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# On the Approximability of Robust Network Design 

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#### Abstract

Given the dynamic nature of traffic, we investigate the variant of robust network design where we have to determine the capacity to reserve on each link so that each demand vector belonging to a polyhedral set can be routed. The objective is either to minimize congestion or a linear cost. Routing is assumed to be fractional and dynamic (i.e., dependent on the current traffic vector). We first prove that the robust network design problem with minimum congestion cannot be approximated within any constant factor. Then, using the ETH conjecture, we get a $\Omega\left(\frac{\log n}{\log \log n}\right)$ lower bound for the approximability of this problem. This implies that the well-known $O(\log n)$ approximation ratio established by Räcke in 2008 is tight. Using Lagrange relaxation, we obtain a new proof of the $O(\log n)$ approximation. An important consequence of the Lagrange-based reduction and our inapproximability results is that the robust network design problem with linear reservation cost cannot be approximated within any constant ratio. This answers a long-standing open question of Chekuri (2007). We also give another proof of the result of Goyal\&al (2009) stating that the optimal linear cost under static routing can be $\Omega(\log n)$ more expensive than the cost obtained under dynamic routing. Finally, we show that even if only two given paths are allowed for each commodity, the robust network design problem with minimum congestion or linear cost is hard to approximate within some constant.


Keywords: Approximability, PCP, ETH, Robust Network Design.

[^0]
## 1. Introduction

Network optimization [1, 2] plays a crucial role for telecommunication operators since it permits to carefully invest in infrastructures, i.e. reduce capital expenditures. As Internet traffic is ever increasing, the network's capacity needs to be expanded through careful investments every year or even half-year. However, the dynamic nature of the traffic due to ordinary daily fluctuations, long term evolution and unpredictable events requires to consider uncertainty on the traffic demand when dimensioning network resources.

Ideally, the network capacity should follow the demand. When the traffic demand can be precisely known, several approaches have been proposed to solve the capacitated network design problem using for instance decomposition methods and cutting planes [3, 4, 5]. But in practice, perfect knowledge of future traffic is not available at the time the decision needs to be taken. The dynamic nature of the traffic due to ordinary daily fluctuations, long term evolution and unpredictable events requires to consider uncertainty on traffic demands when dimensioning network resources. While overestimated traffic forecasts could be used to solve a deterministic optimization problem, it is likely to yield to a costly over-provisioning of the network capacities, which is not acceptable. Therefore, robust optimization under uncertainty sets is a must for the design of network capacities. In this context, our paper presents new approximability results on two tightly related variants of the robust network design problem, the minimization of either the congestion or a linear cost.

Let's consider an undirected graph $G=(V(G), E(G))$ representing a communication network. The traffic is characterized by a set of commodities $h \in \mathcal{H}$ associated to different node pairs. And the routing of a commodity can be represented by a flow $f^{h} \in \mathbb{R}^{E(G)}$ of intensity $d_{h}$. To take into account the changing nature of the demand, $d$ is assumed to be uncertain and more precisely to belong to a polyhedral set $\mathcal{D}$. The polyhedral model was introduced in $[6,7]$ as an extension of the hose model $[8,9]$, where limits on the total traffic going into (resp. out of) a node are considered.

When solving a robust network design problem, several objective functions can be considered. Given a capacity $c_{e}$ for each edge $e$, one might be interested in minimizing the congestion given by $\max _{e \in E(G)} \frac{u_{e}}{c_{e}}$ where $u_{e}$ is the
reserved capacity on edge $e$. Another common objective function is given by the linear reservation cost $\sum_{e \in E(G)} \lambda_{e} u_{e}$. This can also represent the average congestion by taking $\lambda_{e}=\frac{1}{c_{e}} \cdot h \in \mathcal{H} f_{e}^{h}(d)$. A popular approach is to minimize the maximum edge utilization given by $\max _{e 1 n E(G)} \frac{u_{e}}{c_{e}}$. The goal is to choose a reservation vector $u$ so that the network is able to support any demand vector $d \in \mathcal{D}$, i.e., there exists a (fractional) routing serving every commodity such that the total flow on each edge $e$ is less than the reservation $u_{e}$.

The robust network design variant that we are focusing on in this paper, is referred to as dynamic routing in the literature since the network is optimized such that any realization of traffic matrix in the uncertainty set has its own routing. The robust network design problem where a linear reservation cost is minimized was proved to be co-NP hard in [10] when the graph is directed. A stronger co-NP hardness result is given in [11] where the graph is undirected (this implies the directed case result). Some exact solution methods for robust network design have been considered in [12, 13]. Some special cases where dynamic routing is easy to compute have been described in $[14,15,16]$.

Routing with uncertain demands has received a significant interest from the community. As opposed to dynamic routing, static routing or stable routing was introduced in [6]: it consists in choosing a fixed flow $x^{h}$ of value 1 for each commodity $h$. The actual flow $f^{h}(d)$ for the demand scenario $d$ will then be scaled by the actual demand $d_{h}$ of commodity $h$, i.e. $f^{h}(d)=$ $d_{h} x^{h}$. Static routing is also called oblivious routing in [17, 18]. In this case, polynomial-time algorithms to compute optimal static routing (with respect to either congestion or linear reservation cost) have been proposed $[6,7,17$, 18] based on either duality or cutting-plane algorithms. When the routing for each commodity is restricted to be on a single path, the problem becomes $N P$-hard even in the deterministic case, this follows from the $N P$-hardness of the edge-disjoint path computation problem [19].

To further improve solutions of static routing and overcome complexity issues related to dynamic routing, a number of restrictions on routing have been considered to design polynomial-time algorithms (see [20, 21] for a complete survey). This includes, for example, the multi-static approach, introduced in [22], where the uncertainty set is partitioned using an hyperplane and routing is restricted to be static over each partition. This idea has been generalized in [23] to unrestricted covers of the uncertainty set and an extension to share the demand between routing templates, called volume routing, has been proposed in [24]. [25] applied affine routing for robust network design,
based on affine adjustable robust counterparts introduced in [26], restricting the recourse to be an affine function of the uncertainties. The performance of this framework has been extensively compared to the static and dynamic routing, both theoretically and empirically [27, 21]. In practice, affine routing provides a good approximation of the dynamic routing while it can be solved in reasonable time thanks to polynomial-time algorithms. Finally, an approach encompassing the previous approaches is the multipolar approach proposed in [28, 29].

In this work, we will only focus on the complexity of the robust network design problem under dynamic routing, while minimizing either congestion or some linear cost. To close this section, let us summarize the main contributions of the paper and review some related work.

### 1.1. Our contributions

- We first prove that the robust network design problem with minimum congestion cannot be approximated within any constant factor. The reduction is based on the PCP theorem and some connections with the Gap-3-SAT problem [30]. The same reduction also allows to show inapproximability within $\Omega\left(\log \frac{n}{\Delta}\right)$ where $\Delta$ is the maximum degree in the graph and $n$ is the number of vertices.
- Using the ETH conjecture $[31,32]$, we prove a $\Omega\left(\frac{\log n}{\log \log n}\right)$ lower bound for the approximability of the robust network design problem with minimum congestion. This implies that the well-known $O(\log n)$ approximation ratio that can be obtained using the result in [33] is tight.
- We show that any $\alpha$-approximation algorithm for the robust network design problem with linear costs directly leads to an $\alpha$-approximation for the problem with minimum congestion. The proof is based on Lagrange relaxation. We obtain that robust network design with minimum congestion can be approximated within $O(\log n)$. This was already proved in [33] in a different way.
- An important consequence of the Lagrange-based reduction and our inapproximability results is that the robust network design problem with linear reservation cost cannot be approximated within any constant ratio. This answers a long-standing open question stated in [34].

Another consequence is a new proof for the existence of instances for which the optimal static solution can be $\Omega(\log n)$ more expensive than
a solution based on dynamic routing, when a linear cost is minimized. This was already proved in [35] in a different way.

