



**HAL**  
open science

# A characterization of necessary and possible interaction among more than two criteria in a Choquet integral model

Paul Alain Kaldjob Kaldjob, Paul Alain Kaldjob Kaldjob, Brice Mayag, Denis Bouyssou

## ► To cite this version:

Paul Alain Kaldjob Kaldjob, Paul Alain Kaldjob Kaldjob, Brice Mayag, Denis Bouyssou. A characterization of necessary and possible interaction among more than two criteria in a Choquet integral model. 2022. hal-03557426

**HAL Id: hal-03557426**

**<https://hal.science/hal-03557426>**

Preprint submitted on 4 Feb 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A characterization of necessary and possible interaction among more than two criteria in a Choquet integral model

Paul Alain Kaldjob Kaldjob<sup>a,\*</sup>, Brice Mayag<sup>a</sup>, Denis Bouyssou<sup>a</sup>

<sup>a</sup>Paris-Dauphine University, PSL Research University, CNRS, LAMSADE, UMR 7243, 75016 Paris, France

---

## Abstract

This paper studies the notion of interaction between criteria in a general Choquet integral model. In the framework of generalized binary alternatives, we give a necessary and sufficient condition for them to be representable by a general Choquet integral model. Using this condition, we show that it is always possible to choose from all the numerical representations, one for which all the Shapley interaction indices are strictly positive. Assuming that there is possibly to have an indifference relation, we give a sufficient condition on ordinal information so that positive interaction is always possible into all subsets of generalized criteria in general Choquet integral model. Outside the framework of binary alternatives, we propose a linear program allowing to test whether the interpretation of the interaction indices is ambivalent or not. We illustrate our results with examples.

**Keywords:** Binary alternatives, Interaction index, General Choquet integral model, Shapley interaction indices.

**JEL Codes:** C44

---

## 1. Introduction

In Multiple Criteria Decision Making (MCDM), the additive value function model is widely used and have solid theoretical foundations (Krantz et al., 1971). The additive model implies the independence hypothesis (Bouyssou and Pirlot, 2016). In some context, this hypothesis is considered to be restrictive (Grabisch and Labreuche, 2004). Thus, several other models that do not require the independence hypothesis have been developed, including the Choquet integral model. Its use in MCDM was popularized through the work of Michel Grabisch (Grabisch, 1995, 1996). The Choquet integral model is now considered as a central tool in MCDM when one wants to escape the independence hypothesis (Grabisch, 2016; Grabisch and Labreuche, 2004, 2010). The use of these functions allow

---

\*corresponding author

*Email addresses:* paul-alain.kaldjob-kaldjob@dauphine.psl.eu (Paul Alain Kaldjob Kaldjob ), brice.mayag@dauphine.psl.eu (Brice Mayag ), denis.bouyssou@dauphine.psl.eu (Denis Bouyssou)

to assume that the recoded criteria are “commensurate” and, hence the application of the Choquet integral model is meaningful (Grabisch et al., 2003).

In the literature, the particular case of 2-additive capacities has received much attention (Grabisch, 1997). This case is often considered as a useful compromise between a additive model implying independence and a general Choquet integral model (i.e., using a capacity that is not restricted to be 2-additive) raising difficult elicitation issues (Grabisch et al., 2008). This model is often used in applications. Such as the evaluation of discomfort (Grabisch et al., 2002), the performance measurement in supply chains (Berrah and Clivillé, 2007; Clivillé et al., 2007), and the complex system design (Pignon and Labreuche, 2007). In this paper, we want to go beyond the 2-additive case, to the general case, although the interpretation remains difficult.

This paper generalizes our preliminary results published in (Kaldjob Kaldjob et al., 2020), by assuming that the preference information set contains possible indifference relations. Outside the framework of binary alternatives, we have also proposed linear programs to test the existence of necessary and possible interactions.

In (Mayag et al., 2011) there are two necessary and sufficient conditions for a preferential information on set of binary alternatives to be represented by a 2-additive Choquet integral model. This result is extended to general model of the Choquet integral. Indeed, our second result shows that the nonexistence of a strict cycle in the preferential information set is a necessary and sufficient condition for it to be representable by a general Choquet integral.

In (Mayag and Bouyssou, 2019) it is proven that in the framework of binary alternatives, if the preferential information contains no indifference, and is representable by a 2-additive Choquet model, then we can choose among these representations one for which all Shapley interaction indices between two criteria are strictly positive. We extend also this result. Indeed, under the conditions of our second result, we show that in the framework of generalized binary alternatives, if the preference information contains no indifference, it is always possible to represent it by a general Choquet integral model which all interaction indices are strictly positive.

We extend our results when it is possible to have an indifference in a set of preference information. Hence, we give a sufficient condition on ordinal information so that positive interaction is always possible into all subsets of criteria in general Choquet integral model. In practice, many cases do not fall under binary alternatives, so we propose a linear program inspired by (Mayag and Bouyssou, 2019) allowing to test whether the interpretation of the interaction indices is ambivalent or not.

This paper is organized as follows. After having recalled in Section 2 some basic elements on the model of the Choquet integral in MCDM, in Section 3, we extend the concept of necessary and possible interaction introduced in (Mayag and Bouyssou, 2019). In Sections 4 and 5, we give our mains results. Outside the framework of binary alter-

natives, Section 6 proposes a linear programming model allowing to test the existence of necessary and possible interactions. We illustrate our results with an example.

## 2. Notations and preliminaries

### 2.1. The framework

Let  $X$  be a set of alternatives evaluated on a set of  $n$  criteria  $N = \{1, 2, \dots, n\}$ . The set of all alternatives  $X$  is assumed to be a subset of a Cartesian product  $X_1 \times X_2 \times \dots \times X_n$ , where  $X_i$  is the set of possible levels on each criterion  $i \in N$ . The criteria are recoded numerically using, for all  $i \in N$ , a function  $u_i$  from  $X_i$  into  $\mathbb{R}$ . Using these functions we assume that the various recoded criteria are commensurate and, hence, the application of the Choquet integral model is meaningful (Grabisch et al., 2003).

We assume that the DM is able to identify on each criterion  $i \in N$  two reference levels  $1_i$  and  $0_i$ :

- the level  $0_i$  in  $X_i$  is considered as a neutral level and we set  $u_i(0_i) = 0$ ,
- the level  $1_i$  in  $X_i$  is considered as a good level and we set  $u_i(1_i) = 1$ .

For all  $x = (x_1, \dots, x_n) \in X$  and  $S \subseteq N$ , we will sometimes write  $u(x)$  as a shorthand for  $(u_1(x_1), \dots, u_n(x_n))$  and we define the alternatives  $a_S = (1_S, 0_{-S})$  of  $X$  such that  $a_i = 1_i$  if  $i \in S$  and  $a_i = 0_i$  otherwise. We assume that the set  $X$  of alternatives contains all the binary alternatives  $a_S, S \subseteq N$ .

### 2.2. Choquet integral

The Choquet integral (Grabisch, 1997, 2016; Grabisch and Labreuche, 2004) is an aggregation function known in MCDM as a tool generalizing the weighted arithmetic mean. The Choquet integral uses the notion of capacity (Choquet, 1954; Pignon and Labreuche, 2007) defined as a function  $\mu$  from the powerset  $2^N$  into  $[0, 1]$  such that:

- $\mu(\emptyset) = 0$ ,
- $\mu(N) = 1$ ,
- For all  $S, T \in 2^N$ ,  $[S \subseteq T \implies \mu(S) \leq \mu(T)]$  (monotonicity).

For an alternative  $x = (x_1, \dots, x_n) \in X$ , the expression of the Choquet integral w.r.t. a capacity  $\mu$  is given by:

$$\begin{aligned} C_\mu(u(x)) &= C_\mu(u_1(x_1), \dots, u_n(x_n)) \\ &= \sum_{i=1}^n \left[ u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)}) \right] \mu(N_{\sigma(i)}), \end{aligned}$$

where  $\sigma$  is a permutation on  $N$  such that:  $N_{\sigma(i)} = \{\sigma(i), \dots, \sigma(n)\}$ ,  $u_{\sigma(0)}(x_{\sigma(0)}) = 0$  and  $u_{\sigma(1)}(x_{\sigma(1)}) \leq u_{\sigma(2)}(x_{\sigma(2)}) \leq \dots \leq u_{\sigma(n)}(x_{\sigma(n)})$ .

Our work is based on the set  $\mathcal{B}^g$  which we define as follows.

**Definition 1.** *The set of **generalized binary alternatives** is defined by*

$$\mathcal{B}^g = \{a_S = (1_S, 0_{-S}) : S \subseteq N\}.$$

**Remark 1.** *For all  $S \subseteq N$ , we have  $C_\mu(u(a_S)) = \mu(S)$ .*

We suppose that the DM can give his preferences by comparing some elements of  $\mathcal{B}^g$ . We then obtain the binary relations  $P$  and  $I$  defined as follows.

**Definition 2.** *An **ordinal preference information**  $\{P, I\}$  on  $\mathcal{B}^g$  is given by:*

$$P = \{(x, y) \in \mathcal{B}^g \times \mathcal{B}^g : \text{DM strictly prefers } x \text{ to } y\},$$

$$I = \{(x, y) \in \mathcal{B}^g \times \mathcal{B}^g : \text{DM is indifferent between } x \text{ and } y\}.$$

We add to this ordinal preference information a binary relation  $M$  modeling the monotonicity relations between generalized binary alternatives, and allowing us to ensure the satisfaction of the monotonicity condition:  $[S \subseteq T \implies \mu(S) \leq \mu(T)]$ .

**Definition 3.** *For all  $a_S, a_T \in \mathcal{B}^g$ ,  $a_S M a_T$  if  $[\text{not}(a_S(P \cup I)a_T) \text{ and } S \supseteq T]$ .*

**Remark 2.**  $a_S M a_T \implies C_\mu(u(a_S)) \geq C_\mu(u(a_T))$ .

In the sequel, we need the following two definitions classic in graph theory (Rudolf and Pilz, 1984).

**Definition 4.** *There exists a **strict cycle** in  $(P \cup M)$  if there exists elements  $x_0, x_1, \dots, x_r$  of  $\mathcal{B}^g$  such that  $x_0(P \cup M)x_1(P \cup M) \dots (P \cup M)x_r(P \cup M)x_0$  and for a least one  $i \in \{0, \dots, r-1\}$ ,  $x_i P x_{i+1}$ .*

**Definition 5.**  $x TC_{I \cup M} y$  if there exists elements  $x_0, x_1, \dots, x_r$  of  $\mathcal{B}^g$  such that  $x = x_0(I \cup M)x_1(I \cup M) \dots (I \cup M)x_r = y$ . Hence,  $TC_{I \cup M}$  is the transitive closure of the binary relation  $I \cup M$ .

### 2.3. Interaction index

In this section, we recall the definition of the interaction index and we give a lemma of decomposition which help us for the proofs.

**Definition 6.** The interaction (Grabisch, 1997) index w.r.t. a capacity  $\mu$  is defined by: for all  $A \subseteq N$ ,

$$I_A^\mu = \sum_{K \subseteq N \setminus A} \frac{(n-k-a)!k!}{(n-a+1)!} \sum_{L \subseteq A} (-1)^{a-\ell} \mu(K \cup L),$$

where  $\ell = |L|$ ,  $k = |K|$  and  $a = |A|$ .

We give a decomposition of the interaction index, it will be useful to us later.

