



HAL
open science

Eyring-Kramers type formulas for some piecewise deterministic Markov processes

Dorian Le Peutrec, Laurent Michel, Boris Nectoux

► **To cite this version:**

Dorian Le Peutrec, Laurent Michel, Boris Nectoux. Eyring-Kramers type formulas for some piecewise deterministic Markov processes. 2022. hal-03553895v1

HAL Id: hal-03553895

<https://hal.science/hal-03553895v1>

Preprint submitted on 3 Feb 2022 (v1), last revised 28 Nov 2023 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

EYRING-KRAMERS TYPE FORMULAS FOR SOME PIECEWISE DETERMINISTIC MARKOV PROCESSES

DORIAN LE PEUTREC, LAURENT MICHEL, AND BORIS NECTOUX

ABSTRACT. In this work, we give sharp asymptotic equivalents in the small temperature regime of the smallest eigenvalues of the generator of some piecewise deterministic Markov processes (including the Zig-Zag process and the Bouncy Particle Sampler process) with refreshment rate α on the one-dimensional torus \mathbb{T} . These asymptotic equivalents are usually called Eyring-Kramers type formulas in the literature. The case when the refreshment rate α vanishes on \mathbb{T} is also considered.

Keywords. Piecewise Deterministic Markov Processes, small temperature regime, Eyring-Kramers formulas, metastability, spectral theory, semiclassical analysis.

AMS classification. 35P15, 47F05, 35P20, 35Q82, 35Q92.

1. INTRODUCTION

1.1. Purpose and motivation. Piecewise Deterministic Markov Processes [9] (PDMP hereafter) have recently attracted a lot of attention for their use within the Markov Chain Monte Carlo methodology. Given a potential $U : \mathbf{M} \rightarrow \mathbb{R}$ on a d -dimensional manifold \mathbf{M} without boundary, such processes are indeed ergodic with respect to the Gibbs measure

$$\pi(dx) = \frac{e^{-\frac{2}{h}U(x)}}{\int_{\mathbf{M}} e^{-\frac{2}{h}U} dx},$$

dx being the Lebesgue measure on the position space \mathbf{M} . Here, the parameter $h > 0$ is proportional to the Boltzmann constant k_B through the relation $h = k_B T$, T being the temperature of the underlying system. When $h > 0$ is fixed, the ergodic properties and the rate of convergence of such processes with or without a refreshment rate α , have for instance been studied in [4, 10, 14, 12, 1, 2, 22, 23] (see also references therein). We also refer to [5] for a spectral analysis of the generator of the one-dimensional zigzag process and its corresponding semigroup when h is fixed.

In many applications in statistical physics where one needs to sample from π , the constant h is very small compared to the energetic barriers of U . Recently, the long time behavior of the semigroups generated by the generators \mathbf{L}_h of these processes on L^2 have been investigated in [16] when $h \ll 1$, where it has also been proved that in the set $\{\operatorname{Re}(z) \geq -\epsilon_0 h\}$, \mathbf{L}_h has exactly n_0 eigenvalues (n_0 being the number of local minima of U), which are non positive, real, and exponentially small as $h \rightarrow 0$. Such results exhibit a metastable behavior of the PDMP when $h \ll 1$, as it is the case for diffusion processes [11]. In this work, we want to push the analysis of the metastability of the PDMP further by proving that each of these n_0 eigenvalues satisfies a so-called Eyring-Kramers type formula when

$h \rightarrow 0$. Such sharp formulas describe completely the successive timescales involved in the convergence of the semigroup generated by $-\mathbf{L}_h$ to π , as it has been done in [8] (see also [17]) for elliptic reversible diffusions.

Let us be more precise on our results. We work in the following setting: $d = 1$ (i.e. the dimension of the position space is equal to 1) and $\mathbf{M} = \mathbb{T}$, the one-dimensional torus. In addition, we work with the operator \mathbf{P}_h defined below in (1.1) which is (up to a multiplication by $-h$) unitary equivalent to \mathbf{L}_h , see (1.2). Thus, all our results are easily translated in terms of \mathbf{L}_h . The purpose of this work is to compute sharp asymptotic equivalents of the n_0 smallest eigenvalues of \mathbf{P}_h (or equivalently, those of \mathbf{L}_h) in the limit $h \rightarrow 0$, see Theorems 2 and 3. In particular these asymptotic equivalents allow us to provide a sharp exponential decay rate of the semigroup associated to $-\mathbf{P}_h$ in L^2 , as $h \ll 1$, see Corollary 1.4. These results hold when (1.4) is satisfied, which implies that the refreshment rate α does not vanish at critical points of U (i.e. when $\partial_x U = 0$), see (1.5). The case $\alpha = 0$ when $\partial_x U = 0$ is investigated in Section 1.4, where we show that the smallest eigenvalues of \mathbf{P}_h satisfy different asymptotic equivalents as $h \rightarrow 0$.

To compute sharp asymptotic equivalents of the n_0 smallest eigenvalues of \mathbf{P}_h we proceed as follows. We first introduce a suitable change of variables to turn the eigenvalue problem $(\mathbf{P}_h - \lambda)u = 0$ into a nonlinear eigenvalue problem: $(\Delta_{V,h} - \lambda W + \lambda^2)g = 0$, where $V = -U$, $\Delta_{V,h}$ is the Witten Laplacian associated with V , and $W = 2|\partial_x U| + \alpha$ (see (2.6) and Lemma 2.1). Using a Grushin problem and known results on the low-lying spectrum of $\Delta_{V,h}$, we prove that if λ is small enough, the kernel of $\Delta_{V,h} + \lambda W + \lambda^2$ is composed of the singularities of a holomorphic function $\lambda \mapsto E_{-+}(\lambda) \in \mathbb{C}^{n_0 \times n_0}$ (see (2.21)). We finally investigate the localization of the singularities of $\lambda \mapsto E_{-+}^{-1}(\lambda)$ to deduce sharp asymptotic equivalents of the n_0 smallest eigenvalues of \mathbf{P}_h .

