, da + \int_{\Gamma_3} F \sqrt{(v_\nu^+)^2 + k_\varepsilon^2} \, da \quad \forall \mathbf{v} \in X. \end{aligned} \quad (58)$$

$$(59)$$

Assume now that  $(\mathbf{u}, \boldsymbol{\sigma})$  represents a regular solution of Problem  $\mathcal{Q}_\varepsilon$ . Then, it is easy to check that, for any  $\mathbf{v} \in K_\varepsilon$ , the following inequality holds:

$$-\sigma_\nu(v_\nu - u_\nu) \leq F \sqrt{(v_\nu^+)^2 + k_\varepsilon^2} - F \sqrt{(u_\nu^+)^2 + k_\varepsilon^2} \quad \text{a.e. on } \Gamma_3.$$

Using this inequality, notation (58), (59) and arguments similar to those used to obtain inequality (53), we deduce the following variational formulation of Problem  $\mathcal{Q}_\varepsilon$ .

**Problem  $\mathcal{Q}_\varepsilon^V$**  Find a displacement field  $\mathbf{u}$  such that

$$\mathbf{u} \in K_\varepsilon, \quad J_\varepsilon(\mathbf{u}) \leq J_\varepsilon(\mathbf{v}) \quad \forall \mathbf{v} \in K_\varepsilon. \quad (60)$$

The analysis of Problem  $\mathcal{Q}^V$  and  $\mathcal{Q}_\varepsilon^V$ , including existence, uniqueness and various convergence results, will be provided in the next section. Here, we restrict ourselves to mention that a function  $\mathbf{u}$  satisfying (53) is called a weak solution of the elastic contact problem (27, 28, 29, 30, 31, 32) and a function  $\mathbf{u}$  satisfying (60) is called a weak solution of the elastic contact problem (38, 39, 40, 41, 42, 43).

## 5 Existence, Uniqueness and Convergence Results

For the results we present in this section, we need some convergence conditions on the data, gathered in the following assumptions.

$$\alpha_\varepsilon \rightarrow \alpha \quad \text{as } \varepsilon \rightarrow 0, \quad (61)$$

$$k_\varepsilon \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad (62)$$

$$\mathbf{f}_{0\varepsilon} \rightharpoonup \mathbf{f}_0 \text{ in } L^2(\Omega)^d, \quad \mathbf{f}_{2\varepsilon} \rightharpoonup \mathbf{f}_2 \text{ in } L^2(\Gamma_2)^d \quad \text{as } \varepsilon \rightarrow 0, \quad (63)$$

$$g_\varepsilon \rightarrow g \quad \text{as } \varepsilon \rightarrow 0. \quad (64)$$

Our first result in this section is the following.

**Theorem 5.1** (i) Under assumptions (45, 46, 47, 48, 49, 50), there exists a unique solution to Problem  $\mathcal{Q}^V$ .

- ii) Under assumptions (45), (46), (49), (54, 55, 56, 57), there exists a unique solution to Problem  $\mathcal{Q}_\varepsilon^V$ .
- iii) Under assumptions (45, 46, 47, 48, 49, 50), (54, 55, 56, 57), (61, 62, 63, 64), the solution of Problem  $\mathcal{Q}_\varepsilon^V$  converges in  $X$  to the solution of Problem  $\mathcal{Q}^V$ , as  $\varepsilon \rightarrow 0$ .

**Proof** i) Since  $g > 0$ , it is clear that the set  $K$  defined by (51) satisfies condition  $(K_2)$ . Moreover, using the properties of the projection operator and the trace operator, it follows that  $J$  is a continuous functional, and therefore, condition  $(J_1)$  holds. On the other hand, using (45), (47), (49) and (44), we see that there exists a positive constant  $c$  such that

$$J(\mathbf{v}) \geq \frac{m\varepsilon}{2} \|\mathbf{v}\|_X^2 - c (\|\mathbf{f}_0\|_{L^2(\Omega)^d} + \|\mathbf{f}_2\|_{L^2(\Gamma_2)^d}) \|\mathbf{v}\|_X \quad \forall \mathbf{v} \in X. \quad (65)$$