**Remark 3.** Given a capacity  $\mu$  and a subset  $A \subseteq N$ , we can rewrite the general interaction index as follows:

$$I_A^\mu = \sum_{K \subseteq N \setminus A} \frac{(n-k-a)!k!}{(n-a+1)!} \Delta_A^{\mu(K)},$$

where  $\ell = |L|$ ,  $k = |K|$  and  $a = |A|$  with  $\Delta_A^{\mu(K)} = \sum_{L \subseteq A} (-1)^{a-\ell} \mu(K \cup L)$ .

The following lemma gives a decomposition of  $\Delta_A^{\mu(K)}$  (we assume that 0 is an even number). We will use it in the proof of the Proposition 3.

**Lemma 1.** For all  $A \subseteq N$ , for all  $K \subseteq N \setminus A$ ,

$$\Delta_A^{\mu(K)} = \sum_{\substack{p=0, \\ p \text{ even}}}^a \left[ \sum_{\substack{L \subseteq A, \\ \ell=a-p}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right].$$

*Proof.* We distinguish two cases, depending on the parity of  $a$ .

- If  $a$  is even.

$$\begin{aligned} \Delta_A^{\mu(K)} &= \sum_{L \subseteq A} (-1)^{a-\ell} \mu(K \cup L) \\ &= \left[ \sum_{\substack{L \subseteq A, \\ \ell=a}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-1}} \mu(K \cup L) \right] + \left[ \sum_{\substack{L \subseteq A, \\ \ell=a-2}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-3}} \mu(K \cup L) \right] \\ &+ \dots + \left[ \sum_{\substack{L \subseteq A, \\ \ell=2}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=1}} \mu(K \cup L) \right] + \left[ \sum_{\substack{L \subseteq A, \\ \ell=0}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=-1}} \mu(K \cup L) \right], \end{aligned}$$

where  $\sum_{\substack{L \subseteq A, \\ \ell=-1}} \mu(K \cup L) = \sum_{L \subseteq \emptyset} \mu(K \cup L) = 0$ .

Hence we have,

$$\Delta_A^{\mu(K)} = \sum_{\substack{p=0, \\ p \text{ even}}}^a \left[ \sum_{\substack{L \subseteq A, \\ \ell=a-p}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right].$$

- If  $a$  is odd.

$$\begin{aligned} \Delta_A^{\mu(K)} &= \sum_{L \subseteq A} (-1)^{a-\ell} \mu(K \cup L) \\ &= \left[ \sum_{\substack{L \subseteq A, \\ \ell=a}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-1}} \mu(K \cup L) \right] + \left[ \sum_{\substack{L \subseteq A, \\ \ell=a-2}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-3}} \mu(K \cup L) \right] \\ &+ \dots + \left[ \sum_{\substack{L \subseteq A, \\ \ell=1}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=0}} \mu(K \cup L) \right]. \end{aligned}$$

Hence we have,

$$\begin{aligned} \Delta_A^{\mu(K)} &= \sum_{\substack{p=0, \\ p \text{ even}}}^{a-1} \left[ \sum_{\substack{L \subseteq A, \\ \ell=a-p}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right] \\ &= \sum_{\substack{p=0, \\ p \text{ even}}}^a \left[ \sum_{\substack{L \subseteq A, \\ \ell=a-p}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right], \end{aligned}$$

since  $a$  is odd but  $p$  is even. □

**Remark 4.** Let  $N$  be the set of criteria,  $A \subseteq N$ ,  $|A| = a$ ,  $K \subseteq N \setminus A$  and  $1 \leq t \leq a$ .

Then we have:  $\sum_{\substack{L \subseteq A, \\ \ell=t}} \sum_{i \in L} \mu(K \cup L \setminus \{i\}) = (a-t+1) \sum_{\substack{L \subseteq A, \\ \ell=t-1}} \mu(K \cup L)$ .

In the next section, we extend the concept of necessary and possible interaction introduced in (Mayag and Bouyssou, 2019) in the case of 2-additive Choquet integral model.

### 3. Necessary and possible interaction

Once the DM has compared a number of alternatives in terms of strict preferences ( $P$ ) or indifference ( $I$ ), the following definition tells us when this ordinal preference information is representable by a Choquet integral model.

**Definition 7.** An ordinal preference information  $\{P, I\}$  on  $X$ , is representable by a general Choquet integral model if we can find a capacity  $\mu$  such that: for all  $x, y \in X$ , we have:

$$x P y \implies C_\mu(u(x)) > C_\mu(u(y));$$

$$x I y \implies C_\mu(u(x)) = C_\mu(u(y)).$$

The set of all capacities that can be used to represent the ordinal preference information  $\{P, I\}$  at hand will be denoted by  $C_{\text{Pref}}(P, I)$ . When there is no ambiguity on the underlying ordinal preference information, we will simply write  $C_{\text{Pref}}$ .

The following definition of necessary and possible interactions will be central in the rest of this text. It is inspired from (Mayag and Bouyssou, 2019) where it was given in the case of 2-additive Choquet integral model.

**Definition 8.** *Let  $A$  be a subset of  $N$  and  $\{P, I\}$  an ordinal preference information. We say that:*

1. *There exists a possible positive (resp. null, negative) interaction among the elements of  $A$  if there exists  $\mu \in C_{\text{Pref}}$  such that  $I_A^\mu > 0$  (resp.  $I_A^\mu = 0$ ,  $I_A^\mu < 0$ );*
2. *There exists a necessary positive (resp. null, negative) interaction among the elements of  $A$  if  $I_A^\mu > 0$  (resp.  $I_A^\mu = 0$ ,  $I_A^\mu < 0$ ) for all  $\mu \in C_{\text{Pref}}$ .*

**Remark 5.** *Let  $A$  be a subset of criteria.*

- *If there exists a necessary positive (resp. null, negative) interaction among the elements of  $A$ , then there exists a possible positive (resp. null, negative) interaction among the elements of  $A$ .*
- *If there is no necessary positive (resp. null, negative) interaction among the elements of  $A$ , then there exists a possible negative or null (resp. positive or negative, positive or null) interaction among the elements of  $A$ .*

Let  $A$  be a subset of  $N$  and  $\{P, I\}$  an ordinal preference information. If we have a possible but not necessary interaction, then the interpretation is difficult because it depends on the capacity chosen in  $C_{\text{Pref}}(P, I)$ . Indeed, the interpretation of the interaction only makes sense in the case of the necessary interaction.

In the next section we present our results when  $\{P, I\}$  contains no indifference.

## 4. Results when $I$ is empty

In all this section, we suppose that the ordinal preference information  $\{P, I\}$  contains no indifference. This condition is likely to be met in most applications: indifference is indeed much less likely between alternative than strict preference, unless alternatives have been



specially designed to be indifferent. We started with the special case  $I$  empty in this section before the general case  $I$  any in next section since the second case uses the results of the first case.

We start this section with a simple observation. In the discrete case, when there is no indifference, there are “holes” between all values of  $C_\mu(u(x))$ . If we slightly modify the capacity  $\mu$ , so as to keep all values  $C_\mu(u(x))$  within these holes, we find that null interactions is never necessary. We formalize this idea by Proposition 1. Its proof is elementary: when there is no indifference, it simply exploits the fact that when the structure has the “holes”, it is possible to slightly modify the representing model while remaining in the “holes”.

**Proposition 1.** *Let  $\{P, I\}$  be an ordinal preference information on a set  $Y \subseteq X$  such that  $\{P, I\}$  is representable by a general Choquet integral model. If the relation  $I$  is empty then there is no necessary null interaction.*

*Proof.* We only give the proof in the special case in which the DM has provided a linear order on a subset  $Y \subseteq X$ . It is easy to modify the proof to cover the other cases. Suppose that we have a general Choquet integral model representing the preference relation  $P$  that linearly orders the set  $Y = \{x^1, x^2, \dots, x^p\}$ . We suppose w.l.o.g. that  $Y = \{x^1, x^2, \dots, x^p\}$  and that  $x^p P x^{p-1} P \dots P x^1$  with  $p \in \mathbb{N}$ ,  $p \geq 1$ .

Let us suppose that this information can be represented by a general Choquet integral model using a capacity  $\mu$  for which  $I_A^\mu = 0$  where  $A$  is a set at a least two criteria. Let us first show that this possible null interaction is not necessary.

Let us define the capacity  $\beta_\varepsilon$  by :

$$\beta_\varepsilon(S) = \begin{cases} \frac{1}{1+\varepsilon} \mu(S) & \text{if } S \subsetneq N, \\ 1 & \text{if } S = N. \end{cases}$$

where  $\varepsilon$  is a strictly positive real number to be determined as follows.

$$\begin{aligned} \text{We have } C_{\beta_\varepsilon}(u(x)) &= \sum_{i=1}^n \left[ u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)}) \right] \beta_\varepsilon(X_{\sigma(i)}) \\ &= u_{\sigma(1)}(x_{\sigma(1)}) \beta_\varepsilon(N) + \sum_{i=2}^n \left( u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)}) \right) \beta_\varepsilon(X_{\sigma(i)}) \\ &= u_{\sigma(1)}(x_{\sigma(1)}) + \frac{1}{1+\varepsilon} \sum_{i=2}^n \left( u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)}) \right) \mu(X_{\sigma(i)}) \\ &= \frac{1}{1+\varepsilon} \left[ u_{\sigma(1)}(x_{\sigma(1)}) + \sum_{i=2}^n \left( u_{\sigma(i)}(x_{\sigma(i)}) - u_{\sigma(i-1)}(x_{\sigma(i-1)}) \right) \mu(X_{\sigma(i)}) \right] + \frac{\varepsilon}{1+\varepsilon} u_{\sigma(1)}(x_{\sigma(1)}) \\ &= \frac{1}{1+\varepsilon} C_\mu(u(x)) + \frac{\varepsilon}{1+\varepsilon} u_{\sigma(1)}(x_{\sigma(1)}). \end{aligned}$$

$$\text{Therefore } C_{\beta_\varepsilon}(u(x)) = \frac{1}{1+\varepsilon} \left[ C_\mu(u(x)) + \varepsilon u_{\sigma(1)}(x_{\sigma(1)}) \right].$$

We then have  $C_{\beta_\varepsilon}(u(x^{i+1})) - C_{\beta_\varepsilon}(u(x^i)) = \frac{1}{1+\varepsilon} \left[ \left( C_\mu(u(x^{i+1})) - C_\mu(u(x^i)) \right) + \varepsilon \left( u_{\sigma(1)}(x_{\sigma(1)}^{i+1}) - u_{\gamma(1)}(x_{\gamma(1)}^i) \right) \right] \forall i = 1, \dots, p-1$ .

We are looking for  $\varepsilon$  such that  $C_{\beta_\varepsilon}(u(x^{i+1})) - C_{\beta_\varepsilon}(u(x^i)) > 0$  for all  $i = 1, \dots, p-1$ .  
 $C_{\beta_\varepsilon}(u(x^{i+1})) - C_{\beta_\varepsilon}(u(x^i)) > 0 \iff \varepsilon \left( u_{\sigma(1)}(x_{\sigma(1)}^{i+1}) - u_{\gamma(1)}(x_{\gamma(1)}^i) \right) > - \left( C_\mu(u(x^{i+1})) - C_\mu(u(x^i)) \right)$ .

We consider the set  $\Omega = \{i = 1, \dots, p-1 : u_{\sigma(1)}(x_{\sigma(1)}^{i+1}) - u_{\gamma(1)}(x_{\gamma(1)}^i) < 0\}$ .

