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| A. Noullez. Vector Alignment and Dimensionality in Turbulence. 2021. hal-03554457

HAL Id: hal-03554457

<https://hal.science/hal-03554457>

Preprint submitted on 3 Feb 2022

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Vector Alignment and Dimensionality in Turbulence

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(Dated: February 3, 2022)

Rapid local directional alignment of different vector quantities has been observed in incompressible turbulence, either in MHD or in fluid turbulence, in both direct numerical simulations or in Solar wind data, and these alignments weaken the nonlinear terms with respect to those expected for Gaussian random fields. Evidence are based on the comparison of the probability distributions of the angle between vectors to those for independent isotropic three-dimensional vectors. We show that this apparent alignment can also be attributed to a local reduction of the dimensionality of the structures towards bidimensionality, leading to an increased probability of finding vectors close to alignment, compared to the spatial three-dimensional case. This interpretation is checked in Helios-2 Fast and Slow Solar wind data.

It has long been observed that different vector fields of turbulence show unexpected alignment or antialignment. This is the case for velocity \mathbf{v} and vorticity $\boldsymbol{\omega} \equiv \nabla \times \mathbf{v}$ in hydrodynamic turbulence [1–3], but also appears for velocity \mathbf{v} and magnetic field \mathbf{b} [4–8], magnetic field \mathbf{b} and current $\mathbf{j} \equiv \nabla \times \mathbf{b}$, or current \mathbf{j} and vorticity $\boldsymbol{\omega}$ in MHD turbulence [9, 10], with measurements both in numerical simulations of decaying MHD turbulence [7, 8], or *in-situ* observations in the Solar wind [4–6, 8, 10]. These alignments were initially thought to be a slow relaxation process, requiring many nonlinear eddy turnover times to drive the turbulent system towards globally aligned Beltrami and/or Alfvénic states that were studied and interpreted as such in [3] or [7], but it has been shown recently that *local* pointwise directional alignment also occurs in turbulence [7–10] and is a fast process, needing less than one eddy turnover time to align vectors in near-independent spatially extended spatial patches. Alignment is dynamically important because it will inhibit the nonlinear terms in the corresponding hydrodynamical equation [3, 11], and slow down the dynamics with respect to what would be expected of random fields.

The main diagnostic to evidence these alignment or antialignment effects is the computation of the probability density functions (PDFs) of the local angle cosine between the two vector fields \mathbf{f} and \mathbf{g}

$$c \equiv \cos \theta = \frac{\mathbf{f} \cdot \mathbf{g}}{|\mathbf{f}| |\mathbf{g}|}, \quad (1)$$

where \mathbf{f} and \mathbf{g} can be \mathbf{v} and $\boldsymbol{\omega}$ in neutral fluid turbulence, or \mathbf{v} and \mathbf{b} , \mathbf{j} and \mathbf{b} , or \mathbf{j} and $\boldsymbol{\omega}$ in MHD turbulence. Compared to a uniform distribution often presented without justification as the one expected for Gaussian uncorrelated vectors, these PDFs are found to become strongly peaked around $c \approx \pm 1$ (aligned vectors) and depleted for $c \sim 0$ (orthogonal vectors). In this Letter, we show that a flat $\cos \theta$ distribution is *not* a signature of Gaussianity, but will occur only for independent isotropic three-dimensional vectors, and breaking *any* of these conditions will produce of a departure of the uni-

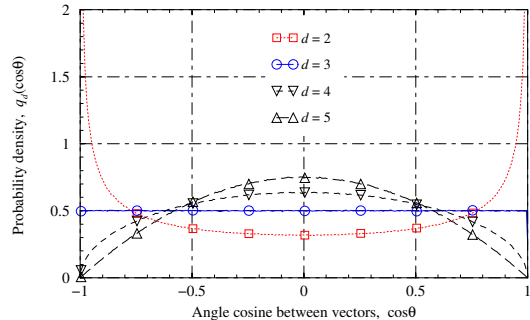


FIG. 1. PDFs of the angle cosine $\cos \theta$ between random independent isotropic vectors for spatial dimensions $d = 2$ to $d = 5$.

form distribution. Moreover, for three-dimensional vectors, any anisotropy common to the two vector fields will induce a reduced dimensionality of the vectors and produce an apparent local directional alignment, even if the vectors are not necessarily preferentially oriented along a common line, but maybe in a common plane. We emphasize that the dynamical alignment effect remains a true feature of turbulence, but its geometrical interpretation and visualization should be done carefully, as vectors in a plane have a higher probability of alignment than vectors in space.

