A Tykhonov-type well-posedness concept for elliptic hemivariational inequalities

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Abstract. In this paper, we introduce a new Tykhonov-type well-posedness concept for elliptic hemivariational inequalities, governed by an approximating function h. We characterize the well-posedness in terms of the metric properties of the family of approximating sets, under various assumptions on h. Then, we use the well-posedness properties in order to obtain convergence results of the solution with respect to the data. The proofs are based on arguments of monotonicity combined with the properties of the Clarke directional derivative. Our results provide mathematical tools in the study of a large number of static problems in Contact Mechanics. To provide an example, we consider a mathematical model which describes the equilibrium of a rod–spring system with unilateral constraints. We prove the unique weak solvability of the model, and then we illustrate our abstract convergence results in the study of this contact problem and provide the corresponding mechanical interpretations.

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1. Introduction

Everywhere in this paper, unless stated otherwise, $(X, \|\cdot\|_X)$ is a real Banach space, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and its dual X^*, K is a nonempty subset of X, $A\colon X \to X^*$, $j\colon X \to \mathbb{R}$ is a locally Lipschitz function and $f \in X^*$. We denote by $j^0(u;v)$ the generalized directional derivative of j at the point u in the direction v, see Definition 2. With these notation, we consider the hemivariational inequality

$$u \in K, \qquad \langle Au, v-u\rangle + j^0(u; v-u) \geq \langle f, v-u\rangle \qquad \forall\, v \in K. \tag{1.1}$$

Such kind of inequalities arise in Contact Mechanics. They model the equilibrium of elastic bodies acted upon the body forces and surface tractions, in frictional or frictionless contact with an obstacle. References in the field are [15,16] and, more recently [2,13,18,22,28,29]. There, existence and uniqueness results for inequality problems of the form (1.1) can be found, under various assumptions on the data. A convergence result for such inequalities was provided in [30] and general results on their numerical analysis of such inequalities can be found in [4–6]. Results on the Tykhonov regularization for hemivariational inequalities can be found in the recent paper [23].

The current paper was inspired by three types of studies related to the hemivariational inequality (1.1): the well-posedness in the sense of Tykhonov, the continuous dependence of the solution with respect to the data, and the perturbation with a convex function, under specific assumptions. We briefly describe in what follows each of these approaches.

First, the concept of well-posedness in the sense of Tykhonov was introduced for minimization problems in [24]. Later, it was extended to variational inequalities in [11,12] and to a particular class of hemivariational inequalities in [3]. References in the field include [8–10,20,21,25,27]. For an inequality of the form (1.1), the Tykhonov well-posedness is based on the following two ingredients: First, it is assumed that inequality (1.1) has a unique solution; second, this solution represents the limit in X, as $\varepsilon \to 0$, of solution sequence, called approximating sequence, to the following problem:

$$u \in K, \qquad \langle Au, v-u\rangle + j^0(u; v-u) + \varepsilon\, \|v-u\|_X \geq \langle f, v-u\rangle \qquad \forall\, v \in K. \tag{1.2}$$

Hemivariational inequalities of the form (1.2) have been considered in [7,19,26], among others. There, necessary and sufficient condition for the well-posedness of inequality (1.1) are provided, under various assumptions on the nonlinear operator A.

Next, we focus on the continuous dependence of the solution with respect to the data and, for simplicity, we restrict to the dependence of the solution with respect to f. To this end, we consider a perturbation $f_\varepsilon \in X^*$ of f, together with the hemivariational inequality

$$u \in K, \qquad \langle Au, v-u\rangle + j^0(u; v-u) \geq \langle f_\varepsilon, v-u\rangle \qquad \forall\, v \in K. \tag{1.3}$$

Then, it is easy to see that the solution of (1.3) satisfies the inequality

$$u \in K, \qquad \langle Au, v-u\rangle + j^0(u; v-u) + \|f_\varepsilon - f\|_{X^*}\|v-u\|_X \geq \langle f, v-u\rangle \quad \forall\, v \in K,$$

and using the notation $h(\varepsilon) = \|f_\varepsilon - f\|_{X^*}$, we deduce that

$$u \in K, \qquad \langle Au, v-u\rangle + j^0(u; v-u) + h(\varepsilon)\|v-u\|_X \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{1.4}$$

It follows from here that the continuous dependence of the solution of (1.1) with respect to f can be deduced from a convergence result for the solutions of (1.4) to the solution of (1.1).

Finally, consider a perturbed version of inequality (1.1) of the form

$$u \in K, \quad \langle Au, v-u\rangle + \varphi(u,v) - \varphi(u,u) + j^0(u; v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K \tag{1.5}$$

where $\varphi : X \times X \to \mathbb{R}$ is a functional. Assume now that $\varphi(u,\cdot) = \varepsilon\, \widetilde{\varphi}(u,\cdot)$ for each $u \in X$ where ε is a positive parameter converging to zero and $\widetilde{\varphi}(u,\cdot) : X \to \mathbb{R}$ is a continuous seminorm. Then, there exists a function $h : X \to \mathbb{R}$ such that

$$\varphi(u,v) - \varphi(u,u) = \varepsilon\, \widetilde{\varphi}(u,v) - \varepsilon\, \widetilde{\varphi}(u,u) \leq \varepsilon\, \widetilde{\varphi}(u, v-u) \leq \varepsilon h(u)\|v-u\|_X$$

for all u, $v \in X$. This implies that the solution of (1.5) satisfies the inequality

$$u \in K, \quad \langle Au, v-u\rangle + j^0(u; v-u) + \varepsilon\, h(u)\|v-u\|_X \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{1.6}$$

Therefore, the convergence of the solution of inequality (1.5) as $\varepsilon \to 0$ can be deduced by studying the convergence of the solution of (1.6) as $\varepsilon \to 0$. Such kind of situations arise in Contact Mechanics, as explained in [13,18]. There, ε is either a friction coefficient or a stiffness coefficient, and establishing such type of convergence results allows us to establish the link between various contact models and to provide various mechanical interpretations.

A careful analysis reveals that inequalities (1.2), (1.4), (1.6) are of the form

$$u \in K, \quad \langle Au, v-u\rangle + j^0(u; v-u) + h(\varepsilon, u)\|v-u\|_X \geq \langle f, v-u\rangle \ \ \forall\, v \in K \tag{1.7}$$

with a convenient choice of the function $h(\varepsilon, u)$, defined for $\varepsilon > 0$ and $u \in X$. Moreover, both the three situations described above require to establish convergence results for the solution of (1.7) to the solution of (1.1), as $\varepsilon \to 0$. It follows from here that the study of the perturbed hemivariational inequalities of the form (1.7) is useful in the study of the hemivariational inequality (1.1), since it could provide its well-posedness, the continuous dependence of the solution with respect to the data and other convergence results.

Motivated by the previous remarks, in this paper we study the link between the solutions of inequalities (1.1) and (1.7), under various assumptions on the data. Our aim is to introduce a new Tykhonov-type well-posedness concept for inequality (1.1), to derive convergence results and to provide mathematical tools useful in the study of mathematical models which describe the contact of deformable bodies and structures.

The rest of the paper is structured as follows. In Sect. 2, we recall some preliminary material which is needed in the rest of the paper. In Sect. 3, we introduce and study a new Tykhonov-type concept of well-posedness which extends that in ([3,19]). In Sect. 4, we use the results in Sect. 3 in order to establish new convergence results of the solution to inequality (1.1) with respect to the data K, A, j and f. Finally, in Sect. 5, we illustrate the results in Sect. 4 in the study of a rod–spring system with unilateral constraints.

2. Preliminaries

Everywhere below, we use $\|\cdot\|_X$ for the norm of space X. Unless stated otherwise, all the limits, upper limits and lower limits below are considered as $n \to \infty$, even if we do not mention it explicitly. The symbols "$\rightharpoonup$" and "$\to$" denote the weak and the strong convergence in the space X.

We start with some definitions related to the operator A and function j.