1.2. Setting. Let $\mathbf{E} = \{(x, v) \in \mathbb{T} \times \{\pm 1\}\}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus. Consider the following unbounded operator \mathbf{P}_h on $L^2(\mathbf{E})$ associated with a smooth function $U : \mathbb{T} \rightarrow \mathbb{R}$ defined by

$$(1.1) \quad \mathbf{P}_h = -v \mathbf{d}_{U,h} + 2(v \partial_x U)_+ (\mathbf{I} - \mathbf{B}) + \alpha (\mathbf{I} - \pi_v)$$

where $\mathbf{d}_{U,h} = h \partial_x + \partial_x U$, $\alpha : \mathbb{T} \rightarrow \mathbb{R}$ is a \mathcal{C}^∞ non-negative function (the refreshment rate), $h > 0$ is a parameter proportional to the temperature of the underlying statistical system, \mathbf{I} is the identity operator, and \mathbf{B} and π_v are defined by

$$\forall (x, v) \in \mathbf{E}, \mathbf{B}f(x, v) = f(x, -v)$$

and

$$\pi_v f(x, v) = \frac{1}{2}(f(x, +1) + f(x, -1))$$

for all $f \in L^2(\mathbf{E})$. Here and in the following, for $u \in L^2(\mathbf{E})$, we denote by $u_\pm = \max(0, \pm u)$. The operator \mathbf{P}_h is linked to the Zig-Zag (or the Bouncy Particle Sampler generator) process generator \mathbf{L}_h where $\mathbf{L}_h = v \partial_x - (\frac{2}{h} v \partial_x U)_+ (\mathbf{I} - \mathbf{B}) - \frac{1}{h} \alpha (\mathbf{I} - \pi_v)$ through the relation

$$(1.2) \quad \mathbf{P}_h = -h e^{-\frac{1}{h}U} \mathbf{L}_h e^{\frac{1}{h}U}.$$

We refer to [3, 24, 7] and references therein for more details on these two processes (see also [15] and [13, Section 3.1]). The space $E = \mathbb{T} \times \{\pm 1\}$ is endowed with the natural scalar product

$$(1.3) \quad \langle f, g \rangle_{L^2(\mathbb{E})} = \frac{1}{2} \sum_{v=\pm 1} \int_{\mathbb{T}} f(x, v) \overline{g(x, v)} dx.$$

Throughout this work, we will assume that

$$U : \mathbb{T} \rightarrow \mathbb{R} \text{ is a smooth Morse function.}$$

By definition, this means that $\partial_x^2 U(x) \neq 0$ when $\partial_x U(x) = 0$, $x \in \mathbb{T}$. In particular, U has a finite number of critical points on \mathbb{T} . It is proved in [16] that \mathbf{P}_h with domain $D(\mathbf{P}_h) = \{f \in L^2(\mathbb{E}), v \partial_x f \in L^2(\mathbb{E})\}$ is maximal accretive. In [16], the authors prove the following spectral result on \mathbf{P}_h in the limit $h \rightarrow 0$.

Theorem 1. ([16, Theorem 1 and Proposition 13]) *Assume that U is a Morse function with n_0 local minimum points. Assume also that*

$$(1.4) \quad \min_{\mathbb{T}} (2|\partial_x U| + \alpha) > 0.$$

Then, there exist $\epsilon_0 > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$, $\sigma(\mathbf{P}_h) \cap \{\operatorname{Re}(z) \leq \epsilon_0 h^2\}$ is made of n_0 real nonnegative eigenvalues $\lambda_{1,h} \leq \dots \leq \lambda_{n_0,h}$ (counted with algebraic multiplicity). Moreover, their algebraic multiplicities equal their geometric multiplicities and there exist $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0]$, $\lambda_{n_0,h} \leq e^{-C/h}$. Finally, $\lambda_{1,h} = 0$ and has algebraic multiplicity 1.

Let us mention that $\dim \operatorname{Ker}(\mathbf{P}_h) = 1$ holds for all $h > 0$ (see [16]). In [16], it is assumed that $\min_{\mathbb{T}} \alpha > 0$, but all the results of [16] still hold if the less stringent assumption (1.4) is satisfied. Indeed, if (1.4) holds and $u \in D(\mathbf{P}_h)$, $\operatorname{Re} \langle \mathbf{P}_h u, u \rangle_{L^2(\mathbb{E})} \geq r_m \|(1 - \pi_v)u\|^2$, where $r_m = \min_{\mathbb{E}} 2|\partial_x U| + \alpha$. This follows from the following computations. By [16, Lemma 5], since $(1 - \mathbf{B}) = 2(1 - \pi_v)$, for all $u \in D(\mathbf{P}_h)$, one has, denoting by $w = (1 - \pi_v)u$ (notice that $|w(\cdot, 1)| = |w(\cdot, -1)|$):

$$\begin{aligned} \operatorname{Re} \langle \mathbf{P}_h u, u \rangle_{L^2(\mathbb{E})} &= \frac{1}{2} \int_{\mathbb{E}} (2v \partial_x U)_+ |(1 - \mathbf{B})u|^2 + \int_{\mathbb{E}} \alpha |(1 - \pi_v)u|^2 \\ &= \int_{\mathbb{E}} 4(v \partial_x U)_+ |w|^2 + \int_{\mathbb{E}} \alpha |w|^2 \\ &= \frac{1}{2} \int_{\mathbb{T}} 4(\partial_x U)_+ |w|^2 + \frac{1}{2} \int_{\mathbb{T}} 4(\partial_x U)_- |w|^2 + \int_{\mathbb{E}} \alpha |w|^2 = \int_{\mathbb{E}} (2|\partial_x U| + \alpha) |w|^2. \end{aligned}$$

Notice that (1.4) implies that α can vanish on \mathbb{T} but not everywhere since (1.4) is equivalent to:

$$(1.5) \quad \text{for any } x \in \mathbb{T}, \quad \partial_x U(x) = 0 \Rightarrow \alpha(x) > 0.$$

The case when there exists $x \in \mathbb{T}$ such that $\partial_x U(x) = \alpha(x) = 0$ is be considered in Section 1.4.