Angle PDFs for random vectors.— The PDF $p_d(\theta)$ of the angle θ between random vectors in any space dimension d can be computed analytically for the special case of isotropic $\langle f_\alpha f_\beta \rangle = \langle g_\alpha g_\beta \rangle = \delta_{\alpha,\beta}$ and independent $\langle f_\alpha g_\beta \rangle = 0$ vectors. The latter property also implies that this PDF is *universal* with respect to the distribution of the vectors components magnitude, and so the Gaussian hypothesis is not needed, as the angle between vectors Eq. (1) does not depend on their length. The isotropy and independence property then implies that the distribution of the intersection of the second vector direction on the d -dimensional sphere is uniform if the first vector direction is fixed. Using a uniform parametrization of the hypersphere with $d - 1$ angles like in [12], and

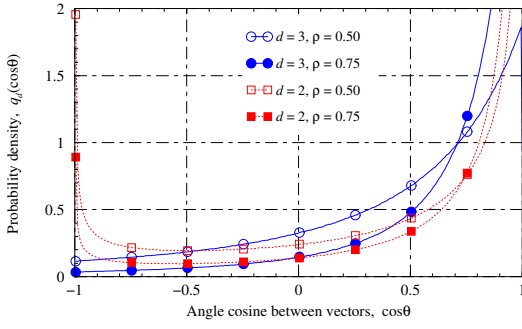


FIG. 2. PDF of the angle cosine $\cos \theta$ between random Gaussian correlated isotropic vectors for $\rho = 0.5$ (empty symbols) and $\rho = 0.75$ (full symbols) and space dimensions $d = 3$ (circles) and $d = 2$ (squares).

integrating over $d-2$ angles gives the sought distribution of the remaining angle θ

$$p_d(\theta) = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right)} (\sin \theta)^{d-2}, \quad (2)$$

and by a change of variable, the PDF $q_d(c)$ of the angle cosine c

$$q_d(c) = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right)} (1-c^2)^{(d-3)/2} \quad (3)$$

shown in Fig. 1 (PDFs of $|\sin \theta| = |\mathbf{f} \times \mathbf{g}| / (|\mathbf{f}| |\mathbf{g}|)$ can be computed similarly for any dimension d). $q_d(c)$ and $p_d(\theta)$ cannot be both uniform and it is a coincidence that $p_2(\theta) = 1/\pi$ is uniform in 2- d and $q_3(c) = 1/2$ uniform in 3- d . In high dimensions, vectors have a higher probability of being close to orthogonal, while in lower dimensions vectors display a tendency for alignment and the PDF $q_d(c)$ becomes peaked near $c \approx \pm 1$ for $d < 3$, even if the vectors are independent. We can quantify this effect, as PDFs can be integrated for integer dimensions to obtain probability distributions, giving $Q_2(c) \equiv \mathcal{P}(|\cos \theta| \geq c) = 2 \arccos(c)/\pi$ and $Q_3(c) = 1 - c$ (substitute $c \equiv \cos a$ to obtain $2P_d(a) \equiv 2\mathcal{P}(\theta < a)$, for instance $P_2(a) = a/\pi$ and $P_3(a) = (1 - \cos(a))/2$). Near-alignment is thus much more probable in 2- d than it is in 3- d , and Fig. 6 of [7] showing aligned patches in a 2- d simulation is misleading as already random vectors have a 50% probability of having $|\cos \theta| \geq 0.7$ in 2- d while it would be only 30% in 3- d . This increased alignment is not an artefact of looking at PDFs of $\cos \theta$ instead of θ , as random vectors have a 50% probability of having $\theta < \pi/4$ or $\theta > 3\pi/4$ in 2- d and only 30% in 3- d .

Correlated random vectors.— We now check if the alignment induced by a reduction of the vector dimensionality persists if the vectors are correlated. In that case, there is no general analytic result for PDFs of the angle, and for nonzero value of the correlation coefficient

$$\rho = \frac{\langle \mathbf{f} \cdot \mathbf{g} \rangle}{\sqrt{\langle |\mathbf{f}|^2 \rangle \langle |\mathbf{g}|^2 \rangle}}, \quad (4)$$