- If  $\Omega = \emptyset$ , then for all  $i = 1, \dots, p-1$ , we have  $u_{\sigma(1)}(x_{\sigma(1)}^{i+1}) - u_{\gamma(1)}(x_{\gamma(1)}^i) \geq 0$ .  
Thus for all  $i = 1, \dots, p-1$ ,  $C_{\beta_\varepsilon}(u(x^{i+1})) - C_{\beta_\varepsilon}(u(x^i)) > 0 \quad \forall \varepsilon > 0$ .
- If  $\Omega \neq \emptyset$ , we choose  $\varepsilon$  such that  $0 < \varepsilon < \min_{i \in \Omega} \left( \frac{C_\mu(u(x^i)) - C_\mu(u(x^{i+1}))}{u_{\sigma(1)}(x_{\sigma(1)}^{i+1}) - u_{\gamma(1)}(x_{\gamma(1)}^i)} \right)$  in such a way that  $C_{\beta_\varepsilon}(u(x^{i+1})) - C_{\beta_\varepsilon}(u(x^i)) > 0$  for all  $i = 1, \dots, p-1$ .

So in both cases we can choose  $\varepsilon = \frac{1}{2} \min_{i \in \Omega} \left( \frac{C_\mu(u(x^i)) - C_\mu(u(x^{i+1}))}{u_{\sigma(1)}(x_{\sigma(1)}^{i+1}) - u_{\gamma(1)}(x_{\gamma(1)}^i)} \right)$  such that the information  $\{P, I\}$  is representable by the Choquet integral model  $C_{\beta_\varepsilon}$ .

Moreover we have :

$$\begin{aligned}
I_A^{\beta_\varepsilon} &= \sum_{K \subseteq N \setminus A} \frac{(n-k-a)!k!}{(n-a+1)!} \sum_{L \subseteq A} (-1)^{a-\ell} \beta_\varepsilon(K \cup L) \\
&= \frac{(n-a)!}{(n-a+1)!} \sum_{L \subseteq A} (-1)^{a-\ell} \beta_\varepsilon((N \setminus A) \cup L) + \sum_{K \subseteq N \setminus A} \frac{(n-k-a)!k!}{(n-a+1)!} \sum_{L \subseteq A} (-1)^{a-\ell} \beta_\varepsilon(K \cup L) \\
&= \frac{(n-a)!}{(n-a+1)!} \beta_\varepsilon(N) + \frac{(n-a)!}{(n-a+1)!} \sum_{L \subseteq A} (-1)^{a-\ell} \beta_\varepsilon((N \setminus A) \cup L) + \\
&\quad \sum_{K \subseteq N \setminus A} \frac{(n-k-a)!k!}{(n-a+1)!} \sum_{L \subseteq A} (-1)^{a-\ell} \beta_\varepsilon(K \cup L) \\
&= \frac{(n-a)!}{(n-a+1)!} + \frac{1}{1+\varepsilon} \frac{(n-a)!}{(n-a+1)!} \sum_{L \subseteq A} (-1)^{a-\ell} \mu((N \setminus A) \cup L) + \\
&\quad \frac{1}{1+\varepsilon} \sum_{K \subseteq N \setminus A} \frac{(n-k-a)!k!}{(n-a+1)!} \sum_{L \subseteq A} (-1)^{a-\ell} \mu(K \cup L) \\
&= \frac{1}{1+\varepsilon} I_A^\mu + \frac{\varepsilon}{1+\varepsilon} \frac{1}{n-a+1}
\end{aligned}$$

As  $I_A^\mu = 0$ , we have  $I_A^{\beta_\varepsilon} = \frac{\varepsilon}{1+\varepsilon} \frac{1}{n-a+1} > 0$ .

Thus there exists a possible positive interaction into  $A$ . Hence there is no null interaction into  $A$ .

Note that, if the modified capacity shows a possible null interaction into a set of criteria  $A$ , the above process can be repeated. This will lead to exhibit a capacity in which there are only positive interactions. Hence, null interactions are never necessary when  $I = \emptyset$ .  $\square$

**Example 1.**  $N = \{1, 2, 3\}$ ,  $X = \{x^1, x^2, x^3, x^4\}$ ,  $x^1 = (6, 11, 9)$ ,  $x^2 = (6, 13, 7)$ ,  $x^3 = (16, 11, 9)$ ,  $x^4 = (16, 13, 7)$  and  $P = \{(x^4, x^3), (x^2, x^1)\}$ .

The ordinal preference information  $\{P, I\}$  is representable by the capacity  $\mu$  (such that  $I_{13}^\mu = 0$ ) given by Table 1 and Choquet integral corresponding is given by Table 2.

$S$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$\mu(S)$	0	0.5	0	1	0	0.5	1

Table 1: A capacity  $\mu \in C_{\text{Pref}}$  such that  $I_{13}^\mu = 0$ .

$x$	$x^4$	$x^3$	$x^2$	$x^1$
$C_\mu(u(x))$	13	11	9.5	8.5

Table 2: Choquet integral corresponding at previous capacity  $\mu$ .

The linear order induced by the Choquet integral is  $x^4 P x^3 P x^2 P x^1$ .

We have  $\Omega = \{3\}$  and  $\varepsilon = \frac{1}{2} \times \frac{C_\mu(u(x^3)) - C_\mu(u(x^4))}{u_{\sigma(1)}(x_{\sigma(1)}^4) - u_{\gamma(1)}(x_{\gamma(1)}^3)} = \frac{1}{2} \times \frac{11 - 13}{7 - 9} = 0.5$

A capacity  $\beta^\mu \in C_{\text{Pref}}$  such that  $I_{13}^{\beta^\mu} > 0$  and Choquet integral corresponding at  $\beta^\mu$  are respectively given by Table 3 and Table 4.

$S$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$\beta^\mu(S)$	0	1/3	0	2/3	0	1/3	1

Table 3: A capacity  $\beta^\mu \in C_{\text{Pref}}$  such that  $I_{13}^{\beta^\mu} > 0$ .

$x$	$x^4$	$x^3$	$x^2$	$x^1$
$C_{\beta^\mu}(u(x))$	11	10.33	8.33	7.66

Table 4: Choquet integral corresponding at previous capacity  $\beta^\mu$ .

Indeed,  $I_{13}^{\beta^\mu} = \frac{1}{1 + \varepsilon} I_{13}^\mu + \frac{\varepsilon}{1 + \varepsilon} \times \frac{1}{3 - 2 + 1} = \frac{0.5}{1.5} \times \frac{1}{2} = \frac{1}{6} > 0$ .

Now we will restrict ourselves to the case of generalized binary alternatives. The following two propositions are still in (Kaldjob Kaldjob et al., 2020). We give the proof again to facilitate the task for the reader. The following proposition gives a necessary and sufficient condition for an ordinal preference information on  $\mathcal{B}^g$  containing no indifference to be representable by a Choquet integral model.

**Proposition 2.** Let  $\{P, I\}$  be an ordinal preference information on  $\mathcal{B}^g$  such that  $I = \emptyset$ . Then,  $\{P, I\}$  is representable by a Choquet integral iff the binary relation  $(P \cup M)$  contains no strict cycle.

*Proof. Necessity.* Suppose that the ordinal preference information  $\{P, I\}$  on  $\mathcal{B}^g$  is representable by a Choquet integral. So there exists a capacity  $\mu$  such that  $\{P, I\}$  is representable by  $C_\mu$ .

If  $P \cup M$  contains a strict cycle, then there exists  $x_0, x_1, \dots, x_r$  on  $\mathcal{B}^g$  such that  $x_0(P \cup M)x_1(P \cup M) \dots (P \cup M)x_r(P \cup M)x_0$  and there exists two elements  $x_i, x_{i+1} \in \{x_0, x_1, \dots, x_r\}$  such that  $x_i P x_{i+1}$ . Since  $\{P, I\}$  is representable by  $C_\mu$ , therefore  $C_\mu(u(x_0)) \geq \dots \geq C_\mu(u(x_i)) > C_\mu(u(x_{i+1})) \geq \dots \geq C_\mu(u(x_0))$ , then  $C_\mu(u(x_0)) > C_\mu(u(x_0))$ , contradiction.

**Sufficiency.** Assume that  $(P \cup M)$  contains no strict cycle, then there exists  $\{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_m\}$  a partition of  $\mathcal{B}^g$ , build by using a suitable topological sorting on  $(P \cup M)$  (Gondran and Minoux, 1995).

We construct a partition  $\{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_m\}$  as follows:

$$\mathcal{B}_0 = \{x \in \mathcal{B}^g : \forall y \in \mathcal{B}^g, \text{ not } [x(P \cup M)y]\},$$

$$\mathcal{B}_1 = \{x \in \mathcal{B}^g \setminus \mathcal{B}_0 : \forall y \in \mathcal{B}^g \setminus \mathcal{B}_0, \text{ not } [x(P \cup M)y]\},$$

$$\mathcal{B}_i = \{x \in \mathcal{B}^g \setminus (\mathcal{B}_0 \cup \dots \cup \mathcal{B}_{i-1}) : \forall y \in \mathcal{B}^g \setminus (\mathcal{B}_0 \cup \dots \cup \mathcal{B}_{i-1}), \text{ not } [x(P \cup M)y]\}, \text{ for all } i = 1, 2, \dots, m.$$

Let us define the mapping  $\phi : \mathcal{B}^g \rightarrow \mathcal{P}(N)$ ,  $f : \mathcal{P}(N) \rightarrow \mathbb{R}$ ,  $\mu : 2^N \rightarrow [0, 1]$  as follows:

$$\phi(a_S) = S \text{ for all } S \subseteq N,$$

$$f(\phi(x)) = \begin{cases} 0 & \text{if } \ell = 0, \\ (2n)^\ell & \text{if } \ell \in \{1, 2, \dots, m\} \end{cases} \text{ for all } \ell \in \{0, 1, \dots, m\}, \forall x \in \mathcal{B}_\ell,$$

$$\mu(S) = \frac{f_S}{\alpha}, \text{ where } f_S = f(\phi(a_S)) \text{ for all } S \subseteq N \text{ and } \alpha = f_N = (2n)^m$$

Let  $a_S, a_T \in \mathcal{B}^g$  such that  $a_S P a_T$ . We show that  $C_\mu(u(a_S)) > C_\mu(u(a_T))$ .

Since  $a_S, a_T \in \mathcal{B}^g$  and  $\{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_m\}$  is a partition of  $\mathcal{B}^g$ , then there exists  $r, q \in \{0, 1, \dots, m\}$  such that  $a_S \in \mathcal{B}_r$ ,  $a_T \in \mathcal{B}_q$ . As  $a_S P a_T$ , then  $r > q$ , thus  $C_\mu(u(a_S)) = \mu(S) = \frac{f_S}{\alpha} = \frac{(2n)^r}{\alpha}$ .

- If  $q = 0$ , then  $a_T \in \mathcal{B}_0$  and  $a_S \in \mathcal{B}_r$  with  $r \geq 1$ .

As  $C_\mu(u(a_T)) = C_\mu(u(a_0)) = \mu(\emptyset) = 0$ , then  $C_\mu(u(a_S)) > C_\mu(u(a_T))$ .

- If  $q \geq 1$ ,  $C_\mu(u(a_T)) = \mu(T) = \frac{f_T}{\alpha} = \frac{(2n)^q}{\alpha}$ , since  $1 \leq q \leq m - 1$ .

