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# GLOBAL STEIN THEOREM ON HARDY SPACES

ALINE BONAMI, SANDRINE GRELLIER AND BENOIT SEHBA

ABSTRACT. Let  $f$  be an integrable function which has integral 0 on  $\mathbb{R}^n$ . What is the largest condition on  $|f|$  that guarantees that  $f$  is in the Hardy space  $\mathcal{H}^1(\mathbb{R}^n)$ ? When  $f$  is compactly supported, it is well-known that the largest condition on  $|f|$  is the fact that  $|f| \in L \log L(\mathbb{R}^n)$ . We consider the same kind of problem here, but without any condition on the support. We do so for  $\mathcal{H}^1(\mathbb{R}^n)$ , as well as for the Hardy space  $\mathcal{H}_{\log}(\mathbb{R}^n)$  which appears in the study of pointwise products of functions in  $\mathcal{H}^1(\mathbb{R}^n)$  and in its dual  $BMO$ .

## 1. INTRODUCTION

The aim of this article is to generalize a well known result by Stein concerning the maximal function. Let us recall it.

**Theorem (Stein)**[8] *Assume that  $f$  is a nonnegative integrable function on  $\mathbb{R}^n$  that is compactly supported. Then its maximal function  $Mf$  is locally integrable if and only if*

$$\int_B f \ln_+(f) dx < \infty.$$

Here, as usual,  $\ln_+ = \max(\ln, 0)$  where  $\ln$  denote the Napierian logarithm.

One may ask about a global version of such a theorem. Of course, it does not make sense without a modification, since the maximal function of such a nonzero function is bounded below by  $\frac{c}{|x|^n}$  at  $\infty$ . But it makes sense if the global maximal function is replaced by the local one, that is, if we consider

$$M^{loc} f(x) = \sup_{0 < r < 1} \frac{1}{r^n} \int_{|y-x| < r} |f(y)| dy.$$

We will give a necessary and sufficient condition on a non negative  $f$  which ensures the integrability of  $M^{loc} f$  on  $\mathbb{R}^n$ , hence a necessary and sufficient condition for a nonnegative  $f$  to belong to the local Hardy space  $\mathfrak{h}^1(\mathbb{R}^n)$ .

A nonnegative integrable function cannot be in  $\mathcal{H}^1(\mathbb{R}^n)$  since functions in  $\mathcal{H}^1(\mathbb{R}^n)$  have mean 0. But one can ask whether  $f - \left( \int f dy \right) \theta$

is in  $\mathcal{H}^1(\mathbb{R}^n)$ . Here  $\theta$  is a fixed bounded function supported in the unit cube  $Q := (-\frac{1}{2}, \frac{1}{2})^n$ , with integral 1. The characterization is simple and reminiscent of Stein Theorem.

**Theorem.** *Assume  $f$  is a nonnegative integrable function on  $\mathbb{R}^n$ . Then  $f - \left(\int f dy\right) \theta$  is in  $\mathcal{H}^1(\mathbb{R}^n)$  if and only if*

$$(1.1) \quad \int_{\mathbb{R}^n} |f| (\ln_+ |f| + \ln_+ (|x|)) dx < \infty.$$

Moreover, condition (1.1) is sufficient, regardless of the condition on the sign of  $f$ .

This answers the question raised in the abstract:

**Corollary.** *The vector space of integrable functions that satisfy (1.1) is the largest space  $S$  with the following property: if  $g$  is in  $S$  and  $f$  is an integrable function such that  $|f| \leq |g|$  and  $\int_{\mathbb{R}^n} f dx = 0$ , then  $f$*

*belongs to  $\mathcal{H}^1(\mathbb{R}^n)$ .*

We will not come back to this corollary later on. So let us deduce it right now from the theorem. First, the theorem implies in particular that the space of functions satisfying condition (1.1) has this property. Let us prove that it is the largest. We choose  $\theta$  nonnegative, bounded and supported in  $Q$ . We know that  $\theta$  is in  $S$  since it satisfies (1.1). Assume that  $g$  is nonnegative and belongs to  $S$ . We want to prove that  $g$  satisfies (1.1). But, since  $S$  is stable through addition (we assumed that it is a vector space),  $g + \left|\int g dy\right| \theta$  is in  $S$ . So the function  $g - \left(\int g dy\right) \theta$  belongs to  $\mathcal{H}^1(\mathbb{R}^n)$ , and (1.1) holds for  $g$ .

The second aspect of our work is the generalization to other Hardy spaces. Recall that one equivalent definition of  $\mathcal{H}^1(\mathbb{R}^n)$  is given in terms of maximal functions  $\mathcal{M}_\varphi$ , where  $\varphi$  is a smooth function supported in the unit ball with nonzero integral. Following, for instance, [10], we define, for  $f$  an integrable function or, more generally, for  $f$  a tempered distribution,

$$(1.2) \quad \mathcal{M}_\varphi f(x) = \sup_{t>0} |\varphi_t * f(x)|,$$

where, as usual, for  $t > 0$   $\varphi_t = t^{-n} \varphi(t^{-1}\cdot)$ . On the other hand, under adequate assumptions on the Musielak function  $\Psi : \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty)$ , we define the Musielak-Orlicz-type space  $L^\Psi(\mathbb{R}^n)$  as the set of all measurable functions  $f$  such that

$$\int_{\mathbb{R}^n} \Psi \left( x, \frac{|f(x)|}{\lambda} \right) dx < \infty$$

for some  $\lambda > 0$ . Typically the class of integrable functions that satisfy (1.1) is a Musielak space with

$$\Psi(x, t) := t(1 + \ln_+(t) + \ln_+ |x|).$$

This is a convex function in the variable  $t$ , and  $L^\Psi$  is a kind of Orlicz space with an Orlicz function that varies with the spatial point  $x$ . Remark that the dual space of this kind of Musielak spaces is well-known (see [7], [6]). For the specific space under consideration, a function  $g$  is in its dual if, for some  $\lambda$ ,

$$\int \frac{e^{\lambda|g(x)|}}{(1 + |x|)^{n+1}} dx < \infty.$$

One recognizes here an easy consequence of the John-Nirenberg inequality for  $BMO$  spaces, which can also be deduced from our results.

Other examples of spaces that are defined in terms of a Musielak functions are Musielak-Orlicz-type Hardy spaces  $\mathcal{H}^\Psi(\mathbb{R}^n)$ . These spaces generalize Hardy spaces  $\mathcal{H}^p(\mathbb{R}^n)$  for  $p \leq 1$  (when  $\Psi$  does not depend on the variable  $x$ , they are the spaces introduced by S. Janson [4]).