For our analysis, it will be convenient to introduce the function

$$V = -U,$$

which is also a Morse function on \mathbb{T} . We shall denote by $\mathbf{U}^{(0)}$ the set of local minima of V and by $\mathbf{U}^{(1)}$ the set of its local maxima. Since we are on the torus, the set $\mathbf{U}^{(0)}$ and $\mathbf{U}^{(1)}$ have the same cardinality n_0 .

Throughout the paper, we will say that a family of complex numbers $(a_h)_{h>0}$ admits a classical expansion in power of h^β (where $\beta > 0$) if there exists a sequence $(a^k)_{k \geq 0}$ such that, for all $K \geq 0$, one has $a_h = \sum_{k=0}^K a^k h^{\beta k} + O(h^{\beta(K+1)})$ when $h \rightarrow 0$. In that case, we will denote $a_h \sim \sum_{k \geq 0} a^k h^{\beta k}$.

1.3. Main results. In this section we give sharp asymptotics of the exponentially small eigenvalues $\lambda_{j,h}$, $j = 1, \dots, n_0$. Observe that if $n_0 = 1$, there is only one small eigenvalue by Theorem 1 which is 0, and there is thus nothing to compute. We then consider the case when $n_0 \geq 2$. We start with the following theorem which gives the result in the simplified setting of a non-symmetric double well potential V (see Theorem 3 below for the general case $n_0 \geq 2$).

Theorem 2. *Let U be a Morse function. Assume that (1.4) holds and that $\mathbf{U}^{(0)}$ is made of two elements \mathbf{m}_1 and \mathbf{m}_2 such that $V(\mathbf{m}_1) < V(\mathbf{m}_2)$. Assume also that the two elements $\mathbf{s}_1, \mathbf{s}_2$ of $\mathbf{U}^{(1)}$ satisfy $V(\mathbf{s}_1) > V(\mathbf{s}_2)$. Let $\epsilon_0 > 0$ be as in Theorem 1. Then, the second smallest eigenvalue $\lambda_{2,h}$ of \mathbf{P}_h satisfies as $h \rightarrow 0$:*

$$\lambda_{2,h} = \zeta_h h e^{-\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))}, \text{ where } \zeta_h \sim \sum_{k \geq 0} h^{\frac{k}{2}} \zeta_k \text{ and } \zeta_0 = \frac{1}{2\pi} \frac{\sqrt{|V''(\mathbf{m}_2)V''(\mathbf{s}_2)|}}{\alpha(\mathbf{m}_2)}.$$

The situation where $V(\mathbf{m}_1) = V(\mathbf{m}_2)$ and/or $V(\mathbf{s}_1) = V(\mathbf{s}_2)$ could be handled easily by constructing adapted quasimodes in the spirit of [21]. Here, we decided to state our result in the above simplified setting in order to lighten the formulas.

Let us now state our result in the general setting $n_0 \geq 2$. To this end, we need to label the local minima and maxima of V in a suitable way. The following construction is inspired from [19] (see also [21] and [20]). In order to simplify, we assume from now that V uniquely attains its maximum at the point $\mathbf{s}_{max} \in \mathbf{U}^{(1)}$, i.e. that

$$(1.6) \quad \operatorname{argmax}_{\mathbb{T}} V = \{\mathbf{s}_{max}\}.$$

Then, set $\underline{\mathbf{U}}^{(1)} = \mathbf{U}^{(1)} \setminus \{\mathbf{s}_{max}\}$. Since $n_0 \geq 2$, $\underline{\mathbf{U}}^{(1)} \neq \emptyset$. We denote the elements of $V(\underline{\mathbf{U}}^{(1)})$ by $\sigma_2 > \sigma_3 > \dots > \sigma_N$, where $N \geq 2$. For convenience, we also introduce a fictive infinite saddle value $\sigma_1 = +\infty$ and we denote $\Sigma = \{\sigma_1, \dots, \sigma_N\}$. Starting from σ_1 , we will recursively associate to each σ_i a finite family of local minima $(\mathbf{m}_{i,j})_j$ and a finite family $(\mathbf{C}_{i,j})_j$ of connected components of $\{V < \sigma_i\}$ in the following way

- * Let $X_{\sigma_1} = \{x \in \mathbb{T}; V(x) < \sigma_1 = +\infty\} = \mathbb{T}$. We let $\mathbf{m}_{1,1}$ be any global minimum of V (not necessarily unique) and $\mathbf{C}_{1,1} = \mathbb{T}$. In the following, we will denote $\underline{\mathbf{m}} = \mathbf{m}_{1,1}$.
- * Next we consider $X_{\sigma_2} = \{x \in \mathbb{T}; V(x) < \sigma_2\}$. This is the union of its finitely many connected components. Exactly one of these components contains $\mathbf{m}_{1,1}$ and the other components are denoted by $\mathbf{C}_{2,1}, \dots, \mathbf{C}_{2,N_2}$. In each component $\mathbf{C}_{2,j}$, we pick up a point $\mathbf{m}_{2,j}$ which is a global minimum of $V|_{\mathbf{C}_{2,j}}$.
- * Suppose now that the families $(\mathbf{m}_{k,j})_j$ and $(\mathbf{C}_{k,j})_j$ have been constructed until rank $k = i - 1$. The set $X_{\sigma_i} = \{x \in \mathbb{T}; V(x) < \sigma_i\}$ has again finitely many connected

components and we label $C_{i,j}$, $j = 1, \dots, N_i$ those of these components which do not contain any $\mathbf{m}_{k,\ell}$ with $k < i$. In each $C_{i,j}$ we pick up a point $\mathbf{m}_{i,j}$ which is a global minimum of $V|_{C_{i,j}}$.

We run the procedure until all the minima have been labeled.