Definition 1. An operator $A\colon X \to X^*$ is said to be:

(a) monotone, if for all u, $v \in X$, we have $\langle Au - Av, u - v\rangle \geq 0$;

(b) strongly monotone, if there exists $m_A > 0$ such that

$$\langle Av_1 - Av_2, v_1 - v_2\rangle \geq m_A\|v_1 - v_2\|_X^2 \qquad \forall\, v_1, v_2 \in X; \tag{2.1}$$

(c) bounded, if A maps bounded sets of X into bounded sets of X^*;

(d) pseudomonotone, if it is bounded and $u_n \rightharpoonup u$ in X with

$$\limsup\, \langle Au_n, u_n - u\rangle \leq 0$$

implies

$$\liminf\, \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle \quad \text{for all } \ v \in X.$$

Definition 2. The generalized (Clarke) directional derivative of a locally Lipschitz function $j : X \to \mathbb{R}$ at the point $u \in X$ in the direction $v \in X$ is defined by

$$j^0(u; v) = \limsup_{x\to u,\ \lambda\downarrow 0} \frac{j(x + \lambda v) - j(x)}{\lambda}.$$

The generalized (Clarke) gradient (subdifferential) of j at u is a subset of the dual space X^* given by

$$\partial j(u) = \{\, \xi \in X^* \mid j^0(u; v) \geq \langle \xi, v\rangle \quad \forall\, v \in X \,\}.$$

Definition 3. A locally Lipschitz function $j : X \to \mathbb{R}$ is said:

(a) to be regular (in the sense of Clarke) at the point $u \in X$ if for all $v \in X$ the one-sided directional derivative $j'(u; v)$ exists and $j^0(u; v) = j'(u; v)$;

(b) to satisfy the relaxed monotonicity condition if there exists $\alpha_j > 0$ such that

$$\langle \xi_1 - \xi_2, u_1 - u_2\rangle \geq -\alpha_j\, \|u_1 - u_2\|_X^2 \qquad \forall\, u_i \in X,\ \xi_i \in \partial j(u_i),\ i = 1, 2.$$

We now recall the following properties related to the directional derivative and Clarke subdifferential.

Proposition 4. *Assume that $j\colon X \to \mathbb{R}$ is a locally Lipschitz function. Then the following holds.*

(a) *For every $u \in X$, the function $X \ni v \mapsto j^0(u; v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $j^0(u; \lambda v) = \lambda j^0(u; v)$ for all $\lambda \geq 0$, $v \in X$ and $j^0(u; v_1 + v_2) \leq j^0(u; v_1) + j^0(u; v_2)$ for all v_1, $v_2 \in X$, respectively.*

(b) *For every u, $v \in X$, we have $j^0(u; v) = \max\{\, \langle \xi, v\rangle \mid \xi \in \partial j(u)\,\}$.*

(c) *The function $X \times X \ni (u, v) \mapsto j^0(u; v) \in \mathbb{R}$ is upper semi-continuous, i.e., for all u, $v \in X$, $\{u_n\}$, $\{v_n\} \subset X$ such that $u_n \to u$ and $v_n \to v$ in X, we have $\limsup j^0(u_n; v_n) \leq j^0(u; v)$.*

For more details on the definitions and properties above, we refer to the books [1,13,15], for instance.
In the study of inequality (1.1), we consider the following assumptions.

(K_1) K is a nonempty closed convex subset of X.
(A_1) $A : X \to X^*$ is a pseudomonotone operator.
(A_2) $A : X \to X^*$ is a strongly monotone operator with constant $m_A > 0$.
(j_1) $j : X \to \mathbb{R}$ is a locally Lipschitz function.
(j_2) There exists $\alpha_j \geq 0$ such that

$$j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j \|v_1 - v_2\|_X^2 \quad \forall\, v_1,\, v_2 \in X.$$

(j_3) $\alpha_j < m_A$.
(j_4) There exist $c_0,\, c_1 \geq 0$ such that $\|\xi\|_{X^*} \leq c_0 + c_1\|v\|_X \quad \forall\, v \in X,\ \xi \in \partial j(v)$.
(f) $f \in X^*$.

It can be proved that for a locally Lipschitz function $j\colon X \to \mathbb{R}$, condition (j_2) is equivalent to the relaxed monotonicity condition introduced in Definition 3(b). A proof of the statement can be found in, e.g., [13]. Note also that if $j\colon X \to \mathbb{R}$ is a convex function, then condition (j_2) holds with $\alpha_j = 0$, since it reduces to the monotonicity of the (convex) subdifferential. Examples of functions which satisfy conditions (j_1), (j_2) and (j_4) can be found in [13,14], for instance. Nevertheless, for the convenience of the reader, we provide below such an example.

Example 5. Let Ω be a bounded domain of $\mathbb{R}^d$ with a smooth boundary Γ and let Γ_0 be a measurable part of Γ such that $meas\,\Gamma_0 > 0$. Let $X = H^1(\Omega)$ be the Sobolev space endowed with its usual Hilbertian structure. Assume that

$$\begin{cases} p\colon \mathbb{R} \to \mathbb{R} \ \text{ is such that} \\ \text{(a) } |p(r_1) - p(r_2)| \leq L_p|r_1 - r_2| \quad \text{for all } r_1, r_2 \in \mathbb{R}, \quad \text{with } L_p > 0; \\ \text{(b) } p(0) = 0. \end{cases} \tag{2.2}$$

Next, consider the functions $q : \mathbb{R} \to \mathbb{R}$ and $j : X \to \mathbb{R}$ defined by

$$q(r) = \int_0^r p(s)\,\mathrm{d}s \qquad \forall\, r \in \mathbb{R}, \tag{2.3}$$

$$j(v) = \int_{\Gamma_0} q(v)\, d\Gamma \qquad \forall\, v \in X. \tag{2.4}$$

Note that here and below, we write v for the trace of the function $v \in X$ to Γ. Using standard arguments ([13, Lemma 3.50 (iii)], for instance), it follows that

$$q^0(r; s) = p(r)\, s \qquad \forall\, r,\, s \in \mathbb{R}, \tag{2.5}$$

where $q^0(r; s)$ denotes the generalized directional derivative of q at the point r in the direction s. Therefore, using (2.2) and [13, Corollary 4.15] it follows that j satisfies conditions (j_1) and (j_4) and, in addition,

$$j^0(u; v) = \int_{\Gamma_0} p(u)v\, d\Gamma \qquad \forall\, u,\, v \in X. \tag{2.6}$$

Hence, given $v_1,\, v_2 \in X$ we have

$$j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) = \int_{\Gamma_0} (p(v_1) - p(v_2))(v_2 - v_1)\, d\Gamma.$$

Moreover, using assumption (2.2) we obtain that

$$j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq L_p\|\gamma\|^2\|v_1 - v_2\|_X^2,$$

where, here and below, $\|\gamma\|$ denotes the norm of the trace operator defined on X with values in $L^2(\Gamma_0)$. This inequality shows that j satisfies condition (j_2) with $\alpha_j = L_p\|\gamma\|^2$.

We now recall the following existence and uniqueness result.

Theorem 6. *Assume that X is a reflexive space and (K_1), (A_1), (A_2), (j_1), (j_2), (j_3), (j_4), (f) hold. Then, there exists a unique solution u to the hemivariational inequality (1.1).*

A proof of Theorem 6 can be found in [18, Chapter 5]. It is carried out in several steps, by using the properties of the subdifferential, a surjectivity result for pseudomonotone multivalued operators and the Banach fixed point argument.