- We show that even if only two given paths are allowed for each commodity, there is a constant $k$ such that the robust network design problem with minimum congestion or linear costs cannot be approximated within $k$.


### 1.2. Related work

Let us first assume that the graph is undirected and a linear cost is minimized. A result attributed to A. Gupta ([34], see also [35] for a more detailed presentation) leads to an $O(\log n)$ approximation algorithm for linear cost under dynamic fractional routing. Furthermore, this approximation is achieved by a routing on a (fixed) single tree. In particular, this shows that the ratios between the dynamic and the static solutions under fractional routing $\left(\frac{L_{n-t a t-f r a c}}{L_{\text {sin }}{ }_{\text {dyn-frac }}}\right)$ (Lin denotes here the optimal linear cost of the solution) and between single path and fractional routing under the static model $\left(\frac{\text { Lin }_{\text {stat-sing }}}{\text { Lin }_{\text {stat-frac }}}\right)$ is $O(\log n)$ and provides an $O(\log n)$ approximation for static single path routing $\operatorname{Lin}_{\text {stat-sing }}$. On the other hand [36] shows that the static single path problem cannot be approximated within a $\Omega\left(\log ^{\frac{1}{4}-\epsilon} n\right)$ ratio unless $N P \not \subset Z P T I M E\left(n^{\operatorname{poly} \log (n)}\right)$. As noticed in [35], this implies (assuming this complexity conjecture) that the gap $\frac{\text { Lin }_{\text {stat-sing }}}{\text { Lin }_{\text {stat-frac }}}$ is $\Omega\left(\log ^{\frac{1}{4}-\epsilon} n\right)$. [35] has shown that the gap $\frac{\text { Lin }_{\text {stat-frac }}}{\text { Lin }_{\text {dyn-frac }}}$ is $\Omega(\log n)$.

For the linear cost and undirected graphs, an extensively studied polyhedron is the symmetric hose model. The demand vector is here not oriented (i.e, there is no distinction between a demand from $i$ to $j$ and a demand from $j$ to $i$ ), and uncertainty is defined by considering an upper-bound limit $b_{i}$ for the sum of demands related to node $i$. A 2 -approximation has been found for the dynamic fractional case $[9,10]$ based on tree routing (where we route
 has been conjectured that this solution resulted in an optimal solution for the static single path routing. This question has been open for some time and has become known as the $V P N$ conjecture. It was finally answered by the affirmative in [37]. The assymetric hose polytope was also considered in many papers. An approximation algorithm is proposed to compute Lin $_{\text {stat-sing }}$ within a ratio of 3.39 [38] (or more precisely 2 plus the best approximation
ratio for the Steiner tree problem). If $\mathcal{D}$ is a balanced asymmetric hose polytope, i.e., $\sum_{v \in V} b_{v}^{o u t}=\sum_{v \in V} b_{v}^{\text {in }}$ where $b_{v}^{i n}$ (resp. $b_{v}^{o u t}$ ) is the upper bound for the traffic entering into (resp. going out of) $v$, then the best approximation factor becomes 2 [38]. Moreover, if we assume that $b_{v}^{o u t}=b_{b}^{i n}$, then $L_{\text {stat-sing }}$ is easy to compute and we get that $\operatorname{Lin}_{\text {stat-tree }}=L_{\text {sin }}^{\text {stat-sing }}$ [39]. In other words, there is some similarity with the case where $\mathcal{D}$ is a symmetric hose polytope.

When congestion is considered, [40] proved the existence of an oblivious (or static) routing with a competitive ratio of $O\left(\log ^{3} n\right)$ with respect to optimum routing of any traffic matrix. Then, [41] improved the bound to $O\left(\log ^{2} n \log \log n\right)$ and gave a polynomial-time algorithm to find such a static routing. Finally, [33] described an $O(\log n)$ approximation algorithm for static routing with minimum congestion. Notice that the bound given by static routing cannot provide a better bound than $O(\log n)$ since a lower bound of $\Omega(\log n)$ is achieved by static routing for planar graphs [42, 43].

It has also been shown in [44] that the gap between the dynamic fractional routing and a dynamic fractional routing restricted to a polynomial number of paths can be $\Omega\left(\frac{\log n}{\log \log n}\right)$.

When a directed graph is considered and congestion is minimized, [18] has shown that the gap between static fractional routing and dynamic fractional routing can be $\Omega(\sqrt{n})$ while [45] proves that the gap is upper-bounded by $O\left(\sqrt{k} n^{\frac{1}{4}} \log n\right)$ (where $k=|\mathcal{H}|$ is the number of commodities). More results can be found in [45] and the references therein.

Using an approximate separation oracle for the dual problem to obtain an approximate solution of the primal is a well-known technique already used in $[46,47,48]$ at least in the context of packing-covering problems. Lagrangian relaxations are also used in [49, 50, 51] to produce dual solutions that are near-optimal.

## 2. From Gap-3-SAT to robust network design with minimum congestion

Given an edge $e$, let $s(e)$ and $t(e)$ be the extremities of $e$. Similarly to edges, for a commodity $h \in \mathcal{H}$, let $s(h)$ and $t(h)$ denote the endpoints of $h$. And let $\mathcal{U}(\mathcal{D})$ be the set of $u \in \mathbb{R}^{E(G)}$ such that each traffic vector $d \in \mathcal{D}$ can be routed on the network when a capacity $u_{e}$ is assigned to edge $e$. Since $\mathcal{D}$ is polyhedral, $\mathcal{U}(\mathcal{D})$ is also polyhedral (see, e.g, [34]).

We are interested in minimizing the congestion under polyhedral uncertainty and dynamic routing: $\min _{u \in \mathcal{U}(\mathcal{D})} \max _{e \in E(G)} \frac{u_{e}}{c_{e}}$.

Given a polytope represented by $A x \leq b$, the size of the polytope denotes the total encoding size of the entries in $A$ and $b$.

Our first main result is related to the inapproximability of the minimum congestion problem within a constant factor.

Theorem 2.1. Unless $P=N P$, the minimum congestion problem cannot be approximated with a polynomial-time algorithm within any constant factor even if $\mathcal{D}$ is given by $\{d: A d+B \psi \leq b\}$ whose size is polynomially bounded by $|V(G)|$.

Notice that it is important to consider polyhedral uncertainty sets that are easy to describe (otherwise the inapproximability results would be a direct consequence of the difficulty to separate from the uncertainty set).

To prove Theorem 2.1, we will need the PCP (Probabilistically Checkable Proof) theorem [30] and an intermediate lemma. For a 3-SAT formula $\varphi$ we note $\operatorname{val}(\varphi)$ the maximum fraction of the clauses which are satisfiable at the same time. In particular, $\operatorname{val}(\varphi)=1$ means that $\varphi$ is satisfiable. The problem where we have to decide if $\operatorname{val}(\varphi)<\rho$ or $\operatorname{val}(\varphi)=1$ for a 3-SAT formula $\varphi$ is called Gap-3-SAT.

The instances such that $\rho \leq \operatorname{val}(\varphi)<1$ do not need to be considered. One way to state PCP theorem is to say that there exists a constant $0<$ $\rho<1$ for which Gap-3-SAT is NP-compete. In other words, it is NP-hard to distinguish between satisfiable 3-SAT formulas and those for which strictly less than a fraction $\rho$ of clauses can be simultaneously satisfied.

To prove the theorem 2.1, we will use the following lemma (where cong denotes the optimal congestion of the corresponding instance).