But  $r > q$  therefore  $C_\mu(u(a_S)) = \frac{(2n)^r}{\alpha} > \frac{(2n)^q}{\alpha} = C_\mu(u(a_T))$ , then  $C_\mu(u(a_S)) > C_\mu(u(a_T))$ .

Hence, in both cases we have  $C_\mu(u(a_S)) > C_\mu(u(a_T))$ . Therefore  $\{P, I\}$  is representable by  $C_\mu$ .  $\square$

Given the ordinal preference information  $\{P, I\}$  on  $\mathcal{B}^g$  under the previous conditions, the following proposition shows that; it is always possible to choose in  $C_{\text{Pref}}(P, I)$ , one capacity allowing all the interaction indices are strictly positive. This result shows that positive interaction is always possible for all subsets of criteria in general Choquet integral model if the ordinal information does not contain indifference. In other words, when there is no indifference, negative and null interactions are not necessary.

**Proposition 3.** *Let  $\{P, I\}$  be an ordinal preference information on  $\mathcal{B}^g$  such that  $I = \emptyset$ , and  $(P \cup M)$  containing no strict cycle.*

*Then there exists a capacity  $\mu$  such that  $C_\mu$  represents  $\{P, I\}$  and for all  $A \subseteq N$ , such that  $|A| \geq 2$ , we have:  $I_A^\mu > 0$ .*

*Proof.* The partition  $\{\mathcal{B}_0, \dots, \mathcal{B}_m\}$  of  $\mathcal{B}^g$  and the capacity  $\mu$  are built as in proof of Proposition 2:  $\mu(S) = \frac{f_S}{\alpha}$ , where  $f_S = f(\phi(a_S))$  and  $\alpha = f_N = (2n)^m$ .

In Proposition 2 we have proved that  $C_\mu$  represents  $\{P, I\}$ .

Let  $A \subseteq N$  such that  $|A| \geq 2$ , to show that  $I_A^\mu > 0$ , we will prove that for all  $K \subseteq N \setminus A$ ,  $\sum_{L \subseteq A} (-1)^{a-\ell} \mu(K \cup L) > 0$ .

Let  $K \subseteq N \setminus A$ . According to Lemma 1 we have:

$$\Delta_A^{\mu(K)} = \sum_{L \subseteq A} (-1)^{a-\ell} \mu(K \cup L) = \sum_{\substack{p=0, \\ p \text{ even}}}^a \left[ \sum_{\substack{L \subseteq A, \\ \ell=a-p}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right].$$

Let  $L \subseteq A$ ,  $|L| = a - p$  with  $p \in \{0, 1, \dots, a\}$  and  $p$  even.

As  $K \cup L \supsetneq K \cup L \setminus \{i\}$  for all  $i \in L$ , then  $a_{K \cup L}(P \cup M) a_{K \cup L \setminus \{i\}}$ , hence there exists  $q \in \{1, 2, \dots, m\}$  such that  $a_{K \cup L} \in \mathcal{B}_q$  and  $\forall i \in L$ , there exists  $r_i \in \{0, 1, 2, \dots, m\}$  such that  $a_{K \cup L \setminus \{i\}} \in \mathcal{B}_{r_i}$  with  $r_i \leq q - 1$ .

Then  $\mu(K \cup L) = (2n)^q = (2n)(2n)^{q-1}$  and  $\mu(K \cup L \setminus \{i\}) = (2n)^{r_i}$ .

So we have  $\sum_{i \in L} \mu(K \cup L \setminus \{i\}) = \sum_{i \in L} (2n)^{r_i} \leq \sum_{i \in L} (2n)^{q-1} = l(2n)^{q-1}$ .

As  $2n > l$ , then  $\mu(K \cup L) > \sum_{i \in L} \mu(K \cup L \setminus \{i\})$ , hence  $\sum_{\substack{L \subseteq A, \\ l=a-p}} \mu(K \cup L) > \sum_{\substack{L \subseteq A, \\ l=a-p}} \sum_{i \in L} \mu(K \cup L \setminus \{i\})$ .

According to Remark 4 (with  $t = a - p$ ), we have

$$\sum_{\substack{L \subseteq A, \\ \ell=a-p}} \sum_{i \in L} \mu(K \cup L \setminus \{i\}) = (p+1) \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}}^a \mu(K \cup L) \geq \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}}^a \mu(K \cup L) \text{ since } \binom{1}{p+1} = p+1 \geq 1.$$

Then  $\sum_{\substack{L \subseteq A, \\ l=a-p}} \mu(K \cup L) > \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}}^a \mu(K \cup L)$ , i.e.,  $\left[ \sum_{\substack{L \subseteq A, \\ l=a-p}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}}^a \mu(K \cup L) \right] > 0$ .

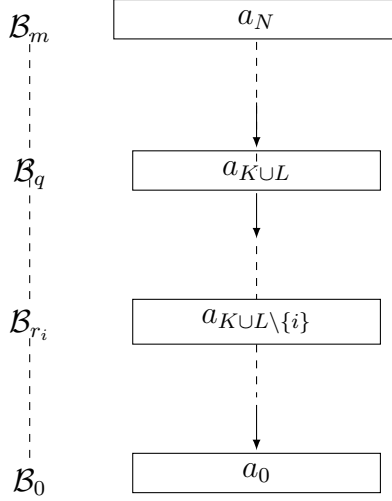


Figure 1: An illustration of the elements  $\mathcal{B}_m$ ,  $\mathcal{B}_q$ ,  $\mathcal{B}_{r_i}$  and  $\mathcal{B}_0$  such that  $m > q > r_i > 0$ .

Therefore  $\sum_{\substack{p=0, \\ p \text{ even}}}^a \left[ \sum_{\substack{L \subseteq A, \\ \ell=a-p}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right] > 0$ , i.e.,  $\Delta_A^{\mu(K)} > 0$ .  $\square$

The following example illustrates this two previous results in this section.

**Example 2.**  $N = \{1, 2, 3, 4\}$ ,  $P = \{(a_{23}, a_1), (a_{234}, a_{123}), (a_2, a_{13})\}$ .

The ordinal preference information  $\{P, I\}$  contains no indifference and the binary relation  $(P \cup M)$  contains no strict cycle, so  $\{P, I\}$  is representable by a general Choquet integral model. A suitable topological sorting on  $(P \cup M)$  is given by:  $\mathcal{B}_0 = \{a_0\}$ ;  $\mathcal{B}_1 = \{a_1, a_3, a_4\}$ ;  $\mathcal{B}_2 = \{a_{13}, a_{14}, a_{34}\}$ ;  $\mathcal{B}_3 = \{a_2\}$ ;  $\mathcal{B}_4 = \{a_{12}, a_{23}, a_{24}\}$ ;  $\mathcal{B}_5 = \{a_{123}, a_{124}, a_{134}\}$ ;  $\mathcal{B}_6 = \{a_{234}\}$  and  $\mathcal{B}_7 = \{a_N\}$ . The preference information  $\{P, I\}$  is representable by the following capacity  $\mu$ :

$S$	$8^7 \times \mu(S)$	$I_S^\mu$
$\{1\}, \{3\}$	8	—
$\{2\}, \{4\}$	$8^3$	—
$\{1, 2\}$	$8^4$	0.33
$\{1, 3\}, \{1, 4\}$	$8^2$	0.33
$\{2, 3\}, \{2, 4\}$	$8^4$	0.33
$\{3, 4\}$	$8^2$	0.33
$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}$	$8^5$	0.49
$\{2, 3, 4\}$	$8^6$	0.50
$N$	$8^7$	0.99

Table 5: The capacity  $\mu$  and the corresponding interaction indices.

We can see that  $I_A^\mu > 0$ ,  $\forall A \subseteq N$  such that  $|A| \geq 2$ .

## 5. Result in the general case

In this section, we assume that the ordinal information may contains indifference and we generalize our previous results. The following proposition gives a necessary and sufficient condition for an ordinal preference information on  $\mathcal{B}^g$  to be representable by a general Choquet integral model.

**Proposition 4.** *Let  $\{P, I\}$  be an ordinal preference information on  $\mathcal{B}^g$ .*

*$\{P, I\}$  is representable by a Choquet integral if and only if the binary relation  $(P \cup M \cup I)$  contains no strict cycle.*

*Proof. Necessity.* Suppose that the ordinal preference information  $\{P, I\}$  on  $\mathcal{B}^g$  is representable by a Choquet integral. So there exists a capacity  $\mu$  such that  $\{P, I\}$  is representable by  $C_\mu$ .

If  $P \cup M$  contains a strict cycle, then there exists  $x_0, x_1, \dots, x_r$  on  $\mathcal{B}^g$  such that  $x_0(P \cup M \cup I)x_1(P \cup M \cup I)\dots(P \cup M \cup I)x_r(P \cup M \cup I)x_0$  and there exists  $x_i, x_{i+1} \in \{x_0, x_1, \dots, x_r\}$  such that  $x_i P x_{i+1}$ . Since  $\{P, I\}$  is representable by  $C_\mu$ , therefore  $C_\mu(u(x_0)) \geq \dots \geq C_\mu(u(x_i)) > C_\mu(u(x_{i+1})) \geq \dots \geq C_\mu(u(x_0))$ , then  $C_\mu(u(x_0)) > C_\mu(u(x_0))$ , contradiction. So,  $P \cup M \cup I$  contains no strict cycle.

**Sufficiency.** We assume that the graph  $(\mathcal{B}^g, P \cup M \cup I)$  does not contain any strict cycle and let us define the relation equivalence  $\mathcal{R}_{IUM}$  on  $\mathcal{B}^g$  by: For all  $x, y \in \text{gen}$ ,  $x \mathcal{R}_{IUM} y$  if and only if  $x T C_{IUM} y$  and  $y T C_{IUM} x$ . Let us pose  $\mathcal{B}' = \mathcal{B}^g / \mathcal{R}_{IUM}$  the quotient set of  $\mathcal{B}^g$  by the equivalence relation  $\mathcal{R}_{IUM}$ . Let us define on the set  $\mathcal{B}'$  the preference relation  $P'$ , the indifference relation  $I'$  and the MOPI relation  $M'$  by: for all  $A, B \in \mathcal{B}'$ ,

- $A P' B \iff \exists a \in A, \exists b \in B : a P b$ ,
- $A I' B \iff \exists a \in A, \exists b \in B : a I b$ .
- $A M' B \iff \exists a \in A, \exists b \in B : a M b$ .

Since the graph  $(\mathcal{B}^g, P \cup M \cup I)$  contains no strict cycle, therefore the graph  $(\mathcal{B}', P' \cup M')$  contains no strict cycle. But the graph  $(\mathcal{B}', P' \cup M')$  contains no indifference (i.e.,  $I' = \emptyset$ ), then there exists  $\{\mathcal{B}'_0, \mathcal{B}'_1, \dots, \mathcal{B}'_m\}$  a partition of  $\mathcal{B}'$ , builds by using a suitable topological sorting on graph  $(\mathcal{B}', P' \cup M')$  (Gondran and Minoux, 1995).