3. Well-posedness results

In this section, we introduce and study a general concept of well-posedness for the hemivariational inequality (1.1). To this end, for each $\varepsilon > 0$, we consider the set $\Omega(\varepsilon)$ defined as follows:

$$\Omega(\varepsilon) = \{\, u \in K \ : \langle Au, v-u\rangle + j^0(u; v-u) + h(\varepsilon, u)\|v-u\|_X \geq \langle f, v-u\rangle \quad \forall\, v \in K \,\} \tag{3.1}$$

where, recall, $h : (0,+\infty)\times X \to \mathbb{R}$ is a given function. We refer to the family of sets $\{\Omega(\varepsilon)\}_{\varepsilon>0}$ as the family of approximating sets. Moreover, we denote by $\mathcal{S}$ the set of solutions of inequality (1.1), i.e.,

$$\mathcal{S} = \{\, u \in K \ : \langle Au, v-u\rangle + j^0(u; v-u) \geq \langle f, v-u\rangle \ \ \forall\, v \in K \,\} \tag{3.2}$$

and we recall that $\mathcal{S}$ is said to be a singleton if $\mathcal{S}$ has a unique element.

Next, we proceed with the following definitions.

Definition 7. A sequence $\{u_n\} \subset X$ is called an approximating sequence for the hemivariational inequality (1.1) if there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}$ such that $0 < \varepsilon_n \to 0$ and $u_n \in \Omega(\varepsilon_n)$, for each $n \in \mathbb{N}$.

Definition 8. The hemivariational inequality (1.1) is said to be well-posed if it has a unique solution and every approximating sequence for (1.1) converges in X to its solution.

Note that this concept of well-posedness above extends that used in [19,26]. Indeed, the later can be recovered in the particular case when

$$h(\varepsilon, u) = \varepsilon \qquad \forall\, \varepsilon > 0,\ u \in X. \tag{3.3}$$

Moreover, this concept is quite different from that introduced in [3] for hemivariational inequalities with constraints.

Our aim in what follows is to characterize the well-posedness of hemivariational inequality (1.1) in terms of the metric properties of the approximating sets $\{\Omega(\varepsilon)\}_{\varepsilon>0}$ and to indicate sufficient conditions on the data which guarantee this well-posedness. To this end, we recall the following definition.

Definition 9. Let Ω be a nonempty subset of X. Then the diameter of Ω, denoted $\mathrm{diam}(\Omega)$, is defined by equality

$$\mathrm{diam}(\Omega) = \sup_{a,\, b\in\Omega} \|a-b\|_X.$$

Moreover, we consider the following additional assumptions.

(K_2) K is a nonempty closed subset of X.
(h_1) $h(\varepsilon, u) \geq 0 \qquad \forall\, u \in X,\ \varepsilon > 0.$
(h_2) $h(\varepsilon_n, u_n) \to 0$ whenever $0 < \varepsilon_n \to 0$ and $\{u_n\} \subset X$ is bounded.
(h_3) $0 < \varepsilon_1 < \varepsilon_2 \quad \Longrightarrow \quad h(\varepsilon_1, u) \leq h(\varepsilon_2, u) \qquad \forall\, u \in X.$
(h_4) There exists $L_h :]0,+\infty[\to \mathbb{R}$ such that

(a) $|h(\varepsilon,u) - h(\varepsilon,v)| \le L_h(\varepsilon)\|u - v\|_X \qquad \forall\, u,\, v \in X,\ \varepsilon > 0,$

(b) $L_h(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Our main result in this section is the following.

Theorem 10. *Let X be a Banach space. The following statements hold.*

(a) *Under assumption (j_1), (h_1) and (f), the hemivariational inequality* (1.1) *is well-posed if and only if its set of solution $\mathcal{S}$ is nonempty and* $\mathrm{diam}(\Omega(\varepsilon)) \to 0$ *as* $\varepsilon \to 0$.

(b) *Under assumptions (K_2), (A_1), (j_1), (h_1), (h_2), (h_3) and (f), the hemivariational inequality* (1.1) *is well-posed if and only if the set $\Omega(\varepsilon)$ is nonempty for each $\varepsilon > 0$ and* $\mathrm{diam}(\Omega(\varepsilon)) \to 0$ *as* $\varepsilon \to 0$.

(c) *Under assumptions (A_2), (j_1), (j_2), (j_3), (h_1), (h_2), (h_4) and (f), the hemivariational inequality* (1.1) *is well-posed if and only if the set $\mathcal{S}$ is a singleton.*

(d) *If X is a reflexive space, then under assumptions (K_1), (A_1), (A_2), (j_1), (j_2), (j_3), (j_4), (h_1), (h_2), (h_4) and (f), the hemivariational inequality* (1.1) *is well-posed.*

Proof. (a) We work under the assumption (j_1), (h_1) and (f) and note that, in this case, for each $\varepsilon > 0$ we have

$$\mathcal{S} \subset \Omega(\varepsilon). \tag{3.4}$$

Assume that (1.1) is well-posed. Then, by definition, $\mathcal{S}$ is a singleton and, therefore, $\mathcal{S} \neq \emptyset$. Arguing by contradiction, we assume that $\mathrm{diam}(\Omega(\varepsilon)) \not\to 0$ as $\varepsilon \to 0$. Then, there exist $\delta_0 \ge 0$, a sequence $\{\varepsilon_n\} \subset \mathbb{R}$ and two sequences $\{u_n\}$, $\{v_n\} \subset X$ such that $0 < \varepsilon_n \to 0$, u_n, $v_n \in \Omega(\varepsilon_n)$ and

$$\|u_n - v_n\|_X \ge \frac{\delta_0}{2} \qquad \forall\, n \in \mathbb{N}. \tag{3.5}$$

Now, since both $\{u_n\}$ and $\{v_n\}$ are approximating sequences for the hemivariational inequality (1.1), the well-posedness of (1.1) implies that $u_n \to u$ and $v_n \to u$ in X where u denotes the unique element of $\mathcal{S}$. This is in contradiction with (3.5). We conclude from here that $\mathrm{diam}(\Omega(\varepsilon)) \to 0$ as $\varepsilon \to 0$.

Conversely, assume that $\mathcal{S}$ is nonempty and $\mathrm{diam}(\Omega(\varepsilon)) \to 0$ as $\varepsilon \to 0$. We claim that $\mathcal{S}$ is a singleton. Indeed, let u, $u' \in \mathcal{S}$ and let $\{u_n\}$ be an approximating sequence for (1.1). Then there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}$ such that $0 < \varepsilon_n \to 0$ and $u_n \in \Omega(\varepsilon_n)$ for all $n \in \mathbb{N}$. Using (3.4) and Definition 9, we have

$$\|u - u'\|_X \le \|u - u_n\|_X + \|u' - u_n\|_X \le 2\,\mathrm{diam}(\Omega(\varepsilon_n)) \to 0,$$

which implies that $u = u'$ and proves the claim. Moreover, for any approximating sequence $\{u_n\}$, we have

$$\|u - u_n\|_X \le \mathrm{diam}(\Omega(\varepsilon_n)) \to 0,$$

which implies that $u_n \to u$ in X and, therefore, (1.1) is well-posed.

(b) We work under the assumptions (K_2), (A_1), (j_1), (h_1), (h_2), (h_3) and (f). Assume that (1.1) is well-posed. Then, we use the part (a) of the theorem and inclusion (3.4) to see that the set $\Omega(\varepsilon)$ is nonempty for each $\varepsilon > 0$ and $\mathrm{diam}(\Omega(\varepsilon)) \to 0$ as $\varepsilon \to 0$.