Remark 1.1. *Since we work on \mathbb{T} , using the terminology of [19], every maximum point $\mathbf{s} \in \underline{U}^{(1)}$ is a separating saddle points (ssp) and \mathbf{s}_{max} is not a ssp. In the case where V attains its global maximum at several distinct points $\mathbf{s}_{max,1}, \dots, \mathbf{s}_{max,k}$, every maximum point is separating and the situation can be handled easily by a modification of the construction above. However, this would lead to a slightly more complicated presentation of the results that we prefer to avoid in this work.*

We now recall some constructions of [21] and [20] that will be useful in the sequel. Throughout we denote $\underline{U}^{(0)} = U^{(0)} \setminus \{\underline{\mathbf{m}}\}$, \mathbf{s}_1 is a fictive saddle point such that $V(\mathbf{s}_1) = \sigma_1 = +\infty$. For any set A , $\mathcal{P}(A)$ denotes the power set of A . From the above labelling we define two mappings

$$C : U^{(0)} \rightarrow \mathcal{P}(\mathbb{R}^d) \quad \text{and} \quad \mathbf{j} : U^{(0)} \rightarrow \mathcal{P}(\underline{U}^{(1)} \cup \{\mathbf{s}_1\}),$$

as follows: for every $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, N_i\}$,

$$(1.7) \quad C(\mathbf{m}_{i,j}) := C_{i,j},$$

and

$$(1.8) \quad \mathbf{j}(\underline{\mathbf{m}}) := \{\mathbf{s}_1\} \quad \text{and} \quad \mathbf{j}(\mathbf{m}_{i,j}) := \partial C_{i,j} \cap \underline{U}^{(1)} \text{ for } i \geq 2.$$

In particular, we have $C(\underline{\mathbf{m}}) = \mathbb{T}$ and for all $i, j \in \{1, \dots, N\}$, one has $\emptyset \neq \mathbf{j}(\mathbf{m}_{i,j}) \subset \{V = \sigma_i\}$. We then define the mappings

$$\sigma : U^{(0)} \rightarrow \Sigma \quad \text{and} \quad S : U^{(0)} \rightarrow (0, +\infty],$$

by

$$(1.9) \quad \forall \mathbf{m} \in U^{(0)}, \quad \sigma(\mathbf{m}) := V(\mathbf{j}(\mathbf{m})) \quad \text{and} \quad S(\mathbf{m}) := \sigma(\mathbf{m}) - V(\mathbf{m}),$$

where, with a slight abuse of notation, we have identified the set $V(\mathbf{j}(\mathbf{m}))$ with its unique element. Note that $S(\mathbf{m}) = +\infty$ if and only if $\mathbf{m} = \underline{\mathbf{m}}$. With the above notations, our last assumption is the following:

- (1.10)
 - ★ Equation (1.6) is satisfied.
 - ★ For any $\mathbf{m} \in U^{(0)}$, \mathbf{m} is the unique global minimum of $V|_{C(\mathbf{m})}$.
 - ★ For all $\mathbf{m}' \in U^{(0)} \setminus \{\mathbf{m}\}$, $\mathbf{j}(\mathbf{m}) \cap \mathbf{j}(\mathbf{m}') = \emptyset$.
 - ★ The map $S : U^{(0)} \rightarrow (0, +\infty]$ is injective.

In particular, (1.10) implies that V uniquely attains its global minimum on \mathbb{T} at $\underline{\mathbf{m}} \in U^{(0)}$. In the following, when (1.10) holds, we label the local minima $\mathbf{m}_1, \dots, \mathbf{m}_{n_0}$ of V such that $(S(\mathbf{m}_j))_{j \in \{1, \dots, n_0\}}$ is decreasing (see (1.9)), that is

$$(1.11) \quad \text{for all } j \in \{1, \dots, n_0 - 1\}, \quad S(\mathbf{m}_{j+1}) < S(\mathbf{m}_j) \text{ and } S(\mathbf{m}_1) = +\infty \text{ (i.e. } \mathbf{m}_1 = \underline{\mathbf{m}}).$$

Remark 1.2. Notice that, in the geometrical setting of Theorem 2, one has by construction of S (see (1.9)) and \mathbf{j} (see (1.8)),

$$S(\mathbf{m}_2) = V(\mathbf{s}_2) - V(\mathbf{m}_2) \text{ and } \mathbf{j}(\mathbf{m}_2) = \{\mathbf{s}_2\}.$$

The main result of this work is the following.

Theorem 3. Let $V = -U$ be a Morse function. Assume that (1.4) and (1.10) are satisfied. Let $\epsilon_0 > 0$ be given by Theorem 1. Then, there exists $h_0 > 0$ such that, for all $h \in]0, h_0]$, the n_0 eigenvalues $\lambda_{1,h} \leq \dots \leq \lambda_{n_0,h}$ of \mathbf{P}_h in $\{\operatorname{Re}(z) \leq \epsilon_0 h^2\}$ (counted with algebraic multiplicity) satisfy: $\lambda_{1,h} = 0$ and, for all $j \in \{2, \dots, n_0\}$,

$$(1.12) \quad \lambda_{j,h} = \zeta_h(\mathbf{m}_j) h e^{-\frac{2}{h}S(\mathbf{m}_j)},$$

where $S : \mathbf{U}^{(0)} \rightarrow (0, +\infty]$ is defined in (1.9) (see also (1.11)) and ζ_h admits a classical expansion $\zeta_h(\mathbf{m}_j) \sim \sum_{k \geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_j)$ with

$$(1.13) \quad \zeta_0(\mathbf{m}_j) = \frac{1}{2\pi} \sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m}_j)} \frac{\sqrt{|V''(\mathbf{m}_j)V''(\mathbf{s})|}}{\alpha(\mathbf{m}_j)},$$

where $\mathbf{j} : \mathbf{U}^{(0)} \rightarrow \mathcal{P}(\mathbf{U}^{(1)} \cup \{\mathbf{s}_1\})$ is defined in (1.8).

Notice that, according to Theorem 3, $\lambda_{j,h}$ is a simple eigenvalue of \mathbf{P}_h for all $j \in \{1, \dots, n_0\}$ and h small enough (that is $\dim(\operatorname{Ker}(\mathbf{P}_h - \lambda_{j,h})^m) = 1$ for every $m \in \mathbb{N}^*$).