We construct a partition  $\{\mathcal{B}'_0, \mathcal{B}'_1, \dots, \mathcal{B}'_m\}$  as follows:

$\mathcal{B}'_0 = \{A \in \mathcal{B}' : \forall C \in \mathcal{B}', \text{ not } [A(P \cup M)C]\}$ ,  $\mathcal{B}'_1 = \{A \in \mathcal{B}' \setminus \mathcal{B}'_0 : \forall C \in \mathcal{B}' \setminus \mathcal{B}'_0, \text{ not } [A(P \cup M)C]\}$ ,

$\mathcal{B}'_i = \{A \in \mathcal{B}' \setminus (\mathcal{B}'_0 \cup \dots \cup \mathcal{B}'_{i-1}) : \forall C \in \mathcal{B}' \setminus (\mathcal{B}'_0 \cup \dots \cup \mathcal{B}'_{i-1}), \text{ not } [A(P \cup M)C]\}$ , for all  $i = 1, 2, \dots, m$ .

Let  $\mathcal{B}_i = \{x \in A : A \in \mathcal{B}'_i\}$  for all  $i = 0, 1, \dots, m$ . Therefore  $\{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_m\}$  is a suitable topological sorting on graph  $(\mathcal{B}^g, P \cup M \cup I)$  since  $\{\mathcal{B}'_0, \mathcal{B}'_1, \dots, \mathcal{B}'_m\}$  is a suitable topological sorting on graph  $(\mathcal{B}', P' \cup M')$ .

Let us define the mapping  $f : \mathcal{B}^g \rightarrow \mathbb{R}$  and  $\mu : 2^N \rightarrow [0, 1]$  as follows: for  $\ell \in \{0, 1, \dots, m\}$ ,  $\forall x \in \mathcal{B}_\ell$ ,  $f(\phi(x)) = \begin{cases} 0 & \text{if } \ell = 0, \\ (2n)^\ell & \text{if } \ell \in \{1, 2, \dots, m\}. \end{cases}$   $\mu(S) = \frac{f_S}{\alpha}$ , where  $f_S = f(\phi(a_S))$  and  $\alpha = f_N = (2n)^m$ .

Let  $a_S, a_T \in \mathcal{B}^g$ .

- Assume that  $a_S I a_T$ , therefore there exists  $A \in \mathcal{B}'$  such that  $a_S, a_T \in A$ . Since  $A \in \mathcal{B}'$ , thus there exists  $\ell \in \{0, 1, \dots, m\}$  such that  $A \in \mathcal{B}'_\ell$ , then we have  $a_S, a_T \in \mathcal{B}_\ell$ .

– If  $\ell = 0$ , therefore  $C_\mu(u(a_S)) = 0 = C_\mu(u(a_T))$ .

– If  $\ell \in \{1, \dots, m\}$ , therefore  $C_\mu(u(a_S)) = \frac{(2n)^\ell}{\alpha} = C_\mu(u(a_T))$ .

In both cases, we have  $C_\mu(u(a_S)) = C_\mu(u(a_T))$ .

- Assume that  $a_S P a_T$ , then there exists  $A \in \mathcal{B}'_r, C \in \mathcal{B}'_q$ , such that  $a_S \in A$ ,  $a_T \in C$  with  $r, q \in \{0, 1, \dots, m\}$  and  $r > q$ . We then have  $C_\mu(u(a_S)) = \mu(S) = \frac{f_S}{\alpha} = \frac{(2n)^r}{\alpha}$ .

– If  $q = 0$ , therefore  $C_\mu(u(a_T)) = \mu(T) = 0 < \frac{(2n)^r}{\alpha} = \mu(S) = C_\mu(u(a_S))$ .

– If  $q \in \{1, \dots, m\}$ , therefore  $C_\mu(u(a_T)) = \mu(T) = \frac{f_T}{\alpha} = \frac{(2n)^q}{\alpha}$ . Since  $r > q$ , therefore  $(2n)^r > (2n)^q$ , then  $\frac{(2n)^r}{\alpha} > \frac{(2n)^q}{\alpha}$ , i.e.,  $C_\mu(u(a_S)) > C_\mu(u(a_T))$ .

Thus  $\{P, I\}$  is representable by  $C_\mu$ . □

**Remark 6.** *In the case where  $I = \emptyset$ , the previous Proposition 4 say that:  $\{P, I\}$  is representable by a Choquet integral model if and only if the binary relation  $(P \cup M)$  contains no strict cycle. Which coincides with proposition 2, therefore Proposition 4 is a generalization of Proposition 2.*

The following proposition gives a sufficient condition on  $\{P, I\}$  such that positive interaction is always possible within all subsets of criteria in general Choquet integral model.

**Proposition 5.** *Let  $\{P, I\}$  be an ordinal preference information on  $\mathcal{B}^g$  such that  $(P \cup M \cup I)$  containing no strict cycle. If not  $(a_S T C_{I \cup M} a_N)$  for all  $S \subsetneq N$ , then there exists a capacity  $\mu$  such that  $C_\mu$  represent  $\{P, I\}$  and  $I_A^\mu > 0$  for all  $A \subseteq N$  such that  $|A| \geq 2$ .*



*Proof.* Partition  $\{\mathcal{B}'_0, \dots, \mathcal{B}'_m\}$  of  $\mathcal{B}'$  and  $\{\mathcal{B}_0, \dots, \mathcal{B}_m\}$  of  $\mathcal{B}^g$  are built as in proof of Proposition 4.

Let us define the mapping  $\phi : \mathcal{B}^g \rightarrow \mathcal{P}(N)$ ,  $f : \mathcal{P}(N) \rightarrow \mathbb{R}$  and  $\mu : 2^N \rightarrow [0, 1]$  as follows: for all  $S \subseteq N$ , for all  $\ell \in \{0, 1, \dots, m\}$ , for all  $x \in \mathcal{B}_\ell$ ,  $\phi(a_S) = S$

$$f(\phi(x)) = \begin{cases} 0 & \text{if } \ell = 0, \\ (2n)^\ell & \text{if } \ell \in \{1, 2, \dots, m-1\}, \\ (2n)^{n+m} & \text{if } \ell = m. \end{cases}$$

$$\mu(S) = \frac{f_S}{\alpha}, \text{ where } f_S = f(\phi(a_S)) \text{ and } \alpha = f_N = (2n)^{n+m}.$$

Let  $a_S, a_T \in \mathcal{B}^g$  such that  $a_S P a_T$ . We show that  $C_\mu(u(a_S)) > C_\mu(u(a_T))$ .

Since  $a_S, a_T \in \mathcal{B}^g$  and  $\{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_m\}$  is a partition of  $\mathcal{B}^g$ , then there exists  $r, q \in \{0, 1, \dots, m\}$  such that  $a_S \in \mathcal{B}_r$ ,  $a_T \in \mathcal{B}_q$ . As  $a_S P a_T$ , then  $r > q$ . We have  $C_\mu(u(a_S)) = \mu(S) = \frac{f_S}{\alpha} = \frac{(2n)^r}{\alpha}$  (if  $1 \leq r \leq m-1$ ) or  $\frac{(2n)^{n+m}}{\alpha}$  (if  $r = m$ ), we then have  $C_\mu(u(a_S)) \geq \frac{(2n)^r}{\alpha}$ .

- If  $q = 0$ , then  $a_T \in \mathcal{B}_0$  and  $a_S \in \mathcal{B}_r$  with  $r \geq 1$ .  
As  $C_\mu(u(a_T)) = C_\mu(u(a_0)) = \mu(\emptyset) = 0$ , then  $C_\mu(u(a_S)) > C_\mu(u(a_T))$ .
- If  $q \geq 1$ ,  $C_\mu(u(a_T)) = \mu(T) = \frac{f_T}{\alpha} = \frac{(2n)^q}{\alpha}$ , since  $1 \leq q \leq m-1$ .  
But  $r > q$  therefore  $C_\mu(u(a_S)) \geq \frac{(2n)^r}{\alpha} > \frac{(2n)^q}{\alpha} = C_\mu(u(a_T))$ , then  $C_\mu(u(a_S)) > C_\mu(u(a_T))$ .

Hence, in both cases we have  $C_\mu(u(a_S)) > C_\mu(u(a_T))$ . Therefore  $\{P, I\}$  is representable by  $C_\mu$ .

Let  $a_S, a_T \in \mathcal{B}^g$  such that  $a_S I a_T$ . We show that  $C_\mu(u(a_S)) = C_\mu(u(a_T))$ .

Since  $a_S I a_T$ , then there exists  $r \in \{0, 1, \dots, m\}$  such that  $a_S, a_T \in \mathcal{B}_r$ , thus  $C_\mu(u(a_S)) = \mu(S) = \mu(T) = C_\mu(u(a_T))$ .

Let  $A \subseteq N$ , according to the Lemma 1 we have:

$$\begin{aligned} (n-a+1)! \times I_A^\mu &= \sum_{K \subseteq N \setminus A} (n-k-a)!k! \sum_{L \subseteq A} (-1)^{a-\ell} \mu(K \cup L) \\ &= \sum_{K \subseteq N \setminus A} (n-k-a)!k! \sum_{\substack{p=0, \\ p \text{ even}}}^a \left[ \sum_{\substack{L \subseteq A, \\ \ell=a-p}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right] \\ &\geq \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \left[ \sum_{\substack{L \subseteq A, \\ \ell=a-p}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right], \text{ since } (n-k-a)!k! \geq 1. \end{aligned}$$

$$\text{Moreover, } \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \left[ \sum_{\substack{L \subseteq A, \\ \ell=a-p}} \mu(K \cup L) - \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right]$$

$$\begin{aligned}
&= \left[ \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p}} \mu(K \cup L) \right] - \left[ \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right] \\
&= \left[ \mu(N) + \sum_{\substack{p=2, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p}} \mu((N \setminus A) \cup L) + \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p}} \mu(K \cup L) \right] - \\
&\quad \left[ \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right] \\
&\geq \mu(N) - \left[ \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right]
\end{aligned}$$

$$\text{Therefore, } (n-a+1)!k \times I_A^\mu \geq \mu(N) - \left[ \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right].$$

$$\text{We still have to prove that } \mu(N) - \left[ \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right] > 0.$$

Let  $K \subseteq N \setminus A$ ,  $p \in \{0, \dots, a\}$  even number and  $L \subseteq A$  with  $\ell = a - p - 1$ . We have  $L \subsetneq A$ , therefore  $K \cup L \subsetneq N$ , then by hypothesis  $\text{not}(a_{K \cup L} TC_{IUM} a_N)$ . Thus  $a_N \in \mathcal{B}_m$  and there exists  $\ell_{K \cup L} \in \{0, 1, \dots, m-1\}$  such that  $a_{K \cup L} \in \mathcal{B}_{\ell_{K \cup L}}$ .

Then  $\mu(K \cup L) = \frac{1}{\alpha} (2n)^{\ell_{K \cup L}}$  or  $\mu(K \cup L) = 0$ , so in both cases we have  $\mu(K \cup L) \leq \frac{1}{\alpha} (2n)^{\ell_{K \cup L}}$ .

$$\begin{aligned}
\text{Therefore, } \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) &\leq \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \frac{1}{\alpha} (2n)^{\ell_{K \cup L}} \\
&\leq \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \frac{1}{\alpha} (2n)^{m-1} = \frac{1}{\alpha} (2)^{n-1} (2n)^{m-1}.
\end{aligned}$$

$$\begin{aligned}
\text{Moreover, we have } \frac{1}{\alpha} (2)^{n-1} (2n)^{m-1} &= \frac{2^{n-1} (2n)^{m-1}}{(2n)^n (2n)^m} = \frac{2^{n-1} (2n)^{m-1}}{2(2)^{n-1} (n)^n (2n) (2n)^{m-1}} \\
&= \frac{1}{4n^{n+1}} < 1 = \mu(N).
\end{aligned}$$

$$\text{Hence we have } \mu(N) > \left[ \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right], \text{ i.e.,}$$

$$\mu(N) - \left[ \sum_{K \subseteq N \setminus A} \sum_{\substack{p=0, \\ p \text{ even}}}^a \sum_{\substack{L \subseteq A, \\ \ell=a-p-1}} \mu(K \cup L) \right] > 0.$$

We can therefore conclude that  $(n-a+1)! \times I_A^\mu > 0$ , i.e.,  $I_A^\mu > 0$ . □

The following lemma gives a simplified condition of previous condition  $\text{not}(a_S TC_{IUM} a_N)$  for all  $S \subsetneq N$ .