The space  $\mathcal{H}^\Psi(\mathbb{R}^n)$  is the space of tempered distributions  $f$  such that  $\mathcal{M}_\varphi f$  belongs to  $L^\Psi(\mathbb{R}^n)$ . We will consider here the space  $\mathcal{H}_{\log}(\mathbb{R}^n)$ , for which

$$\Psi(x, t) := \frac{t}{1 + \ln_+(t) + \ln_+ |x|}.$$

This space has been introduced in [2] in relation with products of functions in  $\mathcal{H}^1(\mathbb{R}^n)$  and  $BMO$  and generalized div-curl lemmas. After the seminal paper of Ky [5], which followed this first paper, one can find a large literature on this type of spaces, which are in particular invariant through singular integrals. Here we will not appeal to the deep properties developed there, but use only elementary properties to stick to the generalization of Stein's theorem. We will see that, for  $\mathcal{H}_{\log}(\mathbb{R}^n)$ , the class of nonnegative functions for which one has the equivalent of the theorem above is defined by

$$(1.3) \quad \int_B |f| (1 + \ln_+(\ln_+(|f|)) + \ln_+(\ln_+(|x|))) dx < \infty.$$

Remark that an integrable function belonging to  $\mathcal{H}_{\log}(\mathbb{R}^n)$  has mean 0. Indeed, since  $f$  is in  $L^1(\mathbb{R}^n)$ , we know that  $\varphi_t * f$  tends to the constant  $\int f dx$  for  $t$  tending to  $\infty$ . By Lebesgue's dominated convergence Theorem, this constant function belongs to  $L_{\log}(\mathbb{R}^n) := L^\Psi(\mathbb{R}^n)$ ,  $\Psi$  defined above, which forces the constant to be 0. So, as for  $\mathcal{H}^1(\mathbb{R}^n)$ , it makes sense to subtract to  $f$  the function  $\left(\int f dx\right) \theta$  in order to characterize  $\mathcal{H}_{\log}(\mathbb{R}^n)$ .

This problem has already been tackled in [1] for  $\mathcal{H}_{\log}(\mathbb{R}^n)$ , for functions with compact support. In other words, the authors of [1] established the analog of Stein's theorem. The interest we found in their paper motivates ours. Here, the new difficulty is the behavior at  $\infty$ . A source of inspiration has also been for the first author the remembrance of discussions with François Bouchut in the nineties, and his manuscript [3], which already contained (1.1) as a sufficient condition.

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**Notations:** In the following, we will write  $A \lesssim B$  (respectively  $A \gtrsim B$ ) whenever there exists a nonnegative constant  $c$  with  $A \leq cB$  (respectively  $A \geq cB$ ) and  $A \simeq B$  whenever  $A \lesssim B$  and  $A \gtrsim B$ . Each time the constants are assumed to be uniform constants, which depend only on the dimension of the space.

## 2. THE HARDY SPACE $\mathcal{H}^1(\mathbb{R}^n)$ AND ITS LOCAL VERSION $\mathfrak{h}^1(\mathbb{R}^n)$

Let us recall that  $\mathcal{H}^1(\mathbb{R}^n)$  is the space of  $L^1(\mathbb{R}^n)$  functions  $f$  such that, for some smooth function  $\varphi$  supported in the unit ball,

$$(2.1) \quad \|f\|_{\mathcal{H}^1} := \|\mathcal{M}_\varphi(f)\|_1 < \infty.$$

The local space  $\mathfrak{h}^1(\mathbb{R}^n)$  is the space of  $L^1$  functions such that

$$(2.2) \quad \|f\|_{\mathfrak{h}^1} := \|\mathcal{M}_\varphi^{loc}(f)\|_1 < \infty.$$

Here  $\mathcal{M}_\varphi^{loc}$  is the local maximal function, which is defined by taking the supremum on  $t < 1$  and not on all positive  $t$ .

$$(2.3) \quad \mathcal{M}_\varphi^{loc} f(x) = \sup_{0 < t < 1} |\varphi_t * f(x)|,$$

**Remark 1.** *It is important to observe that  $\mathcal{M}_\varphi^{loc}(f) \leq CM^{loc}f$  for all  $f$ . Moreover, for a nonnegative function  $f$ , the local maximal Hardy-Littlewood function  $M^{loc}f$  is equivalent to  $\mathcal{M}_\varphi^{loc}$ .*

In the following, we will note

$$Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^n, \quad Q_k := k + Q \text{ for } k \in \mathbb{Z}^n, \text{ and } f_k := f\chi_{Q_k}.$$

Here  $\chi_E$  is the characteristic function of the set  $E$ .

We first consider the case of the local Hardy space. The following statement, which has its own interest, holds.

**Theorem 1.** *Let  $f$  be an integrable function. Then  $M^{loc}f$  belongs to  $L^1(\mathbb{R}^n)$  if and only if  $f = \sum f_k$  satisfies*

$$(2.4) \quad \sum_{k \in \mathbb{Z}^n} \int |f_k(x)| \left(1 + \ln_+ \left(\frac{|f_k(x)|}{\|f_k\|_1}\right)\right) dx < \infty.$$

*In particular, if  $f$  satisfies condition (2.4) then  $f$  belongs to  $\mathfrak{h}^1(\mathbb{R}^n)$ , the reverse being true if  $f$  is nonnegative.*

One would prefer to have global integrals, but unfortunately this is not the case. Remark that the condition

$$(2.5) \quad \int_{\mathbb{R}^n} |f|(1 + \ln_+ |f|) dx < \infty$$

is necessary. We get a sufficient condition when we add the following one

$$(2.6) \quad \sum_k \|f_k\|_1 \ln_+(\|f_k\|_1^{-1}) < \infty.$$

This last condition may be interpreted as an amalgam condition: the sequence  $(\|f_k\|_1)$  belongs to a kind of  $\ell \log \ell$  space of sequences.

*Proof of the sufficient condition.* By subadditivity of the maximal function,

$$\|M^{loc}(f)\|_1 \leq \sum_k \|M^{loc}(f_k)\|_1.$$

We will consider each term separately. We first remark that  $M^{loc}(f_k)$  has support in  $k + 2Q$ , and, on this cube,

$$M^{loc}(f_k) \leq M(f_k).$$

The classical  $L \log L$  inequality (see [8] for instance) written for the normalized function  $\frac{f_k}{\|f_k\|_1}$  gives

$$(2.7) \quad \int_{k+2Q} M f_k dx \lesssim \|f_k\|_1 + \int |f_k| \ln_+ \left( \frac{f_k}{\|f_k\|_1} \right) dx.$$

We conclude at once. □

*Proof of the necessary condition.* As for the sufficient condition, we first prove an estimate for each  $f_k$ . We claim that the following estimate holds for  $v$  supported in  $Q$ ,

$$(2.8) \quad \int_Q M^{loc} v dx \gtrsim \int_Q |v| \ln_+(|v|) dx.$$

The proof is a variant of the proof of [8] for the local maximal function. We write it for the sake of completeness. Consider the family  $\mathcal{Q}$  of dyadic sub-cubes of  $Q$ . The maximal dyadic function is defined on  $\mathcal{Q}$  by

$$(2.9) \quad M^d v(x) := \sup_{x \in R, R \in \mathcal{Q}} \frac{1}{|R|} \int_R |v(y)| dy.$$

The following classical estimate is the key of the proof. For  $v$  with norm 1, and  $s > 1$ ,

$$(2.10) \quad |\{x, M^d(v)(x) > s\}| \geq \frac{1}{s} \int_{|v|>s} |v| dx$$

Let  $\ell(R)$  be the length of the sides of the cube  $R$ . The following lemma compares the two maximal functions.

