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# The Measurement and Analysis of Shapes

## An Application of Hydrodynamics and Probability Theory

James Benn · Stephen Marsland

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**Abstract** A de Rham  $p$ -current can be viewed as a map (the current map) between the set of embeddings of a closed  $p$ -dimensional manifold into an ambient  $n$ -manifold and the set of linear functionals on differential  $p$ -forms. We demonstrate that, for suitably chosen Sobolev topologies on both the space of embeddings and the space of  $p$ -forms, the current map is continuously differentiable, with an image that consists of bounded linear functionals on  $p$ -forms. Using the Riesz Representation Theorem we prove that each  $p$ -current can be represented by a unique co-exact differential form that has a particular interpretation depending on  $p$ .

Embeddings of a manifold can be thought of as shapes with a prescribed topology. Our analysis of the current map provides us with representations of shapes that can be used for the measurement and statistical analysis of collections of shapes. We consider two special cases of our general analysis and prove that: (1) if  $p = n - 1$  then closed, embedded, co-dimension one surfaces are naturally represented by probability distributions on the ambient manifold; and (2) if  $p = 1$  then closed, embedded, one-dimensional curves are naturally represented by fluid flows on the ambient manifold. In each case we outline some statistical applications using an  $\dot{H}^1$  and  $L^2$  metric, respectively.

**Keywords** Shape · Currents · Hodge theory · Sobolev diffeomorphisms · Euler Equations · Probability densities

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## 1 Introduction

This article presents a theoretical setting for the measurement and analysis of shapes. It was motivated by the fact that while the human brain seems to be naturally equipped with a metric that can distinguish between “shapes”, formulating a precise mathematical statement that allows one to formulate judgments like “objects 1 and 2 are the same” or “objects A and B are different” is non-trivial.

Restricting to the simplest possible case for a precise definition of a shape, we consider it as an embedding of a closed manifold  $N$  into a Euclidean space or some well-chosen closed manifold  $M$ ; this gives a set of shapes in  $M$  a prescribed topology – that of  $N$ . Equipping the space of embeddings of  $N$  into  $M$  with an appropriate function space topology gives the space of shapes the structure of a smooth Fréchet, Banach, or Hilbert manifold. From here one can equip the space with a metric and begin studying the metric geometry or Riemannian geometry of shapes. Alternatively, one can study the action of a transformation group on the space of shapes and use a metric that measures the “cost” of deforming one shape into another. Either approach may be used for the purpose of shape classification and recognition, or to give a statistical description of a collection of shapes, a problem that has found potential application in medical imaging.

In [17] Michor and Mumford considered the manifold of  $C^\infty$  closed, embedded curves  $\text{Emb}(S^1, \mathbb{R}^2)$  equipped with a Riemannian metric of Sobolev type. Elements of  $\text{Emb}(S^1, \mathbb{R}^2)$  represent shapes, but many elements represent the same shape – for instance, all that differ only by reparameterisation (i.e., composition on the right) by a circle diffeomorphism  $\eta \in \mathcal{D}(S^1)$  – so a shape space that results by quotienting out this action is commonly used.

The method of Large Deformation Diffeomorphic Metric Mapping (LDDMM) [5, 26] studies the action of the diffeomorphism group of  $M$ ,  $\mathcal{D}(M)$ , on the space of embeddings (which we will refer to as shapes), with applications in the field of medical imaging. Given a metric  $g$  on  $\mathcal{D}(M)$  and a fixed reference shape, a measure of similarity is given by the infimum over all curves connecting the identity diffeomorphism with a diffeomorphism that deforms the shape to the given reference shape. A shape is identical to the reference if there exists an isometry that maps the shape to the reference.

It can be beneficial to transform the shapes in some way prior to analysis. One option is to use de Rham  $p$ -currents, as in [24, 10, 8, 6], which were originally intended to provide a computationally efficient method for shape comparison using LDDMM. Note that some mathematical aspects have been explored in [7, 13] for both currents and varifolds. We suggest that currents are the correct theoretical tool to represent shapes for addressing the general problem and so provide an analysis of shape in the spirit of geometric measure theory [20] by treating the integral as a function of the domain of integration, in this case the shape. Our goal is to derive meaningful invariants of shapes that do not rely on fixed reference shapes or explicit knowledge of the mathematical embedding describing the shape, and then use these invariants to measure the similarity or dissimilarity of shapes. Furthermore, the set of embeddings of a closed manifold  $N_1$  into some ambient space is unrelated to the set of embeddings of a topologically distinct manifold  $N_2$ , so we require invariants that depend only on the dimension of the shape and not on the topology in order to compare (and potentially interpolate) between shapes of different topologies.

We consider the space  $\mathcal{E}_N^s$  consisting of Sobolev  $H^s$  embeddings of a closed,  $p$ -dimensional manifold  $N$  into the  $n$ -dimensional torus  $\mathbb{T}^n$  (note that the torus is used for notational convenience, and matches typical computational implementation, but not necessarily other analytic work) where  $s > \frac{n}{2} + 1$  so that each embedding is at least  $C^1$ . To each embedding in  $\mathcal{E}_N^s$  we assign a bounded linear functional on the space of Sobolev  $H^k$  differential  $p$ -forms over  $\mathbb{T}^n$  via the current map  $\mathcal{C}$  which, for any embedding  $\varphi \in \mathcal{E}_N^s$  and any  $H^k$   $p$ -form  $\alpha \in H^k(\Lambda^p(\mathbb{T}^n))$ , is defined by:

$$\mathcal{C}(\varphi)(\alpha) = \int_{\varphi(N)} \alpha.$$

Observe that  $\mathcal{C}(\varphi)$  is invariant under reparameterisations of  $N$  by Sobolev  $H^s$  diffeomorphisms,  $\mathcal{D}^s(N)$ ; i.e., the current map can be seen as an alternative to Michor and Mumford's shape space  $\mathcal{E}_N^s/\mathcal{D}^s(N)$ . So for us, the bounded linear functional  $\mathcal{C}(\varphi)$  is exactly the shape of the pre-shape  $\varphi$ . In Section 2 we prove that the current map has the following properties: (1) for any  $\varphi \in \mathcal{E}_N^s$ ,  $\mathcal{C}(\varphi)$  is a bounded linear functional on  $H^k(\Lambda^p(\mathbb{T}^n))$  whenever  $k > \frac{n}{2}$ ; (2) the current map is continuous whenever  $k > \frac{n}{2}$ ; and (3) the current map is differentiable whenever  $k > \frac{n}{2} + 1$ . A consequence of property (1) is that we may apply the Riesz Representation Theorem, which guarantees the existence of a unique  $H^k$   $p$ -form  $\beta_{\mathcal{C}(\varphi)}$  representing the shape  $\mathcal{C}(\varphi)$ . In Section 3 we prove that: (a) no two representing forms are collinear with the zero form; and (b) any representing form is always a co-exact form.

Following our general analysis we specialise to the situations where  $p = (n - 1)$ ,  $N$  any  $(n - 1)$ -dimensional manifold, and  $p = 1$ ,  $N = S^1$ , and interpret the results of our general analysis in each of these contexts.

When  $p = (n - 1)$  and  $N$  is any  $(n - 1)$ -dimensional manifold we use property (2) to show that each current of a closed, embedded, co-dimension one surface can be represented by a unique probability density on the ambient space, which can be computed using only the characteristic function of the shape and the Hodge Laplacian. Moreover, we are able to show that that Michor and Mumford's shape space  $\mathcal{E}_N^s/\mathcal{D}^s(N)$  is topologically embedded in the set of  $H^k$  probability densities. We equip the space of positive probability densities with the  $\dot{H}^1$  metric considered in [14] and explain how this can be used as a metric on co-dimension one shapes along with some statistical applications.

When  $p = 1$  and  $N = S^1$ , property (2) of our general results implies that the forms representing currents of closed, embedded curves correspond to divergence-free vector fields on  $\mathbb{T}^n$ . We regard this space as the tangent space to the identity element of the group (and smooth Hilbert manifold) of Sobolev  $H^k$  volume-preserving diffeomorphisms of  $\mathbb{T}^n$ . When equipped with the right-invariant  $L^2$  metric this group becomes a smooth Riemannian manifold with a smooth right-invariant Levi-Civita connection, smooth right-invariant curvature tensor, and smooth exponential map that is a diffeomorphism on a neighbourhood of the identity element [9]. In Section 5.1 we demonstrate how the  $L^2$  exponential map can be used to represent each shape by a unique volume-preserving diffeomorphism and how the  $L^2$  metric can be used as a metric on one dimensional shapes – an interesting counterpoint to the  $L^2$  metric on the space of embeddings of curves studied in [17], which was discovered to be degenerate. We note that Moser's theorem implies that the space of probability densities is equivalent to the ambient

space's diffeomorphism group quotiented by the volumorphism group and in this sense currents of one dimensional shapes are dual to the currents of co-dimension one shapes.

The fact that no two divergence-free vector fields constructed via the current map are collinear with the zero form (property (1)) implies that each closed embedded curve generates a unique local geodesic that exists on an open interval about 0 and constitutes a Lagrangian solution to the Euler equations of hydrodynamics on the ambient space. That is, each one-dimensional shape generates a unique fluid flow on  $\mathbb{T}^n$ , as is explained in Section 5.2. The fluid flows generated by shapes, which we call "shape flows", may also be of interest to researchers working in the field of hydrodynamics. It could be interesting to understand what fluids and their invariants can say about families of shapes in light of this observation. Conversely, it might be of interest to understand what shapes can say about fluids and to understand the kinds of flows they define.

Lastly, we mention that these ideas can be extended to the class of  $H^s$  immersions, which also form a smooth manifold. The difference in this situation is that two immersions that are not equivalent up to a reparameterisation by a diffeomorphism of  $N$  may still have the same point sets (consider a set of nested circles intersecting at a single point and permute the order of traversal of each circle) and therefore have the same current. This still produces the correct result because it does not matter how the shape is described, only how it looks. All of our results continue to hold with obvious alterations to take into account the difference just described; however, Michor and Mumford's shape space  $\mathcal{E}_N^s/\mathcal{D}^s(N)$  is no longer homeomorphic to an open subset of probability densities since the current map is not injective on equivalence classes for the reason given above.

## 2 The Current Map

This section sets out the central tools used in our construction of a framework for shape analysis: Hilbert spaces and the Riesz Representation Theorem in general; the particular Hilbert space of differential forms on which the Riesz Representation Theorem will be applied; the Hilbert manifold of embeddings considered as parameterised shapes; and finally the current map that links shapes with forms.

A real Hilbert space  $H$  is a complete linear space provided with an inner product, denoted  $(x, y)$  for  $x, y \in H$ , satisfying:

$$\begin{aligned} (ax_1 + bx_2, y) &= a(x_1, y) + b(x_2, y), \quad \forall a, b \in \mathbb{R} \\ (x, y) &= (y, x) \\ (x, x) &> 0 \text{ unless } x = 0. \end{aligned}$$

If we fix an element  $y \in H$  then the expression  $F_y(x) = (y, x)$  assigns to each  $x \in H$  a real number. Observe that  $F_y$  is linear in  $x$  and is bounded. We call  $F_y$  a bounded linear functional on  $H$ . It is a theorem of Riesz that these are the only bounded linear functionals on  $H$ , which we will denote collectively by  $H^*$ .

**Theorem.** *For every bounded linear functional  $F \in H^*$  on a (real) Hilbert space  $H$  there exists a unique  $y \in H$  such that:*

$$F(x) = (y, x), \quad \forall x \in H.$$

Furthermore, the Riesz operator, given by the map  $\mathcal{R} : H^* \rightarrow H$  that sends  $F$  to its corresponding  $y$ , is an isometric isomorphism.

The particular Hilbert space considered here is the set of Sobolev  $H^k$  differential  $p$ -forms on  $\mathbb{T}^n$ . The main references for differential forms and their algebra/analysis on which the following summary is based are [1] and [25].

