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Slope of the topological charge, proton spin and the 0^{-+} pseudoscalar di-gluonia spectra

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Abstract

In this paper, we attempt to discuss the topics mentioned in the title by scrutinizing and improving the 0^{-+} pseudoscalar di-gluonia/glueballs sum rules within the standard SVZ-expansion at N2LO *without instantons*.

First, we reconsider the estimate of the slope of the topological charge $\sqrt{\chi'(0)}(Q^2 = 2 \text{ GeV}^2) = 24.3(3.4) \text{ MeV}$ from low degree moments which imply $\int_0^1 dx g_1^P(x) = 0.144(5)$ [data: 0.145(14)] for the first moment of the polarized proton structure function (proton spin) and $G_A^{(0)}(10 \text{ GeV}^2) \equiv \Delta u + \Delta d + \Delta s = 0.340(50)$ [data = 0.330(39)] for the singlet form factor of the axial current.

Second, we work with high degree moments and parametrize the spectral function beyond the minimal duality ansatz: “*One resonance \oplus QCD continuum*” to get the 0^{-+} pseudoscalar di-gluonia spectra. Then, we obtain three groups of gluonia: *The familiar light η_1* [singlet gluon component of the $\eta'(958)$] with $[M_{\eta_1}, f_{\eta_1}] = [825(45), 905(72)] \text{ MeV}$ which is important for understanding the $U(1)_A$ anomaly; *The two new medium gluonia* with $M_{P_{1a}} = 1338(112) \text{ MeV}$ and $[M_{P_{1b}} = 1462(117) \text{ MeV}$ or their mean $[M_{P_1}, f_{P_1}] = [1397(81), 594(144)] \text{ MeV}$ which support the gluonium nature of the excellent experimental candidate $\eta(1405)$ and may bring a small gluon piece to the $\eta(1295)$; their corresponding 1st radial excitations: $M_{P'_{1a}} = 1508(226) \text{ MeV}$ and $M_{P'_{1b}} = 1553(139) \text{ MeV}$ with their mean: $[M_{P'_1}, f_{P'_1}] = [1541(118), 205(282)] \text{ MeV}$ which may be identified (up to some eventual mixings with $\bar{q}q$ states) with the observed $\eta(1475, 1700)$ states; *The heavy gluonium* with the mean mass: $[M_{P_2}, f_{P_2}] = [2751(140), 500(43)] \text{ MeV}$ which can be compared with the lattice results. One can remark the (natural)

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one to one correspondence between the pseudoscalar gluonia and their chiral scalar analogue from Ref. [1]: $\sigma(1) \rightarrow \eta_1$; $G_1(1.55) \rightarrow P_1$; $[\sigma'(1.1), G'(1.56)] \rightarrow P'_{1a,1b}$; $G_2(3) \rightarrow P_2$ which is mainly due to the importance of the QCD PT contributions in the sum rule analysis that are almost equal in these two channels. © 2022 Elsevier B.V. All rights reserved.

Keywords: QCD spectral sum rules; Perturbative and non-perturbative QCD; Exotic hadrons; Masses and decay constants

1. Introduction

Gluonia/Glueballs bound states are expected to be a consequence of QCD [2]. However, despite considerable theoretical and experimental efforts, there are not (at present) any clear indication of their signature. The difficulty is also due to the fact that the observed candidates can be a strong mixing of the gluonia with the $\bar{q}q$ light mesons or with some other exotic mesons (four-quark, hybrid states).

• Experimental facts of the pseudoscalar states below 2 GeV

There are experimental observation of 0^{-+} η -like mesons below 2 GeV [3,4]:

$$\eta(1295), \quad \eta(1405), \quad \eta(1475), \quad \eta(1760), \quad (1)$$

where some of them can be glueball candidates, as they are produced in some gluonia rich channels like e.g. J/ψ radiative decays, $\bar{p}p$ annihilation at rest. The fact that the $\eta(1295)$, $\eta(1405)$ are seen to decay into $\eta(\pi\pi)_{S\text{-wave}}$ where the $(\pi\pi)_{S\text{-wave}}$ may originate from the $\sigma/f_0(550)$ meson can be understood with the $U(1)_A$ gluon vertex [1,5,6] $\langle \eta(1295) | \theta_{\mu}^{\mu} | \eta \rangle$ where the σ is the dilaton associated to the trace of the energy momentum-tensor θ_{μ}^{μ} . This feature can indicate the presence of the gluon component into the wave function of these two states. The absence of the $\eta(1405)$ from $\gamma\gamma$ collisions may indicate that it has a larger gluon component than the $\eta(1295, 1475)$ where the two latter are assumed [4] to be the radial excitations of the η and η' while the $\eta(1405)$ is expected to be an excellent gluonium candidate.

• A short reminder of different theoretical predictions

◇ An earlier qualitative analysis of Novikov et al. [7,8] and a tachyonic gluon mass [9] based on some hadron scale hierarchy arguments expects a high mass (pseudo)scalar gluonium. These arguments seem to be supported by a quantitative analysis [6,10–13] using QCD spectral sum rules (QSSR) à la SVZ [14–24] within one resonance including or not the η' contribution and (or not) direct instanton contribution leading to a gluonium mass from (2.05 ± 0.19) [6] to 2.7 GeV [11]. We shall comment these results later on.

◇ Lattice simulations with one resonance find a mass in the range (2.15–2.72) GeV [25–28].

◇ An holographic model provides a mass of about 2.1 GeV [29].

◇ However, a recent analysis using inverse dispersive problem approach to the spectral function predicts a lower mass around 1.75 GeV [30], while the flux tube [31] and η - η' - G mixing [32] models predictions are around 1.4 GeV in agreement with the previous experimental expectations for the $\eta(1405)$.

- *Projects*

◊ Motivated by the failure of some of the previous approaches including QSSR to explain the data below 2 GeV, we scrutinize previous QSSR results and attempt to improve them by parametrizing the spectral function with more than one resonance and by working with high degree moments.

◊ “En passant”, we revise our NLO estimate of the slope of the topological charge $\chi'(0)$ in the chiral limit [33] and study the effect of this N2LO result on the spin of the proton.

2. The QCD anatomy of the pseudoscalar gluonium two-point correlator

We shall work with two-point correlator:

$$\psi_P(q^2) = (8\pi)^2 i \int d^4x e^{iqx} \langle 0 | Q(x) Q^\dagger(0) | 0 \rangle \quad (2)$$

associated to the divergence of the $U(1)_A$ axial current which reads for n_f quark flavours;

$$\partial_\mu J_5^\mu(x) = \sum_{i=u,d,s} 2m_i \bar{\psi}_i \psi_i + 2n_f Q(x), \quad (3)$$

where:

$$Q(x) \equiv \left(\frac{\alpha_s}{16\pi} \right) \epsilon_{\mu\nu\rho\sigma} G_a^{\mu\nu}(x) G_a^{\rho\sigma}(x) \quad (4)$$

is the topological charge density, $a = 1, \dots, 8$ is the colour index, $\alpha_s \equiv g^2/4\pi$ is the QCD coupling and $G_a^{\mu\nu}(x) \equiv \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + gf_{abc} A^{\mu,b} A^{\nu,c}$ is the Yang-Mills field strength constructed from the gluon fields A_a^μ .

- *The standard SVZ-expansion*

Using the Operator Product Expansion (OPE) à la SVZ, its QCD expression can be written as:

$$\psi_P(q^2) = 2 \sum_{0,1,2,\dots} C_{2n} \langle \mathcal{O}_{2n} \rangle, \quad (5)$$

where C_{2n} is the Wilson coefficients calculable perturbatively while $\langle \mathcal{O}_{2n} \rangle$ is a short-hand notation for the non-perturbative vacuum condensates $\langle 0 | \mathcal{O}_{2n} | 0 \rangle$ of dimension $2n$.

- ◊ *The unit perturbative operator ($n = 0$)*

Its contribution reads:

$$C_0 \equiv -Q^4 L_\mu \left[C_{00} + C_{01} L_\mu + C_{02} L_\mu^2 \right] \text{ with :} \quad (6)$$

$$C_{00} = a_s^2 \left(1 + 20.75a_s + 305.95a_s^2 \right), \quad C_{01} = -a_s^3 \left(\frac{9}{4} + 72.531a_s \right), \quad C_{02} = 5.0625a_s^4,$$

where the NLO (resp. N2LO) contributions have been obtained in [34] (resp. [11,35]). $L_\mu \equiv \text{Log}(Q^2/\mu^2)$ where μ is the subtraction point and $a_s \equiv \alpha_s/\pi$. We shall use for 3 flavours:

$$\Lambda = 340(28) \text{ MeV}, \quad (7)$$

deduced from $\alpha_s(M_Z) = 0.1182(19)$ from $M_{\chi_{c0,b0}} - M_{\eta_c, \eta_b}$ mass-splittings [36,37], τ -decays [38,39] and the world average [3,40]. We shall use the running QCD coupling to order α_s^2 :

$$a_s(\mu) = a_s^{(0)} \left\{ 1 - a_s^{(0)} \frac{\beta_2}{\beta_1} LL_\mu + \left(a_s^{(0)} \right)^2 \left[\left(\frac{\beta_2}{\beta_1} \right)^2 (LL_\mu^2 - LL_\mu - 1) + \frac{\beta_3}{\beta_1} \right] \right\}, \quad (8)$$

where:

$$a_s^{(0)} \equiv \frac{1}{-\beta_1 \text{Log}(\mu/\Lambda)} \quad \text{and} \quad LL_\mu \equiv \text{Log} \left[2 \text{Log}(\mu/\Lambda) \right]. \quad (9)$$

◇ *The dimension-four gluon condensate ($n = 2$)*

Its contribution reads [7,10]:

$$C_4(\mathcal{O}_4) \equiv (C_{40} + L_\mu C_{41}) \langle \alpha_s G^2 \rangle : C_{40} = 2\pi a_s \left(1 + \frac{175}{36} a_s \right), \quad C_{41} = \frac{9}{2} \pi a_s^2. \quad (10)$$

We shall use the value:

$$\langle \alpha_s G^2 \rangle = (6.35 \pm 0.35) \times 10^{-2} \text{ GeV}^4, \quad (11)$$

determined from light and heavy quark systems [36,37,41,42].