**Lemma 1.** *Let  $v$  be an integrable function supported in  $Q$  with norm 1. For  $x \in Q$ ,*

$$(2.11) \quad M^{loc}v(x) \geq c \sup \frac{1}{|R|} \int_R |v(y)| dy,$$

where the supremum is taken on dyadic cubes  $R$  containing  $x$  and such that  $\ell(R)\sqrt{n} \leq 1$ . Here  $c = n^{-n/2}|B(0, 1)|^{-1}$ .

*Proof.* Each dyadic cube  $R$  is contained in the ball  $B(x, \ell(R)\sqrt{n})$ , which has the volume  $n^{n/2}|B(0, 1)||R|$ . So

$$\frac{1}{|B(x, \ell(R)\sqrt{n})|} \int_{B(x, \ell(R)\sqrt{n})} |v| dx > c \frac{1}{|R|} \int_R |v| dx.$$

This ball has radius bounded by 1 when  $\ell(R)\sqrt{n} \leq 1$ . The inequality between suprema follows at once.  $\square$

We claim that the supremum defined in the lemma coincides with  $M^d v(x)$  when  $M^d v(x) > \sqrt{n}^n$ . Indeed, for larger dyadic cubes, we have

$$\frac{1}{|R|} \int_R |v(y)| dy \leq \sqrt{n}^n \int_Q |v| dy = \sqrt{n}^n.$$

So, using the inequality (2.10) for the dyadic maximal function, we get

$$(2.12) \quad |\{x \in Q, M^{loc}(v)(x) > s\}| \gtrsim \frac{1}{s} \int_{|v|>s/c} |v| dx$$

for  $s > c\sqrt{n}^n$ . We integrate both sides from  $c\sqrt{n}^n$  to  $\infty$ , and find

$$\int_Q M^{loc}v dx \gtrsim \int |v| \ln_+ \left( \frac{|v|}{\sqrt{n}^n} \right) dx.$$

Since the norm of  $v$  is 1, the estimate (2.8) on  $|v| \ln_+(|v|)$  follows at once, using the fact that  $\ln_+(t) \leq \ln_+\left(\frac{t}{\sqrt{n}^n}\right) + \ln_+(\sqrt{n}^n)$ .

To conclude for the proof, we remark that  $M^{loc}f \geq M^{loc}f_k$  on  $Q_k$ . Hence, applying the inequality (2.8) to each  $f_k$  and summing on  $k$  allows to get the necessary condition.  $\square$

We now turn to global Hardy space  $\mathcal{H}^1(\mathbb{R}^n)$ . For  $f \in L^1(\mathbb{R}^n)$ , let us define

$$(2.13) \quad T_\theta f := f - \left( \int f dx \right) \theta.$$

Here,  $\theta$  is a fixed bounded function supported in the unit cube with

$$\int \theta = 1.$$

As said in the introduction, it was proved by Bouchut in informal notes (see [3]) that the function  $T_\theta f$  is in  $\mathcal{H}^1(\mathbb{R}^n)$  whenever  $f$  belongs to the Musielak space of functions  $g$  such that

$$\int |g(x)| (1 + \ln_+ |g(x)| + \ln_+ |x|) dx < \infty.$$

The following theorem may be seen as a global Stein's theorem (see [8]) and gives as well the necessary condition.

**Theorem 2.** *Let  $T_\theta$  be defined as in (2.13). Let  $f$  be an integrable function. Then  $T_\theta f$  is a function of  $\mathcal{H}^1(\mathbb{R}^n)$  if*

$$(2.14) \quad \int |f(x)| (1 + \ln_+ |f(x)| + \ln_+ |x|) dx < \infty.$$

Moreover, if  $f$  is nonnegative and  $T_\theta f$  is in  $\mathcal{H}^1(\mathbb{R}^n)$ , then condition (2.14) holds.

In particular, a function  $f$  of integral 0 satisfying (2.14) belongs to  $\mathcal{H}^1(\mathbb{R}^n)$ .

*Proof.* Even if the sufficient condition has already been established by Bouchut (see the remark below), we will give a slightly different but complete proof of it.

Let us remark first that  $f$  belongs to  $\mathfrak{h}^1(\mathbb{R}^n)$ . Indeed, let us prove that the two sufficient conditions given in (2.5), (2.6) are satisfied. It is straightforward for the first one. We now want to prove that condition (2.14) implies  $\sum_{k \neq 0} \mu_k \ln_+(\mu_k^{-1}) < \infty$ , with  $\mu_k = \|f_k\|_1$ . We divide this last sum into two, depending on the fact that  $\mu_k > k^{-(n+1)}$  or not. For the first sum, the inequality comes from the assumption

$$\int |f(x)| (1 + \ln_+ |x|) dx < \infty$$

since

$$\int |f(x)| (1 + \ln_+ |x|) dx \gtrsim \sum_k (1 + \ln_+ |k|) \int |f_k(x)| dx = \sum_k \mu_k (1 + \ln_+ |k|).$$

For  $\mu_k \leq |k|^{-(n+1)}$ ,  $|k| \geq 1$  using that  $x \mapsto x \ln_+(x^{-1})$  is non-decreasing, the sum of the corresponding terms is bounded by

$$\sum_{|k| \geq 1} |k|^{-(n+1)} \ln |k|$$



which is finite. Hence,  $f$  belongs to the local Hardy space  $\mathfrak{h}^1(\mathbb{R}^n)$ . It remains to deal with the non local part.

Without loss of generality, we assume  $\|f\|_1 = 1$ . As before, we write  $f = \sum_k f_k$ , with  $f_k = f\chi_{Q_k}$ ,  $k \in \mathbb{Z}^n$ . We have as well

$$T_\theta f = \sum \left( f_k - \left( \int f_k dx \right) \theta \right).$$

We write

$$T_\theta f_k = f_k - \left( \int f_k dx \right) \chi_{Q_k} + \left( \int f_k dx \right) (\chi_{Q_k} - \theta).$$

We will consider separately the two parts.

We first prove the following proposition.

