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EYRING-KRAMERS TYPE FORMULAS FOR SOME PIECEWISE DETERMINISTIC MARKOV PROCESSES

DORIAN LE PEUTREC, LAURENT MICHEL, AND BORIS NECTOUX

Abstract. In this work, we give sharp asymptotic equivalents in the small temperature regime of the smallest eigenvalues of the generator of some piecewise deterministic Markov processes (including the Zig-Zag process and the Bouncy Particle Sampler process) with refreshment rate α on the one-dimensional torus $\mathbb{T}$. These asymptotic equivalents are usually called Eyring-Kramers type formulas in the literature. The case when the refreshment rate α vanishes on $\mathbb{T}$ is also considered.

Keywords. Piecewise Deterministic Markov Processes, small temperature regime, Eyring-Kramers formulas, metastability, spectral theory, semiclassical analysis.
AMS classification. 35P15, 47F05, 35P20, 35Q82, 35Q92.

1. Introduction

1.1. **Purpose and motivation.** Piecewise Deterministic Markov Processes [9] (PDMP hereafter) have recently attracted a lot of attention for their use within the Markov Chain Monte Carlo methodology. Given a potential $U : \mathsf{M} \to \mathbb{R}$ on a d-dimensional manifold M without boundary, such processes are indeed ergodic with respect to the Gibbs measure

$$\pi(dx) = \frac{e^{-\frac{2}{h}U(x)}}{\int_{\mathsf{M}} e^{-\frac{2}{h}U}} dx,$$

dx being the Lebesgue measure on the position space M. Here, the parameter $h > 0$ is proportional to the Boltzmann constant k_B through the relation $h = k_B T$, T being the temperature of the underlying system. When $h > 0$ is fixed, the ergodic properties and the rate of convergence of such processes with or without a refreshment rate α, have for instance been studied in [4, 10, 14, 12, 1, 2, 22, 23] (see also references therein). We also refer to [5] for a spectral analysis of the generator of the one-dimensional Zig-Zag process and its corresponding semigroup when h is fixed.

In many applications in statistical physics where one needs to sample from π, the positive constant h is very small compared to the energetic barriers of U. Recently, the long time behavior of the semigroups generated by the generators L_h of these processes on L^2 have been investigated in [16] when $h \ll 1$, where it has also been proved that in the subset $\{\mathrm{Re}(z) \geq -\epsilon_0 h\}$ of the complex plane, L_h has exactly n_0 eigenvalues (n_0 being the number of local minima of U), which are non positive, real, and exponentially small as $h \to 0$. Such results exhibit a metastable behavior of the PDMP when $h \ll 1$, as it is the case for diffusion processes (see e.g. [11] and references therein). In this work, we want to push the analysis of the metastability of the PDMP further by proving that each of these n_0

eigenvalues satisfies a so-called Eyring-Kramers type formula when $h \to 0$. Such sharp formulas describe completely the successive timescales involved in the convergence of the semigroup generated by $-\mathsf{L}_h$ to π, as it has been done in [8, 17] for elliptic reversible diffusions.

Let us be more precise on our results. We work in the following setting: $d = 1$ (i.e. the dimension of the position space is equal to 1) and $\mathsf{M} = \mathbb{T}$, the one-dimensional torus. In addition, we work with the operator P_h defined below in (1.1) which is (up to a multiplication by $-h$) unitary equivalent to L_h, see (1.2). Thus, all our results are easily translated in terms of L_h. The purpose of this work is to compute sharp asymptotic equivalents of the n_0 smallest eigenvalues of P_h (or equivalently, those of L_h) in the limit $h \to 0$, see Theorems 2 and 3. In particular these asymptotic equivalents allow us to provide a sharp exponential decay rate of the semigroup associated with $-\mathsf{P}_h$ in L^2 when $h \ll 1$, see Corollary 1.5. These results hold when (1.4) is satisfied, which implies that the refreshment rate α does not vanish at critical points of U (i.e. where $\partial_x U = 0$), see (1.5). The case $\alpha = 0$ when $\partial_x U = 0$ is investigated in Section 1.4, where we show that the smallest eigenvalues of P_h satisfy different asymptotic equivalents as $h \to 0$, see Theorem 4

To compute sharp asymptotic equivalents of the n_0 smallest eigenvalues of P_h, we proceed as follows. We first introduce a suitable change of variables to turn the eigenvalue problem

$$(\mathsf{P}_h - \lambda)u = 0$$

into a nonlinear eigenvalue problem:

$$(\Delta_{V,h} - \lambda W + \lambda^2)g = 0,$$

where $V = -U$, $\Delta_{V,h}$ is the Witten Laplacian associated with V, and $W = 2|\partial_x U| + \alpha$ (see (2.7) and Lemma 2.2). Using a Grushin problem and known results on the low-lying spectrum of $\Delta_{V,h}$, we then prove that when λ is small enough, the kernel of $\Delta_{V,h} + \lambda W + \lambda^2$ is composed of the singularities of a holomorphic function $\lambda \mapsto E_{-+}(\lambda) \in \mathbb{C}^{n_0 \times n_0}$ (see (2.23)). We finally investigate the localization of the singularities of $\lambda \mapsto E_{-+}^{-1}(\lambda)$ to deduce sharp asymptotic equivalents of the n_0 smallest eigenvalues of P_h.

1.2. **Setting.** Let $\mathsf{E} = \{(x, v) \in \mathbb{T} \times \{\pm 1\}\}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus. Consider the following unbounded operator P_h on $L^2(\mathsf{E})$ associated with a smooth function $U : \mathbb{T} \to \mathbb{R}$ defined by

$$\mathsf{P}_h = -v\, \mathsf{d}_{U,h} + 2(v\, \partial_x U)_+ (\mathsf{I} - \mathsf{B}) + \alpha(\mathsf{I} - \pi_v), \tag{1.1}$$

where $\mathsf{d}_{U,h} = h\partial_x + \partial_x U$, $\alpha : \mathbb{T} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ non-negative function (the refreshment rate), $h > 0$ is a parameter proportional to the temperature of the underlying statistical system, I is the identity operator, and B and π_v are defined by

$$\forall (x, v) \in \mathsf{E},\ \mathsf{B}f(x, v) = f(x, -v)$$

and

$$\pi_v f(x, v) = \frac{1}{2}(f(x, +1) + f(x, -1))$$

for all $f \in L^2(\mathsf{E})$. Here and in the following, for $u \in L^2(\mathsf{E})$, we denote by $u_\pm = \max(0, \pm u)$. The operator P_h is linked to the Zig-Zag (or the Bouncy Particle Sampler generator) process generator L_h, where $\mathsf{L}_h = v\partial_x - (\frac{2}{h} v\partial_x U)_+(\mathsf{I}-\mathsf{B}) - \frac{1}{h}\alpha(\mathsf{I}-\pi_v)$, through the relation

$$\mathsf{P}_h = -h\, e^{-\frac{1}{h}U}\, \mathsf{L}_h e^{\frac{1}{h}U}. \tag{1.2}$$

We refer to [3, 24, 7] and references therein for more details on these two processes (see also [15] and [13, Section 3.1]). The space $E = \mathbb{T} \times \{\pm 1\}$ is endowed with the natural scalar product

$$\langle f, g\rangle_{L^2(\mathsf{E})} = \frac{1}{2}\sum_{v=\pm 1}\int_{\mathbb{T}} f(x,v)\overline{g(x,v)}dx. \tag{1.3}$$

Throughout this work, we will assume that

$$U : \mathbb{T} \to \mathbb{R} \text{ is a smooth Morse function.}$$

By definition, this means that $\partial_x^2 U(x) \neq 0$ when $\partial_x U(x) = 0$, $x \in \mathbb{T}$. In particular, U has a finite number of critical points on $\mathbb{T}$. It is proved in [16] that P_h with domain $D(\mathsf{P}_h) = \{f \in L^2(\mathsf{E}),\ v\partial_x f \in L^2(\mathsf{E})\}$ is maximal accretive. The following spectral result on P_h in the limit $h \to 0$ is also proved in [16].

Theorem 1. *([16, Theorem 1 and Proposition 13]) Assume that U is a Morse function with n_0 local minimum points. Assume also that*

$$\min_{\mathbb{T}}\big(2|\partial_x U| + \alpha\big) > 0. \tag{1.4}$$

Then, there exist $\epsilon_0 > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$, $\sigma(\mathsf{P}_h) \cap \{\mathrm{Re}(z) \le \epsilon_0 h^2\}$ is made of n_0 real nonnegative eigenvalues $\lambda_{1,h} \le \ldots \le \lambda_{n_0,h}$ (counted with algebraic multiplicity). Moreover, their algebraic multiplicities equal their geometric multiplicities and there exist $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0]$, $\lambda_{n_0,h} \le e^{-C/h}$. Finally, $\lambda_{1,h} = 0$ and has algebraic multiplicity 1.

Let us mention that $\dim \mathrm{Ker}(\mathsf{P}_h) = 1$ holds for all $h > 0$ (see [16]). In [16], it is assumed that $\min_{\mathbb{T}} \alpha > 0$, but all the results of [16] still hold when the less stringent assumption (1.4) is satisfied. Indeed, if (1.4) holds and $u \in D(\mathsf{P}_h)$, then $\mathrm{Re}\,\langle \mathsf{P}_h u, u\rangle_{L^2(\mathsf{E})} \ge r_m \|(\mathsf{I}-\pi_v)u\|^2$, where $r_m = \min_{\mathsf{E}} 2|\partial_x U| + \alpha$. This follows from the following computations. By [16, Lemma 5], since $(\mathsf{I}-\mathsf{B}) = 2(\mathsf{I}-\pi_v)$, one has for all $u \in D(\mathsf{P}_h)$, denoting by $w = (\mathsf{I}-\pi_v)u$ (notice that $|w(\cdot,1)| = |w(\cdot,-1)|$):

$$\begin{aligned}
\mathrm{Re}\,\langle \mathsf{P}_h u, u\rangle_{L^2(\mathsf{E})} &= \frac{1}{2}\int_{\mathsf{E}} (2v\,\partial_x U)_+ |(\mathsf{I}-\mathsf{B})u|^2 + \int_{\mathsf{E}} \alpha |(\mathsf{I}-\pi_v)u|^2 \\
&= \int_{\mathsf{E}} 4(v\,\partial_x U)_+ |w|^2 + \int_{\mathsf{E}} \alpha |w|^2 \\
&= \frac{1}{2}\int_{\mathbb{T}} 4(\partial_x U)_+ |w|^2 + \frac{1}{2}\int_{\mathbb{T}} 4(\partial_x U)_- |w|^2 + \int_{\mathsf{E}} \alpha |w|^2 = \int_{\mathsf{E}} (2|\partial_x U| + \alpha)|w|^2.
\end{aligned}$$

Notice that (1.4) allows α to vanish on $\mathbb{T}$, but not everywhere since (1.4) is equivalent to:

$$\text{for any } x \in \mathbb{T},\quad \partial_x U(x) = 0 \ \Rightarrow\ \alpha(x) > 0. \tag{1.5}$$

The case when there exists $x \in \mathbb{T}$ such that $\partial_x U(x) = \alpha(x) = 0$ is considered in Section 1.4.

For our analysis, it is convenient to introduce the function

$$V = -U,$$

which is also a Morse function on $\mathbb{T}$. We shall denote by $\mathsf{U}^{(0)}$ the set of local minima of V (i.e. of local maxima of U) and by $\mathsf{U}^{(1)}$ the set of its local maxima (i.e. of local minima of U). Since we are on the torus, the sets $\mathsf{U}^{(0)}$ and $\mathsf{U}^{(1)}$ have the same cardinality n_0.

Remark 1.1. *As mentioned in the introduction, we use the spectral properties of the Witten Laplacian*

$$\Delta_{V,h} = -h^2\Delta + |\partial_x V(x)|^2 - h\partial_x^2 V(x)$$

to study the spectrum of P_h*. Let us emphasize that, considering the Witten Laplacian acting on the full space of differential forms,* $\Delta_{V,h}$ *is simply* $\Delta^{(1)}_{U,h}$ *acting on* 1*-forms (through the identification between functions and* 1*-forms) on the one-dimensional torus. In addition, the Witten complex structure on the torus shows that* $\sigma(\Delta^{(1)}_{U,h}) = \sigma(\Delta_{U,h})$ *and thus*

$$\sigma(\Delta_{V,h}) = \sigma(\Delta_{U,h}), \quad \textit{counting the multiplicities.}$$

Furthermore, $\Delta_{U,h} = \mathsf{d}^*_{U,h}\mathsf{d}_{U,h}$*, where* $\mathsf{d}^*_{U,h} = -h\partial_x + \partial_x U$*, and* $\mathsf{d}_{U,h}$ *induces a one-to-one correspondance between the eigenfunctions of* $\Delta_{U,h}$ *associated with its nonzero eigenvalues (i.e. which are not in* $\mathrm{Span}(e^{-U/h})$*) and those of* $\Delta_{V,h}$ *(i.e. which are not in* $\mathrm{Span}(e^{U/h})$*). In the present setting, the application* Ψ *defined in* (2.8) *realizes a correspondance between the Witten Laplacian and the generator* L_h *of the PDMP.*
We refer e.g. to the introduction of [18] *and references therein on the Witten complex structure.*

Throughout the paper, we will say that a family of complex numbers $(a_h)_{h>0}$ admits a classical expansion in power of h^β (where $\beta > 0$) if there exists a sequence $(a^k)_{k\geq 0}$ such that, for all $K \geq 0$, one has $a_h = \sum_{k=0}^K a^k h^{\beta k} + O(h^{\beta(K+1)})$ when $h \to 0$. In that case, we will denote $a_h \sim \sum_{k\geq 0} a^k h^{\beta k}$.