Let  $g$  be the flat Euclidean metric on  $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$  and  $\mu$  the standard Lebesgue measure normalised so that  $\mu(\mathbb{T}^n) = 1$ , all presented in the standard coordinates  $(x_1, \dots, x_n)$ . If  $\alpha$  is a 1-form on  $\mathbb{T}^n$  then  $\alpha^\sharp$  is the vector field defined through  $\alpha(w) = g(\alpha^\sharp, w)$  for every vector field  $w$ ;  $\flat$  is its inverse. In this way we obtain an  $L^2$  metric on smooth differential 1-forms  $\Lambda^1(\mathbb{T}^n)$  defined by:

$$(\alpha, \beta) = \int_{\mathbb{T}^n} g(\alpha^\sharp, \beta^\sharp) d\mu.$$

For forms of degree  $p$  we first construct a point-wise metric on  $\Lambda^p T_x^* \mathbb{T}^n$  via:

$$g^p(\sigma_1 \wedge \dots \wedge \sigma_p, \omega_1 \wedge \dots \wedge \omega_p)(x) = \sum_{\pi} \text{sign}(\pi) g(\sigma_1^\sharp, \omega_{\pi(1)}^\sharp)(x) \cdots g(\sigma_p^\sharp, \omega_{\pi(p)}^\sharp)(x),$$

where  $\sigma_i, \omega_i \in T_x^* \mathbb{T}^n$ ,  $\wedge$  is the wedge product, and  $\pi$  ranges over the set of permutations of  $\{1, \dots, p\}$ , and then define an  $L^2$  metric on smooth differential  $p$ -forms  $\Lambda^p(\mathbb{T}^n)$  by:

$$(\alpha, \beta)_{L^2} = \int_{\mathbb{T}^n} g^p(\alpha, \beta) d\mu.$$

When  $p = 0$  (i.e., functions) we use the standard  $L^2$  metric on functions. We have the exterior derivative  $d$ , which takes  $p$ -forms to  $(p+1)$ -forms (with the understanding that  $d$  of an  $n$ -form is identically zero), and its formal  $L^2$  dual  $\delta$ , which takes  $(p+1)$ -forms to  $p$ -forms (with the understanding that  $\delta$  of a function is identically zero). In particular, for any smooth  $p$ -form  $\alpha \in \Lambda^p(\mathbb{T}^n)$  and any smooth  $(p+1)$ -form  $\beta \in \Lambda^{p+1}(\mathbb{T}^n)$ , we have:

$$\int_{\mathbb{T}^n} g^{p+1}(d\alpha, \beta) d\mu = \int_{\mathbb{T}^n} g^p(\alpha, \delta\beta) d\mu. \quad (1)$$

The Hodge star operator  $\star$  maps  $p$ -forms to  $(n-p)$ -forms. It is defined through  $g^p(\alpha, \beta) \mu = \alpha \wedge \star\beta$ , and relates  $d$  and  $\delta$  by:

$$\delta = (-1)^{n(p+1)+1} \star d\star, \quad (2)$$

while satisfying:

$$\star\star = (-1)^{p(n-p)}. \quad (3)$$

Furthermore, if  $\{e_1, \dots, e_n\}$  is an oriented basis for  $T_x^* \mathbb{T}^n$  we have:

$$\star(e_{i_1} \wedge \dots \wedge e_{i_p}) = \text{sign}(\pi) e_{j_1} \wedge \dots \wedge e_{j_{n-p}}, \quad (4)$$

where  $\{i_1, \dots, i_p, j_1, \dots, j_{n-p}\} = \{1, \dots, n\}$  and  $\pi$  is the permutation mapping the one ordered set to the other.

The positive-definite,  $L^2$  self-adjoint Hodge Laplacian is given by  $\Delta := d\delta + \delta d$  and commutes with  $d$ ,  $\delta$  and  $\star$ . On functions, the Hodge Laplacian reduces to the usual Laplacian:

$$\Delta = \delta d = -\left(\partial_{x_1}^2 + \dots + \partial_{x_n}^2\right),$$

while for general  $p$ -forms  $\alpha = \sum_{1 \leq i_1 \leq \dots \leq i_p \leq n+1} f_{i_1, \dots, i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$  with coefficient functions  $f_*$ , the operators  $d$ ,  $\delta$ , and  $\star$  can be used to show that:

$$\Delta \alpha = \sum_{1 \leq i_1 \leq \dots \leq i_p \leq n+1} (\Delta f_{i_1, \dots, i_p}) dx_{i_1} \wedge \dots \wedge dx_{i_p}. \quad (5)$$

However, it should be noted that (5) relies on the Euclidean structure of the torus and does not hold on a general Riemannian manifold  $(M, g)$  (see [25]).

Define the Sobolev  $H^k$  inner product on differential  $p$ -forms by:

$$(\alpha, \beta)_{H^k} = \int_{\mathbb{T}^n} g^p (A^k \alpha, \beta) d\mu, \quad (6)$$

where:

$$A^k = \sum_{r=0}^k a_r \Delta^r, \quad (7)$$

with  $a_0$  and  $a_k$  both strictly positive constants and  $a_1, \dots, a_{k-1}$  all non-negative constants. Using (5) and the definition of the  $L^2$  metric on  $p$ -forms, the induced  $H^k$  norm of a  $p$ -form  $\alpha$  can be understood as

$$\|\alpha\|_{H^k}^2 = \sum_{1 \leq i_1 \leq \dots \leq i_p \leq n} \|f_{i_1, \dots, i_p}\|_{H^k}^2 \quad (8)$$

where

$$\|f_{i_1, \dots, i_p}\|_{H^k}^2 = \int_{\mathbb{T}^n} (A^k f_{i_1, \dots, i_p}) \cdot f_{i_1, \dots, i_p} d\mu;$$

that is,  $\alpha$  is a vector whose squared  $H^k$  norm is equal to the sum of the squared  $H^k$  norms of its component functions. Define the Hilbert space of Sobolev  $H^k$   $p$ -forms  $H^k(\Lambda^p(\mathbb{T}^n))$  with inner product (6) as the completion of  $\Lambda^p(\mathbb{T}^n)$  in the  $H^k$  norm (8). This space consists of all those differential  $p$ -forms on  $\mathbb{T}^n$  whose component functions admit square-integrable derivatives up to order  $k$ . The component functions of elements of  $H^k(\Lambda^p(\mathbb{T}^n))$  are not necessarily classically  $C^k$  differentiable. However, if the index  $k > \frac{n}{2} + r$ , then the Sobolev Embedding Theorem guarantees that all component functions are at least  $C^r$  differentiable (see [11]).

On  $H^k(\Lambda^p(\mathbb{T}^n))$  the exterior derivative  $d$  takes  $H^k$   $p$ -forms to  $H^{k-1}$   $(p+1)$ -forms while  $\delta$  takes  $H^k$   $(p+1)$ -forms to  $H^{k-1}$   $p$ -forms. The Hodge decomposition [21] provides an  $L^2$ -orthogonal splitting of  $H^k(\Lambda^p(\mathbb{T}^n))$  into exact, co-exact, and harmonic subspaces

$$H^k(\Lambda^p(\mathbb{T}^n)) = dH^{k+1}(\Lambda^{p-1}(\mathbb{T}^n)) \oplus \delta H^{k+1}(\Lambda^{p+1}(\mathbb{T}^n)) \oplus \mathcal{H}_p^k, \quad (9)$$

where a form  $h$  is harmonic if and only  $dh = \delta h = \Delta h = 0$ . The Hodge Laplacian is an isomorphism between  $(\mathcal{H}_p^k)^\perp$  and  $(\mathcal{H}_p^{k-2})^\perp$  with inverse  $\Delta^{-1}$  (see [21] or [25]) that also commutes with the operators  $d$ ,  $\delta$ , and  $\star$ .

Let  $P$  denote the  $L^2$  orthogonal projection onto the space  $\delta H^{k+1}(\Lambda^{p+1}(\mathbb{T}^n))$  that is given by

$$P = \delta \Delta^{-1} d. \quad (10)$$

The operator  $P$  is continuous and commutes with  $\Delta$ ,  $\Delta^{-1}$ ,  $\delta$ , and  $d$  (see Ebin and Marsden [9]).

Lastly, given a vector field  $v$  on  $\mathbb{T}^n$  the interior product  $\iota_v$  maps a  $p$ -form  $\alpha$  to the  $p-1$  form  $\iota_v \alpha = \alpha(v, \cdot, \dots, \cdot)$  that can be expressed in coordinates for a simple form  $\alpha$  as:

$$\iota_v \alpha = \sum_{r=1}^n f_{i_1, \dots, i_p} (-1)^r v^r dx_{i_1} \wedge \dots \wedge \hat{dx}_r \wedge \dots \wedge dx_{i_p}, \quad (11)$$

where the hat marks that  $dx_r$  no longer appears in the wedge product.

Consider the set of Sobolev  $H^s$  embeddings of a closed  $p$ -dimensional  $N$  into  $\mathbb{T}^n$ :

$$\mathcal{E}_N^s = \{\varphi : N \rightarrow \mathbb{T}^n : \varphi \text{ is an embedding}\},$$

where  $s > \frac{p}{2} + 1$  so that each embedding is at least  $C^1$  [11]. The set  $\mathcal{E}_N^s$  is a connected open subset of the space of all Sobolev maps from  $N$  into  $\mathbb{T}^n$  and inherits the structure of a Hilbert manifold whose tangent space at a point  $\varphi$  consists of  $H^s$  vector fields  $v : N \rightarrow T\mathbb{T}^n$  covering  $\varphi$ .

One can think of  $\mathcal{E}_N^s$  as a space of parameterised shapes. It contains a set of descriptions of the kinds of shapes one might see in  $\mathbb{T}^n$ , with some elements describing the same shape. In order to understand the differences and similarities between shapes, or groups of shapes, we need to observe shapes interacting with other objects. The simplest and most natural interaction one can facilitate is through integrating a  $p$ -form on  $\mathbb{T}^n$  over the image of a shape description  $\varphi$ . So, the behavioural counter-part to  $\mathcal{E}_N^s$  is the set  $H^k(\Lambda^p(\mathbb{T}^n))$  of Sobolev  $H^k$   $p$ -forms on  $\mathbb{T}^n$  introduced above.

The interaction between shape and form is best captured by the current map

$$\mathcal{C} : \mathcal{E}_N^s \rightarrow H^k(\Lambda^p(\mathbb{T}^n))^* \quad (12)$$

which, for any  $p$ -form  $\alpha$ , is defined by:

$$\mathcal{C}(\varphi)(\alpha) = \int_{\varphi(N)} \alpha. \quad (13)$$

Since integration is a well-defined operation, any two descriptions  $\varphi$  and  $\psi$  of a shape that differ by a re-parameterisation of  $N$  will define the same current. So, for us, shape space is the set of currents associated to  $\mathcal{E}_N^s$ :

$$\mathcal{S}_{\mathcal{E}} = \mathcal{C}(\mathcal{E}_N^s).$$

*Remark 1* Contrast this with the approach of Michor and Mumford [17] who considered shape space as the quotient space:

$$\mathcal{B}^s(N, \mathbb{T}^n) = \mathcal{E}_N^s / \mathcal{D}^s(N), \quad (14)$$

where  $\mathcal{D}^s(N)$  the set of orientation preserving Sobolev diffeomorphisms of  $N$ . The space  $\mathcal{B}^s(N, \mathbb{T}^n)$  is known to be a Hausdorff topological space, but it does not appear to be a manifold since pre-composition with a Sobolev  $H^s$  diffeomorphism is only continuous [12]. The current map (12) can be viewed as an alternative way of writing the projection map onto the quotient.



The main properties of the current map that we prove are (1) the image of the current map consists of bounded linear functionals on  $H^k(\Lambda^p(\mathbb{T}^n))$ ; (2) the current map depends continuously on the embedding  $\varphi$  (i.e., continuous dependence on the domain of integration) for sufficiently large index  $k$ ; and (3) the current map depends differentiably on the embedding  $\varphi$  (i.e., differentiable dependence on the domain of integration) for sufficiently large index  $k$ .