**Proposition 1.** *Let  $f_k$  be defined as above and  $h := \sum h_k$ , where  $h_k := f_k - \left( \int f_k dx \right) \chi_{Q_k}$  and assume condition (2.4). Then  $h$  is in  $\mathcal{H}^1(\mathbb{R}^n)$ .*

*Proof.* We prove that the  $h_k$ 's are in  $\mathcal{H}^1(\mathbb{R}^n)$  and that

$$\sum \| \mathcal{M}_\varphi h_k \|_1 < \infty.$$

This is done by a slight modification of the fact that atoms are in  $\mathcal{H}^1(\mathbb{R}^n)$ . Inside  $k+2Q$ , we consider the two terms of  $h_k$  separately. We conclude directly for the part involving the characteristic function of  $Q_k$ , while, for  $f_k$  we use (2.7), which we recall here:

$$(2.15) \quad \int_{k+2Q} M f_k dx \lesssim \left( \|f_k\|_1 + \int |f_k| \ln_+ \left( \frac{f_k}{\|f_k\|_1} \right) dx \right).$$

From condition (2.4), the sum on  $k$  of the right hand side is bounded. It follows that

$$\left\| \sum_k (M h_k) \chi_{k+2Q} \right\|_1 < \infty.$$

It remains to prove that

$$\left\| \sum_k (\mathcal{M}_\varphi h_k) \chi_{(k+2Q)^c} \right\|_1 < \infty.$$

By the zero-mean of  $h_k$ , the maximal function  $\mathcal{M}_\varphi h_k$  is bounded by  $\|h_k\|_1/|x-k|^{n+1}$ . So

$$\int_{\mathbb{R}^n \setminus (k+2Q)} \mathcal{M}_\varphi h_k dx \lesssim \|h_k\|_1 \lesssim \|f_k\|_1.$$

The sum of the corresponding integrals is bounded by some uniform constant. It allows to conclude.  $\square$

We consider now the remaining part  $g := T_\theta f - h$ . Recall that

$$g := \sum_k \int f_k dx (\chi_{Q_k} - \theta).$$

The next lemma gives a sufficient condition for  $g$  to be in  $\mathcal{H}^1(\mathbb{R}^n)$ .

**Lemma 2.** *Let  $a$  be bounded by 1, of mean 0, and assume that  $a$  is supported in  $Q_0 \cup Q_j$ , with  $|j| > 2$ . Then*

$$(2.16) \quad \|a\|_{\mathcal{H}^1(\mathbb{R}^n)} \lesssim 1 + \ln |j|.$$

*Proof.* Remark that  $|j|^{-1}a$  is an atom of  $\mathcal{H}^1(\mathbb{R}^n)$ , and thus has bounded norm. But we need a better estimate. As before, we use the classical inequality

$$(2.17) \quad |\mathcal{M}_\varphi a(x)| \lesssim \frac{|j|}{|x|^{n+1}}$$

for  $|x| > 2|j|$ . For  $|x| \leq 2|j|$  we use the elementary inequality

$$(2.18) \quad |\mathcal{M}_\varphi a(x)| \lesssim M\chi_{Q_0}(x) + M\chi_{Q_j}(x) \lesssim \frac{1}{1+|x|^n} + \frac{1}{1+|x-j|^n}.$$

We get

$$(2.19) \quad \|\mathcal{M}_\varphi a\|_1 \lesssim 2 + \int_{|x|>2|j|} \frac{|j|dx}{|x|^{n+1}} + \int_{1<|x|<2|j|} \frac{dx}{|x|^n} + \int_{1<|x-j|<3|j|} \frac{dx}{|x-j|^n}.$$

The conclusion follows easily that (2.16) holds.  $\square$

We have the following proposition.

**Proposition 2.** *Let  $\lambda_j$  be a sequence of real numbers indexed by  $\mathbb{Z}^n$  and let  $g := \sum g_j$ , with  $g_j := \lambda_j(\chi_{Q_j} - \theta)$ . Then  $g$  is in  $\mathcal{H}^1(\mathbb{R}^n)$  if*

$$(2.20) \quad \sum_j |\lambda_j| (1 + \ln_+ |j|) < \infty.$$

*Moreover, if the  $\lambda_j$ 's are nonnegative, it is a necessary condition for having  $g$  in  $\mathcal{H}^1(\mathbb{R}^n)$ .*

The sufficiency of the condition comes from Lemma 2.

Let us now assume that the  $\lambda_j$ 's are nonnegative and prove the necessity of the condition. We choose  $\varphi$  such that  $0 \leq \varphi \leq 1$ , and  $\varphi = 1$  on the ball  $B(0, 1/2)$ .

We will give a bound below of  $|\varphi_r * g|(x)$  for  $|x| > \sqrt{n}$  and  $r = 4|x|$ . We first prove that

$$-\varphi_r * g(x) = \sum_j \lambda_j (\varphi_r * \theta(x) - \varphi_r * \chi_{Q_j}(x))$$

is nonnegative as a sum of nonnegative terms. Indeed, for  $|x| > \sqrt{n}$  and  $r = 4|x|$ , the support of  $\theta$  is entirely contained in the set of  $y$  for which  $\varphi_r(x - y) = 1$ . So, on one hand

$$\varphi_r * \theta(x) = \frac{1}{r^n}.$$

On the other hand, since 1 is the maximum of  $\varphi$ , the other terms  $\varphi_r * \chi_{Q_j}(x)$  are bounded by  $r^{-n}$ . We have proved our claim on the sign of  $-\varphi_r * g(x)$ . Now, a bound below is given by  $r^{-n} \sum_{j \in J} \lambda_j$  where  $J = \{j \in \mathbb{Z}^n ; B(x, r) \cap Q_j = \emptyset\}$ . Indeed, for such  $j$ 's,  $\varphi_r * \chi_{Q_j}(x) = 0$ . But this set of indices contains all  $j$ 's such that  $|j| \geq 6|x|$  since, under these conditions,

$$\text{dist}(B(x, 4|x|), j) \geq |x| \geq \sqrt{n} = \text{diam}(Q_j).$$

Eventually,

$$|\mathcal{M}_\varphi g(x)| \geq |\varphi_{4|x}| * g(x) \geq \frac{1}{(4|x|)^n} \sum_{|j| \geq 6|x|} \lambda_j.$$

and, integrating over the set  $|x| \geq \sqrt{n}$ ,

$$\|\mathcal{M}_\varphi g\|_1 \gtrsim \sum_{|j| > 6\sqrt{n}} \lambda_j \ln(|j|/6).$$

We conclude at once. □

*End of the proof of Theorem 2.* The sufficiency of the condition

$$\int |f(x)| (1 + \ln_+ |f(x)| + \ln_+(|x|)) dx < \infty$$

is a consequence of Propositions 1 and 2. Indeed, we already mentioned that condition (2.4) holds. Hence, what we called  $h$  in Proposition 1 belongs to  $\mathcal{H}^1(\mathbb{R}^n)$ . To prove that  $g = f - h$  is in  $\mathcal{H}^1(\mathbb{R}^n)$ , one has to prove estimate (2.20) with  $\lambda_j = \int f_j$ . Since  $|x| \simeq |j|$  on  $Q_j$ , it follows easily from the discretization of the integral  $\int |f(x)| (1 + \ln_+(|x|)) dx$ .