Remark 1.3. The assumptions (1.6) and ($S : \mathbf{U}^{(0)} \rightarrow (0, +\infty]$ is injective) in (1.10) are generic. They can be relaxed following the procedure of [20]. The whole assumption (1.10) could also be relaxed by following the strategy of [21].

Let us recall that by the Hille-Yosida Theorem, $-\mathbf{P}_h$ generates a strongly continuous contraction semigroup $(e^{-t\mathbf{P}_h})_{t \geq 0}$ on $L^2(\mathbf{E})$. Let us recall that under the assumptions of Theorem 3, according to [16, Theorem 2], we have, for some $c > 0$ and every $h > 0$ small enough:

$$e^{-t\mathbf{P}_h} = \sum_{j=1}^{n_0} e^{-t\lambda_{j,h}} \Pi_{j,h} + O(e^{-ct h^2}) \quad \text{in } \mathcal{L}(L^2(\mathbf{E})),$$

where, for $j = 1, \dots, n_0$, $\Pi_{j,h}$ is the spectral projector associated with the eigenvalue $\lambda_{j,h}$ of \mathbf{P}_h , and $\Pi_{j,h} = O(1)$ in $\mathcal{L}(L^2(\mathbf{E}))$. Using in addition Theorem 3, we get sharp asymptotic equivalents of the different timescales $1/\lambda_{j,h}$ involved in the return to equilibrium. This leads in particular to the following accurate exponential decay rate in $L^2(\mathbf{E})$ of the semigroup $(e^{-t\mathbf{P}_h})_{t \geq 0}$ as $h \ll 1$.

Corollary 1.4. Let U be a Morse function. Assume that (1.4) and (1.10) are satisfied. Then, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$, it holds for all $t \geq 0$:

$$\|e^{-t\mathbf{P}_h} - \Pi_{1,h}\|_{\mathcal{L}(L^2(\mathbf{E}))} \leq C e^{-t\lambda_{2,h}},$$

where, as $h \rightarrow 0$, $\lambda_{2,h} = \zeta_h(\mathbf{m}_2) h e^{-\frac{2S(\mathbf{m}_2)}{h}}$ with $\zeta_h(\mathbf{m}_2) \sim \sum_{k \geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_2)$ and $\zeta_0(\mathbf{m}_2)$ given by (1.13), and where $\Pi_{1,h}$ is the $L^2(\mathbf{E})$ orthogonal projection on $\operatorname{Span}(e^{-U/h} \mathbf{1}_{\{\pm 1\}})$, where $\mathbf{1}_{\{\pm 1\}}$ is the constant function on $\{\pm 1\}$ which equals 1.

1.4. Extension of the results to the case when (1.4) is not satisfied. In this section, we assume that

$$(1.14) \quad \text{for any } x \in \mathbb{T}, \quad \partial_x U(x) = 0 \Rightarrow \alpha(x) = 0,$$

and we give asymptotic equivalents of the n_0 first eigenvalues of \mathbf{P}_h when (1.14) holds (and then, in particular, when there is no refreshment at all, i.e. when $\alpha = 0$).

Theorem 4. *Let U be a Morse function. Assume that (1.14) and (1.10) are satisfied. Then, for any $c > 0$, there exists $h_0 > 0$ such that, for all $h \in (0, h_0]$, $\sigma(\mathbf{P}_h) \cap D(0, ch^2)$ is made of n_0 real eigenvalues $\lambda_{1,h} \leq \dots \leq \lambda_{n_0,h}$ (counted with geometric multiplicity), which all have geometric multiplicity one. In addition, $\lambda_{1,h} = 0$ and, for all $j \in \{2, \dots, n_0\}$,*

$$(1.15) \quad \lambda_{j,h} = \zeta_h(\mathbf{m}_j) \sqrt{h} e^{-\frac{2}{h} S(\mathbf{m}_j)},$$

where ζ_h admits a classical expansion $\zeta_h(\mathbf{m}_j) \sim \sum_{k \geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_j)$ with

$$(1.16) \quad \zeta_0(\mathbf{m}_j) = \frac{1}{4} \sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m}_j)} \sqrt{\frac{|V''(\mathbf{s})|}{\pi}}.$$

Compared with Theorem 3, we cannot exclude the existence of other eigenvalues in the strip $0 < \text{Re}(z) < ch^2$ with large imaginary part. The reason for this is that Theorem 1 does not apply when α and $\partial_x U$ vanish simultaneously.

The prefactor (1.16) is consistent with the one obtained in [23, Theorem 1.1] for the expected hitting time of a 1-dimensional PDMP with no refreshment. Indeed, when $\alpha = 0$ and V , and thus $U = -V$, is a non-symmetric double well potential (i.e. when $\mathbf{U}^{(0)} = \{\mathbf{m}_1, \mathbf{m}_2\}$ with $V(\mathbf{m}_1) < V(\mathbf{m}_2)$ and $\mathbf{U}^{(1)} = \{\mathbf{s}_1, \mathbf{s}_2\}$ with $V(\mathbf{s}_1) > V(\mathbf{s}_2)$), Theorem 4 and [23, Theorem 1.1] imply that $h \lambda_{2,h} \mathbb{E}[\tau] = 1 + o(1)$ as $h \rightarrow 0$, where τ is the first time the process (X_t, Y_t) with generator \mathbf{L}_h on $\mathbb{T} \times \{\pm 1\}$ hits $\{\mathbf{m}_2\}$ when it starts at $(\mathbf{s}_2, -1)$. Let us also recall here that $h \lambda_{2,h}$ is the first nonzero eigenvalue of $-\mathbf{L}_h$.

Of course, the situation where the refreshing function α vanishes at some critical points $x \in \mathbb{T}$ of U but not at all could easily be handled and would lead, for each eigenvalue $\lambda_{j,h}$, either to the formula given in Theorem 3 or to the formula given in Theorem 4, depending on whether $\alpha(\mathbf{m}_j) > 0$ or $\alpha(\mathbf{m}_j) = 0$.