Conversely, assume that the set $\Omega(\varepsilon)$ is nonempty for each $\varepsilon > 0$ and $\mathrm{diam}(\Omega(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Then, using (3.4), again, and Definition 9 we deduce that the hemivariational inequalities (1.1) admits at most one solution. Let $\{u_n\}$ be an approximating sequence for (1.1). Then there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}$ such that $0 < \varepsilon_n \to 0$ and $u_n \in \Omega(\varepsilon_n)$ for all $n \in \mathbb{N}$. Since $\mathrm{diam}(\Omega(\varepsilon)) \to 0$, for any $\delta > 0$ there exists a positive integer N_δ such that

$$\mathrm{diam}(\Omega(\varepsilon_n)) \le \delta \qquad \forall\, n \ge N_\delta. \tag{3.6}$$

Let n, $m \in \mathbb{N}$ be such that n, $m \ge N_\delta$ and assume that $\varepsilon_m \le \varepsilon_n$. Then using assumption (h_3) we have $h(\varepsilon_m, u_m) \le h(\varepsilon_n, u_m)$ and, therefore (3.1) implies that $u_m \in \Omega(\varepsilon_n)$. On the other hand, Definition 7 guarantees that $u_n \in \Omega(\varepsilon_n)$, too. Therefore, (3.6) implies that

$$\|u_n - u_m\|_X \le \delta.$$

This inequality holds if $\varepsilon_m > \varepsilon_n$, too, since in this case $u_n, u_m \in \Omega(\varepsilon_m)$. We conclude from here that $\{u_n\}$ is a Cauchy sequence in X and, since X is assumed to be a Banach space, there exists $u \in X$ such that

$$u_n \to u \quad \text{in} \quad X. \tag{3.7}$$

This convergence combined with assumption (K_2) yields

$$u \in K. \tag{3.8}$$

We now prove that u solves the hemivariational inequality (1.1) and, to this end, we use a pseudomonotonicity argument. First, the inclusion $u_n \in \Omega(\varepsilon_n)$ implies that

$$\langle Au_n, u_n - v\rangle \le j^0(u_n; v - u_n) + \langle f, u_n - v\rangle + h(\varepsilon_n, u_n)\|u_n - v\|_X \quad \forall\, v \in K,\ n \in \mathbb{N}.$$

Next, we pass to the upper limit as $n \to \infty$ in this inequality and use the convergence (3.7), Proposition 4 (c) and assumption (h_2) to deduce that

$$\limsup \langle Au_n, u_n - v\rangle \le j^0(u; v - u) + \langle f, u - v\rangle \qquad \forall\, v \in K. \tag{3.9}$$

On the other hand, regularity (3.8) allows us to test with $v = u$ in (3.9) to find that

$$\limsup \langle Au_n, u_n - u\rangle \le 0.$$

Therefore, by the pseudomonotonicity of the operator A, guaranteed by assumption (A_1), we obtain

$$\liminf \langle Au_n, u_n - v\rangle \ge \langle Au, u - v\rangle \qquad \forall\, v \in X. \tag{3.10}$$

We now combine (3.8), (3.9) and (3.10) to see that u is a solution to the hemivariational inequality (1.1), which implies that $\mathcal{S}$ is a singleton. This together with (3.7) indicates that any approximating sequence of (1.1) converges to the unique element of $\mathcal{S}$. It follows from here that the hemivariational inequality (1.1) is well-posed, which concludes the proof of (b).

(c) We work under the assumptions (A_2), (j_1), (j_2), (j_3), (h_1), (h_2), (h_3) and (f). Assume that inequality (1.1) is well-posed. Then, by Definition 7 it follows that $\mathcal{S}$ is a singleton.

Conversely, assume that $\mathcal{S}$ is a singleton and denote by $u \in K$ the unique solution of (1.1). Let $\{u_n\} \subset X$ be an approximating sequence for the hemivariational inequality (1.1). Then there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}$ such that $0 < \varepsilon_n \to 0$ and

$$\langle Au_n, v - u_n\rangle + j^0(u_n; v - u_n) + h(\varepsilon_n, u_n)\|v - u_n\|_X \ge \langle f, v - u_n\rangle \ \ \forall\, v \in K. \tag{3.11}$$

Letting $v = u_n$ in inequality (1.1) and $v = u$ in inequality (3.11), we add the resulting inequalities to see that

$$\langle Au_n - Au, u_n - u\rangle \le j^0(u_n; u - u_n) + j^0(u; u_n - u) + h(\varepsilon_n, u_n)\|u_n - u\|_X.$$

We now use assumptions (A_2) and (j_2) to obtain that

$$(m_A - \alpha_j)\|u_n - u\|_X \le h(\varepsilon_n, u_n). \tag{3.12}$$

Next, we use assumption (h_4)(a) to write

$$h(\varepsilon_n, u_n) = h(\varepsilon_n, u_n) - h(\varepsilon_n, u) + h(\varepsilon_n, u) \le L(\varepsilon_n)\|u_n - u\|_X + h(\varepsilon_n, u)$$

and, therefore, (3.12) yields

$$(m_A - \alpha_j - L(\varepsilon_n))\|u_n - u\|_X \le h(\varepsilon_n, u). \tag{3.13}$$

Note that Definition 7 guarantees that $\varepsilon_n \to 0$. Therefore, passing to the upper limit in (3.13) and using conditions (h_4)(b), (h_2) and the smallness condition (j_3) yield

$$\limsup \|u_n - u\|_X \le 0.$$

We deduce from here that $u_n \to u$ in X and conclude the proof of (c).

(d) We work under the assumptions (K_1), (A_1), (A_2), (j_1), (j_2), (j_3), (j_4), (h_1), (h_2), (h_3) and (f). Moreover, we assume that the space X is reflexive. We use Theorem 6 to deduce that inequality (1.1) has a unique solution. Then we apply part (c) of the theorem to conclude the proof. □

We end this section with two remarks. First, the statement (a)–(c) in Theorem 10 provides equivalence results. They do not guarantee the well-posedness of the hemivariational inequality (1.1). In contrast, sufficient conditions which guarantee its well-posedness are provided by the statement (d) of the theorem. Second, the function $h : (0,\infty) \times X \to \mathbb{R}$ defined by (3.3) satisfies assumptions (h_1)–(h_4). Therefore, Theorem 10 works in this particular case and allows us to recover part of the result obtained in [19,26].

4. Convergence results

In this section, we use Theorem 10 in order to prove two convergence results for the solution of inequality (1.1). To this end, for each $\varepsilon > 0$, we consider an operator $A_\varepsilon : X \to X^*$, a function $j_\varepsilon : X \to \mathbb{R}$ and an element $f_\varepsilon \in X^*$. With these data we consider the following perturbed version of inequality (1.1):

$$u \in K, \qquad \langle A_\varepsilon u, v - u\rangle + j_\varepsilon^0(u; v - u) \geq \langle f_\varepsilon, v - u\rangle \qquad \forall\, v \in K. \tag{4.1}$$

Consider now the following assumptions.

(h_A) There exists $h_A :]0,+\infty[\times X \to \mathbb{R}$ such that
 (a) $\|A_\varepsilon u - Au\|_{X^*} \leq h_A(\varepsilon, u) \qquad \forall\, u \in X,\ \varepsilon > 0,$
 (b) h_A satisfies assumptions (h_1), (h_2) and (h_4) with $L_{h_A} :]0,+\infty[\to \mathbb{R}$.
(j_ε) $j_\varepsilon : X \to \mathbb{R}$ is a locally Lipschitz function.
(h_j) There exists $h_j :]0,+\infty[\times X \to \mathbb{R}$ such that
 (a) $j_\varepsilon^0(u; v) - j^0(u; v) \leq h_j(\varepsilon, u)\|v\|_X \qquad \forall\, u,\, v \in X,\ \varepsilon > 0,$
 (b) h_j satisfies assumptions (h_1), (h_2) and (h_4) with $L_{h_j} :]0,+\infty[\to \mathbb{R}$.
(f_ε) $f_\varepsilon \to f$ in X^* as $\varepsilon \to 0$.

We complete these assumptions with the following two examples which will be useful in the next section.

Example 11. An example of operator $A_\varepsilon : X \to X^*$ which satisfies assumption (h_A) is given by $A_\varepsilon u = Au + \varepsilon T u$ for all $u \in X$, $\varepsilon > 0$, where $T : X \to X^*$ is a Lipschitz continuous operator. Indeed, it is easy to see that in this case condition (h_A) is satisfied with $h_A(\varepsilon, u) = \varepsilon\|Tu\|_{X^*}$ and $L_{h_A}(\varepsilon) = \varepsilon L_T$, L_T being the Lipschitz constant of the operator T.