Lemma 2.1. For every $\gamma \in \mathbb{N}$ there is a mapping $f_{\gamma}$ computable in polynomial time from 3-SAT instances to minimum congestion instances defined by an undirected graph $G_{\gamma}$, a set of commodities $\mathcal{H}_{\gamma}$ and a polytope $\mathcal{D}_{\gamma}=\left\{d: A_{\gamma} d+B_{\gamma} \psi_{\gamma} \leq b_{\gamma}\right\}$ such that $\left|V\left(G_{\gamma}\right)\right|=O\left(m^{\gamma}\right),\left|E\left(G_{\gamma}\right)\right|=O\left(m^{\gamma}\right)$ and the size of $\mathcal{D}_{\gamma}$ is $O\left(m^{c \gamma}\right)$ where $c$ is some positive constant and $m$ is the number of clauses. The mapping satisfies the following:

- $\operatorname{val}(\varphi)=1 \Longrightarrow \operatorname{cong}\left(f_{\gamma}(\varphi)\right) \geq 1+\gamma(1-\rho)$
- $\operatorname{val}(\varphi)<\rho \Longrightarrow \operatorname{cong}\left(f_{\gamma}(\varphi)\right) \leq 1$.


Figure 1: $G_{1}$ and $G_{2}$

Proof. of Theorem 2.1 We are going to use Lemma 2.1 and PCP Theorem for the proof. Suppose that congestion can be approximated in polynomial time within a constant approximation factor $\alpha$. We first choose $\gamma$ such that $\alpha<1+\gamma(1-\rho)$.

Starting from a 3-SAT formula such that either $\operatorname{val}(\varphi)<\rho$ or $\operatorname{val}(\varphi)=1$, we construct $f_{\gamma}(\varphi)$ in polynomial time. The optimal congestion will satisfy either $\operatorname{cong}\left(f_{\gamma}(\varphi)\right) \geq 1+\gamma(1-\rho)$ or $\operatorname{cong}\left(f_{\gamma}(\varphi)\right) \leq 1$. Applying the $\alpha-$ approximation to $f_{\gamma}(\varphi)$ provides an approximate value $\tilde{\beta}$ for congestion. If $\tilde{\beta}<1+\gamma(1-\rho)$ holds, then we can deduce that $\operatorname{cong}\left(f_{\gamma}(\varphi)\right) \leq \tilde{\beta}<1+\gamma(1-\rho)$. This implies that $\operatorname{cong}\left(f_{\gamma}(\varphi)\right) \leq 1$ and hence $\operatorname{val}(\varphi)<\rho$. Otherwise, we have $\tilde{\beta} \geq 1+\gamma(1-\rho)$ and $\alpha \times \operatorname{cong}\left(f_{\gamma}(\varphi)\right) \geq \tilde{\beta}$ (since $\tilde{\beta}$ is an $\alpha$-approximation), leading to $\operatorname{cong}\left(f_{\gamma}(\varphi)\right) \geq \frac{1+\gamma(1-\rho)}{\alpha}>1$. We consequently have $\operatorname{cong}\left(f_{\gamma}(\varphi)\right) \geq$ $1+\gamma(1-\rho)$ and $\operatorname{val}(\varphi)=1$. This proves that a constant $\alpha$-approximation for the congestion problem allows the solution of Gap-3-SAT.

Furthermore, as the size of the polytope used in Lemma 2.1 is $O\left(m^{c \gamma}\right)$ while $\left|V\left(G_{\gamma}\right)\right|=O\left(m^{\gamma}\right)$, its size is polynomially bounded in the number of vertices as announced in Theorem 2.1

We are now going to prove Lemma 2.1 by first constructing instances of the congestion problem leading to some inapproximabilty factor. Then, this factor is increased by recursively building larger instances with higher values of $\gamma$.

Proof. of Lemma 2.1, case $\gamma=1$
We start with a 3-SAT formula $\varphi$, with $m$ clauses and $r$ variables. We note $\mathcal{L}=\left\{l_{1}, \ldots, l_{r}, \neg l_{1}, \ldots, \neg l_{r}\right\}$ the set of the literals appearing in formula $\varphi$ and $l_{i, j}$ the literal appearing in the $i$-th clause $C_{i}$ at the $j$-th position for $i=1, \ldots, m$ and $j=1,2,3$ (it is not restrictive to assume that each clause contains exactly 3 literals).

We build as follows a graph $G_{1}$ and a set of commodities $\mathcal{H}_{1}$. For each $i=1, \ldots, m, j=1,2,3$ we add 3 consecutive edges $e_{i, j}$ (i.e. such that $t\left(e_{i, 1}\right)=$ $s\left(e_{i, 2}\right)$ and $\left.t\left(e_{i, 2}\right)=s\left(e_{i, 3}\right)\right)$ and 3 commodities $h_{i, j}$ with $s\left(h_{i, j}\right)=s\left(e_{i, j}\right)$ and $t\left(h_{i, j}\right)=t\left(e_{i, j}\right)$. We impose that all nodes $s\left(e_{i, 1}\right)\left(\right.$ resp. $\left.t\left(e_{i, 3}\right)\right)$ for $i=1, \ldots, m$ are equal to a single node noted $s_{1}$ (resp. $t_{1}$ ) (see Figure 1). We consider an additional commodity $h_{0}$ between $s_{1}$ and $t_{1}$.

We create a polyhedron $\mathcal{D}_{1}$ of the form $\mathcal{D}_{1}=\left\{d: A_{1} d+B_{1} \psi_{1} \leq b_{1}\right\}$ as follows. We consider for each literal $l \in \mathcal{L}$ a non-negative variable $\xi_{l}$ and add for $k=1, \ldots, r$ the constraint $\xi_{l_{k}}+\xi_{\neg l_{k}}=1$. We also consider, for $i=1, \ldots, m$ and $j=1,2,3$, the constraint $d_{h_{i, j}}=\xi_{l_{i, j}}$. A constraint related to $d_{h_{0}}$ is also integrated: $d_{h_{0}} \leq m(1-\rho)$. Finally, the capacity $c_{e}$ of each edge $e$ is here equal to $1\left(c_{e}=1\right)$.

If $\operatorname{val}(\varphi)=1$, then there is a demand vector such that for each path between $s_{1}$ and $t_{1}$ (there is one path corresponding to each clause), at least one commodity whose endpoints are on the path is equal to 1 (a commodity corresponding to a true literal). This implies that all paths are blocked and thus the optimal routing for commodity $h_{0}$ is to equally spread $m(1-\rho)$ between the $m$ paths leading to a congestion of $1+(1-\rho)$.

Let us now assume that $\operatorname{val}(\varphi)<\rho$. Notice first that the components of the extreme points of the polyhedron $\mathcal{D}_{1}$ are integers (except $d_{h_{0}}$ ). This is due to the fact that $\mathcal{D}_{1}$ can be seen as a coordinate projection of the higher dimensional polytope $\left\{(d, \psi) \mid A_{1} d+B_{1} \psi\right\}$ whose extreme points are obviously integers (except $d_{h_{0}}$ ). For such an extreme demand vector $d \in \mathcal{D}_{1}$ there are at least $m(1-\rho)$ free paths to route the demand $d_{h_{0}}$ allowing a congestion less than or equal to 1 . This implies that all demands in $\mathcal{D}_{1}$ can also be routed with a congestion less than or equal to 1

Observe that $\left|V\left(G_{1}\right)\right|=O(m),\left|E\left(G_{1}\right)\right|=O(m), \mathcal{D}_{1}$ has the appropriate form $\left(\mathcal{D}_{1}=\left\{d: A_{1} d+B_{1} \psi_{1} \leq b_{1}\right\}\right)$ and the size of $\mathcal{D}_{1}$ is $O\left(m^{c}\right)$ for some constant $c$.