2. REDUCTION TO A FINITE DIMENSIONAL PROBLEM

In this section, we prove that, if λ is small enough, we can reduce the infinite dimensional problem $(\mathbf{P}_h - \lambda_h)u = 0$ into a finite dimensional nonlinear eigenvalue problem.

2.1. A suitable change of variables. Let us introduce some notation. For $\lambda \in \mathbb{C}$ and $r > 0$ we denote $D(\lambda, r) = \{z \in \mathbb{C}, |z - \lambda| < r\}$. For two families of numbers $a = (a_h)_{h>0}$ and $b = (b_h)_{h>0}$, we say that $a \in \mathcal{E}_{cl}(b)$ if there exists a family $c = (c_h)_{h>0}$ such that $a_h = b_h c_h$ and c admits a classical expansion $c_h \sim \sum_{j \geq 0} c_j h^j$ with $c_0 = 1$ as $h \rightarrow 0$. In all this work, $C > 0$ and $c > 0$ are constant which are independent of h and which can change from one occurrence to another.

Recall that $\mathbf{E} = \mathbb{T} \times \{\pm 1\}$ is endowed with the natural scalar product (1.3). Let $\mathbf{F} = L^2(\mathbb{T}) \times L^2(\mathbb{T})$ and let $\langle \cdot, \cdot \rangle_{\mathbf{F}}$ be the Hilbertian structure induced by the isomorphism

$$(2.1) \quad \begin{aligned} \Omega_1 : L^2(\mathbb{T} \times \{\pm 1\}) &\rightarrow L^2(\mathbb{T}) \times L^2(\mathbb{T}) \\ f &\mapsto \begin{pmatrix} f(\cdot, +1) \\ f(\cdot, -1) \end{pmatrix}. \end{aligned}$$

Then, Ω_1 is unitary from $(\mathbf{E}, \langle \cdot, \cdot \rangle_{L^2(\mathbf{E})})$ onto $(\mathbf{F}, \langle \cdot, \cdot \rangle_{\mathbf{F}})$, and

$$\Omega_1 D(\mathbf{P}_h) = H^1(\mathbb{T}) \times H^1(\mathbb{T}).$$

Direct computations show that

$$\begin{aligned} \Omega_1 v \mathbf{d}_{U,h} \Omega_1^{-1} &= \begin{pmatrix} \mathbf{d}_{U,h} & 0 \\ 0 & -\mathbf{d}_{U,h} \end{pmatrix}, \quad \Omega_1 (v \partial_x U)_+ \Omega_1^{-1} = \begin{pmatrix} (\partial_x U)_+ & 0 \\ 0 & (\partial_x U)_- \end{pmatrix} \\ \Omega_1 \mathbf{B} \Omega_1^{-1} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \Omega_1 \pi_v \Omega_1^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

Combining these identities with (1.1), we get

$$(2.2) \quad \Omega_1 \mathbf{P}_h \Omega_1^{-1} = \begin{pmatrix} -\mathbf{d}_{U,h} & 0 \\ 0 & \mathbf{d}_{U,h} \end{pmatrix} + 2 \begin{pmatrix} (\partial_x U)_+ & -(\partial_x U)_+ \\ -(\partial_x U)_- & (\partial_x U)_- \end{pmatrix} + \frac{\alpha}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

We now change the variables in \mathbf{F} and consider the unitary transformation $\Omega_2 : \mathbf{F} \rightarrow \mathbf{F}$ defined by

$$\Omega_2(f, g) = \frac{1}{\sqrt{2}}(f + g, g - f).$$

Consider the two vectors of \mathbb{R}^2 given by $e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $e_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Then, one has:

$$(2.3) \quad \Omega_1 \mathbf{P}_h \Omega_1^{-1}(f e_1) = \mathbf{d}_{U,h} f e_2.$$

On the other hand,

$$\begin{aligned} 2 \begin{pmatrix} (\partial_x U)_+ & -(\partial_x U)_+ \\ -(\partial_x U)_- & (\partial_x U)_- \end{pmatrix} \begin{pmatrix} -f \\ f \end{pmatrix} &= 4f \begin{pmatrix} -(\partial_x U)_+ \\ (\partial_x U)_- \end{pmatrix} \\ &= 2((\partial_x U)_- - (\partial_x U)_+) \begin{pmatrix} f \\ f \end{pmatrix} + 2((\partial_x U)_- + (\partial_x U)_+) \begin{pmatrix} -f \\ f \end{pmatrix} \\ &= -2\partial_x U f e_1 + 2|\partial_x U| f e_2. \end{aligned}$$

It follows that:

$$(2.4) \quad \begin{aligned} \Omega_1 \mathbf{P}_h \Omega_1^{-1}(f e_2) &= \mathbf{d}_{U,h} f e_1 - 2\partial_x U f e_1 + (\alpha + 2|\partial_x U|) f e_2 \\ &= \mathbf{d}_{-U,h} f e_1 + (\alpha + 2|\partial_x U|) f e_2 \end{aligned}$$

Combining (2.3) and (2.4), we get

$$\Omega_2 \Omega_1 \mathbf{P}_h \Omega_1^{-1} \Omega_2^{-1} = \begin{pmatrix} 0 & \mathbf{d}_{-U,h} \\ \mathbf{d}_{U,h} & \alpha + 2|\partial_x U| \end{pmatrix}.$$

Set $\mathbf{Q}_h := \Omega_2 \Omega_1 \mathbf{P}_h \Omega_1^{-1} \Omega_2^{-1}$ with domain $D(\mathbf{Q}_h) = H^1(\mathbb{T}) \times H^1(\mathbb{T})$ on $L^2(\mathbb{T})^2$, i.e.

$$(2.5) \quad \mathbf{Q}_h = \begin{pmatrix} 0 & \mathbf{d}_{-U,h} \\ \mathbf{d}_{U,h} & W \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{d}_{V,h} \\ -\mathbf{d}_{V,h}^* & W \end{pmatrix}$$

where we recall that $V = -U$ and where we denote

$$W(x) := \alpha(x) + 2|\partial_x U(x)| = \alpha(x) + 2|\partial_x V(x)|, \quad x \in \mathbb{T}.$$