◇ *The dimension-six ($n = 3$) gluon condensate*

Its contribution reads to lowest order [7] (see [10,43] for the α_s correction for $n_f = 0^1$):

$$C_6(\mathcal{O}_6) = (C_{60} + L_\mu C_{61}) \langle g^3 f_{abc} G^a G^b G^c \rangle / Q^2 : C_{60} = -a_s, \quad C_{61} = \frac{29}{4} a_s^2, \quad (12)$$

with [41]:

$$\langle g^3 f_{abc} G^a G^b G^c \rangle = (8.2 \pm 1.0) \text{ GeV}^2 \langle \alpha_s G^2 \rangle, \quad (13)$$

which notably differs from the instanton liquid model estimate $\langle g^3 f_{abc} G^a G^b G^c \rangle \approx (1.5 \pm 0.5) \text{ GeV}^2 \langle \alpha_s G^2 \rangle$ [7,8,14,44] used in [6,10–12]. However, the ratio of $\langle g^3 f_{abc} G^a G^b G^c \rangle$ over $\langle \alpha_s G^2 \rangle$ is in fair agreement with earlier lattice determination [45].

◇ *The dimension-eight ($n = 4$) gluon condensate*

Its contribution reads:

$$C_8(\mathcal{O}_8) = C_{80} \langle \alpha_s^2 G^4 \rangle / Q^4 : C_{80} = 4\pi \alpha_s \quad (14)$$

with:

$$\begin{aligned} \langle \alpha_s^2 G^4 \rangle &\equiv \left[\langle (\alpha_s f_{abc} G_{\mu\rho}^a G_{\nu\rho}^b)^2 \rangle + 10 \langle (\alpha_s f_{abc} G_{\mu\nu}^a G_{\rho\lambda}^b)^2 \rangle \right] \\ &\simeq k \left(\frac{15}{16} \right) \langle \alpha_s G^2 \rangle^2 : k = (1.5 \pm 0.5), \end{aligned} \quad (15)$$

from factorization [7,8] where its validity has been questioned by [46]. Indeed, in the quark channel, the factorization hypothesis has been found to be largely violated for the four-quark condensates [39,47,48]. To be conservative, we assume that the factorization is violated within the k -factor.

¹ C_{61} would be zero for $n_f = 3$ [11]. We shall see in the analysis that the effect of this correction is negligible.

• *Beyond the standard SVZ-expansion*

◇ *The tachyonic gluon mass ($n = 1$)*

To these standard contributions in the OPE, we can consider the one from a dimension-two tachyonic gluon mass contribution which is expected to phenomenologically parametrize the uncalculated large order terms of the perturbative QCD series [9,49–51]. Its existence is supported by some AdS approaches [52–54]. This effect has been calculated explicitly in [9]:

$$C_2\langle O_2 \rangle = -C_{21} L_\mu \lambda^2 Q^2 : \quad C_{21} = 3a_s, \tag{16}$$

where λ^2 is the tachyonic gluon mass determined from $e^+e^- \rightarrow$ hadrons data and the pion channel [9,55,56]:

$$a_s \lambda^2 \simeq -(6.0 \pm 0.5) \times 10^{-2} \text{ GeV}^2. \tag{17}$$

◇ *The direct instanton ($n \geq 5/2$)*

In an instanton liquid model [8,44], the direct instanton contribution is assumed to be dominated by the single instanton-anti-instanton contribution via a non-perturbative contribution to the perturbative Wilson coefficient [10,12]:

$$\psi_P(Q^2)|_{\bar{I}-I} = -32 Q^4 \int \rho^4 \left[K_2(\rho\sqrt{Q^2}) \right]^2 dn(\rho), \tag{18}$$

which is opposite in sign with the one for the gluonium scalar correlator. $K_2(x)$ is the modified Bessel function of the second kind. At this stage this classical field effect is beyond the SVZ expansion where the later assumes that one can separate without any ambiguity the perturbative Wilson coefficients from the non-perturbative condensate contributions.

Besides the fact that it contributes in the OPE as $1/Q^5$, i.e. it acts like other high-dimension condensates not taken into account in the OPE, the above instanton effect depends crucially on the (model-dependent) overall density $\bar{n} = \int_0^\infty d\rho n(\rho)$ and on its average size $\bar{\rho} = (1/\bar{n}) \int_0^\infty d\rho \rho n(\rho)$ which (unfortunately) are not quantitatively under a good control ($\bar{\rho}$ ranges from 5 [8] to 1.94, [41] and 1.65 GeV^{-1} [44], while $\bar{n} \approx (0.5 \sim 1.2) \text{ fm}^{-4}$). They contribute with a high power in ρ to the spectral function $\text{Im}\psi(t)$, which can be found explicitly in [10,12], and behave as:

$$\frac{1}{\pi} \text{Im}\psi_P(t)|_{\bar{I}-I} \begin{matrix} \xrightarrow{t \rightarrow \infty} -n \left(\rho\sqrt{t} \right)^{-5} \\ \xrightarrow{t \rightarrow 0} -n \left(\rho\sqrt{t} \right)^4. \end{matrix} \tag{19}$$

However, it has been noticed in [12] that this negative sign leads to a violation of positivity and some inconsistencies for the sum rule analysis in this pseudoscalar channel. As such effects are quite inaccurate and model-dependent, we shall not consider them explicitly in our analysis. Instead, an eventual deviation of our results within the standard SVZ expansion from some experimental data or/and or some other alternative estimates (Low-Energy Theorems (LET), Lattice calculations, ...) may signal the need of such (beyond the standard OPE) effects in the analysis.

One should mention that the approach within the standard SVZ OPE and without a direct instanton effect used in the:

– $U(1)_A$ channel has predicted successfully the value of the topological charge, its slope and the η' -mass and decay constant [33,57–60].

Table 1
QCD input parameters from recent QSSR analysis based on stability criteria. k measures the violation of factorization.

| Parameter | Value | Ref. |
|---|--|------------|
| $\alpha_s(M_Z)$ | $0.1181(16)(3) \implies \Lambda = (340 \pm 28) \text{ MeV}$ | [36,37] |
| $\lambda^2 \times 10^2 [\text{GeV}^2]$ | $-6.0(5)$ | [55,56] |
| $\langle \alpha_s G^2 \rangle \times 10^2 [\text{GeV}^4]$ | $6.35(35)$ | [41,42] |
| $\langle g^3 G^3 \rangle / \langle \alpha_s G^2 \rangle [\text{GeV}^2]$ | $8.2(1.0)$ | [41,42,45] |
| $\langle \alpha_s^2 G^4 \rangle$ | $k \frac{15}{16} \langle \alpha_s G^2 \rangle^2 \quad : \quad k = (1.5 \pm 0.5)$ | [8] |

– Pseudoscalar pion and kaon channels have reproduced successfully the value of the light quark masses where we have also explicitly shown [61] that the direct instanton effect induces a relatively small correction contrary to some vigorous claims in the literature (see e.g. [62]).

3. The inverse Laplace transform sum rules

From its QCD asymptotic behaviour $\sim (-q^2)^2 \text{Log}(-q^2/\mu^2)$ one can write a twice subtracted dispersion relation:

$$\psi_P(q^2) = \psi_P(0) + q^2 \psi'_P(0) + \frac{q^4}{\pi} \int_0^\infty \frac{dt}{t^2} \frac{\text{Im}\psi_P(t)}{(t - q^2 - i\epsilon)}. \tag{20}$$

• Following standard QSSR techniques [14,21,22], one can derive from it different form of the sum rules. In this paper, we shall work with the Exponential or Borel [14,17,18] or Inverse Laplace transform [16] Finite Energy sum rule (LSR)²:

$$\mathcal{L}_n^c(\tau) = \int_0^{t_c} dt t^n e^{-t\tau} \frac{1}{\pi} \text{Im}\psi_P(t) \quad : \quad n = -2, -1, 0, 1, \dots, 4, \tag{21}$$

and the corresponding ratios of sum rules:

$$\mathcal{R}_{n+l}^c(\tau) \equiv \frac{\mathcal{L}_{n+l}^c(\tau)}{\mathcal{L}_n^c(\tau)}, \tag{22}$$

where τ is the Laplace sum rule variable. In the duality ansatz:

$$\begin{aligned} \frac{1}{\pi} \text{Im}\psi_P(t) &= 2 \sum_P f_P^2 M_P^4 \delta(t - M_G^2) + \theta(t - t_c) \text{“QCD continuum”} : \\ \langle 0 | (8\pi) Q(x) | P \rangle &= \sqrt{2} f_P M_P^2, \end{aligned} \tag{23}$$

where the f_P is the resonance coupling normalized as $f_\pi = 93 \text{ MeV}$; M_P is the mass while the “QCD continuum” comes from the discontinuity $\text{Im}\psi_P(t)|_{QCD}$ of the QCD diagrams from the continuum threshold t_c . In the “One narrow resonance \oplus QCD continuum” parametrization of the spectral function³:

² The name Inverse Laplace transform has been attributed due to the fact that perturbative radiative corrections have this property.

³ Finite width effect has been shown in the example of the large width σ meson to give a negligible correction [5,6] and will not be considered in this paper.

$$\mathcal{R}_{n+l}^c(\tau) \simeq M_P^2. \quad (24)$$

• $\mathcal{L}_{(-2,-1)}$ have been used to estimate the topological charge, its slope and the η' parameter within the minimal duality ansatz which will be reviewed and discussed in Section 4.

• To get \mathcal{L}_0 , we take the Inverse Laplace transform of the 1st superconvergent 3rd derivative of the two-point correlator. In this way, we obtain:

$$\mathcal{L}_0^c(\tau) = \tau^{-2} 2 \sum_{n=0,2,\dots} D_n^0, \quad (25)$$

with [$L_\tau \equiv \text{Log}(\tau\mu^2)$]:

$$\begin{aligned} D_0^0 &= \left[2C_{00} - 2C_{01}(3 - 2\gamma_E - 2L_\tau) \right. \\ &\quad \left. - 6C_{02}[1 - 3\gamma_E + \gamma^2 - \pi^2/6 + (-3 + 2\gamma_E)L_\tau + L_\tau^2] \right] (1 - \rho_2), \\ D_2^0 &= -\frac{C_{21}}{2} \lambda^2 \tau (1 - \rho_1), \\ D_4^0 &= -C_{41} \langle \alpha_s G^2 \rangle \tau^2, \\ D_6^0 &= [C_{60} - C_{61}(\gamma_E - L_\tau)] \langle g^3 f_{abc} G^a G^b G^c \rangle \tau^3, \\ D_8^0 &= C_{80} \left(\frac{15}{16} \right) k \langle \alpha_s G^2 \rangle^2 \tau^4. \end{aligned} \quad (26)$$

• The other higher degrees sum rules $\mathcal{L}_n^c(\tau)$ for $n \geq 1$ can be deduced from the n^{th} τ -derivative of $\mathcal{L}_0^c(\tau)$:

$$\mathcal{L}_n^c(\tau) = (-1)^n \frac{d^n}{d\tau^n} \mathcal{L}_0^c(\tau). \quad (27)$$

• These superconvergent sum rules obey the homogeneous renormalization group equation (RGE):

$$\left\{ -\frac{\partial}{\partial t} + \beta(\alpha_s) \alpha_s \frac{\partial}{\partial \alpha_s} \right\} \mathcal{L}_n^c(e^t \tau, \alpha_s) = 0, \quad (28)$$

where $t \equiv (1/2)L_\tau$. The renormalization group improved (RGI) solution is:

$$\mathcal{L}_n^c(e^t \tau, \alpha_s) = \mathcal{L}_n^c(t=0, \bar{\alpha}_s(\tau)), \quad (29)$$

where $\bar{\alpha}_s(\tau)$ is the QCD running coupling.