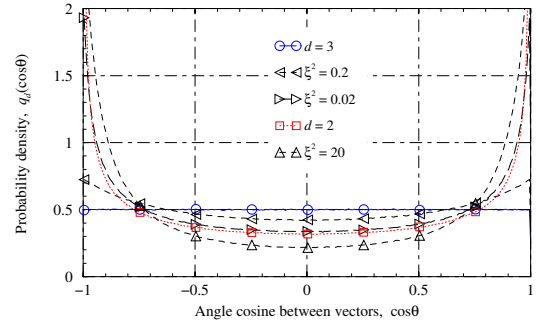


FIG. 3. PDF of the angle cosine $\cos \theta$ between random independent anisotropic vectors for different anisotropic amplitudes $\xi^2 = \langle f_1 f_1 \rangle = \langle g_1 g_1 \rangle$ in space dimension $d = 3$. Isotropic distributions for $d = 3$ (corresponding to $\xi^2 = 1$) and $d = 2$ (corresponding to $\xi^2 = 0$) are shown for comparison. The bottom case $\xi^2 = 20 \gg 1$ displays strong alignment, larger than for isotropic 2- d vectors and close to 1- d isotropic random vectors.

the PDF depends on ρ , and thus on the distribution of the vectors components magnitude. We can however resort to numerical simulations to compute PDFs for correlated Gaussian vectors by generating independent Gaussian isotropic vectors \mathbf{f} and \mathbf{h} and then computing

$$\mathbf{g} = \rho \mathbf{f} + \sqrt{1 - \rho^2} \mathbf{h}, \quad (5)$$

that will be Gaussian, isotropic and correlated with \mathbf{f} : $\langle f_\alpha g_\beta \rangle = \rho \delta_{\alpha, \beta}$ (see [13] for details), and then looking at the PDF of Eq. (1), shown in Fig. 2 for different values of d and ρ . The main characteristics of the isotropic PDF are preserved, that is $q_3(c)$ goes to finite values for both $c = \pm 1$ even for strong values of ρ , while the 2- d PDF $q_2(c)$ still diverges and becomes larger than $q_3(c)$ for $c \rightarrow \pm 1$ but is always flatter than $q_3(c)$ around the orthogonal direction $c \sim 0$.

Anisotropic random vectors.— We now ascertain that anisotropy in the random vectors distribution can induce a reduction in the vectors dimensionality and an apparent alignment of the vectors even if the anisotropy is not purely unidimensional. We generate numerically independent random Gaussian 3- d vectors, with a variance depending on the direction: ξ^2 in direction 1 and unity in the others, for both vectors

$$\langle f_\alpha f_\beta \rangle = \langle g_\alpha g_\beta \rangle = \delta_{\alpha, \beta} [\xi^2 \delta_{\alpha, 1} + (1 - \delta_{\alpha, 1})]. \quad (6)$$

If $\xi^2 > 1$, the anisotropy is one-dimensional, with the two vectors pointing preferentially in direction 1, and trivially aligning together in that direction, with a PDF $q(c)$ tending for $\xi^2 \gg 1$ to the one for 1- d isotropic vectors (two delta functions at $c = \pm 1$), see the bottom curve of Fig. 3. Less obviously, if $\xi^2 < 1$, the anisotropy is two-dimensional, with the vectors lying preferentially in the 23-plane, but there is still an apparent alignment of the two vectors with respect to the

3- d isotropic case, due to the reduced probability of orthogonality and the increased probability of alignment, with the PDF $q(c)$ changing continuously from the 3- d isotropic form to the 2- d one when ξ^2 goes from one to zero (see again Fig. 3). This alignment occurs even though vectors \mathbf{f} and \mathbf{g} are *independent* and in fact point uniformly in any direction in the 23-plane, but is a pure geometrical effect of the vectors sharing preferentially a common plane (but not a common direction). Indeed, it must be stressed that this alignment completely disappears and the PDF $q(c)$ is the 3- d isotropic one if the anisotropic direction(s) are not the same for \mathbf{f} and \mathbf{g} . On the other hand, if the anisotropies are in the same direction(s), both one- ($\xi^2 > 1$) and two-dimensional ($\xi^2 < 1$) anisotropy will lead to an apparent alignment in 3- d .