Since the transformation $\Omega_2\Omega_1$ is unitary from \mathbf{E} onto \mathbf{F} ,

$$\sigma(\mathbf{P}_h) = \sigma(\mathbf{Q}_h).$$

We will in the following study the spectrum of \mathbf{Q}_h to prove Theorems 2 and 3. Introduce the semiclassical Witten Laplacian $\Delta_{V,h}$ associated with V on \mathbb{T} , that is

$$\Delta_{V,h} = \mathbf{d}_{V,h}^* \mathbf{d}_{V,h} = -h^2 \Delta + |\partial_x V(x)|^2 - h \partial_x^2 V(x), \quad \text{with domain } D(\Delta_{V,h}) = H^2(\mathbb{T}).$$

For $\lambda \in \mathbb{C}$, we finally define

$$(2.6) \quad \mathsf{T}_h(\lambda) = \Delta_{V,h} - \lambda W + \lambda^2 \quad \text{with domain } D(\Delta_{V,h}).$$

The following result is the key point of our analysis: it establishes an equivalence between the spectrum of \mathbf{Q}_h and the kernels of the operators $\mathsf{T}_h(\lambda)$, $\lambda \in \mathbb{C}$.

Lemma 2.1. *The operator \mathbf{Q}_h with domain $D(\mathbf{Q}_h) = H^1(\mathbb{T}) \times H^1(\mathbb{T})$ is closed and has compact resolvent. In particular, it has only discrete spectrum. Moreover, $\lambda = 0$ is a simple eigenvalue of \mathbf{Q}_h and $\dim(\text{Ker } \mathsf{T}_h(0)) = \dim(\text{Ker } \mathbf{Q}_h) = 1$. Besides, for every $\lambda \in \mathbb{C} \setminus \{0\}$, the application*

$$(2.7) \quad \begin{aligned} \Psi : \text{Ker}(\mathsf{T}_h(\lambda)) &\longrightarrow \text{Ker}(\mathbf{Q}_h - \lambda) \\ g &\longmapsto \begin{pmatrix} \frac{1}{\lambda} \mathbf{d}_{V,h} g \\ g \end{pmatrix} \end{aligned}$$

is a linear isomorphism. Eventually, any $\mu \in \mathbb{C}$ is a singularity of $\lambda \mapsto \mathsf{T}_h(\lambda)^{-1}$ if and only if there exists $g \in H^2(\mathbb{T})$, $g \neq 0$, such that $\mathsf{T}_h(\mu)g = 0$.

Proof. The operator \mathbf{Q}_h is closed on $L^2(\mathbb{T})^2$ since it is a bounded perturbation of the closed operator $(0, h\partial_x; h\partial_x, 0)$. The resolvent of \mathbf{Q}_h is compact since the injection $H^1(\mathbb{T}) \subset L^2(\mathbb{T})$ is compact. Since \mathbf{Q}_h is unitarily equivalent to \mathbf{P}_h (see (2.5)), it follows from Theorem 1 that $\lambda = 0$ is a simple eigenvalue of \mathbf{Q}_h . Moreover, since $\mathsf{T}_h(0) = \Delta_{V,h}$, one gets $\text{Ker } \mathsf{T}_h(0) = \mathbb{C}e^{-\frac{V}{h}}$ and then $\dim(\text{Ker } \mathsf{T}_h(0)) = 1$.

Let us now consider $\lambda \in \mathbb{C} \setminus \{0\}$. For any $g \in \text{Ker}(\mathsf{T}_h(\lambda))$, one has $g \in H^2(\mathbb{T})$ and thus $(\frac{1}{\lambda} \mathbf{d}_{V,h} g, g) \in D(\mathbf{Q}_h)$. It follows moreover from $\mathsf{T}_h(\lambda)g = 0$ that

$$\mathbf{Q}_h \Psi(g) = \begin{pmatrix} 0 & \mathbf{d}_{V,h} \\ -\mathbf{d}_{V,h}^* & W \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} \mathbf{d}_{V,h} g \\ g \end{pmatrix} = \begin{pmatrix} \mathbf{d}_{V,h} g \\ (-\frac{1}{\lambda} \Delta_{V,h} + W)g \end{pmatrix} = \lambda \Psi(g).$$

This proves that Ψ is well defined. The linearity and the injectivity of Ψ are obvious. To prove its surjectivity, consider $u = \begin{pmatrix} f \\ g \end{pmatrix} \in D(\mathbf{Q}_h)$ such that $\mathbf{Q}_h u = \lambda u$, i.e. such that

$$\begin{cases} \mathbf{d}_{V,h} g = \lambda f, \\ -\mathbf{d}_{V,h}^* f + W g = \lambda g. \end{cases}$$

It follows from the first equation that $g \in H^2(\mathbb{T})$ and, since $\lambda \neq 0$, that $u = \begin{pmatrix} \frac{1}{\lambda} \mathbf{d}_{V,h} g \\ g \end{pmatrix}$. Moreover, applying $\mathbf{d}_{V,h}^*$ to the second equation leads to

$$\Delta_{V,h} g = \lambda \mathbf{d}_{V,h}^* f = \lambda W g - \lambda^2 g.$$

Consequently, $g \in D(\Delta_{V,h})$ and $\mathsf{T}_h(\lambda)g = 0$, which proves the surjectivity.