**Proposition 1** For  $s > \frac{p}{2} + 1$  and  $\varphi \in \mathcal{E}_N^s$ :

1.  $\mathcal{C}(\varphi)$  is a bounded linear functional on  $H^k(\Lambda^p(\mathbb{T}^n))$  and  $\mathcal{C}(\mathcal{E}_N^s) \subseteq H^k(\Lambda^p(\mathbb{T}^n))^*$ , whenever  $k > \frac{n}{2}$ ;
2. the current map (12) is continuous whenever  $k > \frac{n}{2}$ ; and
3. the current map (12) is differentiable, with derivative a bounded linear operator given by:

$$(DC(\varphi) \cdot v)(\alpha) = \int_{\varphi(N)} i_v d\alpha \quad \forall \alpha \in H^k(\Lambda^p(\mathbb{T}^n)) \quad (15)$$

whenever  $k > \frac{n}{2} + 1$ .

*Proof.* The current map is clearly linear in  $\alpha$ , so it suffices to prove the statements for simple  $p$ -forms  $\alpha = f \cdot dx_{i_1} \wedge \dots \wedge dx_{i_p}$  of Sobolev class  $H^k$ . Let  $\varphi : N \rightarrow \mathbb{T}^n$  be an  $H^s$  embedding parameterised by  $N$  and  $\mathcal{A} = \{(U_j, \psi_j)\}$  a finite atlas on  $N$ . If  $x_i$  are the usual Euclidean coordinates on  $\mathbb{T}^n$  and  $y_i$  the coordinates in a chart  $(U_j, \psi_j)$ , then  $dx_{i_1} \wedge \dots \wedge dx_{i_p}$  transforms as  $\frac{\partial(x_{i_1}, \dots, x_{i_p})}{\partial(y_1^j, \dots, y_p^j)} dy_1^j \cdots dy_p^j = J^\varphi d\mathbf{y}$ . If  $x_r$  is the coordinate not appearing in  $J^\varphi$  then we will declare this through the notation  $J_r^\varphi$ .

In addition to the above notation we shall make use of the following facts throughout this proof:

1. the composition of the continuous functions  $f$ ,  $\varphi$ , and  $\psi_j$  is again continuous;
2. the supremum of  $f$  over  $\mathbb{T}^n$  is at least as big as its supremum over the image of  $\varphi$  in a chart  $(U_j, \psi_j)$  so that:

$$\|f \circ \varphi \circ \psi_j\|_{C^0(U_j)} \leq \|f\|_{C^0(\mathbb{T}^n)}; \quad (16)$$

3. the Sobolev Embedding Theorem guarantees the existence of a finite constant  $K^r(k, n)$  such that:

$$\|\alpha\|_{C^r(\Lambda^p(\mathbb{T}^n))} \leq K^r(k, n) \|\alpha\|_{H^k(\Lambda^p(\mathbb{T}^n))} \quad (17)$$

whenever  $k > \frac{n}{2} + r$ .

To prove the first statement of the proposition we observe that  $J^\varphi$  is a polynomial containing  $p!$  terms, each of degree  $p$ , involving derivatives of the component functions of  $\varphi$  up to order at most one and therefore satisfies:

$$|J^\varphi| \leq p! \cdot \|\varphi\|_{C^1}^p \leq p! \cdot \left(K^1(s, n)\right)^p \|\varphi\|_{H^s}^p.$$

Estimating in a coordinate chart using:

$$\begin{aligned}
|\mathcal{C}(\varphi|_{\psi_j(U_j)})(\alpha)| &= \left| \int_{U_j} (f \circ \varphi \circ \psi_j) \cdot J^\varphi \, d\mathbf{y} \right| \\
&\leq \|f \circ \varphi \circ \psi_j\|_{C^0(U_j)} \cdot p! \cdot \left(K^1(s, n)\right)^p \cdot \text{Vol}(N) \cdot \|\varphi\|_{H^s}^p \\
&\leq \|f\|_{C^0(\mathbb{T}^n)} \cdot p! \cdot \left(K^1(s, n)\right)^p \cdot \text{Vol}(N) \cdot \|\varphi\|_{H^s}^p \\
&\leq K^0(k, n) \cdot \|\alpha\|_{H^k(\Lambda^n(\mathbb{T}^n))} \cdot p! \cdot \left(K^1(s, n)\right)^p \cdot \text{Vol}(N) \cdot \|\varphi\|_{H^s}^p,
\end{aligned}$$

and taking the supremum over  $\|\alpha\|_{H^k(\Lambda^p(\mathbb{T}^n))} = 1$  gives:

$$\|\mathcal{C}(\varphi)\|_{H^k(\Lambda^p(\mathbb{T}^n))^*} \leq p! \cdot K^0(k, n) \cdot \left(K^1(s, n)\right)^p \cdot \text{Vol}(N) \cdot \|\varphi\|_{H^s}^p. \quad (18)$$

Consequently, the current map is a bounded linear functional in the  $H^k(\Lambda^p(\mathbb{T}^n))^*$  topology.

Continuity of the integral requires two things: the embeddings need to be at least  $C^1$  close so that the regions enclosed between two  $H^s$ -close embeddings are small, and the forms need to be at least  $C^0$  continuous so that their values on nearby embeddings are close. These two conditions are captured by the requirements that  $k > \frac{n}{2}$  and  $s > \frac{n}{2} + 1$ . Fix  $\varphi_o \in \mathcal{E}_N^s$  and write the current map in the following way:

$$\begin{aligned}
&|(\mathcal{C}(\varphi_o|_{\psi_j(U_j)}) - \mathcal{C}(\varphi|_{\psi_j(U_j)}))(\alpha)| \\
&= \left| \int_{U_j} (f \circ \varphi_o \circ \psi_j) \cdot J^{\varphi_o} - (f \circ \varphi \circ \psi_j) \cdot J^\varphi \, d\mathbf{y} \right| \\
&\leq \int_{U_j} |(f \circ \varphi_o \circ \psi_j) - (f \circ \varphi \circ \psi_j)| \cdot |J^{\varphi_o}| \, d\mathbf{y} \\
&+ \int_{U_j} |(f \circ \varphi \circ \psi_j)| \cdot |J^{\varphi_o} - J^\varphi| \, d\mathbf{y},
\end{aligned} \quad (19)$$

where we have added and subtracted  $(f \circ \varphi \circ \psi_j) \cdot J^{\varphi_o}$  and then grouped the terms.

Consider the term  $|f \circ \varphi_o \circ \psi_j - f \circ \varphi \circ \psi_j|$ . Since  $N$  is compact, the composition of  $f$  with  $\varphi$  is uniformly continuous and we can write the definition of continuity in a chart  $(U_j, \psi_j)$ : for any  $\epsilon_f > 0$  we can find a  $\delta_f > 0$  such that:

$$|f \circ \varphi_o \circ \psi_j - f \circ \varphi \circ \psi_j| < \epsilon_f \quad (20)$$

whenever:

$$\|\varphi_o \circ \psi_j - \varphi \circ \psi_j\|_{C^0} < \delta_f. \quad (21)$$

Now consider the term  $|J^{\varphi_o} - J^\varphi|$ . The determinant is a continuous function with respect to the entries of the corresponding matrix, and since  $N$  is compact and  $\varphi$  is  $C^1$  it follows that  $J^\varphi$  is uniformly continuous. We can also write the definition of continuity in a chart here: for any  $\epsilon_J > 0$  we can find a  $\delta_J > 0$  such that:

$$|J^{\varphi_o} - J^\varphi| < \epsilon_J \quad (22)$$

whenever:

$$\|\varphi_o \circ \psi_j - \varphi \circ \psi_j\|_{C^1} < \delta_J. \quad (23)$$

Using (20) and (22) (we will specify the conditions on  $\delta_f$  and  $\delta_J$  momentarily) we have:

$$\begin{aligned} & |(\mathcal{C}(\varphi_o|_{\psi_j(U_j)}) - \mathcal{C}(\varphi|_{\psi_j(U_j)}))(\alpha)| \\ & \leq \text{Vol}(N) \cdot p! \cdot \|\varphi_o\|_{C^1}^p \cdot \epsilon_f + \text{Vol}(N) \cdot \|f \circ \varphi \circ \psi_j\|_{C^0} \cdot \epsilon_J \\ & \leq \text{Vol}(N) \cdot p! \cdot \left(K^1(s, p)\right)^p \|\varphi_o\|_{H^s}^p \cdot \epsilon_f + \text{Vol}(N) \cdot \|f\|_{C^0(\mathbb{T}^n)} \cdot \epsilon_J \\ & \leq \text{Vol}(N) \cdot p! \cdot \left(K^1(s, p)\right)^p \|\varphi_o\|_{H^s}^p \cdot \epsilon_f + \text{Vol}(N) \cdot K^0(k, n) \|\alpha\|_{H^k(\Lambda^p(\mathbb{T}^n))} \cdot \epsilon_J. \end{aligned}$$

Taking the supremum over  $\|\alpha\|_{H^k(\Lambda^p(\mathbb{T}^n))} = 1$  gives:

$$\begin{aligned} \left\| (\mathcal{C}(\varphi_o|_{\psi_j(U_j)}) - \mathcal{C}(\varphi|_{\psi_j(U_j)})) \right\|_{H^k(\Lambda^p(\mathbb{T}^n))^*} & \leq \text{Vol}(N) \cdot p! \cdot \left(K^1(s, p)\right)^p \|\varphi_o\|_{H^s}^p \cdot \epsilon_f \\ & \quad + \text{Vol}(N) \cdot K^0(k, n) \cdot \epsilon_J. \end{aligned}$$

Let  $\epsilon > 0$ ; set  $\epsilon_f = \frac{\epsilon}{2 \cdot \text{Vol}(N) \cdot p! \cdot (K^1(s, p))^p \|\varphi_o\|_{H^s}^p}$  and  $\epsilon_J = \frac{\epsilon}{2 \cdot \text{Vol}(N) \cdot K^0(k, n)}$  with corresponding  $\delta_f$  and  $\delta_J$ , respectively, and let  $\delta = \frac{1}{K^1(s, p)} \cdot \min\{\delta_f, \delta_J\}$  with:

$$\|\varphi_o \circ \psi_j - \varphi \circ \psi_j\|_{H^s} < \delta.$$

Then, since:

$$\begin{aligned} \|\varphi_o \circ \psi_j - \varphi \circ \psi_j\|_{C^0} & \leq \|\varphi_o \circ \psi_j - \varphi \circ \psi_j\|_{C^1} \\ & \leq K^1(s, p) \|\varphi_o \circ \psi_j - \varphi \circ \psi_j\|_{H^s} \\ & < \min\{\delta_f, \delta_J\} \end{aligned}$$

it follows that:

$$\left\| (\mathcal{C}(\varphi_o|_{\psi_j(U_j)}) - \mathcal{C}(\varphi|_{\psi_j(U_j)})) \right\|_{H^k(\Lambda^p(\mathbb{T}^n))^*} < \epsilon$$

whenever:

$$\|\varphi_o \circ \psi_j - \varphi \circ \psi_j\|_{H^s} < \delta.$$

To prove that the current map is differentiable for  $k > \frac{n}{2} + 1$  we will compute the derivative formally and then justify the expression. Let  $\psi(t)$  be a  $C^1$  curve in  $\mathcal{E}_N^s$  with  $\psi(0) = \varphi$  and  $\partial_t|_{t=0}\psi(t) = v$ . Then:

$$\begin{aligned} (DC(\varphi) \cdot v)(\alpha) & = \partial_t|_{t=0} \mathcal{C}(\psi(t))(\alpha) \\ & = \left( \partial_t|_{t=0} \int_N \psi(t)^* \alpha \right) \\ & = \int_N \varphi^* \mathcal{L}_v \alpha \\ & = \int_{\varphi(N)} i_v d\alpha \end{aligned}$$

where we have used Stokes' Theorem in the last line. We now prove that  $DC(\varphi)$  is a bounded operator by estimating  $|(DC(\varphi) \cdot v)(\alpha)|$ .