We now prove the necessity for a nonnegative function  $f$  with  $T_\theta f \in \mathcal{H}^1(\mathbb{R}^n)$ . As  $\mathcal{H}^1(\mathbb{R}^n)$  is contained in  $\mathfrak{h}^1(\mathbb{R}^n)$ ,  $T_\theta f$  belongs to  $\mathfrak{h}^1(\mathbb{R}^n)$ . Since  $\theta$  is also in  $\mathfrak{h}^1(\mathbb{R}^n)$ , the function  $f$  itself is in  $\mathfrak{h}^1(\mathbb{R}^n)$ . So the condition (2.4) is satisfied. By Proposition 1,  $h$  is in  $\mathcal{H}^1(\mathbb{R}^n)$ . So  $g = f - h$  is in  $\mathcal{H}^1(\mathbb{R}^n)$ . By Proposition 2, this implies

$$\sum_{|k| > 1} \mu_k (1 + \ln_+ |k|) < \infty$$

with  $\mu_k = \int f_k$ . As before, this last condition reads

$$\int |f(x)| (1 + \ln_+(|x|)) dx < \infty.$$

Eventually, combined with the estimate (2.5), one gets

$$\int |f(x)|(1 + \ln_+ |x| + \ln_+(|f(x)|)) dx < \infty.$$

□

**Remark.** The precise statement of Bouchut in [3] is the following.

Let  $T$  be a singular operator given by the convolution by  $K$ . Then the operator  $R_K$ , defined by

$$R_K(f)(x) = T(f)(x) - \left( \int f dy \right) K(x)\chi_{|x|>1}$$

maps the space of functions that satisfy (1.1) into  $L^1(\mathbb{R}^n)$ .

This may be obtained as a consequence of Theorem 2. Indeed, let  $f$  satisfy (1.1). We know that  $T_\theta f$  is in  $\mathcal{H}^1(\mathbb{R}^n)$ . Since singular operators map  $\mathcal{H}^1(\mathbb{R}^n)$  into itself and since the integral of  $T_\theta f$  is zero,  $R_K(T_\theta f)$  is integrable. So it is sufficient to see that  $R_K(\theta)$  is also in  $L^1(\mathbb{R}^n)$ . Because of  $L^p$  estimates for singular operators,  $T(\theta)$  is locally integrable. The integrability on  $\{|x| > 2\}$  follows from the estimate

$$\int_{|x|>2} \int_{|y|<1} |K(x-y) - K(x)|\theta(y) dy dx < \infty$$

since the kernel  $K$ , which is the kernel of a singular integral, satisfies

$$\int_{|x|>2} |K(x-y) - K(x)| dx < \infty$$

for  $|y| < 1$ .

Conversely, by using the characterization of  $\mathcal{H}^1(\mathbb{R}^n)$  through Riesz transforms, it is easy to see that the statement of Bouchut implies the sufficient condition in Theorem 2.

### 3. THE LOCAL HARDY SPACE $\mathfrak{h}_{\log}$

We first consider the local Hardy space  $\mathfrak{h}_{\log}(\mathbb{R}^n)$ . The following result gives the analogue of Theorem 1.

**Theorem 3.** *Let  $f$  be an integrable function. Then  $M^{loc} f$  is in  $L_{\log}(\mathbb{R}^n)$  if and only if*

$$(3.1) \quad \int |f(x)| \left( 1 + \ln_+ \frac{\ln(e + |f(x)|)}{\ln(e + |x|)} \right) dx < \infty.$$

As a consequence, if  $f$  satisfies this condition then  $f$  is in  $\mathfrak{h}_{\log}(\mathbb{R}^n)$ . Conversely, if  $f$  is nonnegative and  $f$  is in  $\mathfrak{h}_{\log}(\mathbb{R}^n)$ , then condition (3.1) holds.

*Proof.* It will be convenient to take an equivalent function for  $\Psi$ , which leads to the same spaces. Namely, let

$$\bar{\Psi}(x, t) := \frac{t}{\ln(e + t) + \ln(e + |x|)}.$$

We omit the bar from now on. The function  $\Psi$  is non-decreasing in  $t$ . An explicit computation gives

$$(3.2) \quad \frac{1}{2} \frac{\Psi(x, t)}{t} \leq \frac{d}{dt} \Psi(x, t) \leq \frac{\Psi(x, t)}{t}.$$

It is also doubling:

$$\Psi(x, 2t) \lesssim \Psi(x, t).$$

We first establish the following lemma, which may be seen as a precised version of the corresponding result in [1].

**Lemma 3.** *Let  $f_k$  be supported in  $k + Q$  with  $\mu_k := \|f_k\|_1 \leq 1$ . One has*

$$(3.3) \quad \int_{k+2Q} \Psi(x, M(f_k)) dx \lesssim \frac{\mu_k}{\ln(e + |k|)} + \frac{\mu_k}{\ln(e + |k|)} \ln_+ \left( \frac{1}{\mu_k} \right) + \int_{k+Q} |f_k(x)| \left( 1 + \ln_+ \frac{\ln(e + |f_k(x)|)}{\ln(e + |k|)} \right) dx.$$

*Proof.* We have

$$(3.4) \quad \Psi(x, t) \simeq \Psi(|k|, t) \quad \text{for } x \in k + 2Q.$$

We write, as it is classical (see for instance [8], Chapter 1)

$$\begin{aligned} \int_{k+2Q} \Psi(k, M(f_k)(x)) dx &\simeq \int_0^\infty \Psi'(k, t) |\{x \in k + 2Q ; M f_k(x) > t\}| dt \\ &\lesssim \Psi(k, \|f_k\|_1) + \int_{\|f_k\|_1}^\infty \frac{\Psi(k, t)}{t} |\{x \in k + 2Q ; M f_k(x) > t\}| dt. \end{aligned}$$

Since  $\Psi(x, t) \leq \frac{t}{\ln(e + |x|)}$ , the first term is bounded by  $\frac{\mu_k}{\ln(e + |k|)}$ .

Using the maximal theorem, one has the inequality

$$|\{x \in k + 2Q ; M f_k(x) > t\}| \lesssim \frac{1}{t} \int_{|f_k| \geq t/2} |f_k(x)| dx.$$

So, using this last inequality and exchanging the integrals, we get

$$\int_{k+2Q} \Psi(k, M(f_k)(x)) dx \lesssim \frac{\mu_k}{\ln(e + |k|)} + \int_{2|f_k(x)| > \mu_k} |f_k(x)| \int_{\mu_k}^{2|f_k(x)|} \frac{\Psi(k, t)}{t^2} dt dx.$$

We cut the last integral into the integral below 1 and above 1. Since  $\Psi$  is doubling, it is straightforward to see that such integrals are also doubling, namely

$$(3.5) \quad \int_1^{2s} \frac{\Psi(x, s)}{s^2} ds \lesssim \int_1^s \frac{\Psi(x, s)}{s^2} ds$$

for  $s > 2$ . We will use the following properties satisfied by  $\Psi$ .