• In the following analysis, we shall work with the family of sum rules having degrees less or equal to 4. In so doing, we shall select the sum rules which present stability (minimum or inflexion point) in the sum rule variable τ and in the continuum threshold t_c such that we can extract optimal information from the analysis.

4. The η' mass and decay constant

Keeping only the η' and/or the lowest mass gluonium contributions in the parametrization of the spectral function (Eq. (23)), \mathcal{L}_{-1}^c and \mathcal{L}_0^c have been used earlier to look for the gluon component of the η' mass and coupling and to estimate the topological charge $\chi(0)$ and its slope $\chi'(0)$ [33,57–60,63]. The LSR results in pure Yang-Mills fairly agree with the ones from large

N_c [64–66] and lattice calculations [67]. Here, we use the sum rule \mathcal{L}_{-1}^c in the chiral limit where $\chi(0) \equiv \psi_P(0)/(8\pi) = 0$:

$$\mathcal{L}_{-1}^c \equiv \int_0^{t_c} \frac{dt}{t} e^{-t\tau} \frac{1}{\pi} \text{Im} \psi_P(t) = \mathcal{L}_{-1}^c|_{\text{QCD}} + \psi_P(0), \quad (30)$$

where:

$$\mathcal{L}_{-1}^c|_{\text{QCD}} = \tau^{-2} 2 \sum_{n=0,2,\dots} D_n^{-2}, \quad (31)$$

with:

$$\begin{aligned} D_0^{-1} &= \left[C_{00} + 2C_{01}(1 - \gamma_E - L_\tau) + 3C_{02}[(\gamma_E + L_\tau)^2 - 2(\gamma_E + L_\tau) - \frac{\pi^2}{6}] \right] (1 - \rho_1), \\ D_2^{-1} &= \tau C_{21} \lambda^2 (1 - \rho_0), \\ D_4^{-1} &= -\tau^2 \left[C_{40} - C_{41}(\gamma_E + L_\tau) \right] \langle \alpha_s G^2 \rangle, \\ D_6^{-1} &= -\tau^3 \left[C_{60} + C_{61}(1 - \gamma_E - L_\tau) \right] \langle g^3 f_{abc} G^a G^b G^c \rangle, \\ D_8^{-1} &= -\frac{\tau^4}{2} C_{80} \langle \alpha_s G^2 \rangle^2, \end{aligned} \quad (32)$$

for extracting the decay constant f_{η_1} . We have attempted to extract the mass M_{η_1} using the low degree ratio of moments \mathcal{R}_{0-1} but fail due to the absence of τ stability. Then, we use the GMO mass formula quoted in [33] derived in the massless pion limit and assuming a SU3 symmetry for the decay constants from [65]:

$$M_{\eta_1}^2 \simeq M_{\eta'}^2 - \frac{2}{3} M_K^2 = (870 \text{ MeV})^2. \quad (33)$$

An alternative derivation of M_{η_1} in pure Yang-Mills from $1/N_c$ expansion by [64–66] and corrected in [5] by including 3 massless quarks loops contribution through the β function [$\mathcal{O}(n_f/N_c)$ correction] leads to:

$$M_{\eta_1}^{n_f=3} \simeq \left(\frac{\beta_1^{n_f=3}}{\beta_1^{n_f=0}} \right)^{1/2} \times M_{\eta_1}^{YM} \simeq 779 \text{ MeV} \quad : \quad \beta_1^{n_f} = -\frac{1}{2} \left(11 - \frac{2}{3} n_f \right), \quad (34)$$

with:

$$\begin{aligned} M_{\eta_1}^{YM} &\simeq \frac{1}{f_\pi} \left(6 \frac{\partial}{\partial \theta} \langle Q(x) \rangle_{\theta=0} \equiv -6\chi(0)|_{YM} \right)^{1/2} : \\ \chi(0)|_{YM} &\simeq -(180 \text{ MeV})^4 \quad \text{and} \quad f_\pi = 92.2 \text{ MeV}. \end{aligned} \quad (35)$$

In the following analysis, we shall use the mean of the two determinations:

$$M_{\eta_1} \simeq 825(45) \text{ MeV}, \quad (36)$$

with the (conservative) error from its distance to the two former. The analysis for N2LO is shown in Fig. 1 where, at the τ -minimum: $\tau = (0.68; 0.74) \pm 0.04 \text{ GeV}^{-2}$ corresponding to $t_c = 8$ (beginning of τ -stability) to 14 GeV^2 (t_c -stability), one deduces the estimate:

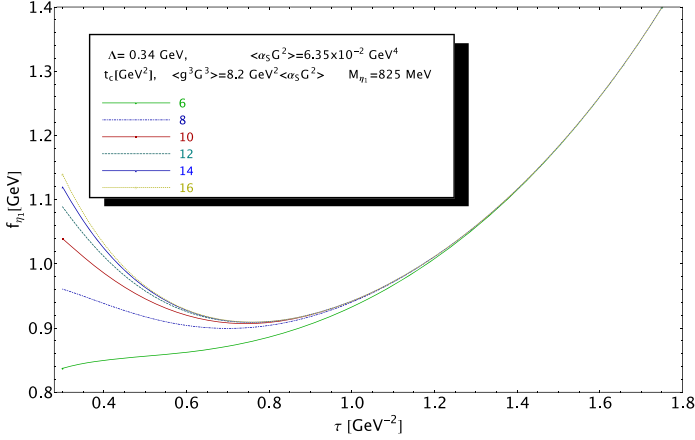


Fig. 1. The decay constant f_{η_1} from \mathcal{L}_{-1}^c as a function of τ at N2LO using $M_{\eta_1} = 825$ MeV and the QCD inputs in Table 1. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

$$f_{\eta_1} = 905(5)_{t_c}(1)_{\tau}(43)_{\Lambda}(53)_{\lambda^2}(3)_{G^2}(2)_{G^3}(0)_{G^4}(24)_{M_{\eta_1}} = 905(72) \text{ MeV}, \quad (37)$$

where one should note that f_{η_1} here is not exactly the one from e.g. $J/\psi \rightarrow \gamma \eta'$, $\eta' \rightarrow \gamma \gamma$ [68]. One can notice that:

◇ The N2LO correction is large and has increased the value of f_{η_1} from NLO by about a factor 2. The large N2LO effects to f_{η_1} require a further control of the size of the higher order PT corrections. We have considered in [9] that the tachyonic gluon mass gives an estimate of such uncalculated higher order terms of PT or it is equivalent to say that the PT series grows geometrically [49]. One can check that the tachyonic gluon mass decreases f_{η_1} by 53 MeV which we consider (here and in the following) as an estimate of the errors due to the truncation of the PT series but we do not include it in the central value of f_{η_1} . Considering this error estimate at its face value, one may consider that the PT series reach its asymptotic value at N2LO.

◇ The value of M_{η_1} and of f_{η_1} in Eqs. (36) and (37) will be used as inputs in the rest of the paper.

5. The slope $\chi'(0)$ of the topological charge and the proton spin in the chiral limit

• Sum rule estimate of $\chi'(0)$ at N2LO in the chiral limit

$\chi'(0)$ has been estimated in pure Yang-Mills to be [60]:

$$\chi'(0)|_{YM} = -(7 \pm 3) \text{ MeV}^2 \quad (38)$$

in agreement with the lattice result $-(9.8 \pm 0.9) \text{ MeV}^2$ [67]. In the following, we shall update its NLO estimate in the chiral limit [33] by including N2LO PT corrections and NLO ones to the condensate contributions. We shall use the previous value of M_{η_1} and the value or the QCD expression of f_{η_1} into the twice subtracted sum rule:

$$\psi'_P(0)|_{\text{LSR}} = \int_0^{t_c} \frac{dt}{t^2} e^{-t\tau} \frac{1}{\pi} \text{Im} \psi_P(t) - \mathcal{L}_{-2}^c|_{QCD} + \tau \psi_P(0), \quad (39)$$

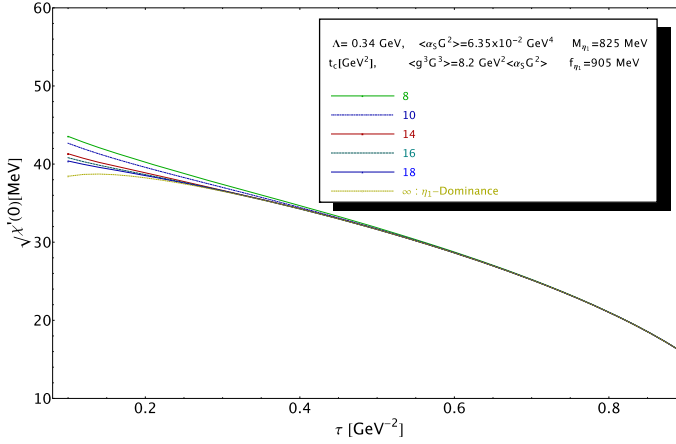


Fig. 2. The slope of the topological charge $\chi'(o)$ from \mathcal{L}_{-2}^c as a function of τ at N2LO using $f_{\eta_1} = 881$ MeV.

where:

$$\mathcal{L}_{-2}^c|_{QCD} = \tau^{-1} 2 \sum_{n=0,2,\dots} D_n^{-2}, \quad (40)$$

with:

$$\begin{aligned} D_0^{-2} &= \left[C_{00} - 2C_{01}(\gamma_E + L_\tau) + 3C_{02}[(\gamma_E + L_\tau)^2 - \frac{\pi^2}{6}] \right] (1 - \rho_0), \\ D_2^{-2} &= -\tau C_{21} \gamma_E \lambda^2, \\ D_4^{-2} &= \tau^2 \left[C_{40} + C_{41}(1 - \gamma_E) \right] \langle \alpha_s G^2 \rangle, \\ D_6^{-2} &= \frac{\tau^3}{2} \left[C_{60} + C_{61} \left(\frac{3}{2} - \gamma_E - L_\tau \right) \right] \langle g^3 f_{abc} G^a G^b G^c \rangle, \\ D_8^{-2} &= \frac{\tau^4}{6} C_{80} \langle \alpha_s G^2 \rangle^2. \end{aligned} \quad (41)$$