Example 12. An example of functions j_ε, j which satisfy assumption (h_j) can be constructed by using the notation presented in Example 5 in the one-dimensional case. Let $L > 0$, $\Omega = (0, L)$ and let

$$X = \{\, v \in H^1(0, L) \mid v(0) = 0 \,\} \tag{4.2}$$

which is a real Hilbert space with the inner product

$$(u, v)_X = \int_0^L u'v'\, \mathrm{d}x \qquad \forall\, u,\, v \in X. \tag{4.3}$$

and the associated norm $\|\cdot\|_X$. Here and below the prime denotes the derivative with respect to $x \in (0, L)$, i.e., $u' = \frac{du}{dx}$. Assume that p is a function which satisfies condition (2.2) and let $\Gamma_0 = \{L\}$. Then, using (2.4), we deduce that $j(v) = q(v(L))$ for all $v \in X$ where, recall, q is given by (2.3). Moreover, (2.6) implies that

$$j^0(u; v) = p(u(L))v(L) \qquad \forall\, u,\, v \in X. \tag{4.4}$$

Let $\varepsilon > 0$ and replace the function p by the function

$$p_\varepsilon(r) = p(r) + \varepsilon r \qquad \forall\, r \in \mathbb{R}. \tag{4.5}$$

Then, it is easy to see that the function p_ε satisfies condition (2.2). Moreover, the corresponding function j_ε is given by $j_\varepsilon(v) = q_\varepsilon(v(L))$ for all $v \in X$ where

$$q_\varepsilon(r) = \int_0^r p_\varepsilon(s)\,\mathrm{d}s \qquad \forall\, r \in \mathbb{R}. \tag{4.6}$$

In addition,

$$j_\varepsilon^0(u; v) = p_\varepsilon(u(L))v(L) \qquad \forall\, u,\, v \in X. \tag{4.7}$$

Finally, an elementary calculation shows that

$$|v(L)| \le \sqrt{L}\, \|v\|_X \qquad \forall\, v \in X. \tag{4.8}$$

We now use (4.4), (4.7) and (4.5), (4.8) to see that the functions j_ε, j satisfy conditiontion (h_j) with $h_j(\varepsilon, u) = L\varepsilon\|u\|_X$ and $L_{h_j}(\varepsilon) = L\varepsilon$.

We have the following existence, uniqueness and convergence result.

Theorem 13. *Assume that X is a reflexive space and (K_1), (A_1), (A_2), (j_1), (j_2), (j_3), (j_4), (f), (h_A), (h_j), (j_ε) and (f_ε) hold. Moreover, assume that for each $\varepsilon > 0$ the element u_ε is a solution to inequality (4.1) and let be u the solution of inequality (1.1) provided in Theorem 6. Then*

$$u_\varepsilon \to u \quad \text{in} \quad X \quad \text{as} \quad \varepsilon \to 0. \tag{4.9}$$

Proof. The proof is structured in three steps, as follows.

(i) *The perturbed hemivariational inequality.* Let $\varepsilon > 0$. We claim that if w is a solution to inequality (4.1) then w satisfies the inequality of form (1.7) with $h :]0, +\infty) \times X \to \mathbb{R}$ given by

$$h(\varepsilon, u) = h_A(\varepsilon, u) + h_j(\varepsilon, u) + \|f_\varepsilon - f\|_{X^*} \qquad \forall\, u \in X,\ \varepsilon > 0. \tag{4.10}$$

Indeed, assume that w is a solution of (4.1) and let $v \in K$. Then,

$$\begin{aligned}&\langle Aw, v - w\rangle + \langle A_\varepsilon w - Aw, v - w\rangle + j^0(w; v - w)\\ &\quad + j_\varepsilon^0(w; v - w) - j^0(w; v - w) + \langle f - f_\varepsilon, v - w\rangle \ge \langle f, v - w\rangle,\end{aligned}$$

which imples that

$$\begin{aligned}&\langle Aw, v - w\rangle + \|A_\varepsilon w - Aw\|_{X^*}\|v - w\|_X + j^0(w; v - w)\\ &\quad + j_\varepsilon^0(w; v - w) - j^0(w; v - w) + \|f - f_\varepsilon\|_{X^*}\|v - w\|_X \ge \langle f, v - w\rangle.\end{aligned}$$

We now use assumptions (h_A)(a) and (h_j)(a) to see that

$$\begin{aligned}&\langle Aw, v - w\rangle + h_A(\varepsilon, w)\|v - w\|_X + j^0(w; v - w)\\ &\quad + h_j(\varepsilon, w)\|v - w\|_X + \|f - f_\varepsilon\|_{X^*}\|v - w\|_X \ge \langle f, v - w\rangle\end{aligned}$$

and, combining this inequality with the definition (4.10) of the function h we obtain that w is a solution of (1.7), as claimed.

(ii) *Properties of the function h.* We claim that the function h defined by (4.10) satisfies conditions (h_1), (h_2) and (h_4). Indeed, this statement is a direct consequence of the definition of h combined with assumptions (h_A)(b), (h_j)(b) and (f_ε).

(iii) *End of proof.* In this step, we assume that for each $\varepsilon > 0$ the element u_ε is a solution to inequality (4.1) and show that $u_\varepsilon \to u$ in X as $\varepsilon \to 0$. In fact, let $\Omega(\varepsilon)$ be the set defined by (3.1) with the function (4.10). Then, the step (i) shows that

$$u_\varepsilon \in \Omega(\varepsilon). \tag{4.11}$$

Assume now that $\{\varepsilon_n\}$ is a sequence of positive numbers such that $\varepsilon_n \to 0$. It follows from inclusion (4.11) that $u_{\varepsilon_n} \in \Omega(\varepsilon_n)$ for each $n \in \mathbb{N}$ and, therefore, Definition 7 shows that $\{u_{\varepsilon_n}\}$ is an approximating sequence for the inequality (1.1). On the other hand, step (ii) guarantees that all the assumptions of Theorem 10 (d) are satisfied and, therefore, we deduce that inequality (1.1) is well-posed with respect to the family $\{\Omega(\varepsilon)\}_{\varepsilon>0}$. We now use Definition 8 to see that $u_{\varepsilon_n} \to u$ in X as $n \to \infty$ which proves the convergence (4.9). □

We now move to a second convergence result and, to this end, for each $\varepsilon > 0$ we consider a set K_ε together with the following perturbed version of inequality (1.1):

$$u \in K_\varepsilon \qquad \langle Au, v-u\rangle + j^0(u; v-u) \geq \langle f_\varepsilon, v-u\rangle \qquad \forall\, v \in K_\varepsilon. \tag{4.12}$$

Consider now the following assumptions.

$(\widetilde{A})$ There exists $L_A > 0$ such that

$$\|Au - Av\|_{X^*} \leq L_A \|u - v\|_X \quad \forall\, u,\, v \in X.$$

$(\widetilde{j})$ There exists $L_j > 0$ such that

$$j^0(u; w) - j^0(v; w) \leq L_j \|u - v\|_X \|w\|_X \quad \forall\, u,\, v,\, w \in X.$$

$(\widetilde{K})$ There exists $\theta \in X$ and for each $\varepsilon > 0$ there exist $c_\varepsilon > 0$, $d_\varepsilon \in \mathbb{R}$ such that

(a) $K_\varepsilon = c_\varepsilon K + d_\varepsilon \theta$,

(b) $c_\varepsilon \to 1, \quad d_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.$

Our second result in this section is the following.

Theorem 14. *Assume that X is a reflexive space and (K_1), $(\widetilde{K})$, $(\widetilde{A})$, (A_2), (j_1), (j_2), (j_3), (j_4), (f), $(\widetilde{j})$ hold. Then the hemivariational inequality* (1.1) *has a unique solution $u \in K$ and, for each $\varepsilon > 0$ the hemivariational inequality* (4.12) *has a unique solution $u_\varepsilon \in K_\varepsilon$. Moreover, $u_\varepsilon \to u$ in X as $\varepsilon \to 0$.*

Proof. It is well known that any monotone Lipschitz continuous operator is pseudomonotone, and therefore, assumptions $(\widetilde{A})$, (A_2) imply that A satisfies condition (A_1). On the other hand, it is easy to check that assumptions (K_1), $(\widetilde{K})$(a), imply that for any $\varepsilon > 0$ the set K_ε is non empty, closed and convex, i.e., it satisfies condition (K_1). The existence and uniqueness of the solution to the hemivariational inequalities (1.1) and (4.12) is now a direct consequence of the Theorem 6.