Proof. of Lemma 2.1, case $\gamma \geq 2$
For $\gamma \geq 2$, having constructed $G_{\gamma-1}, \mathcal{H}_{\gamma-1}, \mathcal{D}_{\gamma-1}$, we build $G_{\gamma}, \mathcal{H}_{\gamma}, \mathcal{D}_{\gamma}$ as follows. We will construct the graph $G_{\gamma}$, by taking the graph $G_{1}$ and replacing each edge by a copy of the graph $G_{\gamma-1}$ denoted by $G_{\gamma-1}^{i, j}$. Each copy $G_{\gamma-1}^{i, j}$ contains a node $s_{\gamma-1}$ that is identified with $s\left(e_{i, j}\right)$ and a node $t_{\gamma-1}$ identified with $t\left(e_{i, j}\right)$ (see Figure 1). All commodities related to $G_{\gamma-1}^{i, j}$ (belonging to $\mathcal{H}_{\gamma-1}$ ) are also considered as commodities of $\mathcal{H}_{\gamma}$. Let us use $d^{i, j} \in \mathbb{R}^{\mathcal{H}_{\gamma-1}}$ to denote the related demand vector. $\mathcal{H}_{\gamma}$ also contains a non-
negative commodity $h_{0, \gamma}$ constrained by $d_{h_{0, \gamma}} \leq m^{\gamma}(1-\rho)$. Thus $\left|\mathcal{H}_{\gamma}\right|=$ $1+3 m \times\left|\mathcal{H}_{\gamma-1}\right|$.

We are going to build an uncertainty set $\mathcal{D}_{\gamma}$ as a coordinate projection of a higher-dimensional polyhedron $\Xi_{\gamma}$, involving demand variables in addition to auxiliary non-negative variables $\xi_{l}$ related to literals, and also auxiliary variables $\psi_{\gamma-1}^{i, j}$ related to $G_{\gamma-1}^{i, j}$ and the description of $\mathcal{H}_{\gamma-1}$. We gradually explain the construction. For $k=1, \ldots, r$, we add the constraint $\xi_{l_{k}}+\xi_{\neg l_{k}}=1$. And for $e_{i, j} \in E\left(G_{1}\right)$, we impose that $d^{i, j} \in \xi_{l_{i, j}} \mathcal{D}_{\gamma-1}:=\left\{\xi_{l_{i j}} d_{0} \mid d_{0} \in \mathcal{D}_{\gamma-1}\right\}$. Let us explain how this can be done. By induction, we know that $\mathcal{D}_{\gamma-1}=\{d$ : $\left.A_{\gamma-1} d+B_{\gamma-1} \psi_{\gamma-1} \leq b_{\gamma-1}\right\}$ and this representation includes (among others) non-negativity constraints of all variables in addition to constraints implying that all variables are upper-bounded. Then by writing $A_{\gamma-1} d^{i, j}+B_{\gamma-1} \psi^{i, j} \leq$ $\xi_{l_{i, j}} b_{\gamma-1}$, we can ensure that $\xi_{l_{i, j}}=0$ implies $d^{i, j}=0$, while $\xi_{l_{i, j}}>0$ leads to $\frac{1}{\xi_{l, j}} d^{i, j} \in \mathcal{D}_{\gamma-1}$. In particular when $\xi_{l_{i, j}}=0$, from outside, the whole subgraph corresponding to $G_{\gamma-1}^{i, j}$ acts like a single edge of capacity $m^{\gamma-1}$.
$\mathcal{D}_{\gamma}$ can be seen as the projection of a polytope $\Xi_{\gamma}=\left\{\left(d, \psi_{\gamma}\right) \mid A_{\gamma} d+B_{\gamma} \psi_{\gamma} \leq\right.$ $\left.b_{\gamma}\right\}$ where $\psi_{\gamma}$ contains the auxiliary variables appearing in all levels. More precisely, $\Xi_{\gamma}$ is defined by:

$$
\begin{align*}
d_{h_{0, \gamma}} & \leq m^{\gamma}(1-\rho) \\
-d_{h_{0, \gamma}} & \leq 0, \\
-\xi_{l} & \leq 0, \quad \forall l \in \mathcal{L} \\
\xi_{l_{k}}+\xi_{\neg l_{k}} & \leq 1, \quad \forall k=1, \ldots, r \\
-\xi_{l_{k}}-\xi_{\neg l_{k}} & \leq-1, \quad \forall k=1, \ldots, r \\
A_{\gamma-1} d^{i, j}+B_{\gamma-1} \psi_{\gamma-1}^{i, j}-\xi_{l_{i, j}} b_{\gamma-1} & \leq 0, \quad \forall i=1, \ldots, m, j=1,2,3 . \tag{1}
\end{align*}
$$

By simple induction, we have $\left|V\left(G_{\gamma}\right)\right|=O\left(m^{\gamma}\right),\left|E\left(G_{\gamma}\right)\right|=O\left(m^{\gamma}\right)$ and the size of $\mathcal{D}_{\gamma}$ is $O\left(m^{c \gamma}\right)$ where $c$ is some positive constant.

We observe that all extreme points of $\Xi_{\gamma}$ are such that $\xi_{l} \in\{0,1\}$ for $l \in \mathcal{L}$. To verify that, we first recall that constraints (1) are equivalent to $d^{i, j} \in \xi_{l_{i, j}} \mathcal{D}_{\gamma-1}$ (in this way, the vectors $\psi_{\gamma-1}^{i, j}$ can be ignored). Second, let $\mathcal{L}_{+}$ be the set of literals appearing in positive form. We observe that variables $\xi_{l}$ for $l \in \mathcal{L}_{+}$are pairwise independent. Only variables $d^{i, j}$ such that either $l_{i, j}=$ $l$ or $l_{i, j}=\neg l$ depend on $\xi_{l}$ since $d^{i, j} \in \xi_{l} \mathcal{D}_{\gamma-1}$ in the first case and $d^{i, j} \in(1-$ $\left.\xi_{l}\right) \mathcal{D}_{\gamma-1}$ in the second case. This immediately implies that given some arbitrary real vectors $q_{i, j}$ and $f$, minimizing $\sum_{i=1, . ., m ; j=1,2,3} q_{i, j}^{T} d^{i, j}+\sum_{l \in \mathcal{L}_{+}} f_{l} \xi_{l}$ is equiv-
alent to minimizing $\sum_{l \in \mathcal{L}_{+}} \xi_{l}\left(f_{l}+\sum_{i, j: l_{i, j}=l} \min _{d^{i, j} \in \mathcal{D}_{\gamma-1}} q_{i, j}^{T} d^{i, j}-\sum_{i, j: l_{i, j}=-l} \min _{d^{i, j} \in \mathcal{D}_{\gamma-1}} q_{i, j}^{T} d^{i, j}\right)$.
It is then clear that optimal $\xi_{l}$ values will be either 0 or 1 . Since this holds for an arbitrary linear objective function, we get the wanted result about extreme points.

Let us now show that $\operatorname{val}(\varphi)<\rho \Longrightarrow \operatorname{cong}\left(f_{\gamma}(\varphi)\right) \leq 1$. Assume that $\operatorname{val}(\varphi)<\rho$. We prove by induction that the congestion of $\left(G_{\gamma}, \mathcal{H}_{\gamma}, \mathcal{D}_{\gamma}\right)$ is 1. Suppose that this is true for some $\gamma-1$. If $\xi_{l_{i, 1}}=\xi_{l_{i, 2}}=\xi_{l_{i, 3}}=0$ for some $i$, a flow of value $m^{\gamma-1}$ can be routed between $s_{\gamma}$ and $t_{\gamma}$ by sending a flow of value 1 on each edge of $G_{\gamma-1}^{i, j}$ for $j=1,2,3$. Since $\operatorname{val}(\varphi)<\rho$, there are necessarily at least $m(1-\rho)$ such $i$, thus we can send the whole demand $m^{\gamma-1} m(1-\rho)=m^{\gamma}(1-\rho)$ this way. For the indices $i, j$ such that $\xi_{l_{i, j}}=1$, by the induction hypothesis $\left(\operatorname{cong}\left(f_{\gamma-1}(\varphi)\right) \leq 1\right)$, the demands inside $G_{\gamma-1}^{i, j}$ can be routed without sending more than one unit of flow on each edge of $G_{\gamma-1}^{i, j}$.