1.3. **Main results.** In this section we give sharp asymptotics of the exponentially small eigenvalues $\lambda_{j,h}$, $j = 1, \dots, n_0$. Observe that if $n_0 = 1$, there is only one small eigenvalue by Theorem 1 which is 0, and there is thus nothing to compute. We then consider the case when $n_0 \geq 2$. We start with the following theorem, which gives the result in the simplified setting of a non-symmetric double well potential V, or equivalently of a non-symmetric double well potential U (see Theorem 3 below for the general case $n_0 \geq 2$).

Theorem 2. *Let* U *be a Morse function. Assume that* (1.4) *holds and that* $\mathsf{U}^{(0)}$ *is made of two elements* $\mathbf{m}_1$ *and* $\mathbf{m}_2$ *such that* $V(\mathbf{m}_1) < V(\mathbf{m}_2)$*. Assume also that the two elements* $\mathbf{s}_1, \mathbf{s}_2$ *of* $\mathsf{U}^{(1)}$ *satisfy* $V(\mathbf{s}_1) > V(\mathbf{s}_2)$*. Let* $\epsilon_0 > 0$ *be as in Theorem 1. Then, the second smallest eigenvalue* $\lambda_{2,h}$ *of* P_h *satisfies as* $h \to 0$*:*

$$\lambda_{2,h} = \zeta_h\, h\, e^{-\frac{2}{h}(V(\mathbf{s}_2)-V(\mathbf{m}_2))}, \ \textit{where } \zeta_h \sim \sum_{k\geq 0} h^{\frac{k}{2}}\zeta_k \ \textit{and } \zeta_0 = \frac{1}{\pi}\frac{\sqrt{V''(\mathbf{m}_2)\,|V''(\mathbf{s}_2)|}}{\alpha(\mathbf{m}_2)}.$$

The situation where $V(\mathbf{m}_1) = V(\mathbf{m}_2)$ and/or $V(\mathbf{s}_1) = V(\mathbf{s}_2)$ could be handled easily by constructing adapted quasimodes in the spirit of [21]. Here, we decided to state our result in the above simplified setting in order to lighten the formulas.

Note that, in terms of U, $\mathbf{s}_1$ and $\mathbf{s}_2$ are the local minima and $U(\mathbf{s}_1) < U(\mathbf{s}_2)$, while $\mathbf{m}_1$ and $\mathbf{m}_2$ are the local maxima and $U(\mathbf{m}_1) > U(\mathbf{m}_2)$. In addition, the energetic barrier $V(\mathbf{s}_2) - V(\mathbf{m}_2)$ for V is the energetic barrier $U(\mathbf{m}_2) - U(\mathbf{s}_2)$ for U. Moreover,

$$\zeta_0 = \frac{1}{\pi} \frac{\sqrt{|U''(\mathbf{m}_2)|\, U''(\mathbf{s}_2)}}{\alpha(\mathbf{m}_2)}.$$

Let us now state our results in the general setting $n_0 \geq 2$. To this end, we need to label the local minima and maxima of V in a suitable way. The following construction is inspired by [19] (see also [21] and [20]). In order to simplify, we assume from now that V uniquely attains its maximum at the point $\mathbf{s}_1 \in \mathsf{U}^{(1)}$, i.e. that

$$\operatorname{argmax}_{\mathbb{T}} V = \{\mathbf{s}_1\}. \tag{1.6}$$

Then, set $\underline{\mathsf{U}}^{(1)} = \mathsf{U}^{(1)} \setminus \{\mathbf{s}_1\}$. Since $n_0 \geq 2$, $\underline{\mathsf{U}}^{(1)} \neq \varnothing$. We denote the elements of $V(\underline{\mathsf{U}}^{(1)})$ by $\sigma_2 > \sigma_3 > \ldots > \sigma_N$, where $N \geq 2$. For convenience, we also introduce a fictive infinite saddle value $\sigma_1 = +\infty$ and we denote $\Sigma = \{\sigma_1, \ldots, \sigma_N\}$. Starting from σ_1, we will recursively associate to each σ_i a finite family of local minima $(\mathbf{m}_{i,j})_j$ and a finite family $(\mathsf{C}_{i,j})_j$ of connected components of $\{V < \sigma_i\}$ in the following way

- ⋆ Let $X_{\sigma_1} = \{x \in \mathbb{T};\ V(x) < \sigma_1 = +\infty\} = \mathbb{T}$. We let $\mathbf{m}_{1,1}$ be an arbitrary global minimum of V and $\mathsf{C}_{1,1} = \mathbb{T}$.
- ⋆ Next we consider $X_{\sigma_2} = \{x \in \mathbb{T};\ V(x) < \sigma_2\}$. This is the union of its finitely many connected components. Exactly one of these components contains $\mathbf{m}_{1,1}$ and the other components are denoted by $\mathsf{C}_{2,1}, \ldots, \mathsf{C}_{2,N_2}$. In each component $\mathsf{C}_{2,j}$, we pick up a point $\mathbf{m}_{2,j}$ which is a global minimum of $V_{|\mathsf{C}_{2,j}}$.
- ⋆ Suppose now that the families $(\mathbf{m}_{k,j})_j$ and $(\mathsf{C}_{k,j})_j$ have been constructed until rank $k = i-1$. The set $X_{\sigma_i} = \{x \in \mathbb{T};\ V(x) < \sigma_i\}$ has again finitely many connected components and we label $\mathsf{C}_{i,j}$, $j = 1, \ldots, N_i$, those of these components which do not contain any $\mathbf{m}_{k,\ell}$ with $k < i$. In each $\mathsf{C}_{i,j}$, we pick up a point $\mathbf{m}_{i,j}$ which is a global minimum of $V_{|\mathsf{C}_{i,j}}$.

We run the procedure until all the minima have been labeled.

Remark 1.2. *Since we work on $\mathbb{T}$, using the terminology of [19], every maximum point $\mathbf{s} \in \underline{\mathsf{U}}^{(1)}$ is a separating saddle points whereas $\mathbf{s}_1$ is not. In the case where V attains its global maximum at several distinct points $\mathbf{s}_{1,1}, \ldots, \mathbf{s}_{1,k}$, every maximum point is separating and the situation can be handled easily by a modification of the above construction. However, this would lead to a slightly more complicated presentation of the results that we prefer to avoid in this work.*

We now recall some constructions of [21] and [20] that will be useful in the sequel. Throughout, we denote by $\mathbf{s}_\infty$ a fictive saddle point such that $V(\mathbf{s}_\infty) = \sigma_1 = +\infty$. For any set A, $\mathcal{P}(A)$ denotes the power set of A. From the above labelling, we define two mappings

$$\mathsf{C} : \mathsf{U}^{(0)} \to \mathcal{P}(\mathbb{R}^d) \qquad \text{and} \qquad \mathbf{j} : \mathsf{U}^{(0)} \to \mathcal{P}(\underline{\mathsf{U}}^{(1)} \cup \{\mathbf{s}_\infty\})$$

as follows: for every $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, N_i\}$,

$$\mathsf{C}(\mathbf{m}_{i,j}) := \mathsf{C}_{i,j}, \tag{1.7}$$

and

$$\mathbf{j}(\mathbf{m}_{1,1}) := \{\mathbf{s}_\infty\} \qquad \text{and} \qquad \mathbf{j}(\mathbf{m}_{i,j}) := \partial \mathsf{C}_{i,j} \cap \underline{\mathsf{U}}^{(1)} \text{ for } i \geq 2. \tag{1.8}$$

In particular, we have

$$\mathsf{C}(\mathbf{m}_{1,1}) = \mathbb{T} \quad \text{and} \quad \varnothing \neq \mathbf{j}(\mathbf{m}_{i,j}) \subset \{V = \sigma_i\} \text{ for all } i, j \in \{1, \dots, N\}.$$

Note also that, in our 1-dimensional setting, the cardinality of every $\mathbf{j}(\mathbf{m})$, $\mathbf{m} \in \mathsf{U}^{(0)}$, is 1 or 2.

We then define the mappings

$$\boldsymbol{\sigma} : \mathsf{U}^{(0)} \to \Sigma \qquad \text{and} \qquad S : \mathsf{U}^{(0)} \to (0, +\infty],$$

by

$$\forall \mathbf{m} \in \mathsf{U}^{(0)}, \qquad \boldsymbol{\sigma}(\mathbf{m}) := V(\mathbf{j}(\mathbf{m})) \qquad \text{and} \qquad S(\mathbf{m}) := \boldsymbol{\sigma}(\mathbf{m}) - V(\mathbf{m}), \tag{1.9}$$

where, with a slight abuse of notation, we have identified the set $V(\mathbf{j}(\mathbf{m}))$ with its unique element. Note that $S(\mathbf{m}) = +\infty$ if and only if $\mathbf{m} = \mathbf{m}_{1,1}$. With the above notations, our last assumption is the following:

(1.10)
- ⋆ Equation (1.6) is satisfied.
- ⋆ For any $\mathbf{m} \in \mathsf{U}^{(0)}$, $\mathbf{m}$ is the unique global minimum of $V_{|\mathsf{C}(\mathbf{m})}$.
- ⋆ For all $\mathbf{m}' \neq \mathbf{m} \in \mathsf{U}^{(0)}$, $\mathbf{j}(\mathbf{m}) \cap \mathbf{j}(\mathbf{m}') = \varnothing$.
- ⋆ The map S: $\mathsf{U}^{(0)} \to (0, +\infty]$ is injective.

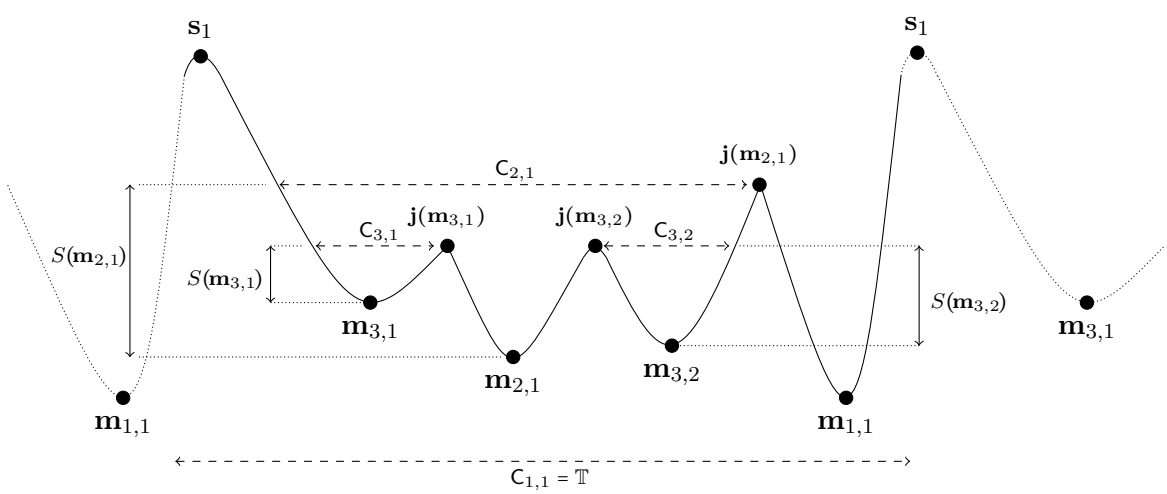

FIGURE 1.1. Schematic representation of the maps C and $\mathbf{j}$. In this example, the function V satisfies (1.10) and $V(\mathbf{j}(\mathbf{m}_{3,1})) = V(\mathbf{j}(\mathbf{m}_{3,2}))$.