**Lemma 2.** *The previous condition  $[\text{not}(a_S TC_{IUM} a_N)$  for all  $S \subsetneq N]$  is equivalent at  $[\text{not}(a_{N \setminus \{i\}} TC_{IUM} a_N)$  for all  $i \in N]$ .*

*Proof. Necessity.* Suppose that  $\text{not}(a_S TC_{I \cup M} a_N)$  for all  $S \subsetneq N$ , then  $\text{not}(a_{N \setminus \{i\}} TC_{I \cup M} a_N)$  for all  $i \in N$  since  $N \setminus \{i\} \subsetneq N$  for all  $i \in N$ .

**Sufficiency.** Assume that  $\text{not}(a_{N \setminus \{i\}} TC_{I \cup M} a_N)$  for all  $i \in N$ . Let be  $S \subsetneq N$ , then there exists  $i \in N \setminus S$ . We have  $S \subseteq N \setminus \{i\}$ , therefore  $a_{N \setminus \{i\}} M a_S$ . Thus  $a_{N \setminus \{i\}} M a_S$  and  $\text{not}(a_{N \setminus \{i\}} TC_{I \cup M} a_N)$ , so  $\text{not}(a_S TC_{I \cup M} a_N)$ .  $\square$

The following example illustrates our two previous results from this section.

**Example 3.**  $N = \{1, 2, 3, 4\}$ ,  $P = \{(a_{23}, a_1), (a_{234}, a_{123}), (a_2, a_{13})\}$  and  $I = \{(a_{14}, a_{23})\}$ .

*The ordinal preference information  $\{P, I\}$  contains one indifference and the binary relation  $(P \cup M \cup I)$  contains no strict cycle, so  $\{P, I\}$  is representable by a Choquet integral model.*

*A suitable topological sorting on  $(P \cup M)$  is given by:*

$\mathcal{B}_0 = \{a_0\}$ ;  $\mathcal{B}_1 = \{a_1, a_3, a_4\}$ ;  $\mathcal{B}_2 = \{a_{13}, a_{34}\}$ ;  $\mathcal{B}_3 = \{a_2\}$ ;  $\mathcal{B}_4 = \{a_{12}, a_{14}, a_{23}, a_{24}\}$ ;  
 $\mathcal{B}_5 = \{a_{123}, a_{124}, a_{134}\}$ ;  $\mathcal{B}_6 = \{a_{234}\}$  and  $\mathcal{B}_7 = \{a_N\}$ .

*The ordinal preference information  $\{P, I\}$  is representable by a following capacity and the corresponding interaction indices are given by the following table  $\mu$ :*

$S$	$8^7 \times \mu(S)$	$I_S^\mu$
$\emptyset$	0	–
$\{1\}, \{3\}, \{4\}$	8	–
$\{2\}$	$8^3$	–
$\{1, 2\}$	$8^4$	0.29
$\{1, 3\}$	$8^2$	0.29
$\{1, 4\}$	$8^4$	0.29
$\{2, 3\}, \{2, 4\}$	$8^4$	0.34
$\{3, 4\}$	$8^2$	1.04
$\{1, 2, 3\}, \{1, 2, 4\}$	$8^5$	0.42
$\{1, 3, 4\}$	$8^5$	0.43
$\{2, 3, 4\}$	$8^6$	0.53
$N$	$8^7$	0.83

Table 6: Capacity and the corresponding interaction indices.

We can see that  $I_A^\mu > 0$ ,  $\forall A \subseteq N$  such that  $|A| \geq 2$ .

The following lemma gives a more simplified version of monotonicity condition. We will use this lemma in our linear programs, in order to reduce the number of constraints.

**Lemma 3.** *The monotonicity condition is equivalent to the following condition For all  $S \subsetneq N$ ,  $\forall i \in N \setminus S$ ,  $\mu(S) \leq \mu(S \cup \{i\})$ .*

*Proof. Necessity.* Assume that for all  $S, T \in 2^N$ ,  $[S \subseteq T \implies \mu(S) \leq \mu(T)]$ .

Therefore  $\forall S \subsetneq N$ ,  $\forall i \in N \setminus S$ , we have  $\mu(S) \leq \mu(S \cup \{i\})$ , since  $\forall S \subsetneq N$ ,  $\forall i \in N \setminus S$ , we have  $S \subsetneq S \cup \{i\}$ .

**Sufficiency.** Suppose that for all  $S \subsetneq N$ ,  $\forall i \in N \setminus S$ ,  $\mu(S) \leq \mu(S \cup \{i\})$ .

Let  $S, T \in 2^N$  such that  $S \subseteq T$ , we show that  $\mu(S) \leq \mu(T)$ .

- If  $S = T$ , then  $\mu(S) = \mu(T) \leq \mu(T)$ .
- If  $S \subsetneq T$ , then  $T \setminus S = \{i_1, \dots, i_r\} \neq \emptyset$ , with  $r \geq 1$ . We have:

$$\begin{aligned}
\mu(T) &= \mu(S \cup (T \setminus S)) \\
&= \mu(S \cup \{i_1, \dots, i_r\}) \\
&\geq \mu(S \cup \{i_1, \dots, i_{r-1}\}) \\
&\vdots \\
&\geq \mu(S \cup \{i_1\}) \\
&\geq \mu(S).
\end{aligned}$$

In both cases we have  $\mu(S) \leq \mu(T)$ . □

## 6. A LP model testing for necessary interaction

This section builds on (Mayag and Bouyssou, 2019). We drop the hypothesis that we only ask preference information on binary alternatives. We show how to test for the existence necessary positive and negative interactions on the basis of preference information given on a subset of  $X$  that is not necessarily  $\mathcal{B}^g$ .

Assume that the DM provides a strict preference  $P$  and an indifference  $I$  relations on a subset of  $X$ . Let  $A$  be a subset of at least two criteria. Our approach consists in testing first, in two steps, the compatibility of this preference information with a general Choquet integral model, and then, in the third step, the existence of necessary positive or negative interaction into  $A$ .

### 6.1. The process

**Step 1.** The following linear program ( $PL_1$ ) models each preference of  $\{P, I\}$  by introducing two nonnegative slack variables  $\alpha_{xy}^+$  and  $\alpha_{xy}^-$  in the corresponding constraint (Equation (1a) or (1b)). Equation (1c) (resp. (1d)) ensures the normalization (resp. monotonicity) of capacity  $\mu$ . The objective function  $Z_1$  minimizes all the nonnegative variables introduced in (1a) and (1b).

$$\text{Minimize } Z_1 = \sum_{(x,y) \in P \cup I} (\alpha_{xy}^+ + \alpha_{xy}^-) \quad (PL_1)$$

Subject to

$$C_\mu(u(x)) - C_\mu(u(y)) + \alpha_{xy}^+ - \alpha_{xy}^- \geq \varepsilon \quad \forall x, y \in X \text{ such that } x P y \quad (1a)$$

$$C_\mu(u(x)) - C_\mu(u(y)) + \alpha_{xy}^+ - \alpha_{xy}^- = 0 \quad \forall x, y \in X \text{ such that } x I y \quad (1b)$$

$$\alpha_{xy}^+ \geq 0, \quad \alpha_{xy}^- \geq 0 \quad \forall x, y \in X \text{ such that } x(P \cup I)y$$

$$\mu(N) = 1 \quad (1c)$$

$$\mu(S \cup \{i\}) \geq \mu(S) \quad \forall S \subsetneq N, \forall i \in N \setminus S. \quad (1d)$$

$$\varepsilon \geq 0$$

The linear program ( $PL_1$ ) is always feasible due to the introduction of the nonnegative variables  $\alpha_{xy}^+$  and  $\alpha_{xy}^-$ . There are two possible cases:

1. If the optimal solution of ( $PL_1$ ) is  $Z_1^* = 0$ , then we can conclude that, depending on the sign of the variable  $\varepsilon$ , the preference information  $\{P, I\}$  may be representable by a general Choquet integral. The next step of the procedure, Step 2 hereafter, will confirm or not this possibility.

2. If the optimal solution of  $(PL_1)$  is  $Z_1^* > 0$ , then there is no general Choquet integral model compatible with  $\{P, I\}$ .

**Step 2.** Here, the linear program  $(PL_2)$  ensures the existence of a general Choquet integral model compatible with  $\{P, I\}$ , when the optimal solution of  $(PL_1)$  is  $Z_1^* = 0$ . Compared to the previous linear program, in this formulation, we only removed the non-negative variables  $\alpha_{xy}^+$  and  $\alpha_{xy}^-$  (or put them equal to zero) and change the objective function by maximizing the value of the variable  $\varepsilon$ , in order to satisfy the strict preference relation.

$$\text{Maximize } Z_2 = \varepsilon \tag{PL_2}$$

Subject to

$$C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \quad \forall x, y \in X \text{ such that } x P y \tag{1a}$$

$$C_\mu(u(x)) - C_\mu(u(y)) = 0 \quad \forall x, y \in X \text{ such that } x I y \tag{1b}$$

$$\mu(N) = 1 \tag{1c}$$

$$\mu(S \cup \{i\}) \geq \mu(S) \quad \forall S \subsetneq N, \forall i \in N \setminus S. \tag{1d}$$

$$\varepsilon \geq 0$$

Notice that  $(PL_2)$  is solved only if  $Z_1^* = 0$ . Hence, the linear program  $(PL_2)$  is always feasible and it does not have an unbounded solution (it is not restrictive to suppose that  $C_\mu(u(x)) \in [0, 1]; \forall x \in X$ ). Hence, we have one of the following two cases:

1. If the linear program  $(PL_2)$  is feasible with optimal solution  $Z_2^* = 0$ , then there is no general Choquet integral model compatible with  $\{P, I\}$ .
2. If the optimal solution of  $(PL_2)$  is  $Z_2^* > 0$ , then ordinal information  $\{P, I\}$  is representable by a general Choquet integral model.