Notice that to show that all traffic vectors of $\mathcal{D}_{\gamma}$ can be routed with congestion 1, we considered demand vectors corresponding with $\{0,1\} \xi$ variables. The result shown above about extreme points is useful here since it allows us to say that each extreme point of $\mathcal{D}_{\gamma}$ can be routed with congestion less than or equal to 1 implying that each demand vector inside $\mathcal{D}_{\gamma}$ can also be routed with congestion less than or equal to 1 .

Let us now show that $\operatorname{val}(\varphi)=1 \Longrightarrow \operatorname{cong}\left(f_{\gamma}(\varphi)\right) \geq 1+\gamma(1-\rho)$. We are going to use induction to build a cut $\delta\left(C_{\gamma}\right)$ where $C_{\gamma}$ is set of vertices of $V\left(G_{\gamma}\right)$ containing $s_{\gamma}$ and not containing $t_{\gamma}$. The number of edges of the cut will be $m^{\gamma}$ and each edge has a capacity equal to 1 . We also show the existence of a demand vector $d \in \mathcal{D}_{\gamma}$ such that the sum of the demands traversing the cut is greater than or equal to $m^{\gamma}(1+\gamma(1-\rho))$. This would show that there is at least one edge that carries at least $1+\gamma(1-\rho)$ units of flow.

Since $\varphi$ is satisfiable, there is a truth assignment represented by $\xi$ variables (the auxiliary variables) such that for each $i=1, \ldots, m$ there is a $j(i)$ such that $\xi_{l_{i, j(i)}}=1$. By considering the graph $G_{\gamma-1}^{i, j(i)}$ and using the induction hypothesis, we can build a cut $\delta\left(C_{\gamma-1}^{i}\right)$ separating the node $s\left(e_{i, j(i)}\right)$ and $t\left(e_{i, j(i)}\right)$ and containing $m^{\gamma-1}$ edges. We also build a demand vector $d^{i, j(i)} \in$ $\mathcal{D}_{\gamma-1}$ such that the sum of demands traversing the cut is greater than or equal
to $m^{\gamma-1}(1+(\gamma-1)(1-\rho))$ (still possible by induction). By taking the union of these $m$ disjoint cuts we get a cut $\delta\left(C_{\gamma}\right)$ that is separating $s_{\gamma}$ and $t_{\gamma}$ having the required number of edges. A demand vector $d$ can be built by combining the vectors $d^{i, j(i)}$ and the demand $d_{h_{0}, \gamma}$ taken equal to $m^{\gamma}(1-\rho)$. Since the demand from $s_{\gamma}$ to $t_{\gamma}$ is also traversing the cut, the total demand through $\delta\left(C_{\gamma}\right)$ is greater than or equal to $m^{\gamma}(1-\rho)+m \cdot m^{\gamma-1}(1+(\gamma-1)(1-\rho))=$ $m^{\gamma}(1+\gamma(1-\rho))$.

Lemma 2.1 can be further exploited in different ways since there are many possible connections between the value $1+\gamma(1-\rho)$ and the characteristics of the undirected graph built in the proof of the lemma. Observe, for example, that by a simple induction we get that the number of vertices $\left|V\left(G_{\gamma}\right)\right|=$ $2+2 m \frac{(3 m)^{\gamma}-1}{3 m-1}$ leading to $\left|V\left(G_{\gamma}\right)\right| \simeq 2 \times 3^{\gamma-1} m^{\gamma}$ (when $m$ goes to infinity). We also have $\Delta\left(G_{\gamma}\right)$ equal to $m^{\gamma}$ where $\Delta($.$) denotes the maximum degree$ in the graph. Consequently, $\log \left(\frac{\left|V\left(G_{\gamma}\right)\right|}{\Delta\left(G_{\gamma}\right)}\right) \simeq \gamma \log 3+\log 2 / 3$. Then by taking any constant $k$ such that $k \times \log 3<(1-\rho)$ where $\rho$ is the constant in the PCP Theorem we get a lower bound of the approximability ratio. This is stated in the following corollary.

Corollary 2.1. Under conditions of Theorem 2.1, for any constant $k<$ $\frac{1-\rho}{\log 3}$, it is not possible to approximate the minimum congestion problem in polynomial time within a ratio of $k \log \frac{|V(G)|}{\Delta(G)}$.

## 3. A $\Omega\left(\frac{\log n}{\log \log n}\right)$ approximability lower bound

To get an approximability lower bound, we will use the well-known ETH conjecture that is recalled below.

Conjecture 3.1 (Exponential Time Hypothesis). [31, 32] There is a constant $\delta$ such that no algorithm can solve 3-SAT instances in time $O\left(2^{\delta m}\right)$, where $m$ is the number of clauses.

Let us use $n$ to denote the number of vertices of the graph.
Theorem 3.1. Under Conjecture 3.1, there exists a constant $k$ such that no polynomial-time algorithm can solve the minimum congestion problem with the approximation ratio $k \frac{\log n}{\log \log n}$.

Proof. The combination of PCP Theorem and ETH Conjecture 3.1 implies that distinguishing between 3-SAT instances such that $\operatorname{val}(\varphi)<\rho$ and $\operatorname{val}(\varphi)=1$ cannot be done in time $O\left(2^{m^{\beta}}\right)$ for some constant $\beta>0$ (a slightly better bound is $O\left(2^{m / \log ^{c} m}\right)$ for some constant $c$, but this will not help us to improve the lower bound of Theorem 3.1).

Suppose that there is an algorithm that solves the minimum congestion problem with an approximation factor $\alpha(n)$ and a running time $O\left(n^{c_{1}}\right)$. Given a 3-SAT instance and a function $\gamma: \mathbb{N} \rightarrow \mathbb{N}$ we can construct a minimum congestion instance $f_{\gamma(m)}(\varphi)$ as in Lemma 2.1 in time $O\left(m^{c_{2} \gamma(m)}\right)$ and where the number of vertices of the instance is $m^{\gamma(m)}$. Then by running the approximation algorithm for minimum congestion we get a total time of $O\left(m^{c_{3} \gamma(m)}\right)$ where $c_{3}=\max \left\{c_{1}, c_{2}\right\}$. Thus by choosing $\gamma(m)=\frac{m^{\beta}}{c_{3} \log m}$ we get an algorithm that runs in time $O\left(2^{m^{\beta}}\right)$. And if the approximation factor $\alpha(n)$ is small enough, that is if $\alpha\left(m^{\gamma(m)}\right)<1+(1-\rho) \gamma(m)$ for a big enough $m$, we get an algorithm solving Gap-3-SAT and thus contradicting Conjecture 3.1. This is the case for $k \frac{\log n}{\log \log n}$ for some constant $k$. To see this, we can observe that:

$$
\frac{1+(1-\rho) \gamma(m)}{\alpha(m \gamma(m))}=\frac{1+(1-\rho) \frac{m^{\beta}}{c_{3} \log m}}{k^{\frac{m^{\beta} / \operatorname{cog} m}{\beta-\log c_{3}}}} \simeq \frac{\beta(1-\rho)}{k} \text {. By taking } k<\beta(1-\rho) \text { we get the }
$$ wanted inapproximability result.

## 4. From minimum congestion to linear costs

Given any $\lambda \geq 0$, the robust network design problem with linear costs is simply the following where $\mathcal{U}(\mathcal{D})$ is the set of possible capacity vectors defined in Section 2:

$$
\begin{equation*}
\min _{u \in \mathcal{U}(\mathcal{D})} \lambda^{T} u \tag{2}
\end{equation*}
$$

Assume that there exists a number $\alpha \geq 1$ such that Problem (2) can be solved in polynomial time within an approximation ratio $\alpha$. More precisely, we have a polynomial-time oracle that takes as input a non-negative linear $\operatorname{cost} \lambda \in \mathbb{R}^{E(G)}$ and outputs a $u^{a p}(\lambda) \in \mathcal{U}(\mathcal{D})$ such that $\lambda^{T} u(\lambda) \leq \lambda^{T} u^{a p}(\lambda) \leq$ $\alpha \lambda^{T} u(\lambda)$ where $u(\lambda) \in \mathcal{U}(\mathcal{D})$ is the optimal solution of (2).