In particular, (1.10) implies that V uniquely attains its global minimum on $\mathbb{T}$ at $\mathbf{m}_{1,1}$. Moreover, the dimension being 1, (1.10) actually implies that every $\mathbf{j}(\mathbf{m})$, $\mathbf{m} \in \mathsf{U}^{(0)}$, has cardinality 1. This follows for example from [20, Remark 1.7], by taking into account the

dimension. Hence, identifying every $\mathbf{j}(\mathbf{m})$ with its unique element, the map $\mathbf{j} : \mathsf{U}^{(0)} \to \mathcal{P}(\underline{\mathsf{U}}^{(1)} \cup \{\mathbf{s}_1\})$ defined in (1.8) becomes an injective function

$$\mathbf{j} : \mathsf{U}^{(0)} \to \underline{\mathsf{U}}^{(1)} \cup \{\mathbf{s}_\infty\} \text{ such that } \mathbf{j}(\mathbf{m}_{1,1}) = \mathbf{s}_\infty. \tag{1.11}$$

In the following, when (1.10) holds, we label the local minima $\mathbf{m}_1, \dots, \mathbf{m}_{n_0}$ of V such that $(S(\mathbf{m}_j))_{j\in\{1,\dots,n_0\}}$ is decreasing (see (1.9)), that is such that

$$\text{for all } j \in \{1, \dots, n_0 - 1\},\ S(\mathbf{m}_{j+1}) < S(\mathbf{m}_j) \text{ and } S(\mathbf{m}_1) = +\infty \text{ (i.e. } \mathbf{m}_1 = \mathbf{m}_{1,1}). \tag{1.12}$$

Remark 1.3. *Note that, in the geometrical setting of Theorem 2, one has by construction of S (see (1.9)) and of $\mathbf{j}$ (see (1.11)),*

$$S(\mathbf{m}_2) = V(\mathbf{s}_2) - V(\mathbf{m}_2) \quad \textit{and} \quad \mathbf{j}(\mathbf{m_2}) = \mathbf{s_2}.$$

The main result of this work is the following.

Theorem 3. *Let $V = -U$ be a Morse function. Assume that (1.4) and (1.10) are satisfied. Let $\epsilon_0 > 0$ be given by Theorem 1. Then, there exists $h_0 > 0$ such that, for all $h \in]0, h_0]$, the n_0 eigenvalues $\lambda_{1,h} \le \dots \le \lambda_{n_0,h}$ of P_h in $\{\mathrm{Re}(z) \le \epsilon_0 h^2\}$ (counted with algebraic multiplicity) satisfy: $\lambda_{1,h} = 0$ and, for all $j \in \{2, \dots, n_0\}$,*

$$\lambda_{j,h} = \zeta_h(\mathbf{m}_j)\, h\, e^{-\frac{2}{h} S(\mathbf{m}_j)}, \tag{1.13}$$

where $S : \mathsf{U}^{(0)} \to (0, +\infty]$ is defined in (1.9) (see also (1.12)) and ζ_h admits a classical expansion $\zeta_h(\mathbf{m}_j) \sim \sum_{k\ge 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_j)$ with

$$\zeta_0(\mathbf{m}_j) = \frac{1}{\pi} \frac{\sqrt{V''(\mathbf{m}_j)\, |V''(\mathbf{j}(\mathbf{m}_j))|}}{\alpha(\mathbf{m}_j)}, \tag{1.14}$$

where $\mathbf{j} : \mathsf{U}^{(0)} \to \underline{\mathsf{U}}^{(1)} \cup \{\mathbf{s}_\infty\}$ is defined in (1.11).

Notice that, according to Theorem 3, $\lambda_{j,h}$ is a simple eigenvalue of P_h for all $j \in \{1, \dots, n_0\}$ and h small enough (that is $\dim\big(\mathrm{Ker}(\mathsf{P}_h - \lambda_{j,h})^m\big) = 1$ for every $m \in \mathbb{N}^*$).

Remark 1.4. *The assumptions (1.6) and $\big(S\colon \mathsf{U}^{(0)} \to (0, +\infty]$ is injective$\big)$ in (1.10) are generic. They can be relaxed following the procedure of [20]. The whole assumption (1.10) could also be relaxed by following the strategy of [21].*

Let us recall that by the Hille-Yosida Theorem, $-\mathsf{P}_h$ generates a strongly continuous contraction semigroup $(e^{-t\mathsf{P}_h})_{t\ge 0}$ on $L^2(\mathsf{E})$. Let us recall that under the assumptions of Theorem 3, according to [16, Theorem 2], we have, for some $c > 0$ and every $h > 0$ small enough:

$$e^{-t\mathsf{P}_h} = \sum_{j=1}^{n_0} e^{-t\lambda_{j,h}} \Pi_{j,h} + O(e^{-ct h^2}) \quad \text{in } \mathcal{L}(L^2(\mathsf{E})),$$

where, for $j = 1, \dots, n_0$, $\Pi_{j,h}$ is the spectral projector associated with the eigenvalue $\lambda_{j,h}$ of P_h, and $\Pi_{j,h} = O(1)$ in $\mathcal{L}(L^2(\mathsf{E}))$. Using in addition Theorem 3, we get sharp asymptotic equivalents of the different timescales $1/\lambda_{j,h}$ involved in the return to equilibrium. This leads in particular to the following accurate exponential decay rate in $L^2(\mathsf{E})$ of the semigroup $(e^{-t\mathsf{P}_h})_{t\ge 0}$ as $h \ll 1$.

Corollary 1.5. *Let U be a Morse function. Assume that (1.4) and (1.10) are satisfied. Then, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$, it holds for all $t \geq 0$:*

$$\left\| e^{-t\mathsf{P}_h} - \Pi_{1,h} \right\|_{\mathcal{L}(L^2(\mathsf{E}))} \leq C e^{-t\lambda_{2,h}},$$

where, as $h \to 0$, $\lambda_{2,h} = \zeta_h(\mathbf{m}_2)\, h\, e^{-\frac{2S(\mathbf{m}_2)}{h}}$ with $\zeta_h(\mathbf{m}_2) \sim \sum_{k\geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_2)$ and $\zeta_0(\mathbf{m}_2)$ given by (1.14), and where $\Pi_{1,h}$ is the $L^2(\mathsf{E})$ orthogonal projection on $\mathrm{Span}(e^{-U/h}\mathbb{1}_{\{\pm 1\}})$, where $\mathbb{1}_{\{\pm 1\}}$ is the constant function on $\{\pm 1\}$ which equals 1.

1.4. **Extension of the results to the case when (1.4) is not satisfied.** In this section, we assume that

$$\text{for any } x \in \mathbb{T}, \quad \partial_x U(x) = 0 \;\Rightarrow\; \alpha(x) = 0, \tag{1.15}$$

and we give asymptotic equivalents of the n_0 first eigenvalues of P_h when (1.15) holds (and then, in particular, when there is no refreshment at all, i.e. when $\alpha = 0$).

Theorem 4. *Let U be a Morse function. Assume that (1.15) and (1.10) are satisfied. Then, for any $c > 0$, there exists $h_0 > 0$ such that, for all $h \in (0, h_0]$, $\sigma(\mathsf{P}_h) \cap D(0, ch^2)$ is made of n_0 real eigenvalues $\lambda_{1,h} \leq \ldots \leq \lambda_{n_0,h}$ (counted with geometric multiplicity), which all have geometric multiplicity one. In addition, $\lambda_{1,h} = 0$ and, for all $j \in \{2, \ldots, n_0\}$,*

$$\lambda_{j,h} = \zeta_h(\mathbf{m}_j)\sqrt{h}\, e^{-\frac{2}{h}S(\mathbf{m}_j)}, \tag{1.16}$$

where ζ_h admits a classical expansion $\zeta_h(\mathbf{m}_j) \sim \sum_{k\geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_j)$ with

$$\zeta_0(\mathbf{m}_j) = \frac{1}{2\sqrt{\pi}} \sqrt{|V''(\mathbf{j}(\mathbf{m}_j))|}, \tag{1.17}$$

where $\mathbf{j} : \mathsf{U}^{(0)} \to \underline{\mathsf{U}}^{(1)} \cup \{\mathbf{s}_\infty\}$ is defined in (1.11).

Compared with Theorem 3, we cannot exclude the existence of other eigenvalues in the strip $0 < \mathrm{Re}(z) < ch^2$ with large imaginary part. The reason for this is that Theorem 1 does not apply when α and $\partial_x U$ vanish simultaneously.

The prefactor (1.17) is consistent with the one obtained in [23, Theorem 1.1] for the expected hitting time of a 1-dimensional PDMP with no refreshment. Indeed, when $\alpha = 0$ and V, and thus $U = -V$, is a non-degenerate double well potential (i.e. when $\mathsf{U}^{(0)} = \{\mathbf{m}_1, \mathbf{m}_2\}$ with $V(\mathbf{m}_1) < V(\mathbf{m}_2)$ and $\mathsf{U}^{(1)} = \{\mathbf{s}_1, \mathbf{s}_2\}$ with $V(\mathbf{s}_1) > V(\mathbf{s}_2)$), Theorem 4 and [23, Theorem 1.1] imply that $h^{-1}\lambda_{2,h}\mathbb{E}[\tau] = 1 + o(1)$ when $h \to 0$, where τ is the first time X_t reaches $\{\mathbf{m}_2\}$ when $(X_0, Y_0) = (\mathbf{s}_2, -1)$, where (X_t, Y_t) is the process with generator L_h on $\mathbb{T} \times \{\pm 1\}$. Let us recall here that $h^{-1}\lambda_{2,h}$ is the first nonzero eigenvalue of $-\mathsf{L}_h$ (see (1.2)).

Of course, the situation where the refreshing function α vanishes at some critical points $x \in \mathbb{T}$ of U but not at all could easily be handled and would lead, for each eigenvalue $\lambda_{j,h}$, either to the formula given in Theorem 3 or to the formula given in Theorem 4, depending on whether $\alpha(\mathbf{m}_j) > 0$ or $\alpha(\mathbf{m}_j) = 0$.

2. Reduction to a finite dimensional problem

In this section, we prove that, if λ is small enough, we can reduce the infinite dimensional problem $(\mathsf{P}_h - \lambda_h)u = 0$ into a finite dimensional nonlinear eigenvalue problem.

2.1. **A suitable change of variables.** Let us introduce some notation. For $\lambda \in \mathbb{C}$ and $r > 0$ we denote $D(\lambda, r) = \{z \in \mathbb{C}, |z - \lambda| < r\}$. For two families of numbers $a = (a_h)_{h>0}$ and $b = (b_h)_{h>0}$, we say that $a \in \mathcal{E}_{cl}(b)$ if there exists a family $c = (c_h)_{h>0}$ such that $a_h = b_h c_h$ and c admits a classical expansion $c_h \sim \sum_{j\geq 0} c_j h^j$ with $c_0 = 1$ as $h \to 0$. In all this work, $C > 0$ and $c > 0$ are constant which are independent of h and which can change from one occurence to another.

Recall that $\mathsf{E} = \mathbb{T} \times \{\pm 1\}$ is endowed with the natural scalar product (1.3). Let $\mathsf{F} = L^2(\mathbb{T}) \times L^2(\mathbb{T})$ and let $\langle , \rangle_{\mathsf{F}}$ be the Hilbertian structure induced by the isomorphism

$$\begin{aligned} \Omega_1 : L^2(\mathbb{T}\times\{\pm 1\}) &\to L^2(\mathbb{T}) \times L^2(\mathbb{T}) \\ f &\longmapsto \begin{pmatrix} f(.,+1) \\ f(.,-1) \end{pmatrix}. \end{aligned} \tag{2.1}$$

Then, Ω_1 is unitary from $(\mathsf{E}, \langle , \rangle_{L^2(\mathsf{E})})$ onto $(\mathsf{F}, \langle , \rangle_{\mathsf{F}})$, and

$$\Omega_1 D(\mathsf{P}_h) = H^1(\mathbb{T}) \times H^1(\mathbb{T}).$$

Direct computations show that

$$\Omega_1\, v\, \mathsf{d}_{U,h}\, \Omega_1^{-1} = \begin{pmatrix} \mathsf{d}_{U,h} & 0 \\ 0 & -\mathsf{d}_{U,h} \end{pmatrix}, \quad \Omega_1\, (v\, \partial_x U)_+\, \Omega_1^{-1} = \begin{pmatrix} (\partial_x U)_+ & 0 \\ 0 & (\partial_x U)_- \end{pmatrix}$$

$$\Omega_1 \mathsf{B} \Omega_1^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \Omega_1 \pi_v \Omega_1^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Combining these identities with (1.1), we get

$$\Omega_1 \mathsf{P}_h \Omega_1^{-1} = \begin{pmatrix} -\mathsf{d}_{U,h} & 0 \\ 0 & \mathsf{d}_{U,h} \end{pmatrix} + 2\begin{pmatrix} (\partial_x U)_+ & -(\partial_x U)_+ \\ -(\partial_x U)_- & (\partial_x U)_- \end{pmatrix} + \frac{\alpha}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.2}$$

We now change the variables in F and consider the unitary transformation $\Omega_2 : \mathsf{F} \to \mathsf{F}$ defined by

$$\Omega_2(f, g) = \frac{1}{\sqrt{2}}(f + g, g - f).$$

Consider the two vectors of $\mathbb{R}^2$ given by $e_1 = \binom{1}{1}$ and $e_2 = \binom{-1}{1}$. Then, one has:

$$\Omega_1 \mathsf{P}_h \Omega_1^{-1}(f e_1) = \mathsf{d}_{U,h} f e_2. \tag{2.3}$$