We show the N2LO analysis from \mathcal{L}_{-2}^c where, in Fig. 2, we use the value of f_{η_1} in Eq. (37) and in Fig. 3 its QCD expression from Eq. (30) where one should note that the sum of higher states contributions to the spectral function is (automatically) included in this second option. One can notice a better τ -stability (clearer inflexion point) for the 2nd case at $\tau \simeq (0.50 \pm 0.04) \text{ GeV}^{-2}$ and for $t_c \simeq (14 \sim 18) \text{ GeV}^2$, at which, we deduce:

$$\begin{aligned} \sqrt{\chi'(0)}|_{\text{N2LO}} &\equiv \frac{1}{8\pi} \sqrt{\psi'_p(0)} \\ &= 24.3(2.8)_\tau(0.2)_{t_c}(1.6)_\Lambda(0.8)_{\lambda^2}(0.2)_{G^2}(0.4)_{G^3}(0.1)_{G^4}(0.4)_{M_{\eta_1}} \\ &= 24.3(3.4) \text{ MeV}. \end{aligned} \quad (42)$$

The NLO analysis gives curves very similar to the N2LO ones. At the corresponding inflexion point $\tau \simeq (0.44 \pm 0.04) \text{ GeV}^{-2}$, one obtains:

$$\begin{aligned} \sqrt{\chi'(0)}|_{\text{NLO}} &= 18.2(2.5)_\tau(0.2)_{t_c}(1.1)_\Lambda(0.9)_{\lambda^2}(0.4)_{G^2}(0.6)_{G^3}(0.1)_{G^4}(0.4)_{M_{\eta_1}} \\ &= 18.2(3.0) \text{ MeV}. \end{aligned} \quad (43)$$

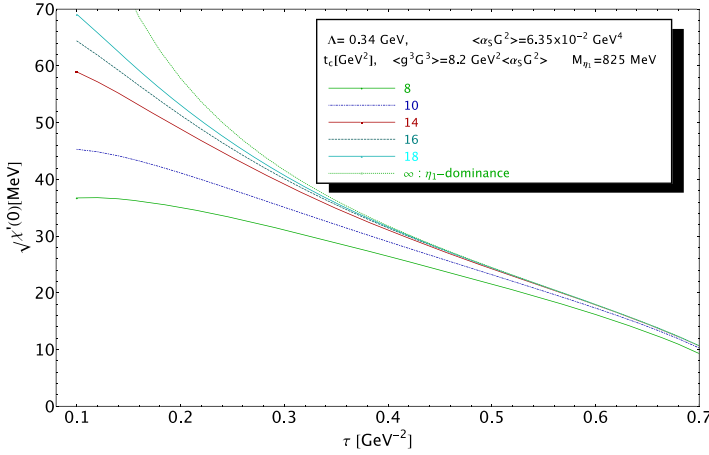


Fig. 3. The slope of the topological charge $\chi'(o)$ from \mathcal{L}^c_{-2} as a function of τ at N2LO using the QCD expression of f_{η_1} given in Eqs. (30) to (32).

These new values agree within the errors with the NLO one 22.3(4.8) MeV obtained from Laplace sum rule (LSR) in [33] using different values of the QCD input parameters and a slightly different QCD expression due to some errors in the α_s PT corrections giving by [34] explaining that the $\chi'(0)$ curves present minimum there.

• *Effect on the proton spin*

It has been shown by [63,68] (see also [69]) that the singlet form factor $G_A^{(0)}(Q^2)$ appearing in the first moment of the polarised proton structure function g_1^P (Ellis-Jaffe sum rule [70]) can be related in the chiral limit ($m_q = 0$) to the slope of the topological charge $\chi'(0)$ as:

$$G_A^{(0)}(Q^2)\bar{u}\gamma_5u = \frac{1}{2M_p}(2n_f)\sqrt{\chi'(0)}\Gamma_{\Phi_{5R}\bar{P}P}, \tag{44}$$

where Φ_{5R} is the renormalized bilinear quark current and $\Gamma_{\Phi_{5R}\bar{P}P}$ the renormalization group (RG) invariant scale independent proper vertex. In the chiral limit and using the OZI approximation, one would obtain [33]:

$$\begin{aligned} \Gamma_{\Phi_{5R}\bar{P}P}|_{\text{OZI}} &= \sqrt{2}g_{\eta_8 PP}(\bar{u}\gamma_5u), & \sqrt{\chi'(0)}|_{\text{OZI}} &= f_\pi/\sqrt{6} = 38 \text{ MeV}, \\ G_A^{(0)}|_{\text{OZI}} &= 0.579 \pm 0.021, \end{aligned} \tag{45}$$

while from our analysis:

$$G_A^{(0)}|_{\text{LSR}}(Q^2 = 10 \text{ GeV}^2) = G_A^{(0)}|_{\text{OZI}} \frac{\sqrt{\chi'(0)}|_{\text{LSR}}}{\sqrt{\chi'(0)}|_{\text{OZI}}} = (0.340 \pm 0.050), \tag{46}$$

after running $\chi'(0)|_{\text{LSR}}$ from 2 to 10 GeV^2 implying:

$$\sqrt{\chi'(0)}|_{\text{LSR}}(Q^2 = 10 \text{ GeV}^2) = (22.5 \pm 3.1) \text{ MeV}. \tag{47}$$

Using the previous value of $G_A^{(0)}|_{\text{LSR}}$ and: $G_A^{(3)} = 0.625 \pm 0.004$, $G_A^{(8)} = (0.167 \pm 0.006)$, from hyperon and β -decays [73], the first moment of the polarised proton structure function gives:

$$\Gamma_1^p(Q^2 = 10 \text{ GeV}^2) \equiv \int_0^1 dx g_1^p(x, Q^2) = (0.144 \pm 0.005), \quad (48)$$

in excellent agreement with the earlier data [71] and experimental world average [72]:

$$\Gamma_1^p(Q^2 = 10 \text{ GeV}^2)|_{\text{exp}} = 0.145 \pm 0.014 \implies G_A^{(0)}|_{\text{exp}} = 0.35 \pm 0.12, \quad (49)$$

and more recent data from COMPASS [74] and HERMES [75]:

$$G_A^{(0)}|_{\text{exp}} = 0.330 \pm 0.039, \quad (50)$$

which are lower than the OZI value by a factor 1.76.

6. On some previous gluonium mass from the ratio of moments \mathcal{R}_{10}

- The extraction of the lowest gluonium mass from the sum rule within the minimal duality ansatz “*One resonance \oplus QCD continuum*” parametrization of the spectral function often comes from the ratio of moments \mathcal{R}_{10} [6,10,59] with the most recent result to N2LO within the standard SVZ-expansion from Laplace sum rule [10] and without including the η' contribution:

$$M_{P_2}|_{\text{N2LO}} = (2.3 \pm 0.2) \text{ GeV} \quad f_{P_2} = (0.21 \pm 0.04) \text{ GeV} \quad \text{for } \Lambda = (0.15 \pm 0.05) \text{ GeV}. \quad (51)$$

Similar results using Laplace sum rules have been obtained in [12] by adding the contribution of instanton-anti-instanton in the OPE and by including the contribution of the η' :

$$M_{P_2}|_{\text{N2LO}} = (2.2 \pm 0.2) \text{ GeV}, \quad f_{P_2} = (0.42 \pm 0.18) \text{ GeV} \quad \text{for } \Lambda = 0.20 \text{ GeV}. \quad (52)$$

The Gaussian sum rule including instanton-anti-instanton but without the η' leads to [11]:

$$M_{P_2}|_{\text{N2LO}} = (2.65 \pm 0.33) \text{ GeV}, \quad \Gamma_{P_2} \leq 0.54 \text{ GeV} \quad \text{for } \Lambda = 0.3 \text{ GeV}. \quad (53)$$

The previous N2LO results can be compared with the LO one (independent of the Λ value) [59]:

$$M_{P_2}|_{\text{LO}} = (1.70 \pm 0.06) \text{ GeV}, \quad (54)$$

and to the NLO one for $\Lambda = (0.375 \pm 0.125) \text{ GeV}$ [6]:

$$M_{P_2}|_{\text{NLO}} = (2.05 \pm 0.19) \text{ GeV}. \quad (55)$$

- We update the N2LO results in the literature using the present values of the QCD parameters given in Table 1 without including the η' -contribution. We show the results of the analysis versus τ in Fig. 4 for different values of t_c . We obtain for $\Lambda = (340 \pm 28) \text{ MeV}$:

$$M_{P_2}|_{\text{N2LO}} = 1470(39)_{t_c}(46)_\tau(83)_\Lambda(44)_{\lambda^2}(19)_{G^2}(11)_{G^3}(31)_{G^4} = 1470(118) \text{ MeV}, \quad (56)$$

where we have taken the values at the inflexion points: $\tau = (0.80 \pm 0.04) \text{ GeV}^{-2}$ and t_c from 7 to 14 GeV^2 . We have considered the contribution of the tachyonic gluon mass squared λ^2 as an estimate of the non-calculated higher order terms as expected from [9,49–51].

◇ For $\Lambda = (0.15 \pm 0.05) \text{ GeV}$, we have a minimum in τ (Fig. 5). At the minimum, we obtain using the values of the condensates given in Table 1 and for t_c from 7 to 14 GeV^2 :

$$M_{P_2}|_{\text{N2LO}}^{0.15} = 2057(53)_{t_c}(127)_\Lambda(44)_{\lambda^2}(14)_{G^2}(43)_{G^3}(70)_{G^4} = 2057(166) \text{ MeV}, \quad (57)$$

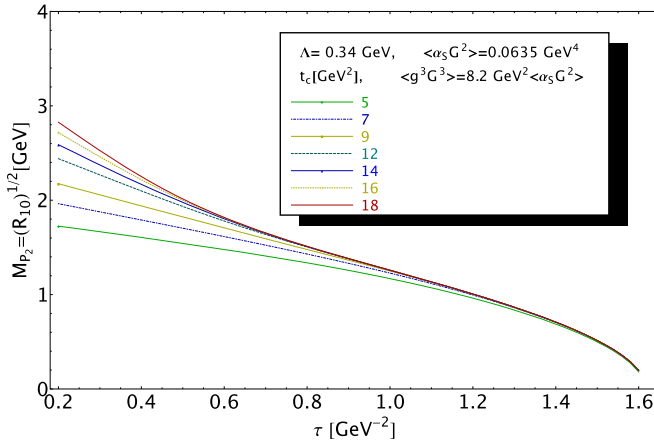


Fig. 4. M_{P_2} from \mathcal{R}_{10}^c as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV.