Recall that the congestion problem is given by

$$
\begin{align*}
& \min _{\beta, u} \beta  \tag{3}\\
& u_{e} \leq c_{e} \beta, \forall e \in E(G) \\
& u \in \mathcal{U}(\mathcal{D})
\end{align*}
$$

where $\beta$ and $u$ are optimization variables.
Let us consider a Lagrange relaxation of (3) by dualizing the capacity constraints and using $\lambda$ for the dual multipliers. The dual problem is then given by $\max _{\lambda \geq 0} \min _{\beta, u \in \mathcal{U}(\mathcal{D})} \beta+\sum_{e \in E(G)} \lambda_{e}\left(u_{e}-\beta c_{e}\right)$ (where $\beta$ is an optimization variable). If $\lambda$ is chosen such $\sum_{e} \lambda_{e} c_{e} \neq 1$, then the value of the inner minimum would be $-\infty$. Thus in an optimal solution, we will always have $\sum_{e} \lambda_{e} c_{e}=1$. The problem is then equivalent to:

$$
\begin{equation*}
\max _{\substack{\lambda \geq 0 \\ e \in(G) \\ e \in E(G)}} \min _{u \in c_{e}=1} \sum_{u \in \mathcal{U}(\mathcal{D})} \sum_{e \in E(G)} \lambda_{e} u_{e}=\max _{\substack{\lambda \geq 0 \\ e \in \in(G) \\ \lambda_{e} c_{e}=1}} \lambda^{T} u(\lambda) . \tag{4}
\end{equation*}
$$

Since $\mathcal{U}(\mathcal{D})$ is polyhedral and all constraints and the objective function are linear, there is is no duality gap between (3) and (4).

Observe that (4) can be expressed as follows:

$$
\begin{align*}
& \max _{\beta, \lambda \geq 0} \beta  \tag{5a}\\
& \beta \leq \sum_{e \in E(G)} \lambda_{e} u_{e}, \forall u \in \mathcal{U}(\mathcal{D})  \tag{5b}\\
& 1=\sum_{e \in E(G)} \lambda_{e} c_{e} \tag{5c}
\end{align*}
$$

We are going to approximately solve (5) using a cutting-plane algorithm where inequalities (5b) are iteratively added by using the $\alpha$-approximation oracle. Let ( $\beta^{\prime}, \lambda^{\prime}$ ) be a potential solution of (5), we can run the $\alpha$-approximation of robust network design problem (2) with the cost vector $\lambda^{\prime}$ to get a solution $u^{a p}\left(\lambda^{\prime}\right)$. If $\beta^{\prime}>\sum_{e \in E(G)} \lambda_{e}^{\prime} u_{e}^{a p}\left(\lambda^{\prime}\right)$ we return the inequality $\beta \leq \sum_{e \in E(G)} \lambda_{e} u_{e}^{a p}\left(\lambda^{\prime}\right)$, otherwise the algorithm stops and returns ( $\beta^{\prime}, \lambda^{\prime}$ ). We know from the separationoptimization equivalence theorem [52] that (5) can be solved by making a polynomial number of calls to the separation oracle leading a globally
polynomial-time algorithm. Notice that this happens if the separation oracle is exact. In our case, the oracle is only an approximate one, implying that the cutting plane algorithm might be prematurely interrupted before obtaining the true optimum of (5). Observe however that this implies that the computing time is polynomially bounded. Let $(\tilde{\beta}, \tilde{\lambda})$ be the solution returned by the cutting-plane algorithm. Let $\left(\beta^{*}, \lambda^{*}\right)$ be the true optimal solution of (5). The next lemma states that the returned solution is an $\alpha$-approximation of the optimal solution.

Lemma 4.1. The cutting-plane algorithm computes in polynomial time a solution $\tilde{\beta}$ satisfying:

$$
\begin{equation*}
\beta^{*} \leq \tilde{\beta} \leq \alpha \beta^{*} . \tag{6}
\end{equation*}
$$

Proof. Observe that $\beta^{*}=\lambda^{* T} u\left(\lambda^{*}\right)$. Moreover, since (5) is equivalent to (4), we get that $\lambda^{* T} u\left(\lambda^{*}\right)=\beta^{*} \geq \tilde{\lambda}^{T} u(\tilde{\lambda})$. From the approximation factor of the oracle, one can write that $\tilde{\lambda}^{T} u^{a p}(\tilde{\lambda}) \leq \alpha \tilde{\lambda}^{T} u(\tilde{\lambda})$. Using the fact that no inequalities can be added for $(\tilde{\beta}, \tilde{\lambda})$, we get that $\tilde{\beta} \leq \tilde{\lambda}^{T} u^{a p}(\tilde{\lambda})$. Finally, since $\left(\beta^{*}, \lambda^{*}\right)$ is feasible for (5), we obviously have $\tilde{\beta} \geq \beta^{*}$. Combining the 4 previous inequalities leads to (6).

The above lemma has many consequences.
Theorem 4.1. Unless $P=N P$, the robust network design problem with linear costs cannot be approximated in polynomial time within any constant ratio. Unless the ETH conjecture is false, the robust network design problem with linear costs cannot be approximated within $\Omega\left(\frac{\log n}{\log \log n}\right)$.

Proof. The result is an immediate consequence of Theorems 2.1, 3.1 and Lemma 4.1.

The theorem above answers a long-standing open question of [34]. All other inapproximability results proved for the congestion problem directly hold for the robust network design problem with linear cost.

Another important consequence is that the congestion problem can be approximated within $O(\log n)$. This result was already proved in [33] using other techniques. In our case, the result is an immediate consequence of the $O(\log n)$-approximation algorithm for the robust network design problem with linear cost provided by $[53,54]$ and fully described in $[34,35]$.

Theorem 4.2. [33] Congestion can be approximated within $O(\log n)$.
Notice that Theorem 3.1 tells us that the ratio $O(\log n)$ is tight.
Starting from the results of $[42,43]$ showing the existence of instances for which the ratio $\frac{\text { cong }_{\text {stat-frac }}}{\text { cong } g_{\text {dyn }} \text { frac }}$ is $\Omega(\log n)$, one can also use the reduction above to prove, differently from [35], the existence of instances for which the ratio $\frac{\text { Lin }_{\text {stat-frac }}}{\text { Lin }_{\text {dyn-frac }}}$ is $\Omega(\log n)$ where a linear cost is minimized.