We write the expression for the derivative in a chart  $(U_j, \psi_j)$  as:

$$(DC(\varphi) \cdot v)(\alpha) = \sum_{l=1}^n \sum_{r=1}^n (-1)^r \int_{U_j} (v_r \circ \psi_j) \cdot \left( \frac{\partial f}{\partial x^l} \circ \varphi \circ \psi_j \right) \cdot J_r^\varphi \, d\mathbf{y}.$$

Estimating just as in the proof of Proposition 1 gives:

$$\begin{aligned} |(DC(\varphi) \cdot v)(\alpha)| &\leq \sum_l \sum_r \|v_r \circ \psi_j\|_{C^0(U_j)} \cdot \left\| \frac{\partial f}{\partial x^l} \circ \varphi \circ \psi_j \right\|_{C^0(U_j)} \int_{U_j} J_r^\varphi \, d\mathbf{y} \\ &\leq C \cdot \|v\|_{H^s(U_j)} \cdot \|\alpha\|_{H^k(\Lambda^p(\mathbb{T}^n))} \cdot \|\varphi\|_{H^s}^p. \end{aligned}$$

Choosing the constant  $C$  so that this estimate holds in every chart we obtain:

$$|(DC(\varphi) \cdot v)(\alpha)| \leq C \cdot \|v\|_{H^s} \cdot \|\alpha\|_{H^k(\Lambda^p(\mathbb{T}^n))} \cdot \|\varphi\|_{H^s}^p.$$

Taking the supremum over  $\|\alpha\|_{H^k(\Lambda^p(\mathbb{T}^n))} = 1$  yields:

$$\|DC(\varphi) \cdot v\|_{H^k(\Lambda^p(\mathbb{T}^n))^*} \leq C \cdot \|v\|_{H^s} \cdot \|\varphi\|_{H^s}^p.$$

□

### 3 Riesz Representations of Currents

From here on we will assume that  $k > \frac{n}{2} + 1$  so that all of our forms are at least  $C^1$  and properties 1 – 3 of Proposition 1 all hold. Property 1 of Proposition 1 allows us to apply the Riesz Representation Theorem, which guarantees the existence of a unique form  $\beta_{\mathcal{C}(\varphi)} \in H^k(\Lambda^p(\mathbb{T}^n))$  associated to each shape  $\mathcal{C}(\varphi)$  that satisfies:

$$\mathcal{C}(\varphi)(\alpha) = (\beta_{\mathcal{C}(\varphi)}, \alpha)_{H^k} \quad (24)$$

for all  $\alpha \in H^k(\Lambda^p(\mathbb{T}^n))$ .

**Lemma 1** *No two Riesz representers that correspond to different shapes are collinear with the zero form.*

*Proof.* Suppose that  $\beta_{\mathcal{C}(\varphi)} = c \cdot \beta_{\mathcal{C}(\psi)}$  for two different shapes  $\mathcal{C}(\varphi)$  and  $\mathcal{C}(\psi)$  with  $c$  a non-zero constant. Then:

$$c \cdot \int_{\psi(N)} \alpha = c \cdot (\beta_{\mathcal{C}(\psi)}, \alpha)_{H^k} = (\beta_{\mathcal{C}(\varphi)}, \alpha)_{H^k} = \int_{\varphi(N)} \alpha,$$

and this identity must hold for all one-forms  $\alpha$ . Since there is a point  $x$  in  $\psi(N)$  that is not in  $\varphi(N)$  there is a neighbourhood of  $x$  that does not intersect  $\varphi(N)$  either. Choose a one-form  $\alpha$  that has support on this neighbourhood. Since both  $\varphi$  and  $\psi$  are embeddings, the form  $\alpha$  can be chosen so that the integral on the left-hand side of the above identity is non-zero, while the right-hand side of the identity is identically zero, which is a contradiction since  $c$  is non-zero. □

**Lemma 2** *For any  $\varphi \in \mathcal{E}_N^s$ , the Riesz representer  $\beta_{\mathcal{C}(\varphi)}$  of  $\mathcal{C}(\varphi)$  satisfies:*

$$\|\beta_{\mathcal{C}(\varphi)}\|_{H^k(\Lambda^p(\mathbb{T}^n))} \leq K^0(k, n) \cdot \left( K^1(s, n) \right)^p \cdot \text{Vol}(N) \cdot \|\varphi\|_{H^s}^p.$$

*Proof.* This follows directly from the Riesz Representation Theorem and (18).  $\square$

**Lemma 3** For  $\varphi \in \mathcal{E}_N^s$ , the Riesz representer  $\beta_{\mathcal{C}(\varphi)}$  of  $\mathcal{C}(\varphi)$  is given by:

$$\beta_{\mathcal{C}(\varphi)} = \delta\gamma + h$$

for some  $H^{k+1}$   $(p+1)$ -form  $\gamma$  and some harmonic  $p$ -form  $h$ .

*Proof.* By the Hodge decomposition, each representer  $\beta_{\mathcal{C}(\varphi)}$  takes the form:

$$\beta_{\mathcal{C}(\varphi)} = d\sigma + \delta\gamma + h, \quad (25)$$

where  $h$  is some harmonic form. It will be shown that the term  $d\sigma$  is always zero.

Using the projection  $P$  given in (10) and the Hodge star we have:

$$P \star \beta_{\mathcal{C}(\varphi)} = P \star d\sigma + P \star \delta\gamma + P \star h.$$

Since the Hodge star commutes with the Hodge Laplacian the Hodge star maps harmonic forms to harmonic forms and therefore the projection  $P \star h$  is zero. Using formulas (2) and (3):

$$\star\delta\gamma = (-1)^{n(p+2)+1} \star \star d \star \gamma = (-1)^{n(p+2)+1+p(n-p)} d \star \gamma,$$

and so the projection of this term to the image of  $\delta$  is also zero. Using (2) and (3) we can write  $\star d\sigma = (-1)^{n(p+2)+1+p(n-p)} \delta \star \sigma$  and hence the projection  $P$  acts like the identity on this term. More concretely, since  $\delta$  commutes with  $\Delta^{-1}$  and  $\delta \circ \delta = 0$  (which is implied by formulas (2), (3), and nilpotence of the exterior derivative  $d$ ):

$$\begin{aligned} P \star d\sigma &= \delta \Delta^{-1} d \star d\sigma = (-1)^{n(p+2)+1+p(n-p)} \Delta^{-1} \delta d \star \sigma \\ &= (-1)^{n(p+2)+1+p(n-p)} \Delta^{-1} (\delta d + d\delta) \star \sigma \\ &= (-1)^{n(p+2)+1+p(n-p)} \Delta^{-1} \Delta \delta \star \sigma \\ &= (-1)^{n(p+2)+1+p(n-p)} \delta \star \sigma \\ &= \star d\sigma. \end{aligned}$$

Therefore:

$$P \star \beta_{\mathcal{C}(\varphi)} = \star d\sigma.$$

On the other hand, for any  $(n-p)$ -form  $\alpha$ :

$$\star P\alpha = \star \delta \Delta^{-1} d\alpha = (-1)^{n((n-p+1)+1)+1} \star \star d \star \Delta^{-1} d\alpha = (-1)^{n^2-p^2+1} d \star \Delta^{-1} d\alpha.$$

Then, using self-adjointness of the projection  $P$  and the Hodge star, we have for any  $(n-p)$ -form  $\alpha$ :

$$\begin{aligned} (P \star \beta_{\mathcal{C}(\varphi)}, \alpha)_{H^k} &= (\beta_{\mathcal{C}(\varphi)}, \star P\alpha)_{H^k} = \int_{\varphi(N)} \star P\alpha \\ &= (-1)^{n^2-p^2+1} \int_{\varphi(N)} d \star \Delta^{-1} d\alpha \\ &= (-1)^{n^2-p^2+1} \int_{\partial\varphi(N)} \star \Delta^{-1} d\alpha \\ &= 0, \end{aligned}$$

where the penultimate equality follows from Stokes' Theorem and the last equality from the fact that  $\varphi(N)$  has empty boundary. As this holds for any  $(n-p)$ -form  $\alpha$  we have  $\pm \star d\sigma = P \star \beta_{\mathcal{C}(\varphi)} = 0$  so that  $d\sigma = 0$ .  $\square$

#### 4 The Distribution of the Shape: Probability and a Non-Degenerate $\dot{H}^1$ Metric on Co-Dimension One Shapes

We now demonstrate how the general results of Sections 2 and 3 may be applied to the situation in which  $N$  is any closed  $(n - 1)$ -dimensional manifold. Using Lemma 3 we show that each shape  $\mathcal{C}(\varphi)$  can be uniquely represented by a positive  $H^k$  probability density which can be calculated from the characteristic function of the embedding and the operator (7) whenever the coefficients satisfy a simple condition. Furthermore, we demonstrate that the shape space of Michor and Mumford  $\mathcal{B}^s(N, \mathbb{T}^n) = \mathcal{E}_N^s / \mathcal{D}^s(N)$  is topologically embedded in the manifold of positive  $H^k$  probability densities:

$$\text{Dens}^k(\mathbb{T}^n) = \left\{ \nu \in H^k(\Lambda^n(\mathbb{T}^n)) : \nu > 0, \int_{\mathbb{T}^n} d\nu = 1 \right\}.$$

In [14], the authors studied the homogeneous space of  $H^k$  probability densities of a closed manifold  $M$  equipped with a right-invariant  $\dot{H}^1$  Riemannian metric. With this metric they showed that the space of densities is isometrically diffeomorphic to a convex, open subset of an infinite-dimensional sphere in  $L^2(M)$  whose  $\dot{H}^1$  geodesic equation corresponds to a natural generalisation of the one-dimensional Hunter-Saxton equation of nematic liquid crystals. Furthermore, they demonstrated that: the  $\dot{H}^1$  metric induces the Fisher-Rao information metric [2] and the related Amari-Chentsov  $\alpha$ -connections of information geometry (see [15] for full details); the induced Riemannian distance coincides with a spherical analogue of the non-degenerate Hellinger distance in probability and mathematical statistics and admits a simple closed form expression; and the Bhattacharya coefficient of two normalised densities can be realised as the  $L^2$  inner product of the square roots of the corresponding Radon-Nikodym derivatives.

We show that the above  $\dot{H}^1$  metric induces a distance metric on shapes. A consequence of the differentiability of the current map is that, given a collection of evolving shapes one can now construct probability measures from this set that vary differentiably with the shapes. Furthermore, one can perform analysis on the measures and reconstruct the shapes to inspect the results. This has important applications in the study of shape evolution and analysis. The Riemannian distance induced by the aforementioned metric can be easily used as a quantitative measure of difference between two shapes, while the Bhattacharya distance provides an additional measure of similarity or ‘‘affinity’’ between them. The significance of our results is that all of the above statistical tools are now available for the analysis of codimension one shapes of any genus and of any dimension. That is, given a collection of shapes with corresponding distributions one can now employ the traditional tools of hypothesis testing, inference, and model fitting to answer questions of the form ‘‘are these distributions, and hence the shapes, statistically the same or are some of them different?’’. Answers to questions of this form are regularly sought in the field of medical imaging and are relevant to the problem of shape classification. Furthermore, shape space is now homeomorphic to a relatively open set of a sphere with antipodal points excluded so that local and global methods can be employed easily.

#### 4.1 Shapes and Probability Densities

The main result of this section is that from each Riesz representer  $\beta_{\mathcal{C}(\varphi)}$  we are able to construct a unique probability density on  $\mathbb{T}^n$  representing the unparameterised shape  $\mathcal{C}(\varphi)$ . According to Lemma 3, each  $\beta_{\mathcal{C}(\varphi)}$  has the form:

$$\beta_{\mathcal{C}(\varphi)} = \delta(\Omega_\varphi \cdot \mu) + h,$$

where  $\Omega_\varphi$  is an  $H^{k+1}$  function,  $\mu$  the Lebesgue measure, and  $h$  an harmonic  $(n-1)$ -form. We will first show that the harmonic part of each representer  $\beta_{\mathcal{C}(\varphi)}$  is zero and then show how one can obtain a unique  $n$ -form from each  $\beta_{\mathcal{C}(\varphi)}$ . Theorem 1 below gives conditions under which this  $n$ -form is a probability measure and shows how this probability measure can be uniquely constructed from the shape.

Let us set some notation: for  $\varphi \in \mathcal{E}_N^s$ , let  $M_\varphi$  denote the union of  $\varphi(N)$  with the region enclosed by it and let  $\chi_{M_\varphi}$  denote the characteristic (indicator) function of  $M_\varphi$  (see [16] for details on the Jordan-Brouwer Separation Theorem for hypersurfaces).