This inequality shows that condition  $(J_2)$  holds, too. Finally, note that the function

$$\mathbf{v} \mapsto \frac{\alpha}{2} \int_{\Omega} \|\boldsymbol{\epsilon}(\mathbf{v}) - P_B \boldsymbol{\epsilon}(\mathbf{v})\|^2 dx$$

is convex since it is Gâteaux differentiable on  $X$  and its gradient, given by

$$\mathbf{v} \mapsto \alpha(\boldsymbol{\epsilon}(\mathbf{v}) - P_B \boldsymbol{\epsilon}(\mathbf{v})) \quad \forall \mathbf{v} \in X,$$

is a monotone operator. The details of the proof can be found in [34]. In addition, using the convexity of the function  $r \mapsto r^+$ , it follows that the function

$$\mathbf{v} \mapsto - \int_{\Omega} \mathbf{f}_0 \cdot \mathbf{v} dx - \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v} da + \int_{\Gamma_3} F v_v^+ da$$

is a convex function, too. Moreover, a simple calculation based the properties of the tensor  $\mathcal{E}$  shows that

$$\begin{aligned} & (1-t) \int_{\Omega} \mathcal{E} \boldsymbol{\epsilon}(\mathbf{v}) \cdot \boldsymbol{\epsilon}(\mathbf{v}) dx + t \int_{\Omega} \mathcal{E} \boldsymbol{\epsilon}(\mathbf{v}) \cdot \boldsymbol{\epsilon}(\mathbf{v}) dx \\ & - \int_{\Omega} \mathcal{E} \boldsymbol{\epsilon}((1-t)\mathbf{u} + t\mathbf{v}) \cdot \boldsymbol{\epsilon}((1-t)\mathbf{u} + t\mathbf{v}) dx = t(1-t) \int_{\Omega} \mathcal{E} \boldsymbol{\epsilon}(\mathbf{v} - \mathbf{u}) \cdot \boldsymbol{\epsilon}(\mathbf{v} - \mathbf{u}) dx \\ & \geq m_\varepsilon t(1-t) \|\mathbf{u} - \mathbf{v}\|_X^2 \end{aligned} \quad (66)$$

for all  $\mathbf{u}, \mathbf{v} \in X$  and  $t \in [0, 1]$ . We gather all these properties to see that the functional  $J$  satisfies condition  $(J_4)$ . It follows from here that  $J$  satisfies condition  $(J_3)$ , too. The unique solvability of Problem  $\mathcal{Q}^V$  is now a consequence of Theorem 2.1 (ii).

(ii) The unique solvability of Problem  $\mathcal{Q}_\varepsilon^V$  follows from arguments similar as those used above. First, we use the continuity of the function

$$\mathbf{v} \mapsto \int_{\Gamma_3} F \sqrt{(v_v^+)^2 + k_\varepsilon^2} da \quad (67)$$

to see that condition  $(J_{1\varepsilon})$  is satisfied. Moreover, using the trace inequality (44), we deduce that there exists  $c > 0$  such that

$$J_\varepsilon(\mathbf{v}) \geq \frac{m\varepsilon}{2} \|\mathbf{v}\|_X^2 - c(\|\mathbf{f}_{0\varepsilon}\|_{L^2(\Omega)^d} + \|\mathbf{f}_{2\varepsilon}\|_{L^2(\Gamma_2)^d})\|\mathbf{v}\|_X \quad (68)$$

for all  $\mathbf{v} \in X$ . This shows that condition  $(J_{2\varepsilon})$  hold, too. On the other hand, the strict convexity of the functional  $J_\varepsilon$ , guaranteed by inequality (66), and the convexity of the function (67) show that  $J_\varepsilon$  satisfies condition  $(J_3)$ . Since condition  $(K_{1\varepsilon})$  is obviously satisfied, the existence of a unique solution to Problem  $\mathcal{Q}_\varepsilon^V$  follows from Theorem 2.1 ii).

(iii) We use Theorem 3.2 ii). To this end, we recall that, as seen above, conditions  $(K_2)$ ,  $(J_1)$ ,  $(J_4)$ ,  $(K_{1\varepsilon})$ ,  $(J_{1\varepsilon})$ ,  $(J_{2\varepsilon})$  are satisfied. Moreover, condition  $(J_{3\varepsilon})$  is a direct consequence of inequality (68) combined with the convergence (62).