(1) for  $0 < t < 1$ ,

$$\int_t^1 \frac{\Psi(x, s)}{s^2} ds \simeq \frac{\ln_+(1/t)}{\ln(e + |x|)},$$

(2) for  $t > 1$ ,

$$\ln_+ \frac{\ln(e + t)}{\ln(e + |x|)} \lesssim \int_1^t \frac{\Psi(x, s)}{s^2} ds \lesssim 1 + \ln_+ \frac{\ln(e + t)}{\ln(e + |x|)}.$$

Indeed, we write, on one hand for  $0 < t < 1$ ,

$$\int_t^1 \frac{\Psi(x, s)}{s^2} ds = \int_t^1 \frac{1}{s(\ln(e + s) + \ln(e + |x|))} ds \simeq \int_t^1 \frac{1}{s \ln(e + |x|)} ds$$

on the other hand, for  $t > 1$ ,

$$\begin{aligned} \int_1^t \frac{\Psi(x, s)}{s^2} ds &= \int_1^t \frac{1}{s(\ln(e + s) + \ln(e + |x|))} ds \\ &\simeq \int_1^t \frac{1}{(s + e)(\ln(e + s) + \ln(e + |x|))} ds \leq \ln \left( 1 + \frac{\ln(e + t)}{\ln(e + |x|)} \right). \end{aligned}$$

Using these properties, we get the inequalities

$$\int_{\mu_k}^1 \frac{\Psi(k, t)}{t^2} dt \lesssim \frac{1}{\ln(e + |k|)} \ln_+ \left( \frac{1}{\mu_k} \right),$$

and

$$\int_1^{2|f_k(x)|} \frac{\Psi(k, t)}{t^2} dt \lesssim 1 + \ln_+ \left( \frac{\ln(e + |f_k(x)|)}{\ln(e + |k|)} \right).$$

It follows that

$$\int_{k+2Q} |f_k(x)| \int_{\mu_k}^{2f_k(x)} \frac{\Psi(k, t)}{t^2} dt dx \lesssim \frac{\mu_k}{\ln(e + |k|)} + \frac{\mu_k}{\ln(e + |k|)} \ln_+ \left( \frac{1}{\mu_k} \right) + \int_{k+Q} |f_k(x)| \left( \ln_+ \left( \frac{\ln(e + |f_k(x)|)}{\ln(e + |k|)} \right) \right) dx.$$

This ends the proof of the Lemma.  $\square$

Let us prove the local theorem 3. Let  $f$  be an integrable function. Without loss of generality, we can assume  $\|f\|_1 = 1$ . We first prove the sufficiency of condition (3.1) to have  $M^{loc}(f) \in L_{\log}(\mathbb{R}^n)$ . We have to estimate

$$\int_{\mathbb{R}^n} \Psi(x, M^{loc}(f)(x)) dx.$$

We write  $f$  as  $\sum f_k$  with  $f_k$  supported in  $k + Q$ ,  $k \in \mathbb{Z}$ ,  $\mu_k := \|f_k\|_1$ . At this point, we recall that  $L_{\log}$  is only a quasi Banach space, so that we need to be careful. We will use the following property, valid for nonnegative functions  $g_j$ , that

$$(3.6) \quad \int_{\mathbb{R}^n} \Psi(x, \sum g_j(x)) dx \lesssim \sum \int_{\mathbb{R}^n} \Psi(x, g_j(x)) dx.$$

This is an easy consequence of the fact that the function  $t \mapsto \frac{\Psi(x, t)}{t}$  is nonincreasing. We apply Lemma 3 to  $f_k$  to get,

$$\begin{aligned} \int \Psi(x, M^{loc}(f_k)) dx &\leq \int_{k+2Q} \Psi(x, M(f_k)) dx \\ &\lesssim \frac{\mu_k}{\ln(e + |k|)} \left( 1 + \ln_+ \left( \frac{1}{\mu_k} \right) \right) + \int_{k+Q} |f_k(x)| \ln_+ \frac{\ln(e + |f_k(x)|)}{\ln(e + |k|)} dx \end{aligned}$$

It remains to sum over  $k$ . The sums corresponding to the two first terms are bounded by the norm of  $f$  in  $L^1$  (we cut the second sum into two parts as in the preceding proof by comparing  $\mu_k$  to  $|k|^{-(n+1)}$ ). Eventually, using that  $|x| \simeq |k|$  on  $k + Q$ ,

$$\int_{\mathbb{R}^n} \Psi(x, M^{loc}(f)(x)) dx \lesssim \|f\|_1 + \int |f(x)| \ln_+ \frac{\ln(e + |f(x)|)}{\ln(e + |x|)} dx.$$

It allows to conclude for the sufficient condition. For the necessary condition, we use Stein Inequality (2.12) as for the characterization of

$\mathfrak{h}^1(\mathbb{R}^n)$ . It gives

$$\begin{aligned} \int \Psi(x, M^{loc}(f_k)) dx &\gtrsim \int_{(k+Q) \cap \{|f_k(x)| > 1\}} |f_k(x)| \int_1^{e^{-1}|f_k(x)|} \frac{\Psi(|k|, s)}{s^2} ds dx \\ &\gtrsim \int_{k+Q} |f_k(x)| \ln_+ \frac{\ln_+(|f_k(x)|)}{\ln(e + |k|)} dx \end{aligned}$$

Summing on  $k$  gives the result as before. Hence, we proved that  $M^{loc}f$  belongs to  $L_{\log}(\mathbb{R}^n)$  if and only if condition (3.1) holds. The result on  $\mathfrak{h}_{\log}(\mathbb{R}^n)$  is an easy consequence of the remark we did before that  $\mathcal{M}_{\varphi}^{loc}(f) \leq CM^{loc}f$  while, when  $\varphi$  and  $f$  are nonnegative, a reverse inequality is also valid.  $\square$

Before leaving the local estimates, let us answer a natural question. What can we say for functions that are only locally integrable? We will see that there exists locally integrable functions, which are not integrable but belong to  $\mathfrak{h}_{\log}(\mathbb{R}^n)$ . Thanks to the differentiation Lebesgue Theorem  $|f| \leq M^{loc}f$ , hence it is clear that  $f \in L_{\log}(\mathbb{R}^n)$  is a necessary condition for a nonnegative function  $f$  to belong to  $\mathfrak{h}_{\log}(\mathbb{R}^n)$ . The following lemma goes into this direction.