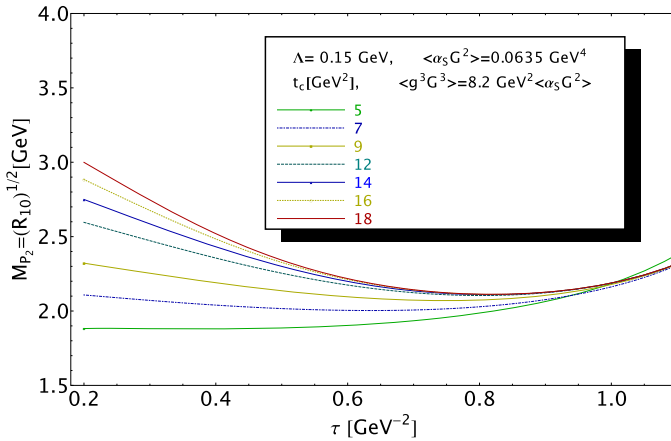


Fig. 5. M_{P_2} from \mathcal{R}_{10}^c as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.15$ GeV [10].

which reproduces within the error the result of [10] quoted in Eq. (51) and indicates that the result is quite sensitive to the value of Λ . The estimate of the HO non-calculated term is quantified by the contribution of λ^2 in the quoted error. We have not repeated the analysis for the Gaussian sum rules which is expected to reproduce the result from LSR at the stability point [11].

◊ We repeat the analysis for $\Lambda = 0.2$ GeV, where we have an inflexion point in τ (Fig. 6). We obtain within the standard SVZ expansion and for $\tau = (0.84 \pm 0.04)$ GeV^{-2} :

$$M_{P_2}|_{\text{N2LO}}^{0,20} \simeq 1867(38)_{t_c}(100)_{\tau}(-)_{\Lambda}(60)_{\lambda^2}(6)_{G^2}(51)_{G^3}(94)_{G^4} = 1867(163) \text{ MeV}, \quad (58)$$

indicating that the eventual instanton contribution, with the parameters used in [12], increases the prediction by about 330 MeV when compared with the one in Eq. (52).

◊ We notice that, for a given truncation of the PT series (N2LO here), the shape of the curves of the LSR changes with the value of Λ (minimum for $\Lambda = 0.15$ GeV and inflexion points for $\Lambda = 0, 20, 0.34$ GeV). In the same time the value of the mass is very affected by the one of Λ .

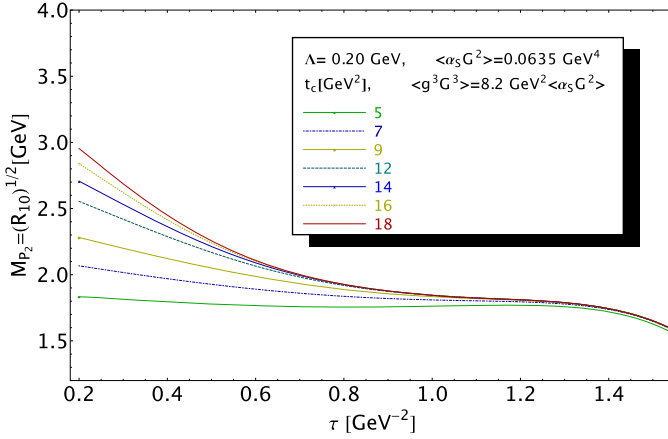


Fig. 6. M_{P_2} from \mathcal{R}_{10}^c as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.20$ GeV [12].

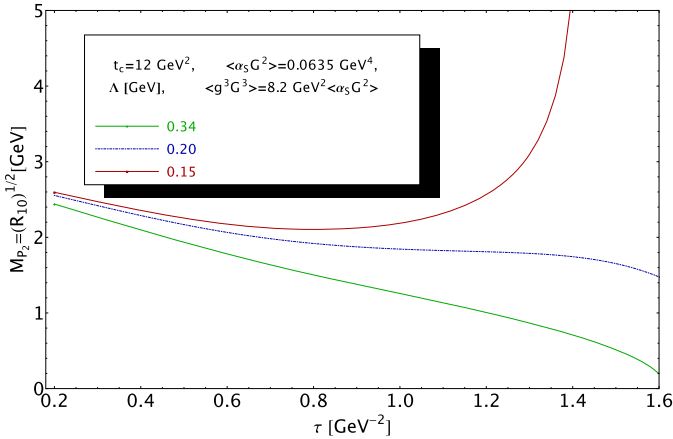


Fig. 7. M_{P_2} from \mathcal{R}_{10}^c as a function of τ at N2LO for different values of Λ and for $t_c = 12$ GeV².

◊ In Fig. 7, we show the results versus τ for three different values of $\Lambda = 0.15, 0.20$ and 0.34 GeV and for a fixed value of $t_c = 12$ GeV². The analysis shows the sensitivity of the results versus Λ .

◊ Comparing the optimal result for each truncation of the PT series, one has (in units of MeV):

$$M_{P_2} = (2653 \pm 327) \text{ (LO)} \xrightarrow{-30\%} (1862 \pm 18) \text{ (NLO)} \xrightarrow{-21\%} (1470 \pm 39) \text{ (N2LO)}, \tag{59}$$

where the quoted error comes only from the range of t_c -values from 7 to 14 GeV². One can notice a slow convergence of the result. In the rest of the paper, we re-emphasize that we shall estimate the contribution of the remaining uncalculated higher order (HO) terms of the series from the contribution of the tachyonic gluon mass which we shall add to the errors in the result for M_P and f_P .

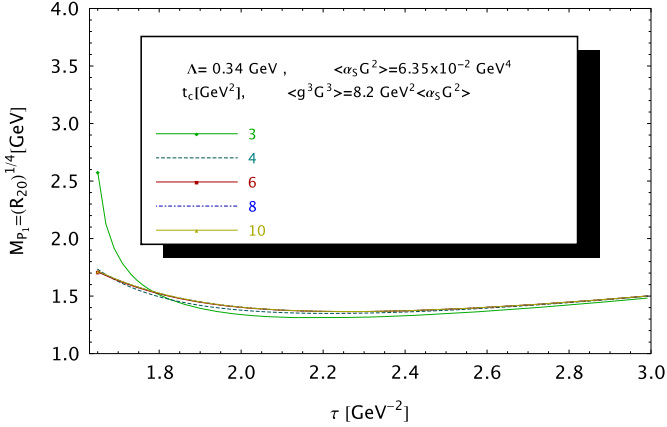


Fig. 8. M_{P_1} from \mathcal{R}_{20}^c as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV where the η_1 contribution is included.

7. Gluonia masses from \mathcal{R}_{mn}^c

• The analysis

In the following, we shall systematically extract the gluonia masses using different ratios of moments \mathcal{R}_{mn}^c ($n, m = -1, 0, \dots, 4$) within τ - and t_c -stability criteria. In order to show explicitly these stability regions specific for a given moment where the optimal results are obtained, we think that is important to show the different figures shown in Fig. 8 to 16. The results are summarized in Table 2 and the different sources of errors are given in Table 3 where we denote:

- ◇ $P_{1,2}$: One resonance: P_1 or P_2 .
- ◇ $S_{1\eta,2\eta} \equiv P_{1,2} \oplus \eta_1$: One resonance $\oplus \eta_1$ where M_{η_1} and f_{η_1} (gluon component of the η') are used as inputs from Eqs. (36) and (37) which have been deduced from previous sum rules analysis of $\mathcal{L}_{-1,-2}$.
- ◇ $S_2 \equiv P_2 \oplus S_{1\eta}$: Two resonances ($P_1 \oplus P_2$) $\oplus \eta_1$.
- ◇ $S'_2 \equiv P'_1 \oplus S_2$: Three resonances ($P'_1 \oplus P_1 \oplus P_2$) $\oplus \eta_1$.

8. The digluonium decay constants f_P

The di-gluonium decay constant is defined in Eq. (23) as $f_\pi = 93$ MeV. We shall use the moments $\mathcal{L}_2, \mathcal{L}_3$ (see Figs. 17 to 19) for extracting the decay constants of P_1, P_2 and P'_1 . The results are summarized in Table 2 while the different sources of the errors are given in Table 3.

9. Summary of the results

The results can be grouped into three types of gluonia:

- *Light*: η_1 [33,57,64–66] (see Eqs. (36) and (37)):

$$M_{\eta_1} = 825(45) \text{ MeV}, \quad f_{\eta_1} = 905(72) \text{ MeV}. \quad (60)$$

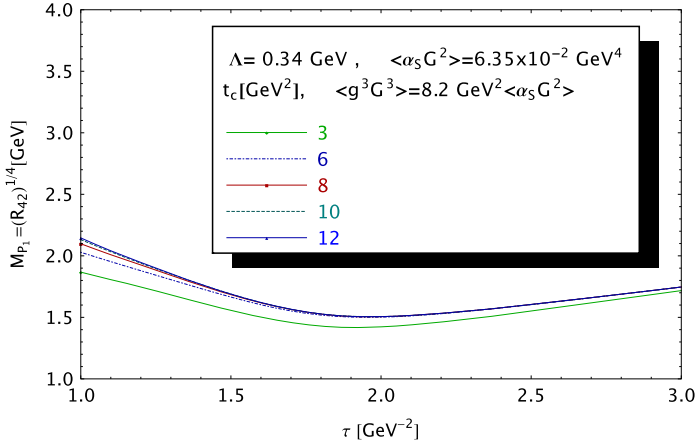


Fig. 9. M_{P_1} from \mathcal{R}_{42}^C as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV where the η_1 contribution is included.