On the other hand,

$$\begin{aligned} 2\begin{pmatrix} (\partial_x U)_+ & -(\partial_x U)_+ \\ -(\partial_x U)_- & (\partial_x U)_- \end{pmatrix}\begin{pmatrix} -f \\ f \end{pmatrix} &= 4f\begin{pmatrix} -(\partial_x U)_+ \\ (\partial_x U)_- \end{pmatrix} \\ &= 2((\partial_x U)_- - (\partial_x U)_+)\begin{pmatrix} f \\ f \end{pmatrix} + 2((\partial_x U)_- + (\partial_x U)_+)\begin{pmatrix} -f \\ f \end{pmatrix} \\ &= -2\partial_x U f e_1 + 2|\partial_x U| f e_2. \end{aligned}$$

It follows that:

$$\begin{aligned}\Omega_1 \mathsf{P}_h \Omega_1^{-1}(fe_2) &= \mathsf{d}_{U,h} fe_1 - 2\partial_x U fe_1 + (\alpha + 2|\partial_x U|) fe_2 \\ &= \mathsf{d}_{-U,h} fe_1 + (\alpha + 2|\partial_x U|) fe_2\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$\Omega_2 \Omega_1 \mathsf{P}_h \Omega_1^{-1} \Omega_2^{-1} = \begin{pmatrix} 0 & \mathsf{d}_{-U,h} \\ \mathsf{d}_{U,h} & \alpha + 2|\partial_x U| \end{pmatrix}.$$

Set $\mathsf{Q}_h := \Omega_2 \Omega_1 \mathsf{P}_h \Omega_1^{-1} \Omega_2^{-1}$ with domain $D(\mathsf{Q}_h) = H^1(\mathbb{T}) \times H^1(\mathbb{T})$ on $L^2(\mathbb{T})^2$, i.e.

$$\mathsf{Q}_h = \begin{pmatrix} 0 & \mathsf{d}_{-U,h} \\ \mathsf{d}_{U,h} & W \end{pmatrix} = \begin{pmatrix} 0 & \mathsf{d}_{V,h} \\ -\mathsf{d}_{V,h}^* & W \end{pmatrix} \tag{2.5}$$

where we recall that $V = -U$ and where we denote

$$W(x) := \alpha(x) + 2|\partial_x U(x)| = \alpha(x) + 2|\partial_x V(x)|, \ x \in \mathbb{T}. \tag{2.6}$$

Since the transformation $\Omega_2 \Omega_1$ is unitary from E onto F,

$$\sigma(\mathsf{P}_h) = \sigma(\mathsf{Q}_h).$$

Remark 2.1. *Note that the identity* $(\mathsf{I} - \mathsf{B}) = 2(\mathsf{I} - \pi_v)$*, which is specific to the one-dimensional setting, is crucial to derive the operator* Q_h*.*

We will in the following study the spectrum of Q_h to prove Theorems 2 and 3. Consider the semiclassical Witten Laplacian $\Delta_{V,h}$ associated with V on $\mathbb{T}$, that is (see Remark 1.1)

$$\Delta_{V,h} = \mathsf{d}_{V,h}^* \mathsf{d}_{V,h} = -h^2 \Delta + |\partial_x V(x)|^2 - h\partial_x^2 V(x), \quad \text{with domain } D(\Delta_{V,h}) = H^2(\mathbb{T}).$$

For $\lambda \in \mathbb{C}$, we finally define

$$\mathsf{T}_h(\lambda) = \Delta_{V,h} - \lambda W + \lambda^2 \text{ with domain } D(\Delta_{V,h}). \tag{2.7}$$

The following result is the key point of our analysis: it establishes an equivalence between the spectrum of Q_h and the kernels of the operators $\mathsf{T}_h(\lambda)$, $\lambda \in \mathbb{C}$.

Lemma 2.2. *The operator* Q_h *with domain* $D(\mathsf{Q}_h) = H^1(\mathbb{T}) \times H^1(\mathbb{T})$ *is closed and has compact resolvent. In particular, it has only discrete spectrum. Moreover,* $\lambda = 0$ *is a simple eigenvalue of* Q_h *and* $\dim(\operatorname{Ker} \mathsf{T}_h(0)) = \dim(\operatorname{Ker} \mathsf{Q}_h) = 1$*. Besides, for every* $\lambda \in \mathbb{C} \setminus \{0\}$*, the application*

$$\begin{array}{rccc} \Psi: & \operatorname{Ker}(\mathsf{T}_h(\lambda)) & \longrightarrow & \operatorname{Ker}(\mathsf{Q}_h - \lambda) \\ & g & \longmapsto & \begin{pmatrix} \frac{1}{\lambda} \mathsf{d}_{V,h} g \\ g \end{pmatrix} \end{array} \tag{2.8}$$

is a linear isomorphism. Eventually, any $\mu \in \mathbb{C}$ *is a singularity of* $\lambda \mapsto \mathsf{T}_h(\lambda)^{-1}$ *if and only if there exists* $g \in H^2(\mathbb{T})$*,* $g \neq 0$*, such that* $\mathsf{T}_h(\mu) g = 0$*.*

Proof. The operator Q_h is closed on $L^2(\mathbb{T})^2$ since it is a bounded perturbation of the closed operator $(0, h\partial_x; h\partial_x, 0)$. The resolvent of Q_h is compact since the injection $H^1(\mathbb{T}) \subset L^2(\mathbb{T})$ is compact. Since Q_h is unitarily equivalent to P_h (see (2.5)), it follows from Theorem 1 that $\lambda = 0$ is a simple eigenvalue of Q_h. Moreover, since $\mathsf{T}_h(0) = \Delta_{V,h}$, one gets $\operatorname{Ker} \mathsf{T}_h(0) = \mathbb{C} e^{-\frac{V}{h}}$ and then $\dim(\operatorname{Ker} \mathsf{T}_h(0)) = 1$.

Let us now consider $\lambda \in \mathbb{C} \setminus \{0\}$. For any $g \in \mathrm{Ker}(\mathsf{T}_h(\lambda))$, one has $g \in H^2(\mathbb{T})$ and thus $(\frac{1}{\lambda}\mathsf{d}_{V,h}g, g) \in D(\mathsf{Q}_h)$. It follows moreover from $\mathsf{T}_h(\lambda)g = 0$ that

$$\mathsf{Q}_h\Psi(g) = \begin{pmatrix} 0 & \mathsf{d}_{V,h} \\ -\mathsf{d}^*_{V,h} & W \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda}\mathsf{d}_{V,h}g \\ g \end{pmatrix} = \begin{pmatrix} \mathsf{d}_{V,h}g \\ (-\frac{1}{\lambda}\Delta_{V,h} + W)g \end{pmatrix} = \lambda\Psi(g).$$

This proves that Ψ is well defined. The linearity and the injectivity of Ψ are obvious. To prove its surjectivity, consider $u = \binom{f}{g} \in D(\mathsf{Q}_h)$ such that $\mathsf{Q}_h u = \lambda u$, i.e. such that

$$\begin{cases} \mathsf{d}_{V,h}g = \lambda f, \\ -\mathsf{d}^*_{V,h}f + Wg = \lambda g. \end{cases}$$

It follows from the first equation that $g \in H^2(\mathbb{T})$ and, since $\lambda \neq 0$, that $u = \binom{\frac{1}{\lambda}\mathsf{d}_{V,h}g}{g}$. Moreover, applying $\mathsf{d}^*_{V,h}$ to the second equation leads to

$$\Delta_{V,h}g = \lambda\mathsf{d}^*_{V,h}f = \lambda Wg - \lambda^2 g.$$

Consequently, $g \in D(\Delta_{V,h})$ and $\mathsf{T}_h(\lambda)g = 0$, which proves the surjectivity.

It remains to prove the last statement of Lemma 2.2. To this end, let us consider $\mu \in \mathbb{C}$. It is a singularity of $\lambda \mapsto \mathsf{T}_h(\lambda)^{-1}$ if and only if $\mathsf{T}_h(\mu) : H^2(\mathbb{T}) \to L^2(\mathbb{T})$ is not invertible. Furthermore, since $\Delta_{V,h} + 1 : H^2(\mathbb{T}) \to L^2(\mathbb{T})$ is invertible,

$$\mathsf{T}_h(\lambda) = \left[1 - (1 + \lambda W - \lambda^2)(\Delta_{V,h} + 1)^{-1}\right](\Delta_{V,h} + 1)$$

is not invertible if and only if $1 - \mathsf{B}(\lambda) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is not invertible, where $\mathsf{B}(\lambda) = (1 + \lambda W - \lambda^2)(\Delta_{V,h} + 1)^{-1}$. Since $(\Delta_{V,h} + 1)^{-1}$: $L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is compact, so is $\mathsf{B}(\lambda)$. The Fredholm alternative then implies that $\mathsf{T}_h(\mu)$ is not invertible if and only if there exists $u \in L^2(\mathbb{T})$, $u \neq 0$ such that $(1 - \mathsf{B}(\lambda))u = 0$, that is if and only if there exists $g \in H^2(\mathbb{T})$, $g \neq 0$ such that $\mathsf{T}_h(\mu)g = 0$. This concludes the proof of Lemma 2.2. □

According to Lemma 2.2, for every $\mu \in \mathbb{C}$:

$$\mu \in \sigma_d(\mathsf{P}_h) = \sigma(\mathsf{P}_h) \text{ if and only if } \mu \text{ is a singularity of } \lambda \mapsto \mathsf{T}_h(\lambda)^{-1} \tag{2.9}$$

and, for such a μ, one has

$$\dim(\mathrm{Ker}(\mathsf{P}_h - \mu)) = \dim(\mathrm{Ker}(\mathsf{T}_h(\mu)). \tag{2.10}$$

To prove Theorems 2 and 3, we will thus investigate the singularities of $\lambda \mapsto \mathsf{T}_h(\lambda)^{-1}$ near 0.

2.2. **Finite dimensional reduction, a Grushin problem.** In this section, we show that the singularities of $\mathsf{T}_h(\lambda)$ near 0 are those of a matrix valued holomorphic function $E_{-+}(\lambda)$. To this end, we will construct a so-called Grushin problem.

This construction relies on well known results on the low-lying spectrum of $\Delta_{V,h}$. From the early works of Witten [26] and Helffer-Sjöstrand [18], we know that there is a one-to-one correspondence between the local minima of V and the smallest eigenvalues of $\Delta_{V,h}$. This correspondence was further investigated by several authors and sharp asymptotic equivalents of these small eigenvalues were finally obtained in [8] and [17] under assumptions on the relative positions of the minima and saddle points, and in [21] in the general case. The

following version gives the general form of these asymptotic equivalents in a non-degenerate setting.

Theorem 5. [8], [17]. *Let $U = -V$ be a Morse function. There exist $\epsilon_* > 0$, $C > 0$, and $h_0 > 0$ such that, for all $h \in]0, h_0]$, the nonnegative self-adjoint operator $(\Delta_{V,h}, H^2(\mathbb{T}))$ admits exactly n_0 eigenvalues (counted with algebraic multiplicity) in $[0, \epsilon_* h]$:*

$$\sigma(\Delta_{V,h}) \cap [0, \varepsilon_* h] = \{0, \mu^{\Delta}_{2,h}, \dots, \mu^{\Delta}_{n_0,h}\}, \tag{2.11}$$

where 0 is a simple eigenvalue of $\Delta_{V,h}$. Let us order $\{\mu^{\Delta}_{2,h}, \dots, \mu^{\Delta}_{n_0,h}\}$ such that $\mu^{\Delta}_{j,h} \le \mu^{\Delta}_{j+1,h}$ for $j = 1, \dots, n_0 - 1$. Then, when (1.10) holds, we have for all $j = 2, \dots, n_0$:

$$\mu^{\Delta}_{j,h} = a_h(\mathbf{m}_j)\, h\, e^{-\frac{2}{h} S(\mathbf{m}_j)}, \tag{2.12}$$

where $S : \mathsf{U}^{(0)} \to (0, +\infty]$ is defined in (1.9) (see also (1.12)) and $a_h(\mathbf{m}_j)$ admits a classical expansion $a_h(\mathbf{m}_j) \sim \sum_{k \ge 0} h^k a_k(\mathbf{m}_j)$ with

$$a_0(\mathbf{m}_j) = \frac{1}{\pi} \sqrt{V''(\mathbf{m}_j)\, |V''(\mathbf{j}(\mathbf{m}_j))|}, \tag{2.13}$$

where $\mathbf{j} : \mathsf{U}^{(0)} \to \underline{\mathsf{U}}^{(1)} \cup \{\mathbf{s}_\infty\}$ is defined in (1.11).