Assume now that $\varepsilon > 0$ is fixed and denote by u_ε the solution of inequality (4.12). Then, using assumption $(\widetilde{K})$, it follows that there exists $\widetilde{u}_\varepsilon \in K$ such that

$$u_\varepsilon = c_\varepsilon \widetilde{u}_\varepsilon + d_\varepsilon \theta. \tag{4.13}$$

Therefore, since $v \in K$ implies that $c_\varepsilon v + d_\varepsilon \theta \in K_\varepsilon$, inequality (4.12) combined with Proposition 4 (a) yield

$$\widetilde{u}_\varepsilon \in K, \quad \langle A(c_\varepsilon \widetilde{u}_\varepsilon + d_\varepsilon\theta), v - \widetilde{u}_\varepsilon\rangle + j^0(c_\varepsilon \widetilde{u}_\varepsilon + d_\varepsilon\theta; v - \widetilde{u}_\varepsilon) \geq \langle f, v - \widetilde{u}_\varepsilon\rangle \quad \forall\, v \in K. \tag{4.14}$$

We now introduce the operator $A_\varepsilon : X \to X^*$, the function $j_\varepsilon : X \to \mathbb{R}$ and the element f_ε defined by

$$A_\varepsilon v = c_\varepsilon A(c_\varepsilon v + d_\varepsilon\theta), \qquad j_\varepsilon(v) = j(c_\varepsilon v + d_\varepsilon \theta) \qquad \forall\, v \in X \tag{4.15}$$

$$f_\varepsilon = c_\varepsilon f. \tag{4.16}$$

Note that the function j_ε is locally Lipschitz and, therefore, condition (j_ε) is satisfied. Moreover, an elementary calculation based on Definition 2 implies that

$$j_\varepsilon^0(u; v) = c_\varepsilon j^0(c_\varepsilon u + d_\varepsilon\theta; v) \qquad \forall\, u,\, v \in X. \tag{4.17}$$

Next, we multiply inequality (4.14) with $c_\varepsilon > 0$, then use equalities (4.15)–(4.17) to deduce that

$$\widetilde{u}_\varepsilon \in K, \quad \langle A_\varepsilon \widetilde{u}_\varepsilon, v - \widetilde{u}_\varepsilon\rangle + j_\varepsilon^0(\widetilde{u}_\varepsilon; v - \widetilde{u}_\varepsilon) \geq \langle f_\varepsilon, v - \widetilde{u}_\varepsilon\rangle \quad \forall\, v \in K. \tag{4.18}$$

Our aim in what follows is to use Theorem 13 to prove that

$$\widetilde{u}_\varepsilon \to u \quad \text{in} \quad X \quad \text{as} \quad \varepsilon \to 0 \tag{4.19}$$

and, to this end, we prove in what follows the validity of conditions (h_A), (h_j) and (f_ε). Below in the proof, we assume that u, $v \in X$ and $\varepsilon > 0$ are given.

First, we use (4.15) to see that

$$A_\varepsilon v - Av = (c_\varepsilon - 1)A(c_\varepsilon v + d_\varepsilon \theta) + A(c_\varepsilon v + d_\varepsilon \theta) - Av$$

and, using assumption $(\widetilde{A})$, we deduce that

$$\|A_\varepsilon v - Av\|_{X^*} \le |c_\varepsilon - 1|\, \|A(c_\varepsilon v + d_\varepsilon \theta)\|_{X^*} + L_A\|(c_\varepsilon - 1)v + d_\varepsilon \theta\|_X.$$

This proves that condition (h_A)(a) holds with function

$$h_A(\varepsilon, v) = |c_\varepsilon - 1|\, \|A(c_\varepsilon v + d_\varepsilon \theta)\|_{X^*} + L_A\|(c_\varepsilon - 1)v + d_\varepsilon \theta\|_X \tag{4.20}$$

which clearly satisfies condition (h_1). Since A is a Lipschitz continuous operator, using assumption $(\widetilde{K})$(b) it is easy to see that the function (4.20) satisfies condition (h_2), too. Finally, using assumption $(\widetilde{A})$, again, it follows that

$$|h_A(\varepsilon, u) - h_A(\varepsilon, v)| \le L_A|c_\varepsilon - 1|(c_\varepsilon + 1)\|u - v\|_X.$$

This inequality combined with the convergence $c_\varepsilon \to 1$ shows that the function h_A defined by (4.18) satisfies condition (h_4) with $L_{h_A} = L_A|c_\varepsilon - 1|(c_\varepsilon + 1)$. We conclude from above that condition (h_A) is satisfied.

Next, we use (4.17) to write

$$j_\varepsilon^0(u; v) - j^0(u; v) = (c_\varepsilon - 1)j^0(c_\varepsilon u + d_\varepsilon \theta; v) + j^0(c_\varepsilon u + d_\varepsilon \theta; v) - j^0(u; v)$$

and, using assumption $(\widetilde{j})$, we find that

$$j_\varepsilon^0(u; v) - j^0(u; v) \le |c_\varepsilon - 1|\, |j^0(c_\varepsilon u + d_\varepsilon \theta; v)| + L_j\|(c_\varepsilon - 1)u + d_\varepsilon \theta\|_X \|v\|_X. \tag{4.21}$$

On the other hand, by Proposition 4 (b) and assumption (j_4), we have

$$\begin{aligned} |j^0(c_\varepsilon u + d_\varepsilon \theta; v)| &\le |\max\{\langle \xi, v\rangle \mid \xi \in \partial j(c_\varepsilon u + d_\varepsilon \theta)\}| \\ &\le (c_0 + c_1\|c_\varepsilon u + d_\varepsilon \theta\|_X)\|v\|_X, \end{aligned}$$

and substituting this inequality in (4.21), we deduce that condition (h_j) holds with function

$$h_j(\varepsilon, u) = (c_0 + c_1\|c_\varepsilon u + d_\varepsilon \theta\|_X)|c_\varepsilon - 1| + L_j\|(c_\varepsilon - 1)u + d_\varepsilon \theta\|_X, \tag{4.22}$$

which satisfies condition (h_1) and (h_2). Note that

$$|h_j(\varepsilon, u) - h_j(\varepsilon, v)| \le (c_1 c_\varepsilon + L_j)|c_\varepsilon - 1|\|u - v\|_X,$$

which shows that the function h_j defined by (4.22) satisfies condition (h_4) with $L_{h_j}(\varepsilon) = (c_1 c_\varepsilon + L_j)|c_\varepsilon - 1|$.

Finally, since $c_\varepsilon \to 1$ and $f_\varepsilon = c_\varepsilon f$, condition (f_ε) is obviously satisfied. We are now in a position to use Theorem 13 in order to see that the convergence (4.19) holds. We now use (4.13), (4.19) and assumption $(\widetilde{K})$(b) to see that $u_\varepsilon \to u$ in X as $\varepsilon \to 0$, which concludes the proof. □

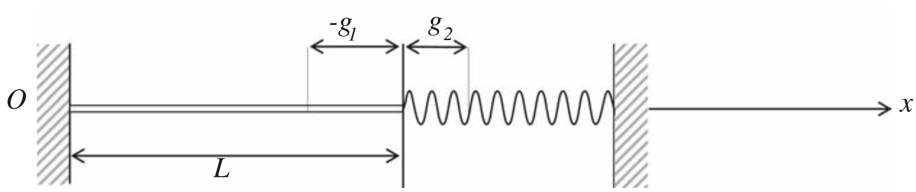

FIG. 1. The rod–spring system with unilateral constraints

5. A spring–rod system with unilateral constraints

The abstract results we present in this paper are useful in the study of various mathematical models which describe the equilibrium of elastic bodies in frictional contact with a foundation. In this section, we present a simple example which illustrate the applicability of these results. The boundary value problem under consideration is the following.