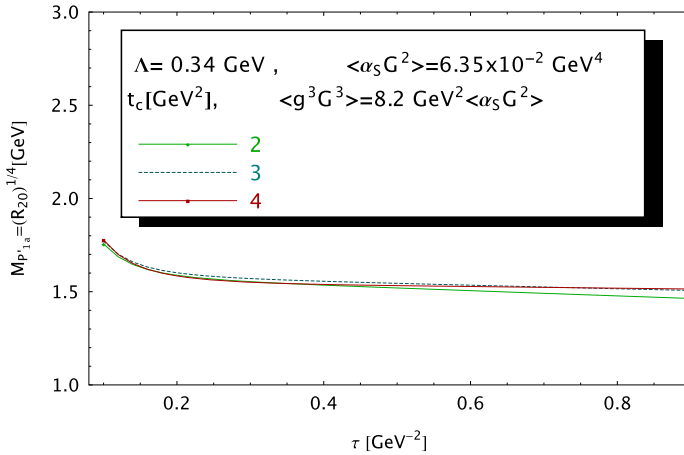


Fig. 10. $M_{P'_{1a}}$ from \mathcal{R}_{20}^C as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV where the η_1 and P_{1a} contributions are included.

- *Medium*: P_1 and the 1st radial excitations P'_1 (see Table 2):

$$\begin{aligned} [M_{P_{1a}}, M_{P_{1b}}] &= [1338(112), 1462(117)] \text{ MeV}, \\ [M_{P'_{1a}}, M_{P'_{1b}}] &= [1508(226), 1553(139)] \text{ MeV}, \end{aligned} \quad (61)$$

with their mean:

$$\begin{aligned} [M_{P_1}, f_{P_1}] &= [1397(81), 594(144)] \text{ MeV}, \\ [M_{P'_1}, f_{P'_1}] &= [1541(118), 205(282)] \text{ MeV}, \end{aligned} \quad (62)$$

where the radial excitation is very closed to the ground state like in the scalar channel [1]. The different sources of the errors are given in Table 3.

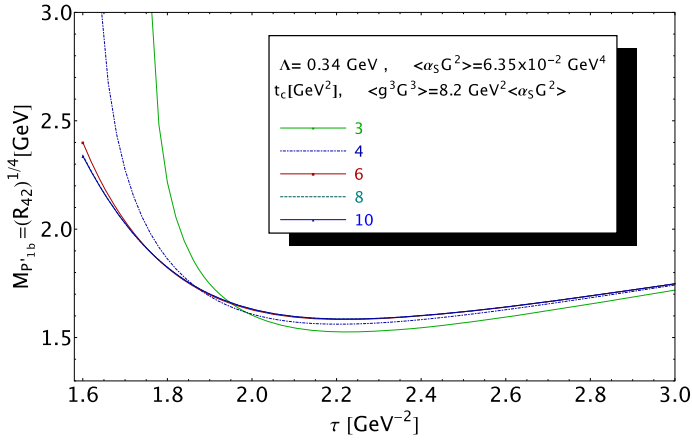


Fig. 11. $M_{P'_{1b}}$ from \mathcal{R}_{42}^C as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV where the η_1 and P_{1b} contributions are included.

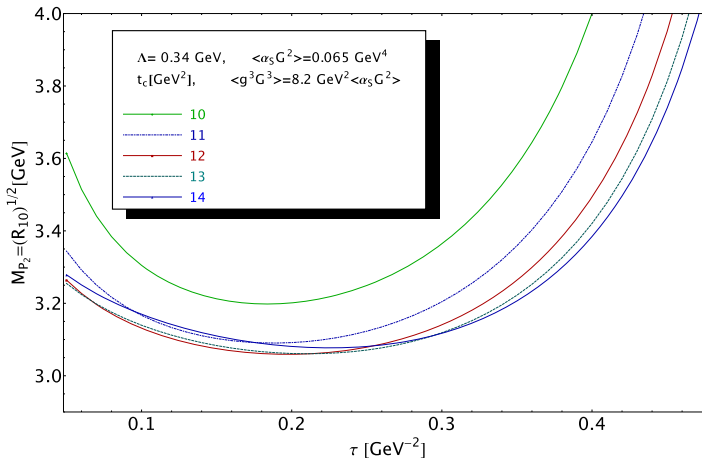


Fig. 12. M_{P_2} from \mathcal{R}_{10}^C as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV where the $\eta_1 \oplus P_1 \oplus P'_1$ contributions are included.

◇ Using the positivity of the spectral function, one can also derive a (rigorous) upper bound at the minimum of the ratio of moments \mathcal{R}_{20} and \mathcal{R}_{42} (equivalent to take $t_c \rightarrow \infty$). Including the η_1 contribution, it gives:

$$M_{P_{1a}} \leq 1365(106) \text{ MeV}, \quad M_{P_{1b}} \leq 1506(104) \text{ MeV} \implies M_{P_1} \leq 1437(74) \text{ MeV}, \tag{63}$$

where one can notice that in each case, the bound is almost saturated.

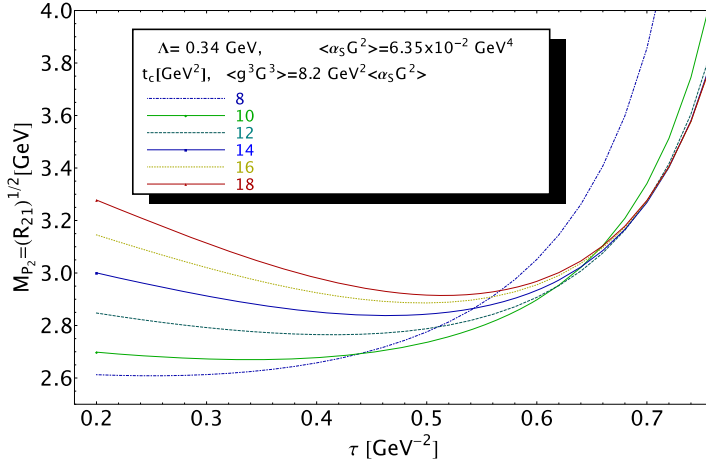


Fig. 13. M_{P_2} from \mathcal{R}_{21}^c as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV where the $\eta_1 \oplus P_1 \oplus P_1'$ contributions are included.

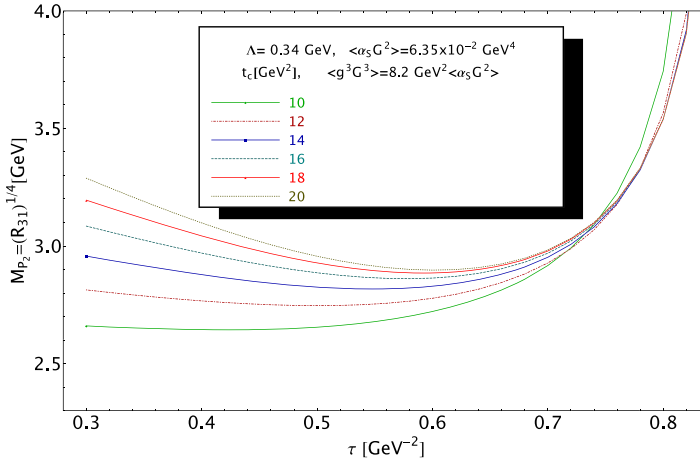


Fig. 14. M_{P_2} from \mathcal{R}_{31}^c as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV where the $\eta_1 \oplus P_1 \oplus P_1'$ contributions are included.

◇ One can notice that the two ratios of moments \mathcal{R}_{20}^c and \mathcal{R}_{42}^c stabilize at relatively large value of $\tau \simeq 2 \text{ GeV}^{-24}$ which make them sensitive to the medium mass M_{P_1} (see Figs. 8 and 9) which are not the cases of the other ratios of moments sensitive to the heavy mass M_{P_2} (see the corresponding values of τ in Table 3).

◇ We check the convergence of the PT series and the OPE at this scale. For a given PT series at N2LO and for different truncation of the OPE and including the η_1 contribution, we have for M_{P_1} (in units of MeV):

⁴ One should mention that \mathcal{R}_{20}^c present a second minimum at smaller value of $\tau \simeq 0.3 \text{ GeV}^{-2}$ which we shall not consider in this case. For an attempt to get M_{P_2} within a two resonances parametrization, this second minimum disappears.

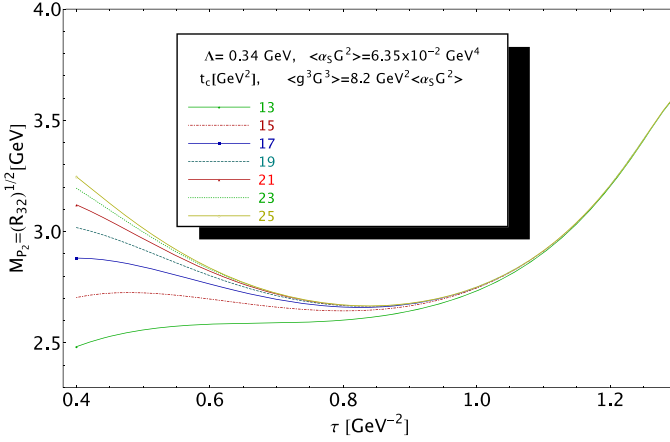


Fig. 15. M_{P_2} from \mathcal{R}_{32}^C as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV where the $\eta_1 \oplus P_1 \oplus P_1'$ contributions are included.

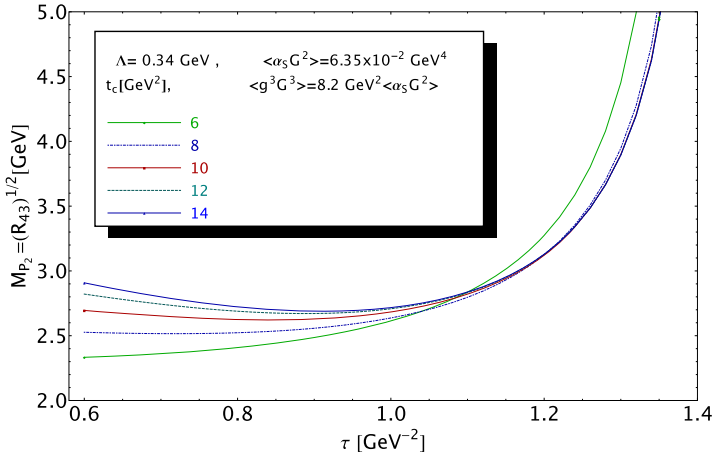


Fig. 16. M_{P_2} from \mathcal{R}_{43}^C as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV where the $\eta_1 \oplus P_1 \oplus P_1'$ contributions are included.

$$\begin{aligned}
 1154 \text{ (N2LO)} &\longrightarrow 1200 \text{ (N2LO} \oplus G^2) \longrightarrow 1683 \text{ (N2LO} \oplus G^2 \oplus G^3) \\
 &\longrightarrow 1338 \text{ (N2LO} \oplus \dots \oplus G^4) : \mathcal{L}_{20}, \\
 1511 \text{ (N2LO)} &\longrightarrow 1530 \text{ (N2LO} \oplus G^2) \longrightarrow 1613 \text{ (N2LO} \oplus G^2 \oplus G^3) \\
 &\longrightarrow 1462 \text{ (N2LO} \oplus \dots \oplus G^4) : \mathcal{L}_{42},
 \end{aligned} \tag{64}$$

and for each truncation of the PT series including the condensates up to G^4 :

$$\begin{aligned}
 1275 \text{ (LO)} &\longrightarrow 1243 \text{ (NLO)} \longrightarrow 1338 \text{ (N2LO)} : \mathcal{L}_{20}, \\
 1298 \text{ (LO)} &\longrightarrow 1412 \text{ (NLO)} \longrightarrow 1462 \text{ (N2LO)} : \mathcal{L}_{42},
 \end{aligned} \tag{65}$$

which show a quite good convergence of the OPE and of the PT series.