**Step 3.** At this step, we suppose the preference information  $\{P, I\}$  is representable by a general Choquet integral model, i.e.,  $Z_2^* > 0$ . In order to know if the interaction into subset of criteria  $A$  is necessarily negative (resp. positive) w.r.t. the provided preference information. At  $(PL_2)$ , we add the constraint (1e) and we obtain the following linear program denoted by  $PL_{NN}^A$  (resp.  $PL_{NP}^A$ ).

$$\text{Maximize } Z_3 = \varepsilon \tag{PL_{NN}^A (resp. PL_{NP}^A)}$$

Subject to

$$C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \quad \forall x, y \in X \text{ such that } x P y \tag{1a}$$

$$C_\mu(u(x)) - C_\mu(u(y)) = 0 \quad \forall x, y \in X \text{ such that } x I y \tag{1b}$$

$$\mu(N) = 1 \tag{1c}$$

$$\mu(S \cup \{i\}) \geq \mu(S) \quad \forall S \subsetneq N, \forall i \in N \setminus S \quad (1d)$$

$$I_A^\mu \geq 0 \text{ (resp. } I_A^\mu \leq 0\text{)}. \quad (1e)$$

$$\varepsilon \geq 0$$

After a resolution of the linear program, we have one of the following three possible conclusions :

1. If  $PL_{NN}^A$  (resp.  $PL_{NP}^A$ ) is not feasible, then there is a necessary negative (resp. positive) interaction into subset  $A$ . Indeed, as the program ( $PL_2$ ) is feasible with an optimal solution  $Z_2^* > 0$ , the contradiction about the representation of  $\{P, I\}$  only comes from the introduction of the constraint  $I_A^\mu \geq 0$  (resp.  $I_A^\mu \leq 0$ ).
2. If  $PL_{NN}^A$  (resp.  $PL_{NP}^A$ ) is feasible and the optimal solution  $Z_3^* = 0$ , then the constraint  $C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \quad \forall x, y \in X$  such that  $x P y$  is satisfied with  $\varepsilon = 0$ , i.e., it is not possible to model strict preference by adding the constraint  $I_A^\mu \geq 0$  (resp.  $I_A^\mu \leq 0$ ) in  $PL_{NN}^A$  (resp.  $PL_{NP}^A$ ). Therefore, we can conclude that there is a necessary negative (resp. positive) interaction into subset of criteria  $A$ .
3. If  $PL_{NN}^A$  (resp.  $PL_{NP}^A$ ) is feasible and the optimal solution  $Z_3^* > 0$ , then there is no necessary negative (resp. positive) interaction into a subset of criteria  $A$ .

The following Table 7 and Table 8 give an idea of the decision variables and the number of constraints of monotonicity.

	Decision variables	Number of constraints of monotonicity
$PL_1$	$\varepsilon, \alpha_{xy}^+, \alpha_{xy}^-, \mu(S) \ (\emptyset \subsetneq S \subsetneq N)$	$n(2^{n-1} - 1)$
$PL_2$	$\varepsilon, \mu(S) \ (\emptyset \subsetneq S \subsetneq N)$	$n(2^{n-1} - 1)$
$PL_{NN}^A$	$\varepsilon, \mu(S) \ (\emptyset \subsetneq S \subsetneq N)$	$n(2^{n-1} - 1)$
$PL_{NP}^A$	$\varepsilon, \mu(S) \ (\emptyset \subsetneq S \subsetneq N)$	$n(2^{n-1} - 1)$

Table 7: Variables and number of constraints of monotonicity.

	Number of variables $\mu(S)$	Number of constraints of monotonicity
$n = 3$	6	9
$n = 4$	14	28
$n = 5$	30	75
$n = 6$	62	186
$n = 7$	126	441
$n = 8$	254	1 016
$n = 9$	510	2 295
$n = 10$	1 022	5 110
$n = 11$	2 046	11 253
$n = 12$	4 094	24 564

Table 8: Number of variables  $\mu(S)$  and number of constraints of monotonicity with  $3 \leq n \leq 12$ .

In practice, the number of criteria generally does not exceed 12. Thus, with a common solver, we are able to solve these linear programs.

## 6.2. Example

In this section, we illustrate our decision procedure with an example, inspired from (Angilella et al., 2010). Let us consider a recruitment problem, where the executive manager of a company looks for engaging a new young employee. The manager takes into account the following four criteria:

1. Educational degree (abbreviated: Ed);
2. Professional experience (abbreviated: Ex);
3. Age (abbreviated: Ag);
4. Job interview (abbreviated: In).

In this example,  $X = \{\text{Arthur, Bernard, Charles, Daniel, Esther, Felix, Germaine, Henry, Irene}\}$  and  $N = \{1, 2, 3, 4\}$ .

The candidates, evaluated by the executive manager and their scores for every criterion on a  $[0, 10]$  scale are presented in Table 9. We suppose that the criteria have to be maximized.

Now, suppose that the executive manager (the DM) on the basis of her preference structure is able to give only the following partial information on the reference actions  $X' = \{\text{Arthur, Bernard, Charles, Germaine, Irene}\}$ :

- (a) The candidate Charles (C) is preferred to candidate Bernard (B);



(b) The candidate Germaine (G) is preferred to candidate Arthur (A);

(c) The candidates Charles (C) and Irene(I) are indifferent.

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>I</i>
8	3	10	5	8	5	8	5	0
6	1	9	9	0	9	10	7	10
7	10	0	2	8	4	5	9	2
5	10	5	9	6	7	7	4	8

Table 9: Evaluation matrix.

**Step 1** Linear program ( $PL_1$ ) with nonnegative slack variables  $\alpha_{CB}^+$ ,  $\alpha_{CB}^-$ ,  $\alpha_{GA}^+$ ,  $\alpha_{GA}^-$ ,  $\alpha_{CI}^+$  and  $\alpha_{CI}^-$ .

Minimize  $Z_1 = \alpha_{CB}^+ + \alpha_{CB}^- + \alpha_{GA}^+ + \alpha_{GA}^- + \alpha_{CI}^+ + \alpha_{CI}^-$ .

Subject to

$$C_\mu(C) - C_\mu(B) + \alpha_{CB}^+ - \alpha_{CB}^- \geq \varepsilon$$

$$C_\mu(G) - C_\mu(A) + \alpha_{GA}^+ - \alpha_{GA}^- \geq \varepsilon$$

$$C_\mu(C) - C_\mu(I) + \alpha_{CI}^+ - \alpha_{CI}^- = 0$$

$$C_\mu(A) = 5 + \mu_{123} + \mu_{13} + \mu_1$$

$$C_\mu(B) = 1 + 2\mu_{134} + 7\mu_{34}$$

$$C_\mu(C) = 5\mu_{124} + 4\mu_{12} + \mu_1$$

$$C_\mu(D) = 2 + 3\mu_{124} + 4\mu_{24}$$

$$C_\mu(E) = 6\mu_{134} + 2\mu_{13}$$

$$C_\mu(F) = 4 + \mu_{124} + 2\mu_{24} + 2\mu_2$$

$$C_\mu(G) = 5 + 2\mu_{124} + \mu_{12} + 2\mu_2$$

$$C_\mu(H) = 4 + \mu_{123} + 2\mu_{23} + 2\mu_3$$

$$C_\mu(I) = 2\mu_{234} + 6\mu_{24} + 2\mu_2$$

$$\mu_{12} \geq \mu_1; \mu_{12} \geq \mu_2$$

$$\mu_{13} \geq \mu_1; \mu_{13} \geq \mu_3$$

$$\mu_{14} \geq \mu_1; \mu_{14} \geq \mu_4$$

$$\mu_{23} \geq \mu_2; \mu_{23} \geq \mu_3$$

$$\mu_{24} \geq \mu_2; \mu_{24} \geq \mu_4$$

$$\mu_{34} \geq \mu_3; \mu_{34} \geq \mu_4$$

$$\mu_{123} \geq \mu_{12}; \mu_{123} \geq \mu_{13}; \mu_{123} \geq \mu_{23}$$

$$\mu_{124} \geq \mu_{12}; \mu_{124} \geq \mu_{14}; \mu_{124} \geq \mu_{24}$$

$$\begin{aligned}
&\mu_{134} \geq \mu_{13}; \mu_{134} \geq \mu_{14}; \mu_{134} \geq \mu_{34} \\
&\mu_{234} \geq \mu_{23}; \mu_{234} \geq \mu_{24}; \mu_{234} \geq \mu_{34} \\
&\mu_{1234} \geq \mu_{123}; \mu_{1234} \geq \mu_{124}; \mu_{1234} \geq \mu_{134}; \mu_{1234} \geq \mu_{234} \\
&\mu_{1234} = 1 \\
&\varepsilon \geq 0 \\
&\alpha_{CB}^+, \alpha_{CB}^-, \alpha_{GA}^+, \alpha_{GA}^-, \alpha_{CI}^+ \text{ and } \alpha_{CI}^- \geq 0.
\end{aligned}$$

The linear program  $(PL_1)$  is feasible and optimal solution of  $(PL_1)$  is  $Z_1^* = 0$ , then we can conclude that, depending on the sign of the variable  $\varepsilon$ , the preference information  $\{P, I\}$  may be representable by a general Choquet integral. The next step of the procedure, Step 2 hereafter, will confirm or not this possibility.

**Step 2** The linear program corresponding to the test of the existence of a capacity  $\mu$  compatible with ordinal information  $\{P, I\}$  is the following:

$$\text{Maximize } Z_2 = \varepsilon$$

Subject to

$$C_\mu(C) - C_\mu(B) \geq \varepsilon$$

$$C_\mu(G) - C_\mu(A) \geq \varepsilon$$

$$C_\mu(C) - C_\mu(I) = 0$$

$$C_\mu(A) = 5 + \mu_{123} + \mu_{13} + \mu_1$$

$$C_\mu(B) = 1 + 2\mu_{134} + 7\mu_{34}$$

$$C_\mu(C) = 5\mu_{124} + 4\mu_{12} + \mu_1$$

$$C_\mu(D) = 2 + 3\mu_{124} + 4\mu_{24}$$

$$C_\mu(E) = 6\mu_{134} + 2\mu_{13}$$

$$C_\mu(F) = 4 + \mu_{124} + 2\mu_{24} + 2\mu_2$$

$$C_\mu(G) = 5 + 2\mu_{124} + \mu_{12} + 2\mu_2$$

$$C_\mu(H) = 4 + \mu_{123} + 2\mu_{23} + 2\mu_3$$

$$C_\mu(I) = 2\mu_{234} + 6\mu_{24} + 2\mu_2$$

$$\mu_{12} \geq \mu_1; \mu_{12} \geq \mu_2$$

$$\mu_{13} \geq \mu_1; \mu_{13} \geq \mu_3$$

$$\mu_{14} \geq \mu_1; \mu_{14} \geq \mu_4$$

$$\mu_{23} \geq \mu_2; \mu_{23} \geq \mu_3$$

$$\mu_{24} \geq \mu_2; \mu_{24} \geq \mu_4$$

$$\mu_{34} \geq \mu_3; \mu_{34} \geq \mu_4$$

$$\mu_{123} \geq \mu_{12}; \mu_{123} \geq \mu_{13}; \mu_{123} \geq \mu_{23}$$

$$\mu_{124} \geq \mu_{12}; \mu_{124} \geq \mu_{14}; \mu_{124} \geq \mu_{24}$$

$$\begin{aligned}
\mu_{134} &\geq \mu_{13}; \mu_{134} \geq \mu_{14}; \mu_{134} \geq \mu_{34} \\
\mu_{234} &\geq \mu_{23}; \mu_{234} \geq \mu_{24}; \mu_{234} \geq \mu_{34} \\
\mu_{1234} &\geq \mu_{123}; \mu_{1234} \geq \mu_{124}; \mu_{1234} \geq \mu_{134}; \mu_{1234} \geq \mu_{234} \\
\mu_{1234} &= 1 \\
\varepsilon &\geq 0
\end{aligned}$$

The linear program ( $PL_2$ ) is feasible and optimal solution of ( $PL_2$ ) is  $Z_2^* = 3.8 > 0$ , then we can conclude that, the preference information  $\{P, I\}$  is representable by a general Choquet integral model. Moreover, the results obtained by solving ( $PL_2$ ) are given by Tables 10 and 11.