Solar wind observations.— We check these results in Solar wind data, using the measurements obtained by the Helios-2 spacecraft during the year 1976. These measurements have been extensively studied [5, 6, 14] and are of a good quality, having been obtained during a Solar minimum. Also, they were for a long time the only data showing the evolution of Solar wind turbulence as the spacecraft moved from a distance of 0.9 AU around day 45 of 1976, down to 0.3 AU around day 110 of 1976. We use data sampled at 81s time resolution to have simultaneous measurements of the 3 components of the velocity \mathbf{v} and magnetic field \mathbf{b} expressed in the RTN reference frame (see [14] for definition), where the R (radial) component points away from the Sun and is nearly aligned with the mean velocity and mean advected Solar magnetic field. Moving on the ecliptic, Helios-2 encountered both Fast and Slow Solar wind streams, known to have different origins and physical properties [14]. To ensure statistical stationarity, we analyzed independently three Slow streams (recognized by $|\mathbf{v}| \ll 500$ km/s), namely 1976:46–48, 1976:72–74 and 1976:98–101, and three Fast streams ($|\mathbf{v}| \gg 500$ km/s), 1976:49–52, 1976:76–78 and 1976:105–108. These datasets are 2–3 days long (≈ 3000 points) and avoid streams edges and boundaries. In each of these streams, we remove the fields means (obtained by time-averaging $\bar{\cdot}$ over the full stream lengths) and build the field fluctuations $\delta\mathbf{v}(t) \equiv \mathbf{v}(t) - \bar{\mathbf{v}}$ and $\delta\mathbf{b}(t) \equiv \mathbf{b}(t) - \bar{\mathbf{b}}$ and construct the local instantaneous angle

$$\cos\theta(t) = \frac{\delta\mathbf{v}(t) \cdot \delta\mathbf{b}(t)}{|\delta\mathbf{v}(t)| |\delta\mathbf{b}(t)|}. \quad (7)$$

PDFs of $\cos\theta(t)$ for the three Slow and three Fast wind streams are shown together in Fig. 4. One can notice that they differ significantly. PDFs for Slow streams are very similar to those for 3- d correlated random vectors (Fig. 2), while those for Fast wind streams are much more strongly aligned. We checked that this alignment can be attributed to a reduced dimensionality of the $\delta\mathbf{v}$ and $\delta\mathbf{b}$ fields by computing the cross-correlation matrix $\langle \delta\mathbf{v} \delta\mathbf{b}^T \rangle / \sqrt{\langle |\delta\mathbf{v}|^2 \rangle \langle |\delta\mathbf{b}|^2 \rangle}$. In the

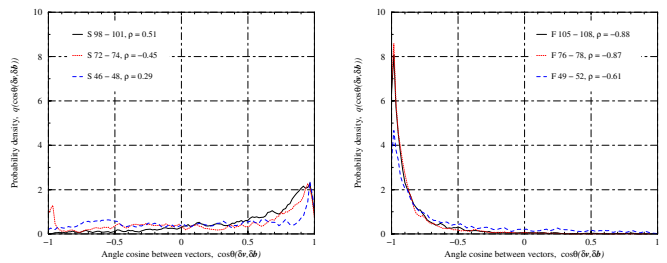


FIG. 4. PDFs of the angle cosine $\cos\theta$ between velocity fluctuations $\delta\mathbf{v}$ and magnetic fluctuations $\delta\mathbf{b}$ in three Solar wind Slow Streams (left; S72–74 histogram mirrored for consistency) and three Fast Streams (right), at the same scale.

Fast stream 1976:105–108, its value is

$$\overline{\delta v_\alpha \delta b_\beta} = \begin{pmatrix} -0.15 & -2 \times 10^{-3} & 6 \times 10^{-2} \\ -2 \times 10^{-2} & -0.34 & -7 \times 10^{-3} \\ 4 \times 10^{-2} & -7 \times 10^{-3} & -0.39 \end{pmatrix}, \quad (8)$$

with similar values for the other Fast streams, and is dominated by the T and N components, suggesting a 2- d anisotropy, a feature already observed by various authors [4, 5]. To verify that this anisotropy is associated to the nature of the stream, we compute the components local variances $\overline{\delta b_\alpha^2}(t) \equiv \overline{(b_\alpha - \bar{b}_\alpha)^2}(t)$, where the time average $\bar{\cdot}(t)$ is now defined as an 8 hours moving average centered on t . The resulting Fig. 5 shows that there is an abrupt change in the components local variances, associated with strong inhomogeneities at the stream boundaries, when switching from Slow stream 1976:98–101 to the Fast stream 1976:105–108. While the three components local variances are nearly identical in the Slow stream, the transverse local variances increase by a factor ~ 3 in the Fast stream, and both fields $\delta\mathbf{b}$ and $\delta\mathbf{v}$ (not shown) become anisotropic with stronger fluctuations in the plane transverse to the R (radial) direction associated to the mean magnetic field $\bar{\mathbf{b}}$ and mean velocity $\bar{\mathbf{v}}$. As pointed out before, this Fast wind bidimensionalization increases the apparent 3- d alignment of the two vector fields. However, field fluctuations $\delta\mathbf{v}$ and $\delta\mathbf{b}$ have very strong Alfvénic (anti)correlations in *all* components, and so the T and N components remain correlated and display an angle PDF corresponding to a strong (anti)alignment between $\delta\mathbf{v}$ and $\delta\mathbf{b}$ in the TN -plane with an asymmetric PDF skewed towards $\cos\theta \approx -1$, $\theta \approx \pi$, this time due to the (anti)correlation $\rho \approx -1$.