**Lemma 4.** *Assume that  $f$  is nonnegative and locally integrable. If*

$$\int \Psi(x, M^{loc}f) dx < \infty,$$

*then the function  $f$  belongs to the weighted space  $L^1(\ln(e + |x|))^{-1}dx$ . In particular, if  $\int \Psi(x, M^{loc}f) dx = 1$ , then*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{\ln(e + |x|)} dx \lesssim 1.$$

*Proof.* If we note as before  $\mu_k := \int f_k dx$ , it is sufficient to prove that

$$(3.7) \quad \sum_k \frac{\mu_k}{\ln(e + |k|)} < \infty.$$

Let  $N$  be the smallest integer such that  $2^{-N}\sqrt{n} \leq 1$ . Using the comparison between  $M^{loc}$  and  $M^d$  given in (2.11), if we consider any sub-cube  $R$  of  $k + Q$  such that  $\ell(R) = 2^{-N}$ , then  $M^{loc}f_k(x) \gtrsim \int_R f_k dy$  for  $x \in R$ . We call  $\mathcal{Q}_{k,N}$  the family of these  $2^{Nn}$  subcubes. It follows



that

$$\begin{aligned} \Psi(k, \mu_k) &\leq \sum_{R \in \mathcal{Q}_{k,N}} \Psi(k, \int_R f_k(y) dy) \\ &\lesssim \sum_{R \in \mathcal{Q}_{k,N}} \int_R \Psi(k, M^{loc} f_k) dx = \int_{k+Q} \Psi(k, M^{loc} f_k) dx. \end{aligned}$$

So

$$\sum_k \Psi(k, \mu_k) \lesssim \sum_k \int_{k+Q} \Psi(x, M^{loc} f_k) dx \leq \int \Psi(x, M^{loc} f) dx.$$

Next, one verifies that  $\frac{\mu_k}{\ln(e + |k|)} \leq 2\Psi(k, \mu_k)$  unless  $\mu_k$  is larger than  $|k|$ , which implies, by monotonicity, that  $\Psi(k, |k|) \leq 2\Psi(k, \mu_k)$ . Since the sequence  $(\Psi(k, \mu_k))_k$  tends to 0, this happens only for a finite number of terms, which proves the convergence of the series (3.7).

Assume now that  $\int \Psi(x, M^{loc} f) dx = 1$ , which implies that

$$\sum_k \Psi(k, \mu_k) \lesssim 1.$$

The sum  $\frac{\mu_k}{\ln(e + |k|)}$ , when restricted to those  $k$  for which  $\mu_k \leq |k|$  is bounded by some uniform constant. The number of the other terms is bounded by a uniform constant. Moreover, for any of them  $\frac{\mu_k}{\ln(e + \mu_k)} \leq 2\Psi(|k|, \mu_k) \lesssim 1$ . It follows that each of these  $\mu_k$  is bounded by some uniform constant. The estimate of the integral follows at once.  $\square$

The function

$$f(x) := (1 + |x|)^{-n} (\ln(e + |x|))^{-1},$$

which varies very slowly, gives an example of non integrable function in  $\mathfrak{h}_{\log}(\mathbb{R}^n)$ . The following proposition revisits Theorem 3 without assuming a priori that  $f$  is integrable. It is much more technical. The notations for  $f_k$  and  $\mu_k$  are the same.

**Proposition 3.** *Let  $f$  be locally integrable. Then we have following inequality.*

$$\begin{aligned} \int \Psi(x, M^{loc} f) dx &\lesssim \sum_k \frac{\mu_k}{\ln(e + |k|)} + \int_{\mathbb{R}^n} |f(x)| \ln_+ \left( 1 + \frac{\ln(e + |f(x)|)}{\ln(e + |x|)} \right) dx \\ (3.8) \quad &+ \sum_k \int_{\mathbb{R}^n} |f_k(x)| \frac{\ln_+ \left( \frac{\min(|f_k(x)|, |k|)}{\mu_k} \right)}{\ln(e + |k|)} dx. \end{aligned}$$

Moreover, the finiteness of the right hand side is a necessary condition for  $M^{loc} f$  to be in  $L_{\log}(\mathbb{R}^n)$ .

**Remark 2.** If  $f$  is such that  $|f_k| \lesssim \mu_k$  and  $|f(x)| \lesssim |x|$  then this condition reads

$$\int \frac{|f(x)|}{\ln(e + |x|)} dx < \infty.$$

This is the case of the example given above

$$f(x) := (1 + |x|)^{-n} (\ln(e + |x|))^{-1}.$$

*Proof.* We will only sketch the proof since it follows the same lines as the proof of Theorem 3. When revisiting this proof, we cut now the integral

$$\int_{\mu_k}^{2|f_k(x)|} \frac{\Psi(k, s)}{s^2} ds = \int_{\mu_k}^{\min(|k|, 2|f_k(x)|)} \frac{\Psi(k, s)}{s^2} ds + \chi_{\{2|f_k(x)| > |k|\}} \int_{|k|}^{2|f_k(x)|} \frac{\Psi(k, s)}{s^2} ds.$$

The second integral is treated as previously. For the first one, we use

$$\frac{\Psi(k, s)}{s^2} \simeq \frac{1}{s \ln(e + |k|)},$$

which leads to the last term.  $\square$

#### 4. THE HARDY SPACE $\mathcal{H}_{\log}$

We finally give conditions for  $\mathcal{H}_{\log}$  and prove the following theorem.

**Theorem 4.** Let  $T_\theta$  be defined as in (2.13). Let  $f$  be an integrable function. Then  $T_\theta f$  is a function of  $\mathcal{H}_{\log}(\mathbb{R}^n)$  if

$$(4.1) \quad \int |f(x)| (1 + \ln_+ \ln(e + |f(x)|) + \ln_+ \ln(e + |x|)) dx < \infty.$$

Moreover, if  $f$  is nonnegative and  $T_\theta f$  is in  $\mathcal{H}_{\log}(\mathbb{R}^n)$ , then (4.1) holds.

As for  $\mathcal{H}^1(\mathbb{R}^n)$ , a function  $f$  of integral 0 and satisfying the condition (4.1) belongs to  $\mathcal{H}_{\log}(\mathbb{R}^n)$ .

*Proof.* We write as before  $T_\theta f = g + h$ . To prove that the function  $h$  belongs to  $\mathcal{H}_{\log}(\mathbb{R}^n)$ , we do as in the proof of Theorem 2 and look first at the  $L_{\log}$  norm of each  $Mh_k$  on  $k + 2Q$ . On this set, we consider the two terms in  $h_k$  separately. We conclude directly for the characteristic function whose sum gives a  $L^1$  term, while, for  $f_k$  we use the estimate (3.3). It remains to look at the integral of  $Mh_k$  outside  $k + 2Q$ . We use the fact that  $\Psi(x, t) \leq t$ , and may consider  $L^1$  norms as in Proposition 1. The proof is identical.

We finally want to estimate  $g$ . In this case, Proposition 2 is replaced by the following one, which may be seen as its generalization.