**Lemma 4** *For any  $\varphi \in \mathcal{E}_N^s$ , the Riesz representer  $\beta_{\mathcal{C}(\varphi)}$  of  $\mathcal{C}(\varphi)$  is given by:*

$$\beta_{\mathcal{C}(\varphi)} = \delta(\Omega_\varphi \cdot \mu),$$

where  $\Omega$  is an  $H^{k+1}$  function and  $\mu$  is the Lebesgue measure.

*Proof.* Let  $Q$  denote the orthogonal projection onto the space of harmonic forms  $\mathcal{H}_p^k$ . Then:

$$(Q\beta_{\mathcal{C}(\varphi)}, \alpha)_{H^k} = (\beta_{\mathcal{C}(\varphi)}, Q\alpha)_{H^k} = \int_{\varphi(N)} Q\alpha = \int_{M_\varphi} dQ\alpha = 0,$$

where the penultimate equality follows from Stokes's theorem and the last follows since  $dQ\alpha = 0$  as  $Q\alpha$  is harmonic. Since the left hand side is zero for any  $\alpha$ ,  $h = Q\beta_{\mathcal{C}(\varphi)} = 0$  and  $\beta_{\mathcal{C}(\varphi)}$  has the representation given in the statement.  $\square$

**Lemma 5** *Each unparameterised codimension one shape  $\mathcal{C}(\varphi)$  can be uniquely represented by a differential  $n$ -form:*

$$\Delta^{-1}d\beta_{\mathcal{C}(\varphi)} = \Omega_\varphi \cdot \mu$$

for some  $H^{k+1}$  function  $\Omega_\varphi$ . Furthermore, no two such  $n$ -forms corresponding to two different unparameterised shapes are a real scalar multiple of one another.

*Proof.* Consider the bounded linear operator:

$$\Delta^{-1}d : \delta H^{k+1}(\Lambda^n(\mathbb{T}^n)) \rightarrow H^{k+1}(\Lambda^n(\mathbb{T}^n)).$$

This is a linear isomorphism between the indicated spaces whose inverse is given by  $\delta$ ; indeed, for any  $\delta\beta \in \delta H^{k+1}(\Lambda^n(\mathbb{T}^n))$  we have:

$$\delta \circ \Delta^{-1}d(\delta\beta) = \Delta^{-1}\delta d(\delta\beta) = \Delta^{-1}(\delta d + d\delta)\delta\beta = \Delta^{-1}\Delta\delta\beta = \delta\beta,$$

where we have used  $\delta \circ \delta = 0$  in the second equality. For any  $\beta \in H^{k+1}(\Lambda^n(\mathbb{T}^n))$  we have:

$$\Delta^{-1}d \circ \delta(\beta) = \Delta^{-1}(d\delta + \delta d)\beta = \Delta^{-1}\Delta\beta = \beta,$$

where we have used the fact that  $d$  of an  $n$ -form is zero in the first equality. Thus  $\Delta^{-1}d$  is a left and right inverse to  $\delta$  on the space of  $n$ -forms. Applying  $\Delta^{-1}d$  to  $\beta_{\mathcal{C}(\varphi)}$  gives:

$$\Delta^{-1}d\beta_{\mathcal{C}(\varphi)} = \Omega_\varphi \cdot \mu.$$

Since  $\Delta^{-1}d$  is an isomorphism and the representer  $\beta_{\mathcal{C}(\varphi)}$  uniquely represents  $\mathcal{C}(\varphi)$ , the differential  $n$ -form  $\Omega_\varphi \cdot \mu$  uniquely represents  $\mathcal{C}(\varphi)$ .

Lemma 3 and linearity of the operators  $\Delta^{-1}d$  and  $\delta$  imply the second statement of the Lemma.  $\square$

**Theorem 1** *Suppose that  $k > \frac{n}{2} + 1$ , so that  $\sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{(2\pi|\xi|)^{2k}} < \infty$ , and let  $2a_o < \frac{a_k}{\sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{(2\pi|\xi|)^{2k}}}$ . Then each shape  $\mathcal{C}(\varphi)$  is represented by a unique  $H^{2k} \subset H^k$  probability measure:*

$$p_\varphi = \frac{1}{\int_{\mathbb{T}^n} \Omega_\varphi \cdot \mu} (\Omega_\varphi \cdot \mu) > 0, \quad (26)$$

obtained as the normalised solution to the PDE:

$$A^k \Omega_\varphi = \chi_{M_\varphi}. \quad (27)$$

*Proof.* By Lemma 5 each unparameterised codimension one shape can be uniquely represented by the differential  $n$ -form  $\Delta^{-1}d\beta_{\mathcal{C}(\varphi)} = \Omega_\varphi \cdot \mu$ . We only need to show that under the assumptions on  $A^k$  the function  $\Omega_\varphi$  is strictly positive on  $\mathbb{T}^n$ . We first derive the PDE for  $\Omega_\varphi$ . Let  $\nu = \Gamma \cdot \mu$  be any smooth  $n$ -form. Then, making use of (24), Stokes' Theorem in the integral over  $\varphi(N)$ , and the commutation of  $\Delta$  and  $\Delta^{-1}$  with the operators  $d$  and  $\delta$  we compute:

$$\begin{aligned} (A^k \Omega_\varphi, \Gamma)_{L^2} &= (A^k \star \Omega_\varphi, \star \Gamma)_{L^2} = (\Omega_\varphi \cdot \mu, \nu)_{H^k} \\ &= (\Delta^{-1}d\beta_{\mathcal{C}(\varphi)}, \nu)_{H^k} \\ &= (\beta_{\mathcal{C}(\varphi)}, \delta \Delta^{-1} \nu)_{H^k} \\ &= \int_{\varphi(N)} \delta \Delta^{-1} \nu \\ &= \int_{M_\varphi} d\delta \Delta^{-1} \nu \\ &= \int_{M_\varphi} \nu \\ &= \int_{\mathbb{T}^n} \chi_{M_\varphi} \cdot \Gamma \cdot \mu \\ &= (\chi_{M_\varphi}, \Gamma)_{L^2}. \end{aligned}$$

As this holds for any  $\Gamma$  we obtain:

$$A^k \Omega_\varphi = \chi_{M_\varphi}.$$



Since the right-hand side is never identically zero ( $\varphi$  is an embedding), neither is  $\Omega_\varphi$  and the Elliptic Regularity Theorem guarantees that  $\Omega_\varphi$  is unique, belongs to  $H^{2k}(\mathbb{T}^n) \subset H^k(\mathbb{T}^n)$ , and is at least  $C^1$  by the Sobolev Embedding Theorem – see [25].

Consider the Fourier series of  $\Omega_\varphi$ :

$$\mathcal{F}(\Omega_\varphi) = \sum_{\xi \in \mathbb{Z}^n} C_\xi \cos(2\pi\xi \cdot x) + D_\xi \sin(2\pi\xi \cdot x),$$

where:

$$C_\xi = (\Omega_\varphi, \cos(2\pi\xi \cdot x))_{L^2}, \quad D_\xi = (\Omega_\varphi, \sin(2\pi\xi \cdot x))_{L^2}.$$

The series converges in the  $L^2$  norm and  $\Omega_\varphi = \mathcal{F}(\Omega_\varphi)$  almost everywhere with respect to  $\mu$ . The Fourier coefficients can be calculated directly from (4.1) and are given by:

$$C_\xi = \frac{(\chi_{M_\varphi}, \cos(2\pi\xi \cdot x))_{L^2}}{\sum_{r=0}^k a_r (2\pi|\xi|)^{2r}}, \quad D_\xi = \frac{(\chi_{M_\varphi}, \sin(2\pi\xi \cdot x))_{L^2}}{\sum_{r=0}^k a_r (2\pi|\xi|)^{2r}}.$$

Under the conditions on the constants  $a_0$  and  $a_k$ , the Fourier series is strictly positive on  $\mathbb{T}^n$ . Indeed, since  $\cos(2\pi\xi \cdot x) \geq -1$  and  $\sin(2\pi\xi \cdot x) \geq -1$ ;  $C_\xi \leq \frac{\text{Vol}(\chi_{M_\varphi})}{\sum_{r=0}^k a_r (2\pi|\xi|)^{2r}}$  and  $D_\xi \leq \frac{\text{Vol}(\chi_{M_\varphi})}{\sum_{r=0}^k a_r (2\pi|\xi|)^{2r}}$ ;  $C_0 = \frac{\text{Vol}(\chi_{M_\varphi})}{a_0} > 0$ ; and  $-\frac{1}{\sum_{r=0}^k a_r (2\pi|\xi|)^{2r}} \geq -\frac{1}{a_k (2\pi|\xi|)^{2k}}$ , we can estimate:

$$\begin{aligned} \mathcal{F}(\Omega_\varphi) &\geq \frac{\text{Vol}(\chi_{M_\varphi})}{a_0} - \sum_{\xi \in \mathbb{Z}^n / \mathbf{0}} \frac{2 \cdot \text{Vol}(\chi_{M_\varphi})}{\sum_{r=0}^k a_r (2\pi|\xi|)^{2r}} \\ &\geq \text{Vol}(\chi_{M_\varphi}) \left( \frac{1}{a_0} - \frac{2}{a_k} \sum_{\xi \in \mathbb{Z}^n / \mathbf{0}} \frac{1}{(2\pi|\xi|)^{2k}} \right) \\ &> 0. \end{aligned}$$

This now implies that  $\Omega_\varphi > 0$  as well; since otherwise, suppose that there exists an  $x_o$  such that  $\Omega_\varphi(x_o) \leq 0 < \mathcal{F}(\Omega_\varphi)$ . Then, as  $\Omega_\varphi$  is at least  $C^1$  there exists a ball of radius  $\epsilon$ , centred at  $x_o$ , on which  $\Omega_\varphi$  is strictly less than  $\mathcal{F}(\Omega_\varphi)$ . This contradicts the fact that  $\Omega_\varphi$  agrees with  $\mathcal{F}(\Omega_\varphi)$  almost everywhere. Therefore:

$$p_\varphi = \frac{1}{\int_{\mathbb{T}^n} \Omega_\varphi \cdot \mu} (\Omega_\varphi \cdot \mu) > 0 \quad (28)$$

is a probability measure on  $\mathbb{T}^n$  associated to the shape  $\mathcal{C}(\varphi)$ . That this scaled  $n$ -form continues to uniquely represent  $\mathcal{C}(\varphi)$  follows from the second statement of Lemma 5. Alternatively, let  $\mathcal{C}(\varphi)$  and  $\mathcal{C}(\psi)$  be two shapes and suppose the two probability measures  $p_\varphi$  and  $p_\psi$  are equal. Applying  $A^k$  to each of them we must have

$$\frac{1}{\int_{\mathbb{T}^n} \Omega_\varphi \cdot \mu} (\chi_{M_\varphi} \cdot \mu) = \frac{1}{\int_{\mathbb{T}^n} \Omega_\psi \cdot \mu} (\chi_{M_\psi} \cdot \mu).$$

From this it is clear that:

$$\frac{1}{\int_{\mathbb{T}^n} \Omega_\varphi \cdot \mu} \cdot \chi_{M_\varphi} = \frac{1}{\int_{\mathbb{T}^n} \Omega_\psi \cdot \mu} \cdot \chi_{M_\psi},$$

which is true if and only if  $\varphi = \psi \circ \eta$  for some diffeomorphism  $\eta$  of  $N$  so that  $\mathcal{C}(\varphi) = \mathcal{C}(\psi)$ .  $\square$

With Theorem 1 we are able to get a little more information about co-dimension one shapes than the general case. A positive probability density is associated to each embedding  $\varphi$  via the composition:

$$\Phi(\varphi) := \mathcal{N} \circ \Delta^{-1} d \circ \mathcal{R} \circ \mathcal{C}(\varphi) = p_\varphi, \quad (29)$$

where  $\mathcal{R}$  is the Riesz operator of the Riesz Representation Theorem given in Section 2 and  $\mathcal{N}$  denotes the operation of normalising a volume form. In particular,  $\mathcal{E}_N^s$  is mapped by  $\Phi$  to the manifold of positive  $H^k$  probability densities:

$$\text{Dens}^k(\mathbb{T}^n) = \left\{ \nu \in H^k(\Lambda^n(\mathbb{T}^n)) : \nu > 0, \int_{\mathbb{T}^n} d\nu = 1 \right\}.$$

**Lemma 6** *Suppose that the conditions of Theorem 1 are satisfied and let  $\Phi(\mathcal{E}_N^s) \subset \text{Dens}^k(\mathbb{T}^n)$  be endowed with the subspace topology. Then  $\Phi : \mathcal{E}_N^s \rightarrow \Phi(\mathcal{E}_N^s) \subset \text{Dens}^k(\mathbb{T}^n)$  is a continuous, relatively open map.*

*Proof.* In view of Lemma 5 and Theorem 1,  $\Phi$  successively maps across the following closed spaces

$$\Phi : \mathcal{E}_N^s \rightarrow \delta H^{k+1}(\Lambda^n(\mathbb{T}^n))^* \rightarrow \delta H^{k+1}(\Lambda^n(\mathbb{T}^n)) \rightarrow H^{k+1}(\Lambda^n(\mathbb{T}^n)) \rightarrow \text{Dens}^k(\mathbb{T}^n)$$

We first show that the current map is a relatively open map between  $\mathcal{E}_N^s$  and  $\delta H^{k+1}(\Lambda^n(\mathbb{T}^n))^*$  by showing that the image of any open set is relatively open. We will then show that each remaining map appearing in the composition defining  $\Phi$  is open between the image of the preceding map and its own, where each successive image has the subspace topology.