Remark 2.3. *We have decided to work with $\Delta_{V,h}$, which appeared naturally in the above derivation of $\mathsf{T}_h(\lambda)$. However, according to Remark 1.1, we could equivalently have worked with $\Delta_{U,h}$.*

We are now in position to construct a Grushin problem. Set

$$\Pi = \pi_{[0, \varepsilon_* h]}(\Delta_{V,h}) \text{ and } \mathsf{E}_0 = \mathrm{Ran}(\Pi), \tag{2.14}$$

where $\pi_{[a,b]}(\Delta_{V,h})$ is the spectral projector associated with $\Delta_{V,h}$ and the interval $[a, b]$. According to Theorem 5, the space E_0 has dimension n_0 for all $h > 0$ small enough. Let $(\Psi_j)_{j=1,\dots n_0}$ be an orthonormal basis of the space E_0. We assume without loss of generality that Ψ_1 is proportional to $e^{-V/h}$. Introduce the operators

$$\begin{aligned} R_- : \mathbb{C}^{n_0} &\to L^2(\mathbb{T}) \\ u &\mapsto \sum_{j=1}^{n_0} u_j \Psi_j \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} R_+ : L^2(\mathbb{T}) &\to \mathbb{C}^{n_0} \\ u &\mapsto (\langle u, \Psi_j \rangle_{L^2})_j. \end{aligned} \tag{2.16}$$

We equip $\mathbb{C}^{n_0}$ with the ℓ_2 norm. Notice that $R_+ R_- = \mathsf{I}_{\mathbb{C}^{n_0}}$ and $R_- R_+ = \Pi$. In addition, $\|R_+\| \le 1$ and $\|R_-\| \le 1$, for all $h > 0$. These inequalities will be used many times in what follows. From now on, we denote

$$\widehat{\Pi} = \mathsf{I} - \Pi.$$

Lemma 2.4. *Let U be a Morse function. For any $\lambda \in \mathbb{C}$, the operator $\widehat{\mathsf{T}}_h(\lambda) := \widehat{\Pi} \mathsf{T}_h(\lambda) \widehat{\Pi}$ acting on $\widehat{\Pi} L^2 := \widehat{\Pi}(L^2(\mathbb{T}))$ with domain $D(\widehat{\mathsf{T}}_h(\lambda)) = \{u \in \widehat{\Pi} L^2,\ u \in H^2(\mathbb{T})\}$ is closed. Moreover, there exist $C, \epsilon_0, h_0 > 0$ such that for all $h \in]0, h_0]$, and for all $\lambda \in D(0, \epsilon_0 h)$, $\widehat{\mathsf{T}}_h(\lambda)$ is invertible, holomorphic with respect to λ, and $\|\widehat{\mathsf{T}}_h(\lambda)^{-1}\| \le C h^{-1}$.*

Proof. The proof is very close to the proof of Lemma 2.1 in [20]. We sketch it for reader's convenience. We first observe that since W is bounded, the operator $\mathsf{T}_h(\lambda)$ with domain $D(\mathsf{T}_h(\lambda)) = D(\Delta_{V,h}) = H^2(\mathbb{T})$ is closed and densely defined. Its adjoint $\mathsf{T}_h(\lambda)^*$ satisfies $D(\mathsf{T}_h(\lambda)^*) = H^2(\mathbb{T})$. Suppose that $(u_n, \widehat{\mathsf{T}}_h(\lambda)u_n) \in D(\widehat{\mathsf{T}}_h(\lambda)) \times L^2(\mathbb{T})$ converges to $(u, v) \in L^2(\mathbb{T}) \times L^2(\mathbb{T})$. For any $j = 1, \dots, n_0$, one has $\Psi_j \in D(\Delta_{V,h}) = D(\mathsf{T}_h(\lambda)^*)$ and hence for all $n \in \mathbb{N}$, one has

$$\Pi \mathsf{T}_h(\lambda)u_n = \sum_{j=1}^{n_0} \langle \mathsf{T}_h(\lambda)u_n, \Psi_j\rangle \Psi_j = \sum_{j=1}^{n_0} \langle u_n, \mathsf{T}_h(\lambda)^*\Psi_j\rangle \Psi_j. \tag{2.17}$$

Consequently, the sequence $(\Pi \mathsf{T}_h(\lambda)u_n)_n$ converges and using the identity

$$\mathsf{T}_h(\lambda)u_n = \Pi \mathsf{T}_h(\lambda)u_n + \widehat{\mathsf{T}}_h(\lambda)u_n,$$

it follows that $(\mathsf{T}_h(\lambda)u_n)$ converges. Since $\mathsf{T}_h(\lambda)$ is closed as a bounded perturbation of a closed operator, and $u_n \in H^2(\mathbb{T}) = D(\mathsf{T}_h(\lambda))$, one deduces that $u \in D(\mathsf{T}_h(\lambda))$ and $\mathsf{T}_h(\lambda)u = \lim \mathsf{T}_h(\lambda)u_n$. Since $\widehat{\Pi} = 1 - \Pi$ is bounded, this implies that $u \in D(\widehat{\mathsf{T}}_h(\lambda))$ and $\widehat{\mathsf{T}}_h(\lambda)u = v$ which proves that $\widehat{\mathsf{T}}_h(\lambda)$ is closed. In addition, the operator $\widehat{\mathsf{T}}_h(\lambda)$ is clearly densely defined on $\widehat{\Pi}L^2$. Let us now study the invertibility of $\widehat{\mathsf{T}}_h(\lambda)$. For any $u \in D(\widehat{\mathsf{T}}_h(\lambda))$, one has by definition

$$\mathrm{Re}\langle \widehat{\mathsf{T}}_h(\lambda)u, u\rangle = \mathrm{Re}\langle (\Delta_{V,h} - \lambda W + \lambda^2)\widehat{\Pi}u, \widehat{\Pi}u\rangle.$$

Using Theorem 5, this implies that for $|\lambda| < \epsilon_0 h$, one has

$$\begin{aligned}
\mathrm{Re}\langle \widehat{\mathsf{T}}_h(\lambda)u, u\rangle &\geq \epsilon_* h\|\widehat{\Pi}u\|^2 - \mathrm{Re}(\lambda)\langle W\widehat{\Pi}u, \widehat{\Pi}u\rangle + \mathrm{Re}(\lambda^2)\|\widehat{\Pi}u\|^2 \\
&\geq (\epsilon_* - \epsilon_0\|W\|_{L^\infty})h\|\widehat{\Pi}u\|^2 - \epsilon_0^2 h^2\|\widehat{\Pi}u\|^2 \\
&\geq \frac{\epsilon_*}{2}h\|\widehat{\Pi}u\|^2
\end{aligned}$$

for ϵ_0 small enough. This proves that $\widehat{\mathsf{T}}_h(\lambda)$ is injective when $|\lambda| < \epsilon_0 h$. We observe that the same proof shows that $\widehat{\mathsf{T}}_h(\lambda)^* = \widehat{\Pi}\mathsf{T}_h(\lambda)^*\widehat{\Pi}$ is injective. Moreover, the Cauchy-Schwarz inequality implies that for all $u \in D(\widehat{\mathsf{T}}_h(\lambda))$,

$$\|\widehat{\mathsf{T}}_h(\lambda)u\| \geq \frac{\epsilon_*}{2}h\|\widehat{\Pi}u\|. \tag{2.18}$$

Let us now prove that $\widehat{\mathsf{T}}_h(\lambda)$ is surjective if $|\lambda| < \epsilon_0 h$. We first observe that $\mathrm{Ran}(\widehat{\mathsf{T}}_h(\lambda))$ is closed since $\widehat{\mathsf{T}}_h(\lambda)$ is closed and the convergence of any sequence $(\widehat{\mathsf{T}}_h(\lambda)u_n)$ implies the convergence of (u_n) thanks to (2.18). Hence it is sufficient to prove that $\mathrm{Ran}(\widehat{\mathsf{T}}_h(\lambda))^\perp = \{0\}$, which is consequence of the injectivity of the operator $\widehat{\mathsf{T}}_h(\lambda)^*$. □

We now introduce the Grushin operator

$$\mathcal{K}_h(\lambda) = \begin{pmatrix} \mathsf{T}_h(\lambda) & R_- \\ R_+ & 0 \end{pmatrix}. \tag{2.19}$$

Proposition 2.5. *Let U be a Morse function. There exist $\epsilon_0 > 0$ and $h_0 > 0$ such that, for all $h \in]0, h_0]$ and all $\lambda \in D(0, \epsilon_0 h)$, the operator $\mathcal{K}_h(\lambda)$ is invertible. Moreover, its inverse*

$\mathcal{E}_h(\lambda)$ writes

$$\mathcal{E}_h(\lambda) = \begin{pmatrix} E(\lambda) & E_+(\lambda) \\ E_-(\lambda) & E_{-+}(\lambda) \end{pmatrix},$$

where E, E_-, E_+, E_{-+} are holomorphic in $D(0,\epsilon_0 h)$ and satisfy the following formulas:

$$E_+(\lambda) \;=\; R_- - \widehat{\mathsf{T}}_h(\lambda)^{-1}\widehat{\Pi}\mathsf{T}_h(\lambda)R_-\,, \quad E_-(\lambda) \;=\; R_+ - R_+\mathsf{T}_h(\lambda)\widehat{\mathsf{T}}_h(\lambda)^{-1}\widehat{\Pi}\,,$$

$$E_{-+}(\lambda) \;=\; -R_+\mathsf{T}_h(\lambda)R_- + R_+\mathsf{T}_h(\lambda)\widehat{\Pi}\widehat{\mathsf{T}}_h(\lambda)^{-1}\widehat{\Pi}\mathsf{T}_h(\lambda)R_- : \mathbb{C}^{n_0} \to \mathbb{C}^{n_0}, \tag{2.20}$$

and

$$E(\lambda) \;=\; \widehat{\mathsf{T}}_h(\lambda)^{-1}\widehat{\Pi}\,. \tag{2.21}$$

Moreover, for $\lambda \in D(0,\epsilon_0 h)$, $\mathsf{T}_h(\lambda)$ is invertible if and only if the matrix $E_{-+}(\lambda)$ is invertible, in which case it holds

$$\mathsf{T}_h(\lambda)^{-1} \;=\; E(\lambda) - E_+(\lambda)E_{-+}^{-1}(\lambda)E_-(\lambda).$$

Proof. Thanks to Lemma 2.4, the proof is reduced to an algebraic computation which is completely analogous to the one used in the proof of [20, Lemma 2.2]. □

We now give some direct consequences of Proposition 2.5 and Lemma 2.4, which will be used in the following. By definition, one has $R_+\mathsf{T}_h(\lambda)\widehat{\Pi} = -\lambda R_+ W\widehat{\Pi}$. Hence (2.20) becomes

$$E_{-+}(\lambda) \;=\; -R_+\mathsf{T}_h(\lambda)R_- + \lambda^2 R_+ W\widehat{\Pi}\widehat{\mathsf{T}}_h(\lambda)^{-1}\widehat{\Pi}WR_-.$$

Introducing the matrices

$$\mathsf{M}_{V,h} = R_+\Delta_{V,h}R_-,\ \mathsf{W}_h = R_+WR_-,\ \text{and}\ \ \mathsf{G}_h(\lambda) = \lambda^2(\mathsf{I}_{\mathbb{C}^{n_0}} - R_+W\widehat{\Pi}\widehat{\mathsf{T}}_h(\lambda)^{-1}\widehat{\Pi}WR_-), \tag{2.22}$$

this rewrites

$$-E_{-+}(\lambda) \;= \mathsf{M}_{V,h} - \lambda\mathsf{W}_h + \mathsf{G}_h(\lambda). \tag{2.23}$$

Moreover, it follows from Lemma 2.4 that $\lambda \in D(0,\epsilon_0 h) \mapsto \mathsf{G}_h(\lambda)$ is holomorphic and that there exists $C > 0$ such that for all h small enough and all $\lambda \in D(0,\epsilon_0 h)$,

$$|\mathsf{G}_h(\lambda)| \le C|\lambda|^2 h^{-1}. \tag{2.24}$$

According to (2.9) and to Proposition 2.5,

$$\lambda \in D(0,\epsilon_0 h) \cap \sigma(\mathsf{P}_h) \quad \text{if and only if} \quad \lambda \in D(0,\epsilon_0 h) \text{ is a singularity of } E_{-+}^{-1}(\lambda). \tag{2.25}$$

To prove Theorems 2 and 3, the strategy consists in studying the singularities of $E_{-+}^{-1}(\lambda)$ in $D(0,\epsilon_0 h)$, which first requires to compute asymptotic equivalents of $\mathsf{M}_{V,h}$ and W_h as $h \to 0$ (see (2.23)).

3. The double well case

In this section we prove Theorem 2. To this end, we assume throughout this section that $\mathsf{U}^{(0)}$ has exactly two elements $\mathbf{m}_1$ and $\mathbf{m}_2$ such that $V(\mathbf{m}_1) < V(\mathbf{m}_2)$. Assume also that the two elements $\mathbf{s}_1, \mathbf{s}_2$ of $\mathsf{U}^{(1)}$ satisfy $V(\mathbf{s}_1) > V(\mathbf{s}_2)$. Recall that in this case, one has by construction of S (see (1.9)) and $\mathbf{j}$ (see (1.11)), $S(\mathbf{m}_2) = V(\mathbf{s}_2) - V(\mathbf{m}_2)$ and $\mathbf{j}(\mathbf{m_2}) = \mathbf{s_2}$ (see Remark 1.3). Thus, according to Theorem 5, it holds

$$\mu_{2,h}^{\Delta} = a_h(\mathbf{m}_2)\, h\, e^{-\frac{2(V(\mathbf{s}_2)-V(\mathbf{m}_2))}{h}}, \tag{3.1}$$

with $a_h(\mathbf{m}_2) \sim \sum_{k\geq 0} h^k a_k(\mathbf{m}_2)$ and $a_0(\mathbf{m}_2) = \frac{1}{\pi}\sqrt{V''(\mathbf{m}_2)\,|V''(\mathbf{s}_2)|}$.