$S$	$\mu(S)$
$\emptyset, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}$	0
$\{2\}, \{2, 3\}, \{2, 4\}, \{2, 3, 4\}$	0.9
$\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, N$	1

Table 10: Capacity compatible with ( $PL_2$ )

$x$	$A$	$B$	$C$	$D$	$E$	$F$	$G$	$H$	$I$
$C_\mu(x)$	6	3	9	8.6	6	8.6	9.8	6.8	9

Table 11: General Choquet integral corresponding at previous capacity  $\mu$

**Step 3** In order to know if the interaction within the subset of criteria  $\{1, 2, 3\}$  is necessarily negative (resp. positive). We obtain the  $PL_{NN}^{123}$  (resp.  $PL_{NP}^{123}$ ) by adding at the previous linear program ( $PL_2$ ) the constraints  $I_{123}^\mu \geq 0$  (resp.  $I_{123}^\mu \leq 0$ ) with  $I_{123}^\mu = \mu_{1234} + \mu_{123} - \mu_{124} - \mu_{134} - \mu_{234} - \mu_{12} - \mu_{13} + \mu_{14} - \mu_{23} + \mu_{24} + \mu_{34} + \mu_1 + \mu_2 + \mu_3 - \mu_4$ .

- The linear program  $PL_{NP}^{123}$  is feasible and the optimal solution is  $Z_3^* = 3.8 > 0$ . Then interaction within  $\{\text{Educational degree, Professional experience, Age}\}$  is not necessary positive. Moreover, the results obtained by solving  $PL_{NP}^{123}$  are given by Tables 12 and 13 (with  $I_{123}^\mu = -2.375 < 0$ ).

$S$	$\mu(S)$
$\emptyset, \{1\}, \{3\}, \{4\}, \{1, 4\}, \{3, 4\}$	0
$\{2\}$	0.6875
$\{2, 4\}$	0.9375
$\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N$	1

Table 12: Capacity compatible with  $PL_{NP}^{123}$

$x$	$A$	$B$	$C$	$D$	$E$	$F$	$G$	$H$	$I$
$C_\mu(x)$	7	3	9	8.75	8	8.25	9.375	7	9

Table 13: Choquet integral corresponding at previous capacity  $\mu$

- The linear program  $PL_{NN}^{123}$  is feasible and the optimal solution is  $Z_3^* = 3.8 > 0$ . Then interaction within {Educational degree, Professional experience, Age} is not necessary negative. Moreover, the results obtained by solving  $PL_{NP}^{123}$  are given by Tables 15 and 14 (with  $I_{123}^\mu = 1 > 0$ ).

$S$	$\mu(S)$
$\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1, 3, 4\}$	0
$\{2, 4\}, \{2, 3, 4\}$	0.125
$\{1, 2, 3\}, \{1, 2, 4\}$	0.2
$N$	1

Table 14: Capacity compatible with  $PL_{NN}^{123}$

$x$	$A$	$B$	$C$	$D$	$E$	$F$	$G$	$H$	$I$
$C_\mu(x)$	5.2	1	1	3.1	0	4.45	5.4	4.2	1

Table 15: Choquet integral corresponding at previous capacity  $\mu$

## 7. Conclusion

This article studies the notion of interaction within a subset of criteria of any size, in the Choquet integral model. We make a restriction in the case where the DM gives preference information on a set of finite number of alternatives, as opposed to the continuous setting used in (Timonin, 2015, 2016a,b). The capacity to represent this preference information is not unique. Moreover, the interpretation of the interaction effects between criteria requires some caution. Indeed, we gave some examples in which the sign of the interaction index depends upon the arbitrary choice of a capacity within the set of all capacities compatible with the preference information. This has led us to we extend concept of necessary and possible interaction introduced in (Mayag and Bouyssou, 2019). Only necessary interactions are robust since their sign and, hence, interpretation, does not vary within the set of all representing capacities.

Our first result prove that null interaction is not necessary in a general Choquet integral model when the preference of the DM contains no indifference. This extends Proposition 1, Page 7 in (Mayag and Bouyssou, 2019).

Our second result gives a necessary and sufficient condition for ordinal preference information contains no indifference on generalized binary alternatives to be representable by a general model of Choquet integral. This extends Theorem 1, Page 305 in (Mayag et al., 2011).

Under the condition of our second result, our third result shows that in the framework of generalized binary alternatives, if the ordinal preference information contains no indifference, it is possible to represent it by a general Choquet integral model which all Shapley interaction indices into a subset of criteria are strictly positive. This extends Theorem 2, Page 10 on (Mayag and Bouyssou, 2019).

We have extended these results assuming that there is possibly to have an indifference relation. Then we have given a sufficient condition on ordinal information so that positive interaction is always possible into all subsets of criteria in general Choquet integral model. We have proposed a linear program inspired by (Mayag and Bouyssou, 2019) allowing to test whether the interpretation of the interaction indices is ambivalent or not.

The subject of this paper offer several avenues for future research. In fact, It would be interesting to solve the dual problem. Indeed, is it always possible to build a capacity relative to which all the interaction indices will be strictly negative?

The notion of interaction would deserve further study. In particular, it would be interesting to have a definition that would not depend on a particular aggregation technique or on a particular index.

It would finally be interesting to study the case of bipolar scales. We are already investigating some of these research avenues.

## References

- S. Angilella, S. Greco, and B. Matarazzo. Non-additive robust ordinal regression: a multiple criteria decision model based on the Choquet integral. *European Journal of Operational Research*, 201(1): 277–288, 2010. ISSN 0377-2217. doi: <https://doi.org/10.1016/j.ejor.2009.02.023>.
- L. Berrah and V. Clivillé. Towards an aggregation performance measurement system model in a supply chain context. *Computers in Industry*, 58(7):709–719, 2007.
- D. Bouyssou and M. Pirlot. *Conjoint Measurement Tools for MCDM*, pages 97–151. Springer New York, New York, NY, 2016. ISBN 978-1-4939-3094-4. doi: 10.1007/978-1-4939-3094-4.4. URL [http://dx.doi.org/10.1007/978-1-4939-3094-4\\_4](http://dx.doi.org/10.1007/978-1-4939-3094-4_4).
- G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1954. doi: 10.5802/aif.53. URL <http://www.numdam.org/articles/10.5802/aif.53/>.
- V. Clivillé, L. Berrah, and G. Mauris. Quantitative expression and aggregation of performance measurements based on the MACBETH multi-criteria method. *International Journal of Production economics*, 105:171–189, 2007.
- M. Gondran and M. Minoux. *Graphes et algorithmes*. Eyrolles, Paris, 3e édition, 1995.
- M. Grabisch. The application of fuzzy integrals in multicriteria decision making. *Fuzzy sets and systems*, 1995.

- M. Grabisch. The application of fuzzy integrals in multicriteria decision making. *European Journal of Operational Research*, 89(3):445–456, March 1996. ISSN 0377-2217. doi: [https://doi.org/10.1016/0377-2217\(95\)00176-X](https://doi.org/10.1016/0377-2217(95)00176-X). URL <https://www.sciencedirect.com/science/article/pii/S037722179500176X>.
- M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92(2):167 – 189, 1997. ISSN 0165-0114. doi: [https://doi.org/10.1016/S0165-0114\(97\)00168-1](https://doi.org/10.1016/S0165-0114(97)00168-1).
- M. Grabisch. *Set Functions, Games and Capacities in Decision Making*. 2016. URL <https://hal.archives-ouvertes.fr/hal-01372911>.
- M. Grabisch and C. Labreuche. Fuzzy measures and integrals in MCDA. In *Multiple Criteria Decision Analysis*, pages 563–608. Kluwer Academic Publishers, 2004. URL <https://halshs.archives-ouvertes.fr/halshs-00268985>.
- M. Grabisch and C. Labreuche. A decade of application of the Choquet and Sugeno integrals in multicriteria decision aid. *Annals of Operations Research*, 175(1):247–290, March 2010. doi: 10.1007/s10479-009-0655-8. URL <https://halshs.archives-ouvertes.fr/halshs-00496558>.
- M. Grabisch, J. Duchêne, F. Lino, and P. Perny. Subjective evaluation of discomfort in sitting position. *Fuzzy Optimization and Decision Making*, 1(3):287–312, 2002.
- M. Grabisch, C. Labreuche, and J-C. Vansnick. On the Extension of Pseudo-Boolean Functions for the Aggregation of Interacting Criteria. *European Journal of Operational Research*, 148:28–47, 2003. URL <https://hal.archives-ouvertes.fr/hal-00272780>.
- M. Grabisch, I. Kojadinovic, and P. Meyer. A review of methods for capacity identification in Choquet integral based multi-attribute utility theory: Applications of the Kappalab R package. *European Journal of Operational Research*, 186(2):766–785, 2008.
- P.A. Kaldjob Kaldjob, B. Mayag, and D. Bouyssou. Necessary and Possible Interaction Between Criteria in a General Choquet Integral Model. In *Information Processing and Management of Uncertainty in Knowledge-Based Systems*, pages 457–466. June 2020. doi: 10.1007/978-3-030-50143-3\36. URL <https://hal.archives-ouvertes.fr/hal-02877291>.
- D.R. Krantz, D.R. Luce, P. Suppes, and A. Tversky. *Foundations of Measurement, Vol. I: Additive and Polynomial Representations*, volume 1 of *Foundations of Measurement*. Academic Press, New York, 1971.
- B. Mayag and D. Bouyssou. Necessary and possible interaction between criteria in a 2-additive Choquet integral model. *European Journal of Operational Research*, 283:308–320, 2019. ISSN 0377-2217. doi: 10.1016/j.ejor.2019.10.036.
- B. Mayag, M. Grabisch, and C. Labreuche. A representation of preferences by the Choquet integral with respect to a 2-additive capacity. *Theory and Decision*, 71(3):297–324, Sep 2011. doi: 10.1007/s11238-010-9198-3.
- J.P. Pignon and C. Labreuche. A methodological approach for operational and technical experimentation based evaluation of systems of systems architectures. In *Int. Conference on Software & Systems Engineering and their Applications (ICSSEA)*, Paris, France, December 4-6 2007.
- L. Rudolf and G. Pilz. *Applied Abstract Algebra*. Springer-Verlag New York, 1984. doi: 10.1007/978-1-4615-6465-2.
- M. Timonin. Axiomatization of the Choquet integral for 2-dimensional heterogeneous product sets. *ArXiv e-prints*, July 2015. <http://adsabs.harvard.edu/abs/2015arXiv150704167T>.
- M. Timonin. Conjoint axiomatization of the Choquet integral for heterogeneous product sets. *ArXiv e-prints*, March 2016a. <http://adsabs.harvard.edu/abs/2016arXiv160308142T>.
- M. Timonin. Conjoint axiomatization of the Choquet integral for heterogeneous product sets. In João Paulo Carvalho, Marie-Jeanne Lesot, Uzay Kaymak, Susana M. Vieira, Bernadette Bouchon-Meunier, and Ronald R. Yager, editors, *Information Processing and Management of Uncertainty in*

*Knowledge-Based Systems - 16th International Conference, IPMU 2016, Eindhoven, The Netherlands, June 20-24, 2016, Proceedings, Part I*, volume 610 of *Communications in Computer and Information Science*, pages 46–57. Springer, 2016b.