Let  $U$  be an open set in  $\mathcal{E}_N^s$ . For any  $f \in \mathcal{C}(U)$  there exists a  $\varphi \in U$  with  $\mathcal{C}(\varphi) = f$ . Let  $T_r$  be a tubular neighbourhood of  $\varphi(N)$  with constant radius  $r$  and volume  $v$ . Since  $U$  is open there exists an  $r$  sufficiently small so that the open ball  $B(\varphi, \frac{r}{K^0(s,n)})$  of radius  $\frac{r}{K^0(s,n)}$  centred at  $\varphi$ , where  $K^0(s,n)$  is the Sobolev embedding constant of  $H^s$  into  $C^0$ , is contained in  $U$ . Now,  $\|\varphi - \psi\|_{C^0} < r$  so that the maximum distance between the image of  $\varphi$  and the image of any  $\psi \in B(\varphi, \frac{r}{K^0(s,n)})$  is less than  $r$  and, therefore, the images of all embeddings in  $B(\varphi, \frac{r}{K^0(s,n)})$  are contained in  $T_r$ . Moreover, the volume of the region between  $\varphi(N)$  and  $\psi(N)$ , for any  $\psi \in B(\varphi, \frac{r}{K^0(s,n)})$ , is strictly less than  $v$ . First:

$$\|\chi_{M_\varphi} - \chi_{M_\psi}\|_{L^2(\mathbb{T}^n)}^2 < v \quad (30)$$

for all  $\psi \in B(\varphi, \frac{r}{K^0(s,n)})$ . Indeed:

$$\begin{aligned} \|\chi_{M_\varphi} - \chi_{M_\psi}\|_{L^2(\mathbb{T}^n)}^2 &= \int_{\mathbb{T}^n} \chi_{M_\varphi}^2 + \chi_{M_\psi}^2 - 2\chi_{M_\varphi} \cdot \chi_{M_\psi} d\mu \\ &= \int_{M_\varphi} d\mu + \int_{M_\psi} d\mu - \int_{M_\varphi \cap M_\psi} d\mu - \int_{M_\varphi \cap M_\psi} d\mu \\ &= \int_{M_\varphi \cup M_\psi} d\mu - \int_{M_\varphi \cap M_\psi} d\mu \\ &< v, \end{aligned}$$

since the penultimate line is precisely the volume of the region between  $\varphi(N)$  and  $\psi(N)$ . Second, recalling that  $\mathcal{C}(\varphi) = f$ , (30) implies that:

$$\|f - \mathcal{C}(\psi)\|_{H^k(\Lambda^{n-1}(\mathbb{T}^n))^*} < \sqrt{v} \quad (31)$$

for all  $\psi \in B(\varphi, \frac{r}{K^0(s,n)})$ . Indeed, let  $\alpha$  be any  $H^k$   $(n-1)$ -form and write  $d\alpha = g \cdot \mu$  for some  $H^{k-1}$  function  $g$ . Then, using Stoke's Theorem and Holder's inequality:

$$\begin{aligned} |f(\alpha) - \mathcal{C}(\psi)(\alpha)| &= \left| \int_{\varphi(N)} \alpha - \int_{\psi(N)} \alpha \right| \\ &= \left| \int_{M_\varphi} d\alpha - \int_{M_\psi} d\alpha \right| \\ &= \left| \int_{\mathbb{T}^n} \chi_{M_\varphi} \cdot d\alpha - \int_{\mathbb{T}^n} \chi_{M_\psi} \cdot d\alpha \right| \\ &\leq \int_{\mathbb{T}^n} |\chi_{M_\varphi} - \chi_{M_\psi}| \cdot |g| d\mu \\ &\leq \|\chi_{M_\varphi} - \chi_{M_\psi}\|_{L^2(\mathbb{T}^n)} \cdot \|g\|_{L^2(\mathbb{T}^n)} \\ &< \sqrt{v} \cdot \|\alpha\|_{H^k(\Lambda^{n-1}(\mathbb{T}^n))}. \end{aligned}$$

Taking the supremum over  $\|\alpha\|_{H^k(\Lambda^{n-1}(\mathbb{T}^n))} = 1$  gives the inequality. Now we can choose  $r$ , and hence  $v$ , small enough so that:

$$\mathcal{C} \left( B\left(\varphi, \frac{r}{K^0(s,n)}\right) \right) =: B^{\text{rel}}(f, \sqrt{v}) \subset \mathcal{C}(U).$$

Moreover,  $B^{\text{rel}}(f, \sqrt{v})$ , whose elements satisfy (31), can be written as:

$$B^{\text{rel}}(f, \sqrt{v}) = B^{H^k(\Lambda^{n-1}(\mathbb{T}^n))^*}(f, \sqrt{v}) \cap \mathcal{C}(\mathcal{E}_N^s)$$

and is therefore open in the subspace topology, where  $B^{H^k(\Lambda^{n-1}(\mathbb{T}^n))^*}$  denotes an open ball in the  $H^k(\Lambda^{n-1}(\mathbb{T}^n))^*$  topology. As this holds for any point in  $\mathcal{C}(U)$  we can cover  $\mathcal{C}(U)$  with balls of the form  $B^{\text{rel}}(f, \sqrt{v})$  and write:

$$\begin{aligned} \mathcal{C}(U) &= \bigcup_{f \in \mathcal{C}(U)} B^{\text{rel}}(f, \sqrt{v}) = \bigcup_{f \in \mathcal{C}(U)} B^{H^k(\Lambda^{n-1}(\mathbb{T}^n))^*}(f, \sqrt{v}) \cap \mathcal{C}(\mathcal{E}_N^s) \\ &= \left( \bigcup_{f \in \mathcal{C}(U)} B^{H^k(\Lambda^{n-1}(\mathbb{T}^n))^*}(f, \sqrt{v}) \right) \cap \mathcal{C}(\mathcal{E}_N^s) \end{aligned}$$

so that  $\mathcal{C}(U)$  is open in the subspace topology.

Now we show that  $\mathcal{R} : \mathcal{C}(\mathcal{E}_N^s) \rightarrow \mathcal{R} \circ \mathcal{C}(\mathcal{E}_N^s)$  is open. Note that the Riesz mapping  $\mathcal{R}$  is an isometric isomorphism between any Hilbert space and its dual and is therefore an open map by the Open Mapping Theorem [23] (Theorem 3.18); in particular it is an isomorphic open mapping between  $\delta H^{k+1}(\Lambda^n(\mathbb{T}^n))^*$  and  $\delta H^{k+1}(\Lambda^n(\mathbb{T}^n))$ . Let  $V$  be open in  $\mathcal{C}(\mathcal{E}_N^s)$  with  $V = \mathcal{O} \cap \mathcal{C}(\mathcal{E}_N^s)$  for some open set  $\mathcal{O}$  in  $\delta H^{k+1}(\Lambda^n(\mathbb{T}^n))^*$ . Then:

$$\mathcal{R}(V) = \mathcal{R}(\mathcal{O} \cap \mathcal{C}(\mathcal{E}_N^s)) \subseteq \mathcal{R}(\mathcal{O}) \cap \mathcal{R}(\mathcal{C}(\mathcal{E}_N^s))$$

by general properties of mappings. On the other hand, if  $\alpha \in \mathcal{R}(\mathcal{O}) \cap \mathcal{R}(\mathcal{C}(\mathcal{E}_N^s))$  then since  $\mathcal{R}$  is an isomorphism there exists a unique  $f_\alpha \in \mathcal{O} \cap \mathcal{C}(\mathcal{E}_N^s) = V$  so that  $\alpha = \mathcal{R}(f_\alpha) \in \mathcal{R}(\mathcal{O} \cap \mathcal{C}(\mathcal{E}_N^s))$  which gives the reverse inclusion. Therefore:

$$\mathcal{R}(V) = \mathcal{R}(\mathcal{O}) \cap \mathcal{R}(\mathcal{C}(\mathcal{E}_N^s))$$

so that  $\mathcal{R}(V)$  is open in  $\mathcal{R} \circ \mathcal{C}(\mathcal{E}_N^s) \subset \delta H^{k+1}(\Lambda^n(\mathbb{T}^n))$  since  $\mathcal{R}(\mathcal{O})$  is open in  $\delta H^{k+1}(\Lambda^n(\mathbb{T}^n))$ .

As was shown in the proof of Lemma 4, the bounded linear operator  $\Delta^{-1}d : \delta H^{k+1}(\Lambda^n(\mathbb{T}^n)) \rightarrow H^{k+1}(\Lambda^n(\mathbb{T}^n))$  between the two closed spaces has an inverse given by  $\delta$  and is therefore an isomorphism. So  $\Delta^{-1}d$  is also an open map by the Open Mapping Theorem and using similar arguments as above we get that  $\Delta^{-1}d(W)$  is open in  $\Delta^{-1}d \circ \mathcal{R} \circ \mathcal{C}(U)$ , where  $W$  is any open set in  $\mathcal{R} \circ \mathcal{C}(\mathcal{E}_N^s)$ .

Finally, the map  $\mathcal{N}$  is a rescaling map and rescales a relatively open set to a relatively open set so that  $\Phi(U) = \mathcal{N} \circ \Delta^{-1}d \circ \mathcal{R} \circ \mathcal{C}(U)$  open in  $\Phi(\mathcal{E}_N^s) \subset \text{Dens}^k(\mathbb{T}^n)$ .