**Proposition 4.** *Let  $\omega$  be a radial nonincreasing function on  $\mathbb{R}^n$ , which satisfies*

$$(4.2) \quad \int_{\mathbb{R}^n} \frac{\omega(x)}{(1+|x|)^{n+1}} dx < \infty.$$

*Let  $\lambda_j$  be a sequence of real numbers indexed by  $\mathbb{Z}^n$ , and let  $g := \sum g_j$ , with  $g_j := \lambda_j(\chi_{Q_j} - \theta)$ . Then  $g$  is in  $\mathcal{H}_\omega^1(\mathbb{R}^n)$  if*

$$(4.3) \quad \sum_j \lambda_j \Omega(|j|) < \infty$$

*where  $\Omega$  is the function*

$$\Omega(R) := \omega(Q_0) + \int_{1 < |y| < R} \frac{\omega(y)}{|y|^n} dy.$$

*Moreover, if the  $\lambda_j$ 's are nonnegative, it is a necessary condition for having  $g$  in  $\mathcal{H}_\omega^1(\mathbb{R}^n)$ .*

*Proof.* We first remark that the function  $\Omega$  is doubling, that is, there exists  $C$  such that  $\Omega(2R) \leq C\Omega(R)$ . Next we adapt Lemma 2. Inequality (2.19) for  $a_j = \chi_{Q_j} - \theta$  is replaced by

$$(4.4) \quad \begin{aligned} \int |\mathcal{M}_\varphi a_j| \omega(x) dx &\leq \omega(Q_0) + \omega(Q_j) + \int_{|x| > 2|j|} \frac{C|j|\omega(x)}{|x|^{n+1}} dx \\ &+ \int_{1 < |x| < 2|j|} \frac{C\omega(x)}{|x|^n} dx + \int_{1 < |x-j| < 3|j|} \frac{C\omega(x)}{|x-j|^n} dx. \end{aligned}$$

From the facts that  $\omega$  is radial and nonincreasing, it follows that  $\omega(Q_0) \geq \omega(Q_j)$  and

$$\int_{1 < |x-j| < 3|j|} \frac{\omega(x)}{|x-j|^n} dx \leq \int_{1 < |x| < 3|j|} \frac{\omega(x)}{|x|^n} dx \lesssim \Omega(|j|).$$

It remains to consider the term

$$\int_{|x| > 2|j|} \frac{|j|\omega(x)}{|x|^{n+1}} dx = \int_{|x| > 2} \frac{\omega(|j|x)}{|x|^{n+1}} dx \lesssim 1.$$

This gives the sufficient condition.

The necessary condition is obtained easily, using only the fact that

$$\int (\mathcal{M}_\varphi a_j) \omega(x) dx \gtrsim \omega(Q_0) + \int_{1 < |x| < |j|/6} \frac{\omega(x)}{|x|^n} dx.$$

□

In particular, when  $\omega(x) = \ln(e+|x|)^{-1}$ , we have  $\Omega(x) \simeq 1 + \ln(\ln(e+|x|))$ , which allows to conclude for the sufficient condition in Theorem 4.

It remains to prove the necessary condition in Theorem 4. Let us assume that  $f$  is a nonnegative function such that  $T_\theta f$  belongs to  $\mathcal{H}_{\log}(\mathbb{R}^n)$ . Then  $T_\theta f \in \mathfrak{h}_{\log}(\mathbb{R}^n)$ , and as  $\theta \in \mathfrak{h}^1(\mathbb{R}^n) \subset \mathfrak{h}_{\log}(\mathbb{R}^n)$ , we conclude that  $f$  itself belongs to  $\mathfrak{h}_{\log}(\mathbb{R}^n)$ . Hence by Theorem 3,  $f$  satisfies (3.1). If we cut  $f$  into  $g + h$  as before, Lemma 3 implies that the function  $h$  is in  $\mathcal{H}_{\log}(\mathbb{R}^n)$  (apply estimate (3.3) to each  $h_k$ ). Hence,  $g$  is also in  $\mathcal{H}_{\log}(\mathbb{R}^n)$ . To conclude, we claim that we can work on weighted inequalities since, as  $|g| \leq 2\|f\|_1 = 2$ , we have

$$\int_{\mathbb{R}^n} \Psi(x, \mathcal{M}_\varphi g) dx \simeq \int_{\mathbb{R}^n} \mathcal{M}_\varphi g(x) \frac{dx}{\ln(e+|x|)}.$$

Hence,  $g$  belongs to  $\mathcal{H}_\omega^1(\mathbb{R}^n)$  where  $\omega(x) = \ln(e+|x|)^{-1}$ . Then, it suffices to apply by Proposition 4, with  $\lambda_k = \int f_k dx$  to get

$$\sum_k \lambda_k (1 + \ln(\ln(e+|k|))) < \infty$$

which is equivalent to

$$\int f(x)(1 + \ln \ln(e+|x|)) dx < \infty.$$

It allows to conclude.  $\square$

## 5. CONCLUDING REMARKS

For the estimates in  $\mathfrak{h}_{\log}(\mathbb{R}^n)$  we have allowed  $f$  not to be integrable. This could also be done for the space  $\mathcal{H}_{\log}(\mathbb{R}^n)$ , even if only  $\left(\int f_k dx\right) \theta$  makes sense for  $f$  locally integrable. Other possible generalizations concern the weighted spaces. Part of this study can be generalized to other Hardy spaces of Musielak type. But formulas seem to be much more complicate and have less interest.

## REFERENCES

- [1] BAKAS, O., POTT, S., RODRIGUEZ-LOPEZ, S., SOLA, A.: Notes on  $H_{\log}$  structural properties, dyadic variants, and bilinear  $H^1 - BMO$  mappings. Arkiv för Matematik, to appear. Available at <https://arxiv.org/abs/2012.02872>.
- [2] BONAMI, A., GRELLIER, S., KY, L. D.: Paraproducts and products of functions in  $BMO(\mathbb{R}^n)$  and  $\mathcal{H}^1(\mathbb{R}^n)$  through wavelets. Journal de math. pures et appl. (2012) Vol. 97, Issue 3, 230–241.
- [3] BOUCHUT, F. : Private communication.
- [4] JANSON, S.: Generalizations of Lipschitz spaces and an application to Hardy spaces and bounded mean oscillation. *Duke Math. J.* **47**, no. 4 (1980), 959 – 982.

- [5] KY, L. D.: New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators. *Integ. Equat. Oper. Theor.* **78** (2014), no. 1, 115-150.
- [6] MAEDA, F.Y., MIZUTA, Y., OHNO, T. ET AL. Duality of Non-Homogeneous Central Herz-Morrey-Musielak-Orlicz Spaces. *Potential Anal* 47, 447-460 (2017).
- [7] MUSIELAK, J. Orlicz Spaces and Modular Spaces, Lecture Notes in Math. book series, Vol **1034**, Springer-Verlag Berlin, 1986.
- [8] STEIN, E. M.: Note on the class  $L_{\log L}$ . *Studia Math.* **32**, no. 3 (1969), 305-310.
- [9] STEIN, E. M.: Singular integrals and differentiability properties of functions. Princeton Math. Series Vol. **30**. Princeton University Press, 2016.
- [10] STEIN, E. M. & MURPHY, T.: Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton University Press, 1993.

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