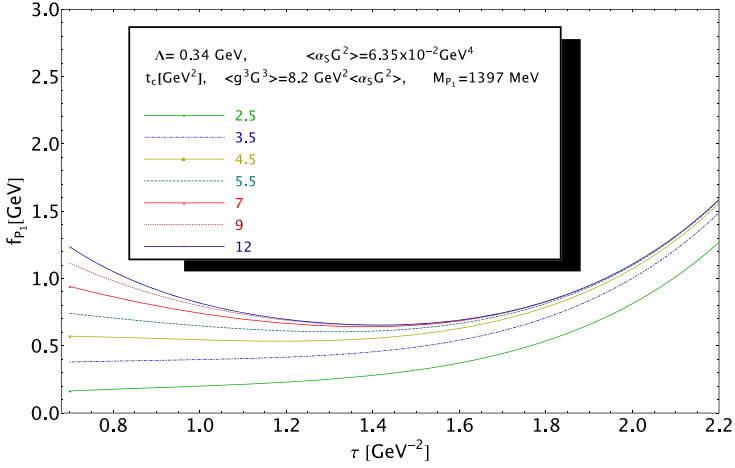


Fig. 17. f_{P_1} from \mathcal{L}_2^c as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV where the η_1 contribution is included.

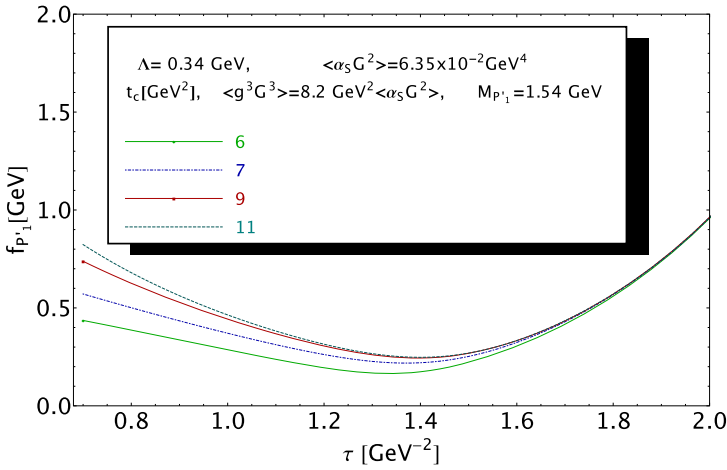


Fig. 18. f_{P_1} from \mathcal{L}_2^c as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV where the $\eta_1 \oplus P_1$ contributions are included.

• *Heavy*: P_2 (see Table 2):

One can notice from the last column of Table 2 that there is a splitting of about 400 MeV for the central value of $M_{P_2} = 3059$ MeV from \mathcal{R}_{10}^c and the ones other P_2 states which may indicate that there can be more than one state in the region above 2 GeV. However, as the error is large we cannot strictly confirm this observation and consider the conservative mean value of M_{P_2} from all determinations:

$$M_{P_2} = 2751(140) \text{ MeV}, \quad f_{P_2} = 500(43) \text{ MeV}, \quad (66)$$

where the errors are mainly due to Λ , f_{P_1} and M_{P_1} (see Table 3).

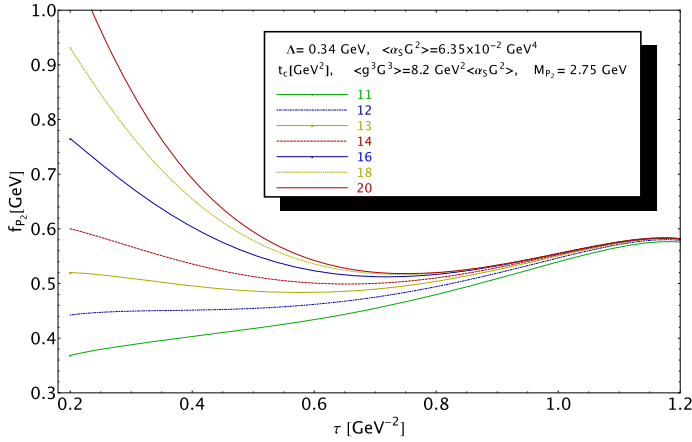


Fig. 19. f_{P_2} from \mathcal{L}_3^c as a function of τ at N2LO for different values of t_c and for $\Lambda = 0.34$ GeV where the $\eta_1 \oplus P_1 \oplus P_1'$ contributions are included.

10. Comparison with some existing estimates and confrontation with the data

• Previous QCD spectral sum rules

– We have already commented some results obtained at N2LO in Section 6 where we have emphasized the large effect of Λ QCD on the results. Using the most recent value of Λ , the N2LO results from [10–12] working with the \mathcal{R}_{10} ratio of moments contribution become $M_{P_2} = 1470(101)$ MeV for “one resonance” and $M_{P_2} = 2097(264)$ MeV for “one resonance $\oplus \eta_1$ ” (see Table 2).

– Comparing the LO result of [13] from \mathcal{R}_{10} within a “One resonance \oplus QCD continuum” parametrization of the spectral function with the one in Eq. (59), one can see that the two LO results agree within the errors though the one of [13] is obtained outside the τ -stability region. The apparent agreement of this LO result for “one resonance” from \mathcal{R}_{10} with the N2LO one with multiple resonances from the same \mathcal{R}_{10} is quite lucky due to the large effect of radiative corrections which decreases the LO result by a large amount for this moment as discussed previously.

• Lattice calculations

◊ Lattice results are in the range (2.15 \sim 2.72) GeV [25–28] which can be compared with the value of $M_{P_2} = 2639(127)$ MeV obtained in Eq. (66) and Table 2. However, a direct comparison of the results from the two approaches cannot be done properly as the lattice does not detect the medium gluonia in Eqs. (61) and (62) though the decay constant f_{P_1} is almost equal to f_{P_2} (see Table 2).

◊ From the present approach, this feature is due to the fact that the resonance contributes to \mathcal{L}_n as:

$$\mathcal{L}_n \simeq \sum_{i=1, \dots} 2f_{P_i}^2 M_{P_i}^{2(n+2)} e^{-M_{P_i}^2 \tau}, \quad (67)$$

Table 2

Pseudoscalar di-gluonium masses from LSR for PT at N2LO and gluon condensates at NLO up to $D = 8$ for: “ $\sum P_i$ resonances \oplus QCD continuum” parametrization of the spectral function. The mean value of $M_{P'}$ is used to get $f_{P'}$ and the one $[M_{P'}, f_{P'}]$ for $[M_{P_2}, f_{P_2}]$ in the last column. The bounds on M_{P_1} in column # 4 are obtained at the minimum of $\mathcal{R}_{20,42}$ using the positivity of the spectral function and including the η_1 contribution. The different sources of errors are given in Table 3. The QCD parameters in Table 1 have been used as inputs.

| Observables | LSR | Fig. # | η_1 | $P_{1,2}$ | $S_{1\eta,2\eta} \equiv P_{1,2} \oplus \eta_1$ | $S_2 \equiv P_2 \oplus S_{1\eta}$ | $S'_1 \equiv P'_1 \oplus S_{1\eta}$ | $S'_2 \equiv P_2 \oplus S'_1$ |
|-------------------------------|----------------------|--------|----------|-----------------|--|-----------------------------------|-------------------------------------|-------------------------------|
| <u>M_P [MeV]</u> | | | | | | | | |
| P_1 | \mathcal{R}_{20}^c | 8 | | 1205(126) | 1338(112) | | | |
| – | – | | | | $\leq 1365(106)$ | | | |
| – | \mathcal{R}_{42}^c | 9 | | 1432(115) | 1462(117) | | | |
| – | – | | | | $\leq 1506(104)$ | | | |
| – | <i>Mean</i> | | | <i>1345(83)</i> | <i>1397(81)</i> | | | |
| – | – | | | | $\leq 1437(74)$ | | | |
| P'_{1a} | \mathcal{R}_{20}^c | 10 | | | | | 1508(226) | |
| P'_{1b} | \mathcal{R}_{42}^c | 11 | | | | | 1553(139) | |
| P'_1 | <i>Mean</i> | | | | | | <i>1541(118)</i> | |
| P_2 | \mathcal{R}_{10}^c | 12 | | 1470(101) | 2097(264) | 2990 (318) | | 3059(385) |
| – | \mathcal{R}_{21}^c | 13 | | 1846(73) | 2005(119) | 2715(285) | | 2778(340) |
| – | \mathcal{R}_{31}^c | 14 | | 1726(71) | 2050(119) | 2699(265) | | 2764(317) |
| – | \mathcal{R}_{32}^c | 15 | | 2096(44) | 2261(39) | 2534(327) | | 2644(391) |
| – | \mathcal{R}_{43}^c | 16 | | 2054(42) | 2080(158) | 2620(205) | | 2661(227) |
| – | <i>Mean</i> | | | <i>1965(25)</i> | <i>2214(34)</i> | <i>2694(119)</i> | | <i>2751(140)</i> |
| <u>f_P [MeV]</u> | | | | | | | | |
| η_1 | \mathcal{L}_{-1}^c | 1 | 905(72) | | | | | |
| P_1 | \mathcal{L}_2^c | 17 | | | 594(144) | | | |
| P'_1 | \mathcal{L}_2^c | 18 | | | | | 205(282) | |
| P_2 | \mathcal{L}_3^c | 19 | | | | 504(39) | | 500(43) |

Table 3

Different sources of errors for the pseudoscalar di-gluonia masses and couplings given in Table 2. The errors from QCD input parameters come from Table 1. The quoted values of τ for P_2 in the cases S_2 and S'_2 correspond to S'_2 .