Problem 15. Find a displacement field $u\colon [0, L] \to \mathbb{R}$ and a stress field $\sigma\colon [0, L] \to \mathbb{R}$ such that

$$\sigma(x) = \mathcal{F}(x, u'(x)) \quad \text{for } x \in (0, L), \tag{5.1}$$

$$\sigma'(x) + f_0(x) = 0 \quad \text{for } x \in (0, L), \tag{5.2}$$

$$u(0) = 0, \tag{5.3}$$

$$\begin{cases} g_1 \le u(L) \le g_2, \\ -\sigma(L) = p(u(L)) \quad \text{if} \quad g_1 < u(L) < g_2, \\ -\sigma(L) \le p(u(L)) \quad \text{if} \quad u(L) = g_1, \\ -\sigma(L) \ge p(u(L)) \quad \text{if} \quad u(L) = g_2. \end{cases} \tag{5.4}$$

Problem 15 represents a mathematical model which describes the equilibrium of a rod–spring system with unilateral constraints, submitted to the action of body forces. In the reference configuration the rod occupies the interval $[0, L]$ on the Ox axis, L being a given positive constant. The physical setting is depicted in Fig. 1. A brief description of the equations and boundary conditions in this problem is the following.

First, equation (5.1) represents the elastic constitutive law in which $\mathcal{F}$ denotes a nonlinear constitutive function. Here and everywhere below the prime denotes the derivative with respect to x, i.e., $u' = \frac{du}{dx}$. Concrete examples of nonlinear elastic constitutive laws of the form (5.1) can be found in [18], for instance. Equation (5.2) is the equilibrium equation in which f_0 denotes the density of body forces acting on the rod. Condition (5.3) represents the displacement condition. We use it here since the rod is assumed to be fixed at the end $x = 0$.

Finally, conditions (5.4) represent the boundary conditions in which g_1 and g_2 are given bounds and p is a real-valued function which will be described below. Such conditions model the physical setting in which the extremity $x = L$ of the rod is attached to a spring which prevents its motion. The spring has an elastic behaviour, as far as the displacement of the point $x = L$, denoted by $u(L)$, belongs to the open interval (g_1, g_2). Its behavior is described with the stiffness function p, assumed to be positive for a positive argument and negative for a negative one. This property of p shows that the spring push the rod when it is in compression and pull it when it is in extension. When $u(L) = g_2$ the spring is completely compressed and when $u(L) = g_1$ the spring is completely extended. In both these cases, it behaves like a rigid and, therefore, it does not allow further extension or compression of the rod, respectively. Mathematical models which describe the equilibrium of spring–rod systems in similar physical settings can be found in [14, 17], together with various mechanical interpretations.

We now turn to the variational formulation of Problem 15, and to this end, we assume that the constitutive function $\mathcal{F}$ and the stiffness function p satisfy the following conditions.

$$\begin{cases} \text{(a)}\mathcal{F} : (0, L) \times \mathbb{R} \to \mathbb{R}. \\ \text{(b)There exists } L_{\mathcal{F}} > 0 \text{ such that} \\ \qquad |\mathcal{F}(x, \varepsilon_1) - \mathcal{F}(x, \varepsilon_2)| \le L_{\mathcal{F}} \, |\varepsilon_1 - \varepsilon_2| \\ \qquad\quad \forall \, \varepsilon_1, \, \varepsilon_2 \in \mathbb{R}, \text{ a.e. } x \in (0, L). \\ \text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that} \\ \qquad (\mathcal{F}(x, \varepsilon_1) - \mathcal{F}(x, \varepsilon_2))(\varepsilon_1 - \varepsilon_2) \ge m_{\mathcal{F}} \, |\varepsilon_1 - \varepsilon_2|^2 \\ \qquad\quad \forall \, \varepsilon_1, \varepsilon_2 \in \mathbb{R}, \text{ a.e. } x \in (0, L). \\ \text{(d)The mapping } x \mapsto \mathcal{F}(x, \varepsilon) \text{ is measurable on}(0, L), \\ \qquad\quad \text{for any} \varepsilon \in \mathbb{R}. \\ \text{(e)The mapping} x \mapsto \mathcal{F}(x, 0) \text{ belongs to } L^2(0, L). \end{cases} \tag{5.5}$$

$$\begin{cases} p : \mathbb{R} \to \mathbb{R} \;\; \text{satisfies condition (2.2) and, moreover,} \\ p(r) > 0 \;\; \text{if } r > 0 \;\; \text{and} \;\; p(r) \le 0 \;\; \text{if } r < 0. \end{cases} \tag{5.6}$$

We also assume that the density of body force has the regularity

$$f_0 \in L^2(0, L) \tag{5.7}$$

and, finally,

$$g_1 < 0 < g_2. \tag{5.8}$$

We use the space (4.2) which is a real Hilbert space with the canonical inner product (4.3) and the associated norm $\| \cdot \|_X$. We denote by X^* and $\langle \cdot, \cdot \rangle$ the dual of X and the duality pairing between X^* and X, respectively and by $q \colon \mathbb{R} \to \mathbb{R}$ the function defined by (2.3). We also define the set K, the operator $A \colon X \to X^*$, the function $j \colon X \to \mathbb{R}$ and the element $f \in X^*$ by equalities

$$K = \{ u \in X \;\; | \;\; g_1 \le u(L) \le g_2 \}, \tag{5.9}$$

$$\langle Au, v \rangle = \int_0^L \mathcal{F}(u') \, v' \, \mathrm{d}x \qquad \forall \, u, v \, \in X, \tag{5.10}$$

$$j(v) = q(v(L)) \qquad \forall \, v \, \in X, \tag{5.11}$$

$$\langle f, v \rangle = \int_0^L f_0 v \, \mathrm{dx} \qquad \forall \, \mathrm{v} \, \in \mathrm{X}. \tag{5.12}$$

Then, the variational formulation of Problem 15, obtained by using standard arguments, is as follows.

Problem 16. Find a displacement field u such that the inequality below holds:

$$u \in K, \qquad (Au, v - u)_X + j^0(u; v - u) \ge (f, v - u)_X \quad \forall v \in K. \tag{5.13}$$

Let

$$k_0 > 0, \tag{5.14}$$

$$K_0 = [-k_0, k_0] \tag{5.15}$$

and denote by $P_0 : \mathbb{R} \to K_0$ the projection operator on K_0, that is

$$P_0 r = \begin{cases} -k_0 & \text{if} \quad r < k_0, \\ \;\; r & \text{if} \quad -k_0 \le r \le k_0, \\ \;\; k_0 & \text{if} \quad r < k_0 \end{cases} \tag{5.16}$$

Moreover, for any $\varepsilon > 0$ define the operator $A_\varepsilon\colon X \to X^*$, the function $j_\varepsilon\colon X \to \mathbb{R}$ and the element $f_\varepsilon \in X^*$ by equalities

$$\langle A_\varepsilon u, v\rangle = \int_0^L \mathcal{F}u'\, v'\, \mathrm{d}x + \varepsilon \int_0^L (u' - P_0 u')v'\, \mathrm{d}x \quad \text{for all } \ u, v \in X, \tag{5.17}$$

$$j_\varepsilon(v) = q(v(L)) + \frac{\varepsilon}{2}\,(v(L))^2 \qquad \text{for all } \ v \in X, \tag{5.18}$$

$$\langle f_\varepsilon, v\rangle = \int_0^L (f_0 + \varepsilon)v\, \mathrm{d}x \qquad \text{for all } \ v \in X. \tag{5.19}$$

With these notation, we consider the following perturbation of Problem 16.