Assume now that  $0 < \varepsilon_n \rightarrow 0$  and  $\{\mathbf{u}_n\} \subset X$  is a weakly convergent sequence. Then, an elementary calculus shows that

$$\begin{aligned} J_{\varepsilon_n}(\mathbf{u}_n) - J(\mathbf{u}_n) &= \frac{1}{2}(\alpha_{\varepsilon_n} - \alpha) \int_\Omega \|\boldsymbol{\epsilon}(\mathbf{u}_n) - P_B \boldsymbol{\epsilon}(\mathbf{u}_n)\|^2 dx \\ &\quad - \int_\Omega (\mathbf{f}_{0\varepsilon_n} - \mathbf{f}_0) \cdot \mathbf{u}_n dx - \int_{\Gamma_2} (\mathbf{f}_{2\varepsilon} - \mathbf{f}_2) \cdot \mathbf{u}_n da \\ &\quad + \int_{\Gamma_3} F(\sqrt{(u_{nv}^+)^2 + k_\varepsilon^2} - u_{nv}^+) da. \end{aligned}$$

We now use the properties of the projection operator  $P_B$ , the compactness of the embedding  $X \subset L^2(\Omega)^d$ , the compactness of the trace operator  $\gamma : X \rightarrow L^2(\Gamma)^d$ , inequality

$$\left| \sqrt{(u_{nv}^+)^2 + k_\varepsilon^2} - u_{nv}^+ \right| \leq k_\varepsilon \quad \text{a.e. on } \Gamma_3$$

and the convergences (61)–(64) to see that each term of the previous equality converges to zero. We deduce from here that  $J_{\varepsilon_n}(\mathbf{u}_n) - J(\mathbf{u}_n) \rightarrow 0$ , and therefore, condition  $(J_{4\varepsilon})$  is satisfied.

Assume now that  $0 < \varepsilon_n \rightarrow 0$  and  $\mathbf{u}_n \rightarrow \mathbf{u}$  in  $X$ . Then, for each  $n \in \mathbb{N}$ , we have

$$\begin{aligned} J_{\varepsilon_n}(\mathbf{u}_n) - J_{\varepsilon_n}(\mathbf{u}) &= \frac{1}{2} \int_\Omega \mathcal{E} \boldsymbol{\epsilon}(\mathbf{u}_n) \cdot \boldsymbol{\epsilon}(\mathbf{u}_n) dx - \frac{1}{2} \int_\Omega \mathcal{E} \boldsymbol{\epsilon}(\mathbf{u}) \cdot \boldsymbol{\epsilon}(\mathbf{u}) dx \\ &\quad + \frac{\alpha_{\varepsilon_n}}{2} \int_\Omega \left( \|\boldsymbol{\epsilon}(\mathbf{u}_n) - P_B \boldsymbol{\epsilon}(\mathbf{u}_n)\|^2 - \|\boldsymbol{\epsilon}(\mathbf{u}) - P_B \boldsymbol{\epsilon}(\mathbf{u})\|^2 \right) dx \\ &\quad - \int_\Omega \mathbf{f}_{0\varepsilon_n} \cdot (\mathbf{u}_n - \mathbf{u}) dx - \int_{\Gamma_2} \mathbf{f}_{2\varepsilon} \cdot (\mathbf{u}_n - \mathbf{u}) da \\ &\quad + \int_{\Gamma_3} F(\sqrt{(u_{nv}^+)^2 + k_\varepsilon^2} - \sqrt{(u_v^+)^2 + k_\varepsilon^2}) da. \end{aligned}$$

We now use the continuity of the bilinear form

$$(\mathbf{u}, \mathbf{v}) \mapsto \int_{\Omega} \mathcal{E}\boldsymbol{\epsilon}(\mathbf{u}) \cdot \boldsymbol{\epsilon}(\mathbf{v}) \, dx \quad \forall \mathbf{u}, \mathbf{v} \in X,$$

the properties of the projection operator  $P_B$ , the compactness of the embedding  $X \subset L^2(\Omega)^d$  and the trace operator  $\gamma : X \rightarrow L^2(\Gamma)^d$ , inequality

$$\left| \sqrt{(u_{nv}^+)^2 + k_\varepsilon^2} - \sqrt{(u_{nv}^+)^2 + k_\varepsilon^2} \right| \leq |u_{nv}^+ - u_v^+| \leq \|u_n - u\| \quad \text{a.e. on } \Gamma_3$$

and the convergences (61, 62, 63, 64) to see that each term of the previous equality converges to zero. We deduce from here that  $J_{\varepsilon_n}(\mathbf{u}_n) - J_{\varepsilon_n}(\mathbf{u}) \rightarrow 0$ , and therefore, condition  $(J_{5\varepsilon})$  holds.