3.1. Asymptotic equivalent of W_h, as $h \to 0$.

Recall that $\mathsf{W}_h = R_+ W R_-$ (see (2.22)), where W, R_+, and R_- are respectively defined in (2.6), (2.15), and (2.16). The operators R_+ and R_- are defined with the help of an orthonormal basis (Ψ_1, Ψ_2) of E_0 (see (2.14)). Hence, in order to compute an asymptotic equivalent of W_h in the limit $h \to 0$, we first consider a suitable basis (Ψ_1, Ψ_2), which is chosen as follows.

We choose Ψ_1 as the $L^2(\mathbb{T})$-normalized solution of $\Delta_{V,h}\Psi_1 = 0$, namely:

$$\Psi_1 = \frac{1}{Z_{1,h}} e^{-(V-V(\mathbf{m}_1))/h} \text{ with } Z_{1,h} = \|e^{-(V-V(\mathbf{m}_1))/h}\|_{L^2(\mathbb{T})},$$

where by Laplace's method: $Z_{1,h} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_1))^{1/4})$, since $\mathbf{m}_1$ is the unique global minimum of V on $\mathbb{T}$. We recall that $Z_{1,h} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_1))^{1/4})$ means that $Z_{1,h} = (\pi h/V''(\mathbf{m}_1))^{1/4} z_h$ where $z_h \sim 1 + \sum_{k\geq 1} w_k h^k$, see the beginning of Section 2.

To construct Ψ_2, we first choose a $L^2(\mathbb{T})$-normalized function φ_2 such that both $\Delta_{V,h}\varphi_2$ and $\langle \Psi_1, \varphi_2\rangle_{L^2(\mathbb{T})}$ are exponentially small as $h \to 0$. Since Ψ_1 concentrates around $\mathbf{m}_1$ and $V(\mathbf{m}_2) > V(\mathbf{m}_1)$, a good candidate is given by:

$$\varphi_2 = \frac{1}{Z_{2,h}} \chi_2 e^{-(V-V(\mathbf{m}_2))/h} \text{ with } Z_{2,h} = \|e^{-(V-V(\mathbf{m}_2))/h}\|_{L^2(\mathbb{T})}, \tag{3.2}$$

where $r > 0$ and $\chi_2 \in \mathcal{C}_c^\infty(\mathbb{T}, [0,1])$ satisfies $1\!\!1_{B(\mathbf{m}_2,r)} \leq \chi_2 \leq 1\!\!1_{B(\mathbf{m}_2,2r)}$. When $r > 0$ is small enough, $\mathbf{m}_2$ is the unique global minimum of V on the closure of $B(\mathbf{m}_2, 2r)$ (because V is a Morse function). Thus, by Laplace's method: $Z_{2,h} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_2))^{1/4})$. In addition, for such a fixed $r > 0$, $V > V(\mathbf{m}_2)$ on $\mathrm{supp}(\nabla\chi_2)$, and therefore, since $\mathrm{supp}([\Delta, \chi_2]) \subset \mathrm{supp}(\nabla\chi_2)$ and $\Delta_{V,h}(e^{-V/h}) = 0$, there exists $C > 0$ such that for h small enough:

$$\Delta_{V,h}\varphi_2 = -h^2[\Delta, \chi_2]\varphi_2 = O(e^{-C/h}) \text{ in } L^2(\mathbb{T}). \tag{3.3}$$

On the other hand, since $\Delta_{V,h}$ is self-adjoint, it follows from the localisation of the spectrum stated in Theorem 5 (with $n_0 = 2$ there) that

$$\forall z \in \partial D(0, \epsilon_* h/2),\ \|(\Delta_{V,h} - z)^{-1}\|_{\mathcal{L}(L^2(\mathbb{T}))} = \frac{1}{d(z, \sigma(\Delta_{V,h}))} = O(h^{-1}). \tag{3.4}$$

By definition of Π and according to Theorem 5,

$$\Pi = \frac{1}{2i\pi} \int_{\partial D(0, \frac{\epsilon_*}{2}h)} (z - \Delta_{V,h})^{-1} dz$$

and, using (3.3) and (3.4) together with the second resolvent identity, it follows that $\Pi\varphi_2$ satisfies

$$\begin{aligned}\Pi\varphi_2-\varphi_2&=\frac{1}{2i\pi}\int_{\partial D(0,\frac{\epsilon_*}{2}h)}\big((z-\Delta_{V,h})^{-1}-z^{-1}\big)\varphi_2 dz\\ &=\frac{1}{2i\pi}\int_{\partial D(0,\frac{\epsilon_*}{2}h)}z^{-1}(z-\Delta_{V,h})^{-1}\Delta_{V,h}\varphi_2 dz=O(e^{-C/h}).\end{aligned}\tag{3.5}$$

Moreover, since $V(\mathbf{m}_2)>V(\mathbf{m}_1)$ and $\Pi\Psi_1=\Psi_1$, one has for h small enough:

$$\langle\Pi\varphi_2,\Psi_1\rangle_{L^2(\mathbb{T})}=\langle\varphi_2,\Psi_1\rangle_{L^2(\mathbb{T})}=O(e^{-C/h}).\tag{3.6}$$

We finally set

$$\Psi_2=\frac{\Pi\varphi_2-\langle\Pi\varphi_2,\Psi_1\rangle_{L^2(\mathbb{T})}\Psi_1}{\|\Pi\varphi_2-\langle\Pi\varphi_2,\Psi_1\rangle\Psi_1\|_{L^2(\mathbb{T})}}.$$

The function Ψ_2 belongs to E_0, is orthogonal to Ψ_1, and $\|\Psi_2\|_{L^2(\mathbb{T})}=1$. From now on, we consider the above orthonormal basis (Ψ_1,Ψ_2) of E_0. Notice that, using (3.5) and (3.6), one has $\langle\Pi\varphi_2,\Psi_1\rangle\Psi_1=O(e^{-C/h})$ and $\|\Pi\varphi_2-\langle\Pi\varphi_2,\Psi_1\rangle\Psi_1\|_{L^2(\mathbb{T})}=1+O(e^{-C/h})$. Consequently,

$$\Psi_2=\varphi_2+O(e^{-C/h})\text{ in }L^2(\mathbb{T}).\tag{3.7}$$

Lemma 3.1. *With the above choice of (Ψ_1,Ψ_2), there exists $C>0$ such that for all h small enough,*

$$\mathsf{W}_h=\begin{pmatrix}\gamma_{1,h}&0\\0&\gamma_{2,h}\end{pmatrix}+O(e^{-C/h}),$$

where we recall that W_h is defined in (2.22) and where, for $i\in\{1,2\}$, $\gamma_{i,h}$ satisfies as $h\to 0$: $\gamma_{i,h}\sim\sum_{k\geq 0}h^k\gamma_{\alpha,k}(\mathbf{m}_i)+\sqrt{h}\sum_{k\geq 0}h^k\gamma_{V,k}(\mathbf{m}_i)$, with

$$\gamma_{\alpha,0}(\mathbf{m}_i)=\alpha(\mathbf{m}_i)\text{ and }\gamma_{V,0}(\mathbf{m}_i)=2\sqrt{\frac{V''(\mathbf{m}_i)}{\pi}}.$$

Proof. Since (Ψ_1,Ψ_2) is an orthonormal family, one has $\mathsf{W}_h=(\langle W\Psi_i,\Psi_j\rangle_{L^2(\mathbb{T})})_{i,j=1,2}$. In addition, $\Psi_1=O(e^{-C/h})$ in $L^2(\mathrm{supp}(\varphi_2))$. Hence, using (3.7), for all $i\neq j$, $\langle W\Psi_i,\Psi_j\rangle_{L^2(\mathbb{T})}=O(e^{-C/h})$. Suppose now that $i\in\{1,2\}$ is fixed. By definition of W, one has $\langle W\Psi_i,\Psi_i\rangle_{L^2(\mathbb{T})}=\langle\alpha\Psi_i,\Psi_i\rangle_{L^2(\mathbb{T})}+2\langle|\partial_x V|\Psi_i,\Psi_i\rangle_{L^2(\mathbb{T})}$. By definition of Ψ_1 and Ψ_2 above, using Laplace's method, one has as $h\to 0$: $\langle\alpha\Psi_i,\Psi_i\rangle_{L^2(\mathbb{T})}\sim\sum_{k\geq 0}h^k\gamma_{\alpha,k}(\mathbf{m}_i)$ with $\gamma_{\alpha,0}(\mathbf{m}_i)=\alpha(\mathbf{m}_i)$. On the other hand, for $i\in\{1,2\}$, one has for $\delta>0$ small enough,

$$\begin{aligned}\langle|\partial_x V|\Psi_i,\Psi_i\rangle_{L^2(\mathbb{T})}&=Z_{i,h}^{-2}\int_{|x-\mathbf{m}_i|<\delta}|\partial_x V(x)|e^{-2(V(x)-V(\mathbf{m}_i))/h}dx+O(e^{-c/h})\\&=Z_{i,h}^{-2}\Big(\int_{\mathbf{m}_i}^{\mathbf{m}_i+\delta}\partial_x V(x)e^{-2(V(x)-V(\mathbf{m}_i))/h}dx\\&\qquad-\int_{\mathbf{m}_i-\delta}^{\mathbf{m}_i}\partial_x V(x)e^{-2(V(x)-V(\mathbf{m}_i))/h}dx\Big)+O(e^{-c/h})\\&=hZ_{i,h}^{-2}+O(e^{-c/h}).\end{aligned}$$

Using the fact that $Z_{i,h}\in\mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_i))^{1/4})$, this ends the proof of the lemma. □

We are now in position to prove Theorem 2.

3.2. **End of the proof of Theorem 2.** Recall that the strategy consists in localizing the singularities of $E_{-+}^{-1}(\lambda)$ in $D(0,\epsilon_0 h)$ (see (2.25)). First of all, $\lambda=0$ is always a singularity of $\lambda\in D(0,\epsilon_0 h)\mapsto E_{-+}^{-1}(\lambda)$, since $\mathsf{T}_h(0)\Psi_1=0$ (see Proposition 2.5). Let us now look for the other singularities of $\lambda\in D(0,\epsilon_0 h)\mapsto E_{-+}^{-1}(\lambda)$. Since $\dim \mathsf{E}_0=2$, $\Delta_{V,h}\Psi_1=0$, and $\Psi_2\in\mathsf{E}_0$ is orthogonal to Ψ_1, it holds: $\Delta_{V,h}\Psi_2=\mu_{2,h}^{\Delta}\Psi_2$ and therefore, $\mathsf{M}_{V,h}=\mathrm{diag}(0,\mu_{2,h}^{\Delta})$. By Lemma 3.1, (2.23), and (2.24), one then has

$$-E_{-+}(\lambda)=\begin{pmatrix}0&0\\0&\mu_{2,h}^{\Delta}\end{pmatrix}-\lambda\begin{pmatrix}\gamma_{1,h}&0\\0&\gamma_{2,h}\end{pmatrix}+\mathsf{r}_h(\lambda),$$

where $\lambda\in D(0,\epsilon_0 h)\mapsto \mathsf{r}_h(\lambda)$ is holomorphic and $\mathsf{r}_h(\lambda)=O(\lambda^2h^{-1}+\lambda e^{-C/h})$ for all $\lambda\in D(0,\epsilon_0 h)$ and h small enough. Set

$$\Gamma_h=\mathrm{diag}(\gamma_{1,h},\gamma_{2,h}).$$

Then, according to Lemma 3.1 and without extra assumption on $\alpha(\mathbf{m}_1)$ and $\alpha(\mathbf{m}_2)$[1],

$$\Gamma_h^{-1}=O(h^{-1/2}). \tag{3.8}$$

Therefore, one deduces that

$$-E_{-+}(\lambda)=\Gamma_h\big(F_h(\lambda)+\mathsf{R}_h(\lambda)\big)\text{ with }F_h(\lambda)=\begin{pmatrix}-\lambda&0\\0&\mu_{2,h}^{\Delta}/\gamma_{2,h}-\lambda\end{pmatrix}, \tag{3.9}$$

where $\mathsf{R}_h(\lambda)$ is holomorphic with respect to $\lambda\in D(0,\epsilon_0 h)$ and $\mathsf{R}_h(\lambda)=O(\lambda^2h^{-\frac{3}{2}}+\lambda h^{-\frac{1}{2}}e^{-C/h})$ for all $\lambda\in D(0,\epsilon_0 h)$ and h small enough. Set

$$\eta_{2,h}:=\mu_{2,h}^{\Delta}/\gamma_{2,h}.$$

According to Lemma 3.1, to (3.1), and to the relation $\alpha(\mathbf{m}_2)>0$ implied by (1.4) (see indeed (1.5)), one has in the limit $h\to 0$:

$$\eta_{2,h}=\zeta_h(\mathbf{m}_2)\,h\,e^{-\frac{2}{h}(V(\mathbf{s}_2)-V(\mathbf{m}_2))}, \tag{3.10}$$

where $\zeta_h(\mathbf{m}_2)\sim\sum_{k\geq 0}h^{\frac{k}{2}}\zeta_k(\mathbf{m}_2)$ and $\zeta_0(\mathbf{m}_2)=\frac{1}{\pi}\sqrt{|V''(\mathbf{m}_2)V''(\mathbf{s}_2)|}/\alpha(\mathbf{m}_2)$. Let $K\geq 2$ be fixed in what follows. Set

$$\mathcal{D}_K=\big\{\lambda\in\mathbb{C},\ |\lambda-\eta_{2,h}|<h^Ke^{-\frac{2}{h}(V(\mathbf{s}_2)-V(\mathbf{m}_2))}\big\},$$

whose closure is by (3.10) included in $D(0,\epsilon_0 h)$ for h small enough. Hence, for any $\lambda\in\overline{\mathcal{D}_K}$, since $K\geq 2$, one has, for all h small enough,

$$C^{-1}he^{-\frac{2}{h}(V(\mathbf{s}_2)-V(\mathbf{m}_2))}\leq|\lambda|\leq Che^{-\frac{2}{h}(V(\mathbf{s}_2)-V(\mathbf{m}_2))}. \tag{3.11}$$