Problem 17. Find a displacement field u such that the inequality below holds:

$$u \in K, \qquad (A_\varepsilon u, v-u)_X + j_\varepsilon^0(u; v-u) \geq (f_\varepsilon, v-u)_X \quad \forall\, v \in K. \tag{5.20}$$

Our main result in this section is the following.

Theorem 18. *Assume* (5.5)–(5.8), (5.14)*, and moreover, assume that*

$$L_p L < m_{\mathcal{F}}. \tag{5.21}$$

Then Problem 16 *has a unique solution u and, for each ε which satisfies the smallness condition*

$$0 < \varepsilon < \frac{m_{\mathcal{F}}}{L} - L_p, \tag{5.22}$$

Problem 17 *has a unique solution, denoted u_ε. In addition, $u_\varepsilon \to u$ in X as $\varepsilon \to 0$.*

Proof. For the existence and uniqueness part we use Theorem 6 on the space X given by (4.2). To this end we use assumption (5.8) to see that the set K given by (5.9) is a nonempty closed convex subset of X, and therefore, it satisfies assumption (K_1). Next, we use assumption (5.5) to see that the operator A defined by (5.10) satisfies the inequalities

$$(Au - Av, u - v)_X \geq m_{\mathcal{F}}\, \|u - v\|_X^2 \qquad \forall\, u, v \in X, \tag{5.23}$$

$$\|Au - Av\|_{X^*} \leq L_{\mathcal{F}} \|u - v\|_X \qquad \forall\, u,\, v \in X. \tag{5.24}$$

This show that A is strongly monotone and Lipshitz continuous and, therefore, conditions (A_1) and (A_2) hold, the second one with $m_A = m_{\mathcal{F}}$. Next, the results presented in Example 5 show that the function j defined by (5.11) satisfies the properties (j_1) (j_2) and (j_4) with $\alpha_j = L_p\|\gamma\|^2$. On the other hand, using inequality (4.8) it is easy to see that $\|\gamma\| \leq \sqrt{L}$ and, therefore, assumption (5.21) guarantees that condition (j_3) is satisfied, too. We are now in a position to use Theorem 6 to obtain the existence of a unique solution to Problem 16.

The unique solvability of Problem 17 follows from similar arguments. In this case, an elementary calculus shows that for each $\varepsilon > 0$ the operator A_ε is Lipschitz continuous with Lipschitz constant $L_{\mathcal{F}} + 2\varepsilon$ and strongly monotone with constant $m_{A_\varepsilon} = m_{\mathcal{F}}$. In addition, since $j_\varepsilon(v) = q_\varepsilon(v(L))$ for all $v \in X$ where q_ε is defined by (4.6), it it easy to see that the function j_ε satisfies conditions (j_1), (j_2) and (j_4) with $\alpha_{j_\varepsilon} = (L_p + \varepsilon)L$. Therefore, if (5.22) holds we deduce that $\alpha_{j_\varepsilon} < m_{A_\varepsilon}$ which shows that condition (j_3) holds in this case, too.

For the convergence part, we use Theorem 13. To this end, we define the operator $T : X \to X^*$ be equality

$$\langle Tu, v\rangle = \int_0^L (u' - P_0 u')v'\, \mathrm{d}x \quad \text{for all } \ u, v \in X, \tag{5.25}$$

and use the properties of the projection map to see that T is a Lipschitz continuous operator. We now use equalities (5.10), (5.25) and (5.17) to see that we are in the framework of Example 11. We conclude from here that condition (h_A) holds. On the other hand, Example 12 guarantees that assumption (h_j) holds, too. Finally, condition (f_ε) is a consequence of equalities (5.12) and (5.19). We are now in a position to use Theorem 13 to see that $u_\varepsilon \to u$ in X as $\varepsilon \to 0$, which concludes the proof. $\square$

Once the displacement field u is known, the stress field σ can be easily obtained by using the constitutive law (5.1). A function u which satisfies (5.1) is called a weak solution to the contact Problem 15. We conclude by Theorem 10 that the Problem 15 has a unique weak solution.

We now illustrate the convergence result in Theorem 14, and to this end, we consider the set

$$K_\varepsilon = \{\, v \in X \ \mid \ g_{1\varepsilon} \le v(L) \le g_{2\varepsilon} \}, \tag{5.26}$$

together with the following assumptions

$$g_{1\varepsilon} < 0 < g_{2\varepsilon}, \tag{5.27}$$

$$g_{1\varepsilon} \to g_1 \quad \text{and} \quad g_{2\varepsilon} \to g_2 \quad \text{as} \quad \varepsilon \to 0. \tag{5.28}$$

We also consider the following perturbation of Problem 16.

Problem 19. *Find a displacement field* $u \in K_\varepsilon$ *such that*

$$u \in K_\varepsilon, \quad (Au, v-u)_X + j^0(u; v-u) \ge (f, v-u)_X \qquad \forall\, v \in K_\varepsilon. \tag{5.29}$$

We have the following existence, uniqueness and convergence result.

Theorem 20. *Assume* (5.5)–(5.8), (5.21), (5.22), (5.27) *and* (5.28)*. Then Problem 16 has a unique solution* $u \in K$ *and, for each* $\varepsilon > 0$*, Problem 19 has a unique solution* $u_\varepsilon \in K_\varepsilon$ *Moreover,* $u_\varepsilon \to u$ *in* X *as* $\varepsilon \to 0$.

Proof. Let $\varepsilon > 0$. We note that assumption (5.8) allows us to define the constants c_ε and d_ε, by equalities

$$c_\varepsilon = \frac{g_{1\varepsilon} - g_{2\varepsilon}}{g_1 - g_2} \quad \text{and} \quad d_\varepsilon = \frac{g_{2\varepsilon} g_1 - g_{1\varepsilon} g_2}{g_1 - g_2}. \tag{5.30}$$

Also, let θ be a function such that

$$\theta \in X \quad \text{and} \quad \theta(L) = 1. \tag{5.31}$$

Assume now that u and v are two elements of X such that

$$v = \frac{g_{1\varepsilon} - g_{2\varepsilon}}{g_1 - g_2}\, u + \frac{g_{2\varepsilon} g_1 - g_{1\varepsilon} g_2}{g_1 - g_2}\, \theta.$$

Then, using (5.31) it is easy to check that $g_1 \le u(L) \le g_2$ if and only if $g_{1\varepsilon} \le v(L) \le g_{2\varepsilon}$ and, therefore, equalities (5.30) show that $u \in K$ if and only if $c_\varepsilon u + d_\varepsilon \theta \in K_\varepsilon$. We conclude from here that

$$K_\varepsilon = c_\varepsilon K + d_\varepsilon \theta. \tag{5.32}$$

On the other hand, assumption (5.28) shows that

$$c_\varepsilon \to 1 \quad \text{and} \quad d_\varepsilon \to 0 \qquad \text{as} \qquad \varepsilon \to 0, \tag{5.33}$$

It follows from (5.32) and (5.33) that condition $(\widetilde{K})$ is satisfied. Theorem 20 is now a direct consequence of Theorem 14. $\square$

We end this section with some mechanical interpretation of our convergence results. First, we note that Problem 17 represents the variational formulation of a problem similar to Problem 15, in which the constitutive law (5.1) was replaced by the constitutive law $\sigma(x) = \mathcal{F}(x, u'(x)) + \varepsilon(u'(x) - P_0 u'(x))$, the stiffness function $r \mapsto p(r)$ was replaced by the function $r \mapsto p(r) + \varepsilon r$ and the density of the body forces f_0 was replaced by $f_0 + \varepsilon$. Thus, the convergence result in Theorem 18 shows that the weak solution of equilibrium problem of the spring–rod system can be approached as close as one wish by the weak solution of the equilibrium problem of the spring–rod system with the above perturbed constitutive law, stiffness function and body forces, for a small parameter ε. In addition, the convergence result in Theorem 18 shows that small perturbations on the data g_1 and g_2, lead to small perturbation of the weak solution of Problem 16.

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