Since  $\Phi$  is a composition of the continuous maps above, it is itself continuous.  $\square$

**Theorem 2** *The map  $\Phi$  is a topological embedding of the shape space  $\mathcal{B}^s(N, \mathbb{T}^n)$  in  $\text{Dens}^k(\mathbb{T}^n)$ ; that is,  $\Phi$  maps  $\mathcal{B}^s(N, \mathbb{T}^n)$  homeomorphically onto its image  $\Phi(\mathcal{B}^s(N, \mathbb{T}^n)) \subset \text{Dens}^k(\mathbb{T}^n)$ .*

*Proof.* Observe that the current map  $\mathcal{C}$  is constant on left cosets of  $\mathcal{B}^s(N, \mathbb{T}^n)$  and is therefore well-defined as a map from  $\mathcal{B}^s(N, \mathbb{T}^n)$ . The current map is injective on left cosets by Lemma 1. By Proposition 1, the current map descends to a continuous, injective map on  $\mathcal{B}^s(N, \mathbb{T}^n)$ . The remaining maps appearing in the composition  $\Phi$  are also continuous, injective maps so that  $\Phi$  descends to a continuous injective mapping of  $\mathcal{B}^s(N, \mathbb{T}^n)$  into  $\text{Dens}^k(\mathbb{T}^n)$ . Finally, Lemma 6 also guarantees the continuity of the inverse map between  $\mathcal{E}_N^s$  and  $\Phi(\mathcal{E}_N^s)$ . That is,  $\mathcal{B}^s(N, \mathbb{T}^n)$  is mapped homeomorphically onto  $\Phi(\mathcal{B}^s(N, \mathbb{T}^n)) \subset \text{Dens}^k(\mathbb{T}^n)$ .  $\square$

#### 4.2 A Non-Degenerate $\dot{H}^1$ Metric on Co-Dimension One Shapes

Let  $\mathcal{D}^k(M)$  denote the group of Sobolev  $H^k$  diffeomorphisms of a closed  $n$ -dimensional Riemannian manifold  $M$  and  $\mathcal{D}_\mu^k$  the subgroup consisting of those diffeomorphisms preserving the volume form  $\mu$  on  $M$ . When  $r > \frac{n}{2} + 1$ , both  $\mathcal{D}^k(M)$  and  $\mathcal{D}_\mu^k(M)$  become smooth Hilbert manifolds. The projection  $\pi : \mathcal{D}^k(M) \rightarrow \mathcal{D}^k(M)/\mathcal{D}_\mu^k(M)$  is a  $C^0$  principal bundle whose base is diffeomorphic to the space  $\text{Dens}(M)$  of normalized  $H^k$  positive densities:

$$\text{Dens}^k(M) = \left\{ \nu \in H^k(\Lambda^n(M)) : \nu > 0, \int_M d\nu = 1 \right\}. \quad (32)$$

Equivalently, if  $\Omega = \frac{d\nu}{d\mu}$  is the Radon-Nikodym derivative of  $\nu$  with respect to  $\mu$  then the base can be regarded as a convex subset of the space of  $H^k$  functions on  $M$ :

$$\mathcal{P} = \left\{ \Omega \in H^k(M, \mathbb{R}_{>0}) : \int_M \Omega d\mu = 1 \right\}. \quad (33)$$

In the latter case the projection  $\pi : \mathcal{D}^k(M) \rightarrow \mathcal{P}$  can be written as  $\pi(\eta) = \text{Jac}_\mu(\eta)$ , where  $\text{Jac}_\mu(\eta)$  is determined by  $\eta^*\mu = \text{Jac}_\mu(\eta) \cdot \mu$ . The fact that  $(\eta \circ \xi)^*\mu = \xi^*\eta^*\mu$  implies that:

$$\text{Jac}_\mu(\eta \circ \xi) = (\text{Jac}_\mu \circ \xi) \cdot \text{Jac}_\mu(\xi).$$

Consequently,  $\pi(\eta \circ \xi) = \pi(\eta)$  whenever  $\xi \in \mathcal{D}_\mu^r(M)$ , so that the projection is constant on the left cosets; that  $\pi$  descends to an *isomorphism* between the quotient space of left cosets to the space of densities is the contents of Moser's Theorem [22] in the Sobolev category (E and M). In [14] the authors consider the homogeneous space of densities  $\text{Dens}(M)$  equipped with the right-invariant metric induced by the  $\dot{H}^1$  inner product:

$$(u \circ \eta, v \circ \eta)_{\dot{H}^1} = \frac{1}{4} \int_M \text{div} u \cdot \text{div} v \, d\mu \quad (34)$$

for any  $u, v \in T_e \mathcal{D}^r(M)$  and  $\eta \in \mathcal{D}^r(M)$ . They found that, when equipped with (34), the space  $\text{Dens}(M)$  is isometric to an open subset of the infinite-dimensional sphere of radius  $\mu(M)$ :

$$S_{\mu(M)}^\infty = \left\{ f \in L^2(M, d\mu) : \int_M f^2 \, d\mu = \mu(M) \right\}.$$

In particular, the  $\dot{H}^1$  geometry of the space of positive  $H^r$  densities is spherical and the Riemannian distance between measures  $\lambda$  and  $\nu$  in  $\text{Dens}(M)$  is given by:

$$\text{dist}_{\dot{H}^1}(\lambda, \nu) = \sqrt{\mu(M)} \arccos \left( \frac{1}{\mu(M)} \int_M \sqrt{\frac{d\lambda}{d\mu} \cdot \frac{d\nu}{d\mu}} \, d\mu \right). \quad (35)$$

The authors of [14] note that their construction could be usefully applied to shapes; our results can be seen as doing precisely that in full generality with only the Hodge Laplacian rather than mollification.

**Theorem 3** *For any two shapes  $\lambda$  and  $\nu$  in  $\mathcal{O} = \Phi(\mathcal{E}_N^s) \subset \mathcal{D}^k(\mathbb{T}^n)/\mathcal{D}_\mu^k(\mathbb{T}^n)$ , the function:*

$$d(\lambda, \nu) = \inf_{\rho(t) \in \mathcal{O}: \rho(0)=\lambda, \rho(1)=\nu} \int_0^1 \|\dot{\rho}(t)\|_{\dot{H}^1} \, dt$$

*is a distance metric on  $\mathcal{O}$ , where  $\|\cdot\|_{\dot{H}^1}$  is the norm induced by (34).*

*Proof.* Symmetry and the triangle inequality are both obvious from the definition. It remains to prove that  $d(\lambda, \nu) = 0$  if and only if  $\lambda = \nu$ . If  $\rho(t)$  is any  $C^1$  curve with  $\rho(0) = \lambda$  and  $\rho(1) = \nu$  then

$$\int_0^1 \|\dot{\rho}(t)\|_{\dot{H}^1} \, dt \geq \text{dist}_{\dot{H}^1}(\lambda, \nu).$$

Constraining  $\rho(t)$  to lie in  $\mathcal{O}$  and taking the infimum over all such curves gives the result.  $\square$

## 5 The Wave of the Shape: Hydrodynamics and a Non-Degenerate $L^2$ Metric on One-Dimensional Shapes

From co-dimension one shapes we now turn our attention to one dimensional shapes in which  $p = 1$  and  $N = S^1$ . In this section we apply the work of Arnold [3] and Ebin and Marsden [9] to show that each one-dimensional shape generates a unique solution to the Euler equations of hydrodynamics on  $\mathbb{T}^n$ . Since Lagrangian solutions of the fluid equations are geodesics of the  $L^2$  metric on the volume-preserving diffeomorphism group  $\mathcal{D}_\mu^k(\mathbb{T}^n)$  of  $\mathbb{T}^n$  we may represent each shape by a unique volume-preserving diffeomorphism and use the  $L^2$  metric to define a distance metric on shapes.

Recall that each co-dimension one shape was uniquely represented by an element of  $\mathcal{D}^k(\mathbb{T}^n)/\mathcal{D}_\mu^k(\mathbb{T}^n)$  so in this sense co-dimension one shapes are dual to one-dimensional shapes.

### 5.1 The $L^2$ Riemannian Metric on $\mathcal{D}_\mu^k(\mathbb{T}^n)$ and Hydrodynamics

Recall the musical isomorphisms of  $\sharp$  and  $\flat$  given in section 2: if  $\beta$  is a 1-form, then  $\beta^\sharp$  is the vector field such that  $\beta(w) = g(\beta^\sharp, w)$  for every vector field  $w$ ;  $\flat$  is its inverse. The divergence of a vector field  $u$ , which is defined through the Lie derivative of the volume form  $\mu$  along the field  $u$  as  $\text{div}u \cdot \mu = \mathcal{L}_u\mu$ , can be written as an operator on 1-forms:  $\text{div}u = -\delta u^\flat$ ; to see this in the particular case of  $\mathbb{T}^n$  first observe that

$$\iota_u\mu = \iota_u dx_1 \wedge \cdots \wedge dx_n = \sum_{r=0}^n (-1)^r u^r dx_1 \wedge \cdots \wedge d\hat{x}_r \wedge \cdots \wedge dx_n = \star \sum_{r=0}^n u^r dx_r = \star u^\flat$$

where we have used the standard coordinates  $(x_1, \dots, x_n)$  on  $\mathbb{T}^n$ , the interior product (11), and the Hodge star (4). Then:

$$\text{div}u = \star(\text{div}u \cdot \mu) = \star\mathcal{L}_u\mu = \star d\iota_u\mu = \star d\star u^\flat = -\delta u^\flat,$$

where we have used Cartan's "magic formula", the fact that the exterior derivative of an  $n$ -form is zero, and (2). The derivation of this formula on a general manifold  $M$  is more involved but see [1] for complete details. Now the Hodge decomposition (9) says that any  $H^k$  divergence-free vector field  $u$  may be written as  $u^\flat = \delta\gamma + h$ , where  $\gamma$  is an  $H^{k+1}$  2-form and  $h$  is a  $C^\infty$  1-form, since  $\delta \circ \delta = 0$  and  $\delta h = 0$  for any harmonic form  $h$ .

If  $\varphi \in \mathcal{E}_{S^1}^s$  then, according to Lemmas 2 and 3, the shape  $\mathcal{C}(\varphi)$  is represented by a unique divergence-free vector field:

$$v_{\mathcal{C}(\varphi)} = \beta_{\mathcal{C}(\varphi)}^\sharp = (\delta\gamma)^\sharp + h^\sharp$$

for some  $H^{k+1}$  2-form  $\gamma$  and harmonic 1-form  $h$ , whose  $H^k$  norm satisfies:

$$\|v_{\mathcal{C}(\varphi)}\|_{H^k} \leq K^0(k, n) \cdot \left(K^1(s, n)\right)^p \cdot \text{Vol}(N) \cdot \|\varphi\|_{H^s}^p.$$

In terms of the Riesz mapping  $\mathcal{R}$  given by the Riesz Representation Theorem in section 2, which sends  $\mathcal{C}(\varphi)$  to  $\beta_{\mathcal{C}(\varphi)}$ , we are able to view  $(\mathcal{R} \circ \mathcal{C}(\mathcal{E}_{S^1}^s))^\sharp$  as a subset of the vector space of  $H^k$  divergence-free vector fields on  $\mathbb{T}^n$ .

In parallel with Lemma 4 we have:

**Lemma 7** For any  $\varphi \in \mathcal{E}_{S^1}^s$  the Riesz representer  $\beta_{\mathcal{C}(\varphi)}$  of  $\mathcal{C}(\varphi)$  corresponds to an exact divergence free vector field:

$$v_{\mathcal{C}(\varphi)} = \beta_{\mathcal{C}(\varphi)}^\sharp = (\delta\gamma)^\sharp$$

for some  $H^{k+1}$  2-form  $\gamma$ .

*Proof.* Let  $Q$  denote the orthogonal projection onto the space of harmonic 1-forms  $\mathcal{H}_1^k$ . If  $S$  is any surface whose boundary is  $\varphi(S^1)$  then Stokes' Theorem gives:

$$(Q\beta_{\mathcal{C}(\varphi)}, \alpha)_{H^k} = (\beta_{\mathcal{C}(\varphi)}, Q\alpha)_{H^k} = \int_{\varphi(S^1)} Q\alpha = \int_S dQ\alpha = 0,$$

where the final equality follows from the fact that  $Q\alpha$  is harmonic. Since this holds for any 1-form  $\alpha$  we must have  $Q\beta_{\mathcal{C}(\varphi)} = 0$ . This, in conjunction with Lemma 3, implies that  $\beta_{\mathcal{C}(\varphi)} = \delta\gamma$  for some  $H^{k+1}$  2-form  $\gamma$  and hence  $\beta_{\mathcal{C}(\varphi)}$  corresponds to an exact divergence free vector field, as stated in the Lemma.  $\square$

Although given in the particular case of the torus  $\mathbb{T}^n$  here, the following referenced definitions, constructions, and results hold for arbitrary closed manifolds  $M$ .

**Definition 1** The *volumorphism group*  $\mathcal{D}_\mu^k$  consists of those diffeomorphisms  $\eta$  of Sobolev class  $H^k$  of  $\mathbb{T}^n$  such that  $\eta^*\mu = \mu$ . When  $k > \frac{n}{2} + 1$ ,  $\mathcal{D}_\mu^k$  is a smooth Hilbert manifold whose tangent space at the identity  $T_e\mathcal{D}_\mu^k$  consists of  $H^k$  divergence-free vector fields on  $\mathbb{T}^n$ .