Proof. Similarly to $\mathcal{U}(\mathcal{D})$ defined when dynamic routing is considered, let $\mathcal{U}_{\text {stat }}(\mathcal{D})$ be the set of capacity vectors for which there exists a static fractional routing satisfying all demand vectors of $\mathcal{D}$. $\mathcal{U}_{\text {stat }}(\mathcal{D})$ is obviously a polyhedral set. The mathematical programs (3), (4) and (5) can be considered in the same way: we only have to replace $\mathcal{U}(\mathcal{D})$ by $\mathcal{U}_{\text {stat }}(\mathcal{D})$. All results stated above about the equivalence of (3), (4) and (5) still hold in the static case. Consider an instance from [42, 43] for which $\frac{\text { cong }_{\text {stat-frac }}}{\text { cong }_{\text {dyn-frac }}}$ is $\Omega(\log n)$. $\operatorname{cong}_{\text {stat-frac }}$ is computed from (5). Then there is at least one vector $\lambda^{s t a t} \geq 0$ and one vector $u^{\text {stat }} \in \mathcal{U}_{\text {stat }}(D)$ such that cong $_{\text {stat-frac }}=\sum_{e \in E(G)} \lambda_{e}^{\text {stat }} u_{e}^{\text {stat }}$ and $\sum_{e \in E(G)} \lambda_{e}^{\text {stat }} c_{e}=1$. This implies that $u^{\text {stat }}$ is an optimal solution of the linear problem where we minimize $\sum_{e \in E(G)} \lambda_{e}^{s t a t} u_{e}$ under the condition $u \in \mathcal{U}_{\text {stat }}(D)$. We consequently have $\operatorname{cong}_{\text {stat-frac }}=$ Lin $_{\text {stat-frac }}$ for the considered instance.
Let $u^{\prime} \in \mathcal{U}(\mathcal{D})$ be an optimal solution minimizing the linear cost $\sum_{e \in E(G)} \lambda_{e}^{\text {stat }} u_{e}$ under dynamic routing. In other words, $\operatorname{Lin}_{d y n-f r a c}=\sum_{e \in E(G)} \lambda_{e}^{s t a t} u_{e}^{\prime}$ when the coefficients of the objective function are $\lambda^{\text {stat }}$. Moreover, we know from (5) that cong dyn-frac is obtained by maximizing through $\lambda$, implying that cong $_{\text {dyn-frac }} \geq \sum_{e \in E(G)} \lambda_{e}^{\text {stat }} u_{e}^{\prime}=$ Lin $_{\text {dyn-frac }}$. Using that $\frac{\text { cong }_{\text {stat-frac }}}{\text { cong }_{\text {dyn }} \text { frac }}$. is $\Omega(\log n)$, we get that $\frac{\text { Lin }_{\text {stat-frac }}}{\text { Lind }_{\text {dyn-frac }}}$ is $\Omega(\log n)$ for the same instance where the linear objective function is defined through $\lambda^{\text {stat }}$.

## 5. Restriction to a constant number of given paths per commodity

First, observe that in the proof of Lemma 2.1, the minimum congestion instances built there are such that some commodities can be routed along many paths. For example, in graph $G_{1}$ (Figure 1), commodity $h_{0}$ (between $s$ and $t$ ) can use up to $m$ paths. Second, consider an instance of the minimum congestion problem where only one path is given for each commodity.


Figure 2: $G^{\prime}$

Then computing the minimum congestion is easy since we only have to compute $\max _{d \in \mathcal{D}} \sum_{h \in \mathcal{H}_{e}} d_{h}$ where $\mathcal{H}_{e}$ denotes the set of commodities routed through $e$. The congestion is just given by $\max _{e \in E(G)} \frac{1}{c_{e}} \max _{d \in \mathcal{D}} \sum_{h \in \mathcal{H}_{e}} d_{h}$. Combining these two observations, one can wonder whether the difficulty of the congestion problem is simply due to the number of possible paths that can be used by each commodity. We will show that the problem is still difficult even if each commodity can be routed along at most two fixed given paths.

Theorem 5.1. Unless $P=N P$, for some positive constant $k$, minimum congestion cannot be approximated within a ratio $k$ even if each commodity can be routed along at most two given paths.

Proof. The proof is a simple modification of the proof of Lemma 2.1 (case $\gamma=1$ ). We are going to slightly modify graph $G_{1}$ in such a way that at most 2 paths are allowed for each commodity. Given a 3-SAT formula $\varphi$ with $m$ clauses, we construct $G^{\prime}, \mathcal{H}^{\prime}, \mathcal{D}^{\prime}$ as follows. We first create two nodes $s_{1}$ and $t_{1}$ and an edge $e_{0}$ between $s_{1}$ and $t_{1}$ of capacity $m \rho$ ( $\rho$ is the constant in PCP theorem). Then for each clause index $i=1, \ldots, m$, as in Lemma 2.1, we create 3 consecutive edges $e_{i, j}(j=1,2,3)$ such that $t\left(e_{i, j}\right)=s\left(e_{i, j+1}\right)$ and a commodity $h_{i, j}$ between $s\left(e_{i, j}\right)$ and $t\left(e_{i, j}\right)$ that is allowed to be routed only through $e_{i, j}$. We also add one edge between $s\left(e_{i, 1}\right)$ and $s_{1}$ and one edge connecting $t_{1}$ and $t\left(e_{i, 3}\right)$ of infinite capacity and a commodity $h_{i, 0}$ between $s\left(e_{i, 1}\right)$ and $t\left(e_{i, 3}\right)$ with a demand $d_{h_{i, 0}}=1$. $h_{i, 0}$ is allowed to be routed only through the path $P_{i}$ containing the edges $\left(e_{i, 1}, e_{i, 2}, e_{i, 3}\right)$ and the path going through $s_{1}, e_{0}$ and $t_{1}$ (see Figure 2). We consider auxiliary variables $\xi_{l}$ for each literal $l$. We add constraints $\xi_{l}+\xi_{\neg l}=1$ and $d_{h_{i, j}}=\xi_{l_{i, j}}$.

If $\operatorname{val}(\varphi)<\rho$ there are at least $m(1-\rho)$ commodities $h_{i, 0}$ that can be routed on the paths $P_{i}$ and the remaining $m \rho$ can be routed on the edge $e_{0}$. This implies that each extreme point of $\mathcal{D}^{\prime}$ can be routed with congestion $\leq 1$. Notice that the observation made in the proof of Lemma 2.1 about extreme points is still valid here: extreme points corresponds to $0-1$ values of the variables $\xi_{l}$.

If $\operatorname{val}(\varphi)=1$, then there is a cut and a demand vector $d$ (corresponding to the truth assignment satisfying $\varphi$ ) such that the capacity of the cut is $m \rho+m$ and the demand that needs to cross the cut is 2 m . There is consequently at least one edge of congestion greater than or equal to $\frac{2 m}{(1+\rho) m}=\frac{2}{1+\rho}$. By taking $k<\frac{2}{1+\rho}$ we get the wanted result.

Finally, observe that the result above can also be stated for the linear cost case using again the Lagrange based reduction of the previous section.

Corollary 5.1. Unless $P=N P$, for some positive constant $k$, robust network design with linear costs is difficult to approximate within a ratio $k$ even if each commodity can be routed along at most two given paths.