Finally, using definitions (51), (58) combined with assumptions (50), (57) and (64), we see that  $K_\varepsilon = \frac{g_\varepsilon}{g} K$  for all  $\varepsilon > 0$ . Based on this equality, it is easy to see that condition  $(M)$  is satisfied.

It follows from above that we are in position to apply Theorem 3.2 ii) to see that Problem  $\mathcal{Q}^V$  is well posed with the family of sets  $\{\Omega(\varepsilon)\}_{\varepsilon>0}$  defined by (7). Now, since both the set of solutions to Problem  $\mathcal{Q}^V$  and the set  $\Omega(\varepsilon)$  are singletons, as noted at the end of Sect. 3, it follows that  $\mathbf{u}_\varepsilon \rightarrow \mathbf{u}$  in  $V$  as  $\varepsilon \rightarrow 0$ , which concludes the proof.  $\square$

Note that Theorem 5.1 (i), (ii) provides the unique weak solvability of Problems  $\mathcal{Q}$  and  $\mathcal{Q}_\varepsilon$ , respectively. Next, in order to provide the mechanical interpretation of the convergence result given by Theorem 5.1 (iii), we denote in what follows by  $\mathbf{u}_\varepsilon(\alpha_\varepsilon, k_\varepsilon, \mathbf{f}_{0\varepsilon}, \mathbf{f}_{2\varepsilon}, g_\varepsilon)$  the solution of Problem  $\mathcal{Q}_\varepsilon^V$  constructed with the data  $\alpha_\varepsilon, k_\varepsilon, \mathbf{f}_{0\varepsilon}, \mathbf{f}_{2\varepsilon}, g_\varepsilon$  satisfying (54, 55, 56, 57). In addition, we denote by  $\mathbf{u}(\alpha, \mathbf{f}_0, \mathbf{f}_2, g)$  the solution of Problem  $\mathcal{Q}$  constructed with the data  $\alpha, \mathbf{f}_0, \mathbf{f}_2, g$  satisfying (47, 48, 49, 50). It follows from Theorem 5.1 iii) that, if the convergences (61, 62, 63, 64) hold, then

$$\mathbf{u}_\varepsilon(\alpha_\varepsilon, k_\varepsilon, \mathbf{f}_{0\varepsilon}, \mathbf{f}_{2\varepsilon}, g_\varepsilon) \rightarrow \mathbf{u}(\alpha, \mathbf{f}_0, \mathbf{f}_2, g) \quad \text{in } X \quad \text{as } \varepsilon \rightarrow 0. \quad (69)$$

On the other hand, a careful analysis based on the definitions (52) and (59) of the functionals  $J$  and  $J_\varepsilon$  reveals that, if  $k_\varepsilon = 0$ , then Problem  $\mathcal{Q}_\varepsilon^V$  reduces to Problem  $\mathcal{Q}^V$ . Therefore,

$$\mathbf{u}_\varepsilon(\alpha_\varepsilon, 0, \mathbf{f}_{0\varepsilon}, \mathbf{f}_{2\varepsilon}, g_\varepsilon) = \mathbf{u}(\alpha_\varepsilon, \mathbf{f}_{0\varepsilon}, \mathbf{f}_{2\varepsilon}, g_\varepsilon). \quad (70)$$

We now take  $k_\varepsilon = 0$  in (69) and use equality (70) to deduce that, if (61), (63) and (64) hold, then

$$\mathbf{u}(\alpha_\varepsilon, \mathbf{f}_{0\varepsilon}, \mathbf{f}_{2\varepsilon}, g_\varepsilon) \rightarrow \mathbf{u}(\alpha, \mathbf{f}_0, \mathbf{f}_2, g) \quad \text{in } X \quad \text{as } \varepsilon \rightarrow 0. \quad (71)$$

In addition to the mathematical interest in the convergence result (71), it is important from mechanical point of view since it shows that the weak solution of the contact

problem  $Q^V$  depends continuously on the densities of the applied force, the yield limit and the thickness of the rigid-plastic layer of the foundation.

Finally, the convergence result (69) shows that, if (62) holds, then

$$u_\varepsilon(\alpha, k_\varepsilon, f_0, f_2, g) \rightarrow u(\alpha, f_0, f_2, g) \text{ in } X \text{ as } \varepsilon \rightarrow 0. \quad (72)$$

The convergence result (72) shows that the weak solution of the contact problem with a rigid body covered by a layer of rigid-plastic material can be approached by the solution of the contact problem with a rigid body covered by a layer of elastic material when the deformability coefficient of this material is small enough.

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