Consequently, for any $\lambda\in\partial\mathcal{D}_K$, the matrix $F_h(\lambda)$ is invertible and

$$F_h(\lambda)^{-1}=O\big(h^{-K}e^{\frac{2}{h}(V(\mathbf{s}_2)-V(\mathbf{m}_2))}\big)\ \text{ on }\partial\mathcal{D}_K. \tag{3.12}$$

On the other hand, it follows from the estimate on R_h below (3.9) that for any $\lambda\in\partial\mathcal{D}_K$,

$$\mathsf{R}_h(\lambda)=O\big(\sqrt{h}\,e^{-4(V(\mathbf{s}_2)-V(\mathbf{m}_2))/h}+\sqrt{h}\,e^{-(2(V(\mathbf{s}_2)-V(\mathbf{m}_2))+C)/h}\big)=O\big(e^{-(2(V(\mathbf{s}_2)-V(\mathbf{m}_2))+C)/h}\big).$$

Hence, by (3.9), $E_{-+}(\lambda)$ is invertible on $\partial\mathcal{D}_K$ for all h small enough and

$$-E_{-+}^{-1}(\lambda)=\big(1+O(e^{-\frac{C}{2h}})\big)F_h(\lambda)^{-1}\Gamma_h^{-1}\text{ on }\partial\mathcal{D}_K,$$

[1]Note that when $\alpha(\mathbf{m}_1)\alpha(\mathbf{m}_2)\neq 0$, it actually holds $\Gamma_h^{-1}=O(1)$.

where we have used (3.12) and the invertibility of Γ_h. Using in addition $\|F_h(\lambda)^{-1}\|\,|\partial\mathcal{D}_K| = O(1)$ and (3.8), this implies that for all h small enough:

$$\begin{aligned}\frac{1}{2i\pi}\int_{\partial\mathcal{D}_K} E_{-+}^{-1}(\lambda)d\lambda &= -\Big[\frac{1}{2i\pi}\int_{\partial\mathcal{D}_K}\begin{pmatrix}-\lambda & 0\\ 0 & \eta_{2,h}-\lambda\end{pmatrix}^{-1}d\lambda\Big]\Gamma_h^{-1} + \|\Gamma_h^{-1}\|O(e^{-\frac{C}{2h}})\\ &= \begin{pmatrix}0 & 0\\ 0 & \gamma_{2,h}^{-1}\end{pmatrix} + O(e^{-\frac{C}{4h}}) \text{ is non trivial.}\end{aligned} \tag{3.13}$$

Hence, for all $K \geq 2$, $\lambda \mapsto E_{-+}^{-1}(\lambda)$ admits at least a singularity $\alpha_{h,K}$ in the disk $\mathcal{D}_K$. In particular, one has for all h small enough,

$$\alpha_{h,K} = \eta_{2,h} + O(h^K e^{-\frac{2}{h}(V(\mathbf{s}_2)-V(\mathbf{m}_2))}), \text{ where } \eta_{2,h} \text{ satisfies (3.10).}$$

But, since (1.4) holds, Theorem 1 implies that for all h small enough, $\sigma(\mathsf{P}_h)\cap\{\mathrm{Re}(z) \leq \epsilon_0 h^2\}$ is composed of two elements 0 and $\lambda_{2,h}$. According to (2.25), the nonzero eigenvalue $\lambda_{2,h}$ then necessarily satisfies $\lambda_{2,h} = \alpha_{h,K}$ for all $K \geq 2$. This ends the proof of Theorem 2.

4. The multiple well case

In this section, we prove Theorem 3. We assume that U is a Morse function and that (1.10) is satisfied. Recall that the local minima $\mathbf{m}_1, \dots, \mathbf{m}_{n_0}$ of V are labeled such that $(S(\mathbf{m}_j))_{j\in\{1,\dots,n_0\}}$ is decreasing (see (1.12)). We then label $\{\Psi_1, \dots, \Psi_{n_0}\}$, the basis of E_0, accordingly.

4.1. **An adapted basis of quasimodes.** In order to compute accurately the matrices $\mathsf{M}_{V,h}$ and W_h, we will build the basis $(\Psi_j)_{j=1,\dots,n_0}$ of E_0 from a family of quasi-modes $(\varphi_j)_{j=1,\dots,n_0}$ constructed in [6]. First, as in the previous section, set, for $h > 0$,

$$\varphi_1 = \frac{1}{Z_{1,h}}e^{-(V-V(\mathbf{m}_1))/h} \text{ with } Z_{1,h} = \|e^{-(V-V(\mathbf{m}_1))/h}\|_{L^2(\mathbb{T})} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_1))^{1/4}),$$

and $\Psi_1 = \varphi_1$. For any $j = 2, \dots, n_0$, let φ_j be the $L^2(\mathbb{T})$-normalized quasi-mode associated with $\mathbf{m}_j$ given in [6, Definition 4.3] applied to the case of the Witten Laplacian $\Delta_{V,h}$. Since φ_i is by definition supported in a neighborhood of $\mathsf{C}(\mathbf{m}_i)$ (see [6, Section 4]), it follows from the second item of Assumption (1.10) that, for all $i \in \{1, \dots, n_0\}$ and $r > 0$ small enough, one has for all h small enough:

$$\varphi_i = \frac{1_{D(\mathbf{m}_i,r)}}{Z_{i,h}}e^{-(V-V(\mathbf{m}_i))/h} + O(e^{-c/h}) \text{ in } L^2(\mathbb{T}), \tag{4.1}$$

where $Z_{i,h} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_i))^{1/4})$. For $j \in \{1, \dots, n_0\}$, we set

$$\kappa_{j,h} := \langle \Delta_{V,h}\varphi_j, \varphi_j\rangle.$$

We have obviously $\kappa_{1,h} = 0$ and, from [6, Proposition 5.1], we have for all $j \in \{2, \dots, n_0\}$,

$$\kappa_{j,h} \in \mathcal{E}_{cl}\Big(he^{-2S(\mathbf{m}_j)/h}\frac{|V''(\mathbf{m}_j)V''(\mathbf{j}(\mathbf{m}_j))|^{\frac{1}{2}}}{\pi}\Big). \tag{4.2}$$

The basis $(\Psi_j)_{j=1,\dots,n_0}$ of $\mathsf{E}_0 = \mathrm{Ran}\,\Pi$ is then constructed from the family $(\varphi_j)_{j=1,\dots,n_0}$ by the following procedure. Set, for $j = 1, \dots, n_0$: $v_j = \Pi\varphi_j$. For h small enough, the family $(v_j)_{j=1,\dots,n_0}$ is then a basis of E_0 (since, according to [6, Proposition 5.3], $\langle v_j, v_i\rangle_{L^2(\mathbb{T})} =$

$\delta_{i,j}+O(e^{-c/h})$, for all $i,j=1,\dots,n_0$). We then consider the family $(\Psi_j)_{j=1,\dots,n_0}$ obtained from $(v_j)_{j=1,\dots,n_0}$ by a Gram-Schmidt procedure as in [6, page 30]. According to [6, Proposition 5.3 and Eq. (5.15)], this orthonormal basis of E_0 satisfies the following properties:

$$\text{for all } j,k=1,\dots,n_0,\quad \langle \Delta_{V,h}\Psi_j,\Psi_k\rangle = \delta_{j,k}\kappa_{j,h}+O\big(h^\infty\sqrt{\kappa_j(h)\kappa_k(h)}\big) \tag{4.3}$$

and

$$\text{for all } j=1,\dots,n_0,\quad \Psi_j=\varphi_j+O(e^{-c/h}) \text{ in } L^2(\mathbb{T}). \tag{4.4}$$

Working in the basis $(\Psi_j)_{j=1,\dots,n_0}$ of E_0, we obtain the following asymptotic equivalent of W_h (see (2.22)).

Lemma 4.1. *There exists $C>0$ such that the matrix W_h satisfies, for h small enough,*

$$\mathsf{W}_h=\Gamma_h+O(e^{-C/h}),$$

where $\Gamma_h=\mathrm{diag}(\gamma_{1,h},\dots,\gamma_{n_0,h})$ and, for $j=1,\dots,n_0$:

$$\gamma_{j,h}\sim\sum_{k\geq 0}h^k\gamma_{\alpha,k}(\mathbf{m}_j)+\sqrt{h}\sum_{k\geq 0}h^k\gamma_{V,k}(\mathbf{m}_j)$$

with $\gamma_{\alpha,0}(\mathbf{m}_j)=\alpha(\mathbf{m}_j)$, and $\gamma_{V,0}(\mathbf{m}_j)=2\sqrt{\frac{V''(\mathbf{m}_j)}{\pi}}$.

Proof. The asymptotic computation of $\langle W\Psi_i,\Psi_i\rangle$ as $h\to 0$ is exactly the same as in Lemma 3.1, using (4.1) and (4.4). The only point to be checked is that for any $i\neq j$, one has $\langle W\Psi_i,\Psi_j\rangle=O(e^{-C/h})$. Thanks to (4.4), this is equivalent to say that for all $i\neq j$, one has $\langle W\varphi_i,\varphi_j\rangle=O(e^{-C/h})$, which follows directly from (4.1). □

4.2. **Proof of Theorem 3.** Let Γ_h be defined by Lemma 4.1 and let $\mathsf{M}_{V,h}$ be given by (2.22). Notice that in the case $n_0\geq 3$, the matrix $\mathsf{M}_{V,h}$ has not to be diagonal and we vill use the following result.

Lemma 4.2. *Introduce the symmetric positive semi-definite matrix*

$$\mathsf{M}^\Gamma_{V,h}=\Gamma_h^{-\frac12}\mathsf{M}_{V,h}\Gamma_h^{-\frac12}.$$

Then, there exists $\epsilon_0>0$ such that, for all h small enough, the n_0 eigenvalues $\beta_{1,h}\leq\dots\leq\beta_{n_0,h}$ of $\mathsf{M}^\Gamma_{V,h}$ satisfy: $\beta_{1,h}=0$ and, for all $k=2,\dots,n_0$, $\beta_{k,h}\in\mathcal{E}_{cl}(\mu^\Delta_{k,h}/\gamma_{k,h})$, where $\mu^\Delta_{k,h}$ is given by (2.11).