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## References

1] N. Wang, K. H. Ho, G. Pavlou, M. Howarth, An overview of routing optimization for internet traffic engineering, IEEE Communications Surveys \& Tutorials 10 (1) (2008) 36-56.
[2] H. P. L. Luna, Network planning problems in telecommunications, in: Handbook of Optimization in Telecommunications, Springer, 2006, pp. 213-240.
[3] A. M. Costa, A survey on benders decomposition applied to fixed-charge network design problems, Computers \& operations research 32 (6) (2005) 1429-1450.
[4] A. Frangioni, B. Gendron, 0-1 reformulations of the multicommodity capacitated network design problem, Discrete Applied Mathematics 157 (6) (2009) 1229-1241.
[5] C. Raack, A. M. Koster, S. Orlowski, R. Wessäly, On cut-based inequalities for capacitated network design polyhedra, Networks 57 (2) (2011) 141-156.
[6] W. Ben-Ameur, H. Kerivin, New economical virtual private networks, Association for Computing Machinery. Communications of the ACM 46 (6) (2003) 69-69.
[7] W. Ben-Ameur, H. Kerivin, Routing of uncertain traffic demands, Optimization and Engineering 6 (3) (2005) 283-313.
[8] N. G. Duffield, P. Goyal, A. Greenberg, P. Mishra, K. K. Ramakrishnan, J. E. van der Merive, A flexible model for resource management in virtual private networks, in: Proceedings of the conference on Applications, technologies, architectures, and protocols for computer communication, 1999, pp. 95-108.
[9] J. A. Fingerhut, S. Suri, J. S. Turner, Designing least-cost nonblocking broadband networks, Journal of Algorithms 24 (2) (1997) 287-309.
[10] A. Gupta, J. Kleinberg, A. Kumar, R. Rastogi, B. Yener, Provisioning a virtual private network: a network design problem for multicommodity flow, in: Proceedings of the thirty-third annual ACM symposium on Theory of computing, 2001, pp. 389-398.
[11] C. Chekuri, F. B. Shepherd, G. Oriolo, M. G. Scutella, Hardness of robust network design, Networks: An International Journal 50 (1) (2007) 50-54.
[12] G. Claßen, A. M. Koster, M. Kutschka, I. Tahiri, Robust metric inequalities for network loading under demand uncertainty, Asia-Pacific Journal of Operational Research 32 (05) (2015).
[13] S. Mattia, The robust network loading problem with dynamic routing, Computational Optimization and Applications 54 (3) (2013) 619-643.
[14] W. Ben-Ameur, M. Żotkiewicz, Robust routing and optimal partitioning of a traffic demand polytope, International Transactions in Operational Research 18 (3) (2011) 307-333.
[15] M. Minoux, Robust network optimization under polyhedral demand uncertainty is NP-hard, Discrete Applied Mathematics 158 (5) (2010) 597603.
[16] A. Frangioni, F. Pascali, M. G. Scutellà, Static and dynamic routing under disjoint dominant extreme demands, Operations research letters 39 (1) (2011) 36-39.
[17] D. Applegate, E. Cohen, Making intra-domain routing robust to changing and uncertain traffic demands: Understanding fundamental tradeoffs, in: Proceedings of the 2003 conference on Applications, technologies, architectures, and protocols for computer communications, 2003, pp. 313-324.
[18] Y. Azar, E. Cohen, A. Fiat, H. Kaplan, H. Räcke, Optimal oblivious routing in polynomial time, Journal of Computer and System Sciences 69 (3) (2004) 383-394.
[19] S. Fortune, J. Hopcroft, J. Wyllie, The directed subgraph homeomorphism problem, Theoretical Computer Science 10 (2) (1980) 111-121.
[20] W. Ben-Ameur, A. Ouorou, M. Zotkiewicz, Robust routing in communication networks, Progress in Combinatorial optimization, Iste, Wiley, Mahjoub Eds. (2012) 353-389.
[21] M. Poss, A comparison of routing sets for robust network design, Optimization Letters 8 (5) (2014) 1619-1635.
[22] W. Ben-Ameur, Between fully dynamic routing and robust stable routing, in: 2007 6th International Workshop on Design and Reliable Communication Networks, IEEE, 2007, pp. 1-6.
[23] M. G. Scutellà, On improving optimal oblivious routing, Operations Research Letters 37 (3) (2009) 197-200.
[24] M. Zotkiewicz, W. Ben-Ameur, More adaptive robust stable routing, in: GLOBECOM IEEE Global Telecommunications Conference, 2009, pp. 1-6.
[25] A. Ouorou, J.-P. Vial, A model for robust capacity planning for telecommunications networks under demand uncertainty, in: International workshop on design and reliable communication networks, IEEE, 2007, pp. 1-4.
[26] A. Ben-Tal, A. Goryashko, E. Guslitzer, A. Nemirovski, Adjustable robust solutions of uncertain linear programs, Mathematical programming 99 (2) (2004) 351-376.
[27] M. Poss, C. Raack, Affine recourse for the robust network design problem: Between static and dynamic routing, Networks 61 (2) (2013) 180198.
[28] W. Ben-Ameur, M. Żotkiewicz, Multipolar routing: where dynamic and static routing meet, Electronic Notes in Discrete Mathematics 41 (2013) 61-68.
[29] W. Ben-Ameur, A. Ouorou, G. Wang, M. Żotkiewicz, Multipolar robust optimization, EURO Journal on Computational Optimization 6 (4) (2018) 395-434.
[30] S. Arora, B. Barak, Computational Complexity: A Modern Approach, 1st Edition, Cambridge University Press, New York, NY, USA, 2009.
[31] R. Impagliazzo, R. Paturi, On the complexity of k-sat, Journal of Computer and System Sciences 62 (2) (2001) 367-375.
[32] R. Impagliazzo, R. Paturi, F. Zane, Which problems have strongly exponential complexity?, Journal of Computer and System Sciences 63 (4) (2001) 512-530.
[33] H. Räcke, Optimal hierarchical decompositions for congestion minimization in networks, in: Proceedings of the fortieth annual ACM symposium on Theory of computing, 2008, pp. 255-264.
[34] C. Chekuri, Routing and network design with robustness to changing or uncertain traffic demands, ACM SIGACT News 38 (3) (2007) 106-129.
[35] N. Goyal, N. Olver, F. B. Shepherd, Dynamic vs. oblivious routing in network design, in: European Symposium on Algorithms, Springer, 2009, pp. 277-288.
[36] N. Olver, F. B. Shepherd, Approximability of robust network design, Mathematics of Operations Research 39 (2) (2014) 561-572.
[37] N. Goyal, N. Olver, F. B. Shepherd, The vpn conjecture is true, Journal of the ACM (JACM) 60 (3) (2013) 1-17.
[38] F. Eisenbrand, F. Grandoni, G. Oriolo, M. Skutella, New approaches for virtual private network design, SIAM Journal on Computing 37 (3) (2007).
[39] N. K. Olver, Robust network design, Ph.D. thesis, McGill University Library (2010).
[40] H. Racke, Minimizing congestion in general networks, in: The 43rd Annual IEEE Symposium on Foundations of Computer Science, 2002. Proceedings., IEEE, 2002, pp. 43-52.
[41] C. Harrelson, K. Hildrum, S. Rao, A polynomial-time tree decomposition to minimize congestion, in: Proceedings of the fifteenth annual ACM symposium on Parallel algorithms and architectures, 2003, pp. 34-43.
[42] B. M. Maggs, F. M. auf der Heide, B. Vocking, M. Westermann, Exploiting locality for data management in systems of limited bandwidth, in: Proceedings 38th Annual Symposium on Foundations of Computer Science, IEEE, 1997, pp. 284-293.
[43] Y. Bartal, S. Leonardi, Ondashline routing in all-optical networks, Theoretical Computer Science 221 (1-2) (1999) 19-39.
[44] M. Hajiaghayi, R. Kleinberg, T. Leighton, Semi-oblivious routing: lower bounds, in: Proceedings of ACM-SIAM symposium on Discrete algorithms, Citeseer, 2007, pp. 929-938.
[45] M. T. Hajiaghayi, R. D. Kleinberg, H. Räcke, T. Leighton, Oblivious routing on node-capacitated and directed graphs, ACM Transactions on Algorithms (TALG) 3 (4) (2007) 51.
[46] L. Fleischer, M. X. Goemans, V. S. Mirrokni, M. Sviridenko, Tight approximation algorithms for maximum general assignment problems, in: Proceedings of ACM-SIAM symposium on Discrete algorithms, 2006, p. 611-620.
[47] R. Carr, S. Vempala, Randomized metarounding, Random Structures \& Algorithms 20 (3) (2002) 343-352.
[48] K. Jain, M. Mahdian, M. R. Salavatipour, Packing steiner trees, in: Proceedings of ACM-SIAM symposium on Discrete algorithms, Vol. 3, 2003, pp. 266-274.
[49] N. Garg, J. Koenemann, Faster and simpler algorithms for multicommodity flow and other fractional packing problems, SIAM Journal on Computing 37 (2) (2007) 630-652.
[50] S. A. Plotkin, D. B. Shmoys, É. Tardos, Fast approximation algorithms for fractional packing and covering problems, Mathematics of Operations Research 20 (2) (1995) 257-301.
[51] E. Neal, Randomized rounding without solving the linear program, in: Proceedings of the sixth annual ACM-SIAM symposium on Discrete algorithms, Vol. 76, 1995, p. 170.
[52] M. Grötschel, L. Lovász, A. Schrijver, Geometric Algorithms and Combinatorial Optimization, Vol. 2 of Algorithms and Combinatorics, Springer, 1988.
[53] A. Gupta, Personal communication (2004).

676 [54] A. Gupta, J. Könemann, Approximation algorithms for network design: A survey, Surveys in Operations Research and Management Science 16 (1) (2011) 3-20.

## Declaration of interests

$\boxtimes$ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
$\square$ The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:



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