| | # Resonances | LSR | t_c [GeV ²] | Δt_c | τ [GeV ⁻²] | $\Delta\tau$ | $\Delta\Lambda$ | $\Delta\lambda^2$ | ΔG^2 | ΔG^3 | ΔG^4 | Δf_{η_1} | ΔM_{η_1} | ΔM_{P_1} | Δf_{P_1} | $\Delta M_{P'_1}$ | $\Delta f_{P'_1}$ | ΔM_{P_2} | |
|--------------------------------|--------------------------------------|----------------------|---------------------------|--------------|-----------------------------|--------------|-----------------|-------------------|--------------|--------------|--------------|---------------------|---------------------|------------------|------------------|-------------------|-------------------|------------------|--|
| <u>ΔM_P</u> | | | | | | | | | | | | | | | | | | | |
| P_1 | $S_{1\eta} \equiv P_1 \oplus \eta_1$ | \mathcal{R}_{20}^c | 3, 6 | 28 | 2.19, 2.27 | 1 | 33 | 15 | 5 | 54 | 79 | 27 | 0 | | | | | | |
| | | \mathcal{R}_{42}^c | 3, 8 | 43 | 1.90, 1.95 | 1 | 99 | 3 | 4 | 6 | 32 | 8 | 15 | | | | | | |
| P'_{1a} | $P'_{1a} \oplus S_{1\eta}$ | \mathcal{R}_{20}^c | 3 | 29 | 0.48 | 4 | 17 | 6 | 3 | 2 | 0 | 25 | 62 | 172 | 126 | | | | |
| P'_{1b} | $P'_{1b} \oplus S_{1\eta}$ | \mathcal{R}_{42}^c | 3, 6 | 29 | 2.22 | 3 | 128 | 4 | 6 | 6 | 32 | 6 | 9 | 4 | 29 | | | | |
| P_2 | $S_{2\eta} \equiv P_2 \oplus \eta_1$ | \mathcal{R}_{10}^c | 7, 14 | 3 | 0.62, 0.80 | 16 | 83 | 44 | 19 | 11 | 21 | 173 | 174 | | | | | | |
| | | \mathcal{R}_{21}^c | 8, 14 | 50 | 0.78, 0.88 | 3 | 44 | 17 | 4 | 14 | 47 | 37 | 59 | | | | | | |
| | | \mathcal{R}_{31}^c | 6, 14 | 93 | 0.84, 1.04 | 1 | 20 | 18 | 5 | 16 | 43 | 59 | 34 | | | | | | |
| | | \mathcal{R}_{32}^c | 13, 19 | 25 | 0.8 | 4 | 26 | 4 | 0 | 0 | 3 | 5 | 13 | | | | | | |
| | | \mathcal{R}_{43}^c | 8, 12 | 72 | 1.16, 1.26 | 4 | 88 | 13 | 2 | 3 | 12 | 4 | 109 | | | | | | |
| P_2 | $S_2 \equiv P_2 \oplus S_{1\eta}$ | \mathcal{R}_{10}^c | 12 | 0 | 0.20 | 12 | 144 | 23 | 6 | 2 | 3 | 45 | 56 | 90 | 250 | 5 | 216 | | |
| | $S'_2 \equiv P'_1 \oplus S_2$ | \mathcal{R}_{21}^c | 10, 16 | 97 | 0.34, 0.50 | 5 | 80 | 23 | 6 | 2 | 3 | 101 | 27 | 83 | 216 | 15 | 185 | | |
| | | \mathcal{R}_{31}^c | 10, 18 | 103 | 0.42, 0.60 | 6 | 74 | 25 | 3 | 0 | 3 | 14 | 28 | 86 | 224 | 15 | 173 | | |
| | | \mathcal{R}_{32}^c | 15, 19 | 5 | 0.82 | 3 | 114 | 31 | 2 | 2 | 11 | 21 | 42 | 146 | 255 | 19 | 213 | | |
| | | \mathcal{R}_{43}^c | 10, 14 | 28 | 0.84, 0.90 | 4 | 46 | 17 | 0 | 0 | 2 | 1 | 6 | 113 | 161 | 15 | 96 | | |
| <u>Δf_P</u> | | | | | | | | | | | | | | | | | | | |
| η_1 | – | \mathcal{L}_{-1}^c | 10, 14 | 21 | 0.66, 0.72 | 2 | 41 | 51 | 3 | 2 | 0 | | | | | | | | |
| P_1 | | \mathcal{L}_2^c | 4.5, 9 | 58 | 1.18, 1.40 | 0 | 114 | 15 | 1 | 3 | 50 | 9 | 21 | 35 | | | | | |
| P'_1 | $P'_1 \oplus S_{1\eta}$ | \mathcal{L}_2^c | 7, 9 | 39 | 1.38, 1.40 | 2 | 139 | 45 | 12 | 14 | 50 | 20 | 67 | 94 | 200 | 8 | | | |
| P_2 | S_2, S'_2 | \mathcal{L}_3^c | 13, 18 | 16 | 0.58, 0.74 | 3 | 32 | 8 | 0 | 0 | 0 | 7 | 0 | 8 | 7 | 2 | 16 | 8 | |

where the exponential factor is expected to kill the high-mass resonance contributions to the sum rule moments (analogous exponential factor plays a similar role in the lattice calculations). Taking M_{P_1, P_2} and $f_{P_1} \approx f_{P_1, P_2}$ in Table 2 and $\tau \approx 0.5 \text{ GeV}^{-2}$ for the moments used to get M_{P_2} , it is easy to check that the suppression due to the exponential weight is not enough to enhance the lowest ground state contribution, where the ratio behaves as:

$$\frac{\mathcal{L}_n(P_2)}{\mathcal{L}_n(P_1)} \approx 0.96 \left(\frac{M_{P_2}}{M_{P_1}} \right)^{2n} : \quad n = 1, 2, \dots, \quad (68)$$

indicating that working with the high moments which stabilize at low values of τ , one can miss the lowest mass P_1 . The opposite situation is obtained for the moments used to get M_{P_1} which stabilizes at larger $\tau \simeq 2 \text{ GeV}^{-2}$ enabling to extract M_{P_1} where the P_2 contribution is negligible.

- *Some other approaches*

- ◊ Holographic approach gives a mass of about 2.1 GeV [29].

- ◊ In contrast to the previous approaches, a recent inverse problem dispersive method found a mass of 1750 MeV [30], while the flux tube and mixing matrix models predict 1.4 GeV [31,32] which is in the range of our mean medium mass predictions $M_{P_1} = 1397(81) \text{ MeV}$ and $M_{P'_1} = 1541(118) \text{ MeV}$ in Eq. (62).

- *Confrontation with the data below 2 GeV*

- ◊ We expect that the new medium gluonia $P_1(1397)$ [or $P_{1a}(1338)$ and $P_{1b}(1462)$] and their first radial excitations [$P'_1(1541)$ (or $P'_{1a}(1508)$ and $P'_{1b}(1553)$] obtained in this paper will bring some light for a much better understanding of the 0^{-+} η -like states found below 2 GeV.

- ◊ $P_1(1397)$ supports the experimental facts that the $\eta(1405)$ is an excellent gluonium candidate. If we assume that the P_{1a} and P_{1b} are different states, we may expect that P_{1a} brings a small gluon component to the $\eta(1295)$ explaining its seen decay to $\eta(\pi\pi)_{S\text{-wave}}$ via (most probably) the σ which is expected to be the lightest scalar gluonium [1,5,6], while P_{1b} can explain the gluonium nature of the $\eta(1405)$ which is also seen to decay into $\eta(\pi\pi)_{S\text{-wave}}$.

- ◊ $P'_1(1541)$ may suggest that the $\eta(1495)$ can possess a gluon component though smaller than the one of the ground state $P_1(1397)$ due to its weaker coupling to the gluonic current: $f_{P'_1} \leq f_{P_1}$. If P'_1 is splitted into P'_{1a} and P'_{1b} , then, P'_{1a} and P'_{1b} may share some gluons to $\eta(1495)$ and $\eta(1760)$.

11. Summary and conclusions

For an attempt to study the topics addressed in the title of this paper, we have scrutinized and improved the 0^{-+} pseudoscalar gluonium sum rules using updated values of the QCD input parameters and using a multiple resonance parametrization of the spectral function beyond the minimal duality ansatz: “one resonance \oplus QCD continuum” and high degree LSR moments

- *Slope of the topological charge and proton spin*

- ◊ The properties of the η_1 singlet piece of the η' as well as the value of the topological charge $\chi(0)$ from $U(1)_A$ large N_c and current algebra approaches are reproduced from the approach [33,57–60].

◇ We have estimated the slope of the topological charge at N2LO and find it to be $\sqrt{\chi'(0)}(Q^2 = 2 \text{ GeV}^2) = 24.3(3.1) \text{ MeV}$ (Eq. (42)). It confirms the previous NLO value of $24.3(3.4) \text{ MeV}$ [33] from LSR updated here, which is still smaller than the OZI value $f_\pi/\sqrt{6} = 38 \text{ MeV}$.

◇ We have used this value of $\sqrt{\chi'(0)}$ to estimate the singlet form factor $G_A^{(0)}$ of the axial current (see Eq. (46)) and the first moment $\int_0^1 dx g_1^P(x)$ of the polarized proton structure function (see Eq. (48)) which remarkably agree with the data [71,74,75].

• 0^{-+} pseudoscalar gluonia

We have used more than one resonance and worked with different high degree moments to analyze the complex spectra of the observed η -like gluonia. We find that:

◇ The lattice results for the pseudoscalar gluonium mass are comparable with the so-called heavy gluonium mass $P_2(2751)$ obtained here from moments ratios which stabilize at smaller values of τ .

◇ New gluonia P_1 -like states with medium masses are found from some ratios of moments which stabilize at larger values of τ and can eventually explain the gluonium nature of some of the four η -like states found below 2 GeV. Our result supports the gluonium nature of the experimental candidate $\eta(1405)$ and suggests some gluon component to the other η -like wave functions.

◇ It is also remarkable to notice that the structure of the pseudoscalar gluonia spectrum is very similar to the one of the scalar gluonia (its chiral partner) studied recently in [1]. There is a (natural) one to one correspondence between the pseudoscalar gluonia and their chiral scalar analogue from Ref. [1]: $\sigma(1) \rightarrow \eta_1$; $G_1(1.55) \rightarrow P_1$; $\sigma'(1.1), G'(1.56) \rightarrow P'_{1a,1b}$; $G_2(3) \rightarrow P_2$ which is mainly due to the importance of the QCD PT contributions that are almost equal in the analysis of these two channels.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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