When equipped with the right-invariant  $L^2$  metric on vector fields:

$$(v, w)_{L^2} = \int_{\mathbb{T}^n} g(v, w) d\mu, \quad (36)$$

$\mathcal{D}_\mu^k(\mathbb{T}^n)$  becomes a smooth Riemannian manifold with a smooth right-invariant Levi-Civita connection  $\nabla^\mu$ , smooth right-invariant curvature tensor  $R^\mu$ , and smooth exponential map  $\exp_e$ , which is a diffeomorphism on a neighbourhood of the identity  $e$ . The importance of these facts resides in an observation of Arnold that  $\eta(t)$  is a geodesic of the  $L^2$  metric if and only if the vector field  $v(t) = \dot{\eta}(t) \circ \eta^{-1}(t)$  solves the Euler equations of hydrodynamics:

$$\begin{aligned} \partial_t v + \nabla_v v &= -\nabla p \\ \operatorname{div} v &= 0 \\ v(0) &= v_o, \end{aligned}$$

where  $\nabla$  is the covariant derivative on  $\mathbb{T}^n$  and  $p$  the pressure of the fluid filling  $\mathbb{T}^n$ . Ebin and Marsden ([9], Theorem 9.2) showed that geodesics of the weak  $L^2$  metric on  $\mathcal{D}_\mu^k$  exist, are unique, and depend smoothly on the initial conditions for any closed Riemannian manifold of dimension  $n$ . This allows us to define a smooth exponential map:

$$\exp_e : T_e\mathcal{D}_\mu^k \rightarrow \mathcal{D}_\mu^k \quad (37)$$

given, for small  $t$ , by:

$$\exp_e(tv_o) = \eta(t),$$

where  $\eta$  is the unique geodesic issuing from the identity with initial velocity  $v_o$ . The exponential map (37) is a local diffeomorphism on a neighbourhood of zero in  $T_e\mathcal{D}_\mu^k$ ; this follows from the fact that its derivative at  $t = 0$  is the identity and the inverse function theorem. The derivative of the exponential map at  $v_o$  is a linear map  $D\exp_e(v_o) : T_e\mathcal{D}_\mu^k \rightarrow T_{\exp_e(v_o)}\mathcal{D}_\mu^k$ , which is bounded as long as  $\exp_e$  is defined and is both injective and surjective when  $\exp_e$  is a diffeomorphism. We refer the reader to [9] and [18] for basic facts regarding the  $L^2$  geometry of diffeomorphism groups.

The next theorem is observational and shows that each shape generates a unique fluid flow on the torus. Hydrodynamics is a well developed field in mathematics and it could be interesting to understand what fluids and their invariants can say about families of shapes in light of this observation. Conversely, it might be of interest to understand what shapes say about fluids and to understand the kinds of flows they define. We refer the reader to Arnold and Khesin [4] for an overview of the geometry of hydrodynamics and fluid invariants.

**Theorem 4** *For  $s > \frac{3}{2}$  and  $k > \frac{n}{2} + 1$ , each shape  $\mathcal{C}(\varphi)$  generates a unique fluid flow  $\eta_\varphi(t)$  on the torus  $\mathbb{T}^n$  defined on a sufficiently small time interval around 0.*

*Proof.* Let  $r$  denote the injectivity radius of  $\exp_e$  so that  $\exp_e : B(0, r) \subset T_e\mathcal{D}_\mu^k(\mathbb{T}^n) \rightarrow \mathcal{D}_\mu^k(\mathbb{T}^n)$  is a diffeomorphism. Let  $\mathcal{C}(\varphi)$  be a shape with representing divergence-free vector field  $v_{\mathcal{C}(\varphi)}$ . Then  $\mathcal{C}(\varphi)$  generates a fluid flow on the torus  $\mathbb{T}^n$  given by:

$$\eta_\varphi(t) = \exp_e \left( t \frac{v_{\mathcal{C}(\varphi)}}{\|v_{\mathcal{C}(\varphi)}\|_{H^k}} \right)$$

which is defined for  $t \in (-r, r)$ . Since no two representing divergence-free vector fields corresponding to two different shapes are collinear with the zero vector field (by Lemma 1) and since the exponential map is a local diffeomorphism the fluid flow is unique. □

*Remark 2* Instead of an  $L^2$  metric, one can also define an  $H^r$  Riemannian metric on  $\mathcal{D}_\mu^k(\mathbb{T}^n)$  for  $0 \leq r \leq k$ . Such metrics are also weak in that they do not generate the  $H^k$  topology (unless  $r = k$ ), but nevertheless carry smooth Levi-Civita connections, curvature tensors, and exponential maps that are local diffeomorphisms (cf. Misiolek and Preston [19]). Although uninteresting from the physical point of view, these higher-order metrics may be of value in the analysis of shapes. In the future we will consider global existence of such shape flows.

In the next section we demonstrate how the  $L^2$  metric can be used to define an intrinsic, non-degenerate distance metric on one dimensional shapes.

## 5.2 A Non-Degenerate $L^2$ Metric on One-Dimensional Shapes

We are interested in an intrinsic distance metric that measures the distance between two representing vector fields using curves that are constrained to lie in the set of vector fields  $g^\sharp\mathcal{R} \circ \mathcal{C}(\mathcal{E}_{S_1}^s)$ . We give the following definition of an intrinsic distance metric:



**Definition 2** The intrinsic distance between two shapes  $\mathcal{C}(\varphi)$  and  $\mathcal{C}(\psi)$  with representing vector fields  $v_\varphi$  and  $v_\psi$  is defined as the infimum over all lengths of  $C^1$  paths  $c(t)$  in  $g^\sharp\mathcal{R} \circ \mathcal{C}(\mathcal{E}_{S^1}^s)$  connecting  $v_\varphi$  with  $v_\psi$ :

$$d(\mathcal{C}(\varphi), \mathcal{C}(\psi)) := \inf_{c(t) \in g^\sharp\mathcal{R} \circ \mathcal{C}(\mathcal{E}_{S^1}^s): c(0)=v_\varphi, c(1)=v_\psi} \int_0^1 \|\dot{c}(t)\|_{L^2} dt$$

**Theorem 5** *The intrinsic distance metric  $d$  is a non-degenerate  $L^2$  distance metric on shape space  $\mathcal{C}(\mathcal{E}_{S^1}^s)$ .*

*Proof.* Symmetry and the triangle inequality are obvious from the definition so we prove non-degeneracy. For a curve  $\gamma : [0, 1] \rightarrow g^\sharp\mathcal{R} \circ \mathcal{C}(\mathcal{E}_{S^1}^s) \subset T_e\mathcal{D}_\mu^k$ , let  $L(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{L^2} dt$  and suppose there exist distinct  $v_\varphi$  and  $v_\psi$  in  $g^\sharp\mathcal{R} \circ \mathcal{C}(\mathcal{E}_{S^1}^s)$  such that

$$d^I(\mathcal{C}(\varphi), \mathcal{C}(\psi)) = \inf_{c(t) \in \mathcal{R} \circ \mathcal{C}(\mathcal{E}_{S^1}^s), c(0)=v_\varphi, c(1)=v_\psi} L(c) = 0.$$

Then, for all  $\epsilon > 0$  there exists a curve  $\tilde{c}(t)$  in  $g^\sharp\mathcal{R} \circ \mathcal{C}(\mathcal{E}_{S^1}^s) \subset T_e\mathcal{D}_\mu^k$  joining  $v_\varphi$  and  $v_\psi$  such that

$$L(\tilde{c}(t)) < \epsilon.$$

However,  $\tilde{c}$  is also a curve in  $T_e\mathcal{D}_\mu^k$  joining the same endpoints and since the  $L^2$  norm is a non-degenerate distance metric choosing  $0 < \epsilon < \|v_\varphi - v_\psi\|_{L^2}$  gives

$$L(\tilde{c}(t)) < \|v_\varphi - v_\psi\|_{L^2}$$

which contradicts the fact that the shortest path joining any two points in a linear space is a straight line. Thus

$$d(\mathcal{C}(\varphi), \mathcal{C}(\psi)) > 0$$

which proves non-degeneracy.  $\square$

### Conflict of interest

The authors declare that they have no conflict of interest.

### Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

## References

1. Abraham, R., Marsden, J.E., Ratiu, T.S.: *Manifolds, Tensor Analysis, and Applications* (3rd Ed.). Springer-Verlag (2007)
2. Amari, S.I., Nagaoka, H.: *Methods of Information Geometry*. AMS (2000)
3. Arnold, V.: *Topological Methods in Hydrodynamics*. Springer (1998)
4. Arnold, V.I., Khesin, B.A.: *Topological methods in hydrodynamics*. Springer (1998)
5. Beg, M.F., Miller, M.I., Trouvé, A., Younes, L.: Computing Large Deformation Metric Mappings via Geodesic Flows of Diffeomorphisms. *International Journal of Computer Vision* **61**(2), 139–157 (2005)
6. Benn, J., Marsland, S., McLachlan, R., Modin, K., Verdier, O.: Currents and finite elements as tools for shape space. *Journal of Mathematical Imaging and Vision* **61**(8), 1197–1220 (2019)
7. Charon, N.: *Analysis of geometric and functional shapes with extensions of currents: applications to registration and atlas estimation*. Ph.D. thesis, Ecole normale supérieure de Cachan (2013)
8. Durrleman, S., Pennec, X., Trouvé, A., Ayache, N.: Statistical Models of Sets of Curves and Surfaces based on Currents. *Medical Image Analysis* **13**(5), 793–808 (2009)
9. Ebin, D.G., Marsden, J.E.: Groups of diffeomorphisms and the motion of an incompressible fluid. *Annals of Mathematics* **92**(1), 102–163 (1970)
10. Glaunès, J., Qiu, A., Miller, M.I., Younes, L.: Large Deformation Diffeomorphic Metric Curve Mapping. *International Journal of Computer Vision* **80**(3), 317–336 (2008)
11. Hebey, E.: *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*. American Mathematical Society (200)
12. Inci, H., Kappeler, T., Topalov, P.: *On the Regularity of the Composition of Diffeomorphisms*. AMS (2013)
13. Kaltenmark, I.: *Geometrical growth models for computational anatomy*. Ph.D. thesis, Université Paris Saclay (2016)
14. Khesin, B., Lennels, J., Misiulek, G., Preston, S.: Geometry of diffeomorphism groups, complete integrability and geometric statistics. *Geometry and Functional Analysis* **23**, 334–366 (2013)
15. Lenells, J., Misiulek, G.: Amari–Chentsov connections and their geodesics on homogeneous spaces of diffeomorphism groups. *Journal of Mathematical Sciences* **196**, 144–151 (2014)
16. Lima, E.: The Jordan-Brouwer separation theorem for smooth hypersurfaces. *American Mathematical Monthly* **95**, 39–42 (1988)
17. Michor, P.W., Mumford, D.: An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach. *Applied and Computational Harmonic Analysis* **23**(1), 74–113 (2007)
18. Misiulek, G.: Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms. *Indiana University Mathematics Journal* **2**, 215–235 (1993)
19. Misiulek, G., Preston, S.: Fredholm properties of Riemannian exponential maps on diffeomorphism groups. *Inventiones Mathematicae* (2010)
20. Morgan, F.: *Geometric Measure Theory: A Beginner’s Guide*, 4th edn. Academic Press, Boston (2009)
21. Morrey, C.: *Multiple integrals in the calculus of variations*. Springer (1966)
22. Moser, J.: On the volume elements on a manifold. *Transactions of the American Mathematical Society* **120**, 286–294 (1965)
23. Schechter, M.: *Principals of Functional Analysis*, second edn. AMS (2002)
24. Vaillant, M., Glaunès, J.: Surface matching via currents. In: *Information processing in medical imaging*, pp. 381–92 (2005)
25. Warner, F.: *Foundations of Differentiable Manifolds and Lie Groups*. Springer (1983)
26. Younes, L.: *Shapes and Diffeomorphisms*. Springer, New York, USA (2010)