Proof. First, observe that $\mathsf{M}_{V,h}$ admits 0 as a simple eigenvalue (since 0 is a simple eigenvalue of $\Delta_{V,h}$), so it is also a simple eigenvalue of $\mathsf{M}^\Gamma_{V,h}$. For $A=(a_{i,j})_{1\leq i,j\leq n_0}$ a matrix, we define $\tilde{A}:=(a_{i,j})_{2\leq i,j\leq n_0}$. We then have:

$$\tilde{\mathsf{M}}^\Gamma_{V,h}=(\tilde\Gamma_h)^{-\frac12}\tilde{\mathsf{M}}_{V,h}(\tilde\Gamma_h)^{-\frac12}.$$

Moreover, by (4.2) and (4.3), the matrix $\tilde{\mathsf{M}}_{V,h}$ writes

$$\tilde{\mathsf{M}}_{V,h}=\Omega_h\big(\mathsf{D}_{V,h}+O(h^\infty)\big)\Omega_h,$$

with $\Omega_h = \mathrm{diag}(e^{-S(\mathbf{m}_2)/h}, \dots, e^{-S(\mathbf{m}_{n_0})/h})$ and $\mathsf{D}_{V,h} = \mathrm{diag}(a_h(\mathbf{m}_2), \dots, a_h(\mathbf{m}_{n_0}))$. Since $\tilde{\Gamma}_h$ is diagonal, one has $\Omega_h \tilde{\Gamma}_h = \tilde{\Gamma}_h \Omega_h$ and it follows that $\tilde{\mathsf{M}}^{\Gamma}_{V,h}$ writes

$$\tilde{\mathsf{M}}^{\Gamma}_{V,h} = \Omega_h \big(\mathsf{D}^{\Gamma}_{V,h} + O(h^\infty) \big) \Omega_h$$

with $\mathsf{D}^{\Gamma}_{V,h} = \mathrm{diag}(\mu^{\Delta}_{j,h}/\gamma_{j,h}, j = 2, \dots, n_0)$. Hence $\tilde{\mathsf{M}}^{\Gamma}_{V,h}$ admits a *graded structure* in the sense of [20, Definition A.1]. We can thus apply [20, Theorem A.4] which yields the result. □

We are now in position to prove Theorem 3. Let U be a Morse function and assume that (1.10) is satisfied. Recall that we look for the singularities of $E^{-1}_{-+}(\lambda)$ and that $\lambda = 0$ is always a singularity of $E^{-1}_{-+}(\lambda)$ (since $\dim \mathrm{Ker}(\mathsf{P}_h) = 1$). Let us now look for the remaining singularities of $E^{-1}_{-+}(\lambda)$ in $D(0, \epsilon_0 h)$. From Lemma 4.2, there exists a unitary change of basis B_h such that $\mathsf{M}^{\Gamma}_{V,h} = B_h^* \mathrm{diag}(\beta_{1,h}, \dots, \beta_{n_0,h}) B_h$ (since $\mathsf{M}^{\Gamma}_{V,h}$ is symmetric and thus diagonalizable in an orthonormal basis). Combined with (2.23), (2.24), and Lemma 4.1, this yields

$$-E_{-+}(\lambda) = \Gamma_h^{1/2} B_h^* \Big[\mathrm{diag}(\beta_{1,h}, \dots, \beta_{n_0,h}) - \lambda \mathsf{I}_{\mathbb{C}^{n_0}} - \mathsf{R}_h(\lambda) \Big] B_h \Gamma_h^{1/2}, \tag{4.5}$$

where, for h small enough, $\mathsf{R}_h(\lambda)$ is holomorphic with respect to $\lambda \in D(0, \epsilon_0 h)$ and

$$\mathsf{R}_h(\lambda) = O(\lambda^2 h^{-\frac{3}{2}} + \lambda h^{-\frac{1}{2}} e^{-C/h}), \tag{4.6}$$

and where we have used that $\Gamma_h^{-1} = O(h^{-1/2})$ by Lemma 4.1. Hence, the singularities of $E^{-1}_{-+}(\lambda)$ in $D(0, \epsilon_0 h)$ are exactly those of L_h^{-1}, where

$$L_h(\lambda) = F_h(\lambda) - \mathsf{R}_h(\lambda), \text{ with } F_h(\lambda) = \mathrm{diag}(\beta_{1,h} - \lambda, \dots, \beta_{n_0,h} - \lambda). \tag{4.7}$$

Recall that $\beta_{1,h} = 0$ and $\beta_{j,h} \in \mathcal{E}_{cl}(\mu^{\Delta}_{j,h}/\gamma_{j,h})$ for all $j \geq 2$. For all $j = 2, \dots, n_0$, using the asymptotic equivalents of $\mu^{\Delta}_{j,h}$ and $\gamma_{j,h}$ given in Theorem 5 and in Lemma 4.1, one has when $h \to 0$, using (1.4) which implies $\alpha(\mathbf{m}_j) > 0$:

$$\beta_{j,h} = \zeta_h(\mathbf{m}_j)\, h\, e^{-\frac{2}{h} S(\mathbf{m}_j)}, \tag{4.8}$$

where $\zeta_h(\mathbf{m}_j) \sim \sum_{k \geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_j)$ and $\zeta_0(\mathbf{m}_j)$ is given by (1.14). Let us now consider $j \in \{2, \dots, n_0\}$ and $K \geq 2$. Denote $\mathcal{D}_{j,K} = D(\beta_{j,h}, h^K e^{-\frac{2}{h} S(\mathbf{m}_j)})$ and let $\lambda \in \partial \mathcal{D}_{j,K}$. Since $S(\mathbf{m}_\ell) < S(\mathbf{m}_k)$ when $\ell \geq k$, for $h > 0$ small enough, the $\mathcal{D}_{j,K}$ are pairwise disjoint, their closures are included in $D(0, \epsilon_0 h)$, and for all $i \in \{2, \dots, n_0\}$:

$$\begin{aligned} &\forall i > j,\ |\lambda - \beta_{i,h}| \geq |\beta_{j,h} - \beta_{i,h}| - |\lambda - \beta_{j,h}| \geq Che^{-\frac{2}{h} S(\mathbf{m}_i)} \geq Che^{-\frac{2}{h} S(\mathbf{m}_j)} \\ &\forall i < j,\ |\lambda - \beta_{i,h}| \geq |\beta_{j,h} - \beta_{i,h}| - |\lambda - \beta_{j,h}| \geq Che^{-\frac{2}{h} S(\mathbf{m}_j)}. \end{aligned} \tag{4.9}$$

Moreover, for h small enough,

$$|\lambda| \geq Che^{-\frac{2}{h} S(\mathbf{m}_j)}. \tag{4.10}$$

Consequently, for $\lambda \in \partial \mathcal{D}_{j,K}$, the matrix $F_h(\lambda)$ is invertible and

$$F_h(\lambda)^{-1} = O(h^{-K} e^{\frac{2}{h} S(\mathbf{m}_j)}) \text{ on } \mathcal{D}_{j,K}.$$

Combining this estimate with (4.6) and reasoning as around (3.13), we prove that for all $j = 2, \dots, n_0$, $K \geq 2$, and h small enough,

$$L_h^{-1}(\lambda) \text{ (and thus } E_{-+}^{-1}(\lambda)) \text{ admits a singularity } \alpha_{j,h,K} \text{ in } \mathcal{D}_{j,K}, \tag{4.11}$$

so in particular $\alpha_{j,h,K} \neq \alpha_{i,h,K}$ when $i \neq j \in \{2, \dots, n_0\}$ (the $\mathcal{D}_{j,K}$ being pairwise disjoint).

In addition, since (1.4) holds, Theorem 1 implies that for all h small enough, $\sigma(\mathsf{P}_h) \cap \{\mathrm{Re}(z) \leq \epsilon_0 h^2\} \setminus \{0\}$ is made of $n_0 - 1$ real eigenvalues $0 < \lambda_{2,h} \leq \dots \leq \lambda_{n_0,h}$ (counted with algebraic multiplicity). It then follows from (2.25) that, for each $j = 2, \dots, n_0$, the eigenvalue $\lambda_{j,h}$ satisfies $\lambda_{j,h} = \alpha_{j,h,K}$ for all $K \geq 2$. Since $\alpha_{j,h,K} = \beta_{j,h} + O(h^K e^{-\frac{2}{h}S(\mathbf{m}_j)})$ for all $K \geq 2$ and $\beta_{j,h} \in \mathcal{E}_{cl}(\mu_{j,h}^{\Delta}/\gamma_{j,h})$, this completes the proof of Theorem 3, using the asymptotic equivalent of $\mu_{j,h}^{\Delta}/\gamma_{j,h}$ as $h \to 0$ (see Theorem 5 and Lemma 4.1).

4.3. **The case when (1.4) is not satisfied.** In this section, we prove Theorem 4, where, compared to Theorem 3, we do no longer assume that (1.4) holds but that (1.15) holds. We can thus no longer use Theorem 1 as we did at the very end of the proof of Theorem 3 to say that P_h admits $n_0 - 1$ nonzero eigenvalues (counted with multiplicity) in $\{\mathrm{Re}(z) \leq \epsilon_0 h^2\}$. We can however make use of all the intermediate results in the proof of Theorem 3 until (4.11) (included), except that one has the following minor scaling changes:

(1) Equation (4.8) must be changed into

$$\beta_{j,h} = \zeta_h(\mathbf{m}_j)\sqrt{h}\, e^{-\frac{2}{h}S(\mathbf{m}_j)}, \tag{4.12}$$

where $\zeta_h(\mathbf{m}_j) \sim \sum_{k \geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_j)$ with $\zeta_0(\mathbf{m}_j)$ satisfying (1.17).

(2) In Equations (4.9) and (4.10), all the Ch must be replaced by $C\sqrt{h}$.

Let $j \in \{2, \dots, n_0\}$ and note that, for h small enough, the sequence $(\alpha_{j,h,K})_{K \geq 2}$ defined in the proof of Theorem 3 is stationary by analyticity of the nontrivial map $\lambda \mapsto E_{-+}(\lambda)$. We denote by $\alpha_{j,h}$ its limit and recall that, according to (2.25), $\{0, \alpha_{2,h}, \dots, \alpha_{n_0,h}\} \subset \sigma(\mathsf{P}_h)$. In addition, the $\alpha_{j,h}$'s are exponentially small and thus belong to $D(0, ch^2)$ for any $c > 0$ and all h small enough. Let us now prove that, for h small enough, 0 and $\alpha_{j,h}$, $j = 2, \dots, n_0$, are the only singularities of $\lambda \mapsto E_{-+}^{-1}(\lambda)$ in $D(0, ch^2)$, that they are all real, and have geometric multiplicity 1 as eigenvalues of P_h. Recall that by (4.5), $\lambda \in D(0, \epsilon_0 h)$ is a singularity of $E_{-+}^{-1}(\lambda)$ if and only if it is a singularity of $L_h(\lambda)^{-1}$ (see (4.7)). Let us denote, for $\lambda \in \mathbb{C}$,

$$d_h(\lambda) = \det L_h(\lambda) = \det(F_h(\lambda) - \mathsf{R}_h(\lambda)),$$

which is holomorphic on $D(0, \epsilon_0 h)$. For all $\lambda \in \partial D(0, ch^2)$, using (4.6), we get $\mathsf{R}_h(\lambda) = O(h^4 h^{-3/2}) = O(h^{5/2})$, and by (4.12),

$$\left|\det F_h(\lambda)\right| = \left|\lambda \times \Pi_{j=2}^{n_0}(\lambda - \beta_{j,h})\right| = c^{n_0} h^{2n_0}(1 + o_h(1)).$$

Thus, for h small enough, it holds uniformly on $\partial D(0, ch^2)$:

$$d_h(\lambda) = \lambda \times \Pi_{j=2}^{n_0}(\lambda - \beta_{j,h}) + O(h^{5/2} h^{2(n_0-1)}) = \det F_h(\lambda)(1 + o_h(1)). \tag{4.13}$$

In particular, for any $\lambda \in \partial D(0, ch^2)$, one has

$$|d_h(\lambda) - \det F_h(\lambda)| < |\det F_h(\lambda)|,$$

which implies by Rouché's theorem that $d_h(\lambda)$ and $\det F_h(\lambda)$ have the same number n_0 of zeros (counted with multiplicity) in $D(0, ch^2)$. These n_0 zeros are thus 0 and $\alpha_{j,h}$, $j = 2, \dots, n_0$. Let us recall that the $\alpha_{j,h}$ are pairwise disctinct and satisfy $\alpha_{j,h} = \beta_{j,h} + O(h^K e^{-\frac{2}{h}S(\mathbf{m}_j)})$ for all $K \geq 2$ and $j = 2, \dots, n_0$. They are then all simple zeros of $d_h(\lambda)$ (and thus $\dim \operatorname{Ker} E_{-+}(\alpha_{j,h}) = 1$ for $j = 2, \dots, n_0$). Using also (2.10) and [25, Equation (2.7)], $\dim \operatorname{Ker}(\mathsf{P}_h - \alpha_{j,h}) = \dim \operatorname{Ker} \mathsf{T}_h(\alpha_{j,h}) = \dim \operatorname{Ker} E_{-+}(\alpha_{j,h}) = 1$, for all $j = \{2, \dots, n_0\}$. Lastly, since the operator P_h has real coefficients, its spectrum is stable by complex conjugation. Hence, since moreover 0 and the $\alpha_{j,h}$, $j = 2, \dots, n_0$, have different asymptotic equivalents as $h \to 0$, one has $\overline{\alpha_{j,h}} = \alpha_{j,h}$ for every $j = 2, \dots, n_0$. To conclude the proof of Theorem 4, it then just remains to use that, for every $j = 2, \dots, n_0$, $\beta_{j,h} \in \mathcal{E}_{cl}(\mu^{\Delta}_{j,h}/\gamma_{j,h})$, and the asymptotic equivalents of $\mu^{\Delta}_{j,h}/\gamma_{j,h}$ as $h \to 0$ given by Theorem 5 and Lemma 4.1.

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