Discussion and conclusions.— The image of aligned vectors that comes naturally to mind is that of vectors that point towards close or nearly opposite directions. But two random vector fields in (hyper)space will also display a higher probability of mutual alignment with respect to the isotropic distribution, if they have been dynamically pushed on a lower dimensional (hyper)plane rather than on a line, even if the two fields have no particular alignment in this plane. The effect is completely

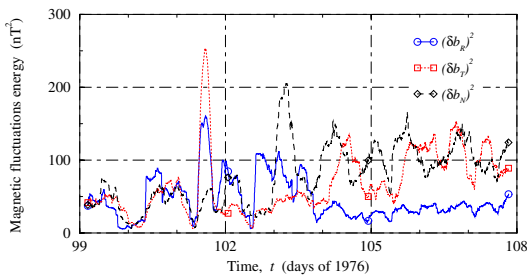


FIG. 5. Time series of the magnetic fluctuation $\delta\mathbf{b}$ components local energy $\overline{\delta b_\alpha^2}$. Note the increased energy by a factor ~ 3 of the components transverse to the radial direction (corresponding to the mean magnetic field), when switching from a Slow to a Fast Stream for $t \approx 102$.

geometrical and comes from the decreased probability of orthogonality and increased likelihood of alignment when reducing the dimension or increasing the anisotropy of the two vector fields. This distinction between linear and planar alignment might seem purely semantic, as lying in a common plane *is* some kind of alignment between the vectors. Also, the dynamical consequences like the reduction of nonlinearity due to decreased values of $\mathbf{f} \times \mathbf{g}$ and increased values of $\mathbf{f} \cdot \mathbf{g}$ remain the same and do not depend on that interpretation. It is however important to realize and understand if the structures that appear locally in 3-*d* fluid or MHD turbulence are preferentially lines or sheets, and we showed that only the analysis of the PDFs of $\cos\theta$ cannot completely distinguish or exclude the two possibilities, as both types of structures will show an alignment of the vectors.

Also, complete dynamical local alignment of any type won't occur over the whole of space, and it has been shown in [7–10] that it occurs in extended spatial patches displaying local alignment or antialignment. The situation is easier and the analysis is simpler in the Solar wind as the radial expansion of the wind and the advected mean magnetic field create a constant direction on which the fluctuations are reduced, so that the anisotropy is 2-*d* with a preserved direction. In homogenous fluid or MHD turbulence, the situation is more complex but the spectra of both \mathbf{v} and \mathbf{b} are IR-divergent and dominated by the large scales, so we can expect that the local mean field can dynamically induce 2-*d* or 1-*d* local anisotropy and a local directional alignment or antialignment, with a slowly time and spatially varying direction. PDFs of $\cos\theta$ will in that case be probably a mixture of the different cases similar to those shown in Fig. 2 and Fig. 3 as neither the correlation ρ nor the anisotropy will be uniform, but PDFs will in any case display alignment of the vector fields if anisotropies of both fields are similarly oriented.

Still, PDFs of $\cos\theta$ can give a hint of the shape and dimension of structures that are effectively present in the vector fields. For instance, one can verify that $\langle \cos^2\theta \rangle = 1/d$ for isotropic independent vectors (this result is modified if vectors are correlated). This defines a kind of

“effective dimension” for isotropic or anisotropic vector fields and for instance PDFs shown in Fig. 3 will have a value of $1/\langle \cos^2\theta \rangle$ that changes continuously from 3 to 2 for increasing 2-*d* anisotropies and down to 1 for increasing 1-*d* anisotropy.

We conclude that local or global vectors alignment is a *geometrical* property that depends on the vectors dimensionality, and that it is favored in low dimensions. So, if dynamics drive the vectors on a lower dimensional variety, the vectors will appear more aligned than if they span the whole space directions. This increased alignment is a real effect, even if the vectors do not necessarily point along the same line, and its consequences on the dynamics through a reduction of the nonlinearity and a slowdown of the nonlinear transfers are also real, so that these aligned anisotropic states will also enjoy a higher probability of being observed once they are realized.

It is a pleasure to thank A. Pouquet, R. Bruno, and L. Sorriso-Valvo for many useful discussions, comments, suggestions and insights on the properties of MHD turbulence and on the many traps and pitfalls lurking in the processing and analysis of Solar wind data.

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