It remains to prove the last statement of Lemma 2.1. To this end, let us consider $\mu \in \mathbb{C}$. It is a singularity of $\lambda \mapsto \mathsf{T}_h(\lambda)^{-1}$ if and only if $\mathsf{T}_h(\mu) : H^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ is not invertible. Furthermore, since $\Delta_{V,h} + 1 : H^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ is invertible,

$$\mathsf{T}_h(\lambda) = [1 - (1 + \lambda W - \lambda^2)(\Delta_{V,h} + 1)^{-1}](\Delta_{V,h} + 1)$$

is not invertible if and only if $1 - \mathsf{B}(\lambda) : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ is not invertible, where $\mathsf{B}(\lambda) = (1 + \lambda W - \lambda^2)(\Delta_{V,h} + 1)^{-1}$. Since $(\Delta_{V,h} + 1)^{-1} : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ is compact, so is $\mathsf{B}(\lambda)$. The Fredholm alternative then implies that $\mathsf{T}_h(\mu)$ is not invertible if and only if there exists $u \in L^2(\mathbb{T})$, $u \neq 0$ such that $(1 - \mathsf{B}(\lambda))u = 0$, that is if and only if there exists $g \in H^2(\mathbb{T})$, $g \neq 0$ such that $\mathsf{T}_h(\mu)g = 0$. This concludes the proof of Lemma 2.1. \square

According to Lemma 2.1, for every $\mu \in \mathbb{C}$:

$$(2.8) \quad \mu \in \sigma_d(\mathsf{P}_h) = \sigma(\mathsf{P}_h) \text{ if and only if } \mu \text{ is a singularity of } \lambda \mapsto \mathsf{T}_h(\lambda)^{-1}$$

and, for such a μ , one has

$$(2.9) \quad \dim(\text{Ker}(\mathsf{P}_h - \mu)) = \dim(\text{Ker}(\mathsf{T}_h(\mu))).$$

To prove Theorems 2 and 3, we will thus investigate the singularities of $\lambda \mapsto \mathsf{T}_h(\lambda)^{-1}$ near 0.

2.2. Finite dimensional reduction, a Grushin problem. In this section, we show that the singularities of $\mathsf{T}_h(\lambda)$ near 0 are those of a matrix valued holomorphic function $E_{-+}(\lambda)$. To this end, we will construct a so-called Grushin problem.

To build the Grushin problem, we first need to recall known results on the low-lying spectrum of $\Delta_{V,h}$. From the early works of Witten [26] and Helffer-Sjöstrand [18], we know that there is a one-to-one correspondence between the local minima of V and the smallest eigenvalues of $\Delta_{V,h}$. This correspondence was further investigated by several authors and sharp asymptotic equivalents of these small eigenvalues were finally obtained in [8] and [17] under assumptions on the relative positions of the minima and saddle points, and in [21] in the general case. The following version gives the general form of these asymptotic equivalents in a non-degenerate setting.

Theorem 5. [8], [17]. *Let $U = -V$ be a Morse function. There exist $\epsilon_* > 0$, $C > 0$, and $h_0 > 0$ such that, for all $h \in]0, h_0]$, the nonnegative self-adjoint operator $(\Delta_{V,h}, H^2(\mathbb{T}))$ admits exactly n_0 eigenvalues (counted with algebraic multiplicity) in $[0, \epsilon_* h]$:*

$$(2.10) \quad \sigma(\Delta_{V,h}) \cap [0, \epsilon_* h] = \{0, \mu_{2,h}^\Delta, \dots, \mu_{n_0,h}^\Delta\},$$

where 0 is a simple eigenvalue of $\Delta_{V,h}$. Let us order $\{\mu_{2,h}^\Delta, \dots, \mu_{n_0,h}^\Delta\}$ such that $\mu_{j,h}^\Delta \leq \mu_{j+1,h}^\Delta$ for $j = 1, \dots, n_0 - 1$. Then, if (1.10) holds, it holds for all $j = 2, \dots, n_0$:

$$(2.11) \quad \mu_{j,h}^\Delta = a_h(\mathbf{m}_j) h e^{-\frac{2}{h}S(\mathbf{m}_j)},$$

where $S : \mathbf{U}^{(0)} \rightarrow (0, +\infty]$ is defined in (1.9) (see also (1.11)) and $a_h(\mathbf{m})$ admits a classical expansion $a_h(\mathbf{m}_j) \sim \sum_{k \geq 0} h^k a_k(\mathbf{m}_j)$ with

$$(2.12) \quad a_0(\mathbf{m}_j) = \frac{1}{2\pi} \sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m}_j)} \sqrt{|V''(\mathbf{m}_j)V''(\mathbf{s})|},$$

where $\mathbf{j} : \mathbf{U}^{(0)} \rightarrow \mathcal{P}(\underline{\mathbf{U}}^{(1)} \cup \{\mathbf{s}_1\})$ is defined in (1.8).

We are now in position to construct a Grushin problem. Set

$$\Pi = \pi_{[0, \varepsilon, h]}(\Delta_{V, h}) \text{ and } \mathbf{E}_0 = \text{Ran}(\Pi),$$

where $\pi_{[a, b]}(\Delta_{V, h})$ is the spectral projector associated with $\Delta_{V, h}$ and the interval $[a, b]$. According to Theorem 5, the space \mathbf{E}_0 has dimension n_0 for all $h > 0$ small enough. Let $(\Psi_j)_{j=1, \dots, n_0}$ be an orthonormal basis of the space \mathbf{E}_0 . We assume without loss of generality that Ψ_1 is proportional to $e^{-V/h}$. Introduce the operators

$$(2.13) \quad \begin{aligned} R_- : \mathbb{C}^{n_0} &\rightarrow L^2(\mathbb{T}) \\ u &\mapsto \sum_{j=1}^{n_0} u_j \Psi_j \end{aligned}$$

and

$$(2.14) \quad \begin{aligned} R_+ : L^2(\mathbb{T}) &\rightarrow \mathbb{C}^{n_0} \\ u &\mapsto (\langle u, \Psi_j \rangle_{L^2})_j. \end{aligned}$$

We equip \mathbb{C}^{n_0} with the ℓ_2 norm. Notice that $R_+ R_- = \text{Id}_{\mathbb{C}^{n_0}}$ and $R_- R_+ = \Pi$. In addition, $\|R_+\| \leq 1$ and $\|R_-\| \leq 1$, for all $h > 0$. These inequalities will be used many times in what follows. From now on, we denote

$$\widehat{\Pi} = \text{Id} - \Pi.$$

Lemma 2.2. *Let U be a Morse function. For any $\lambda \in \mathbb{C}$, the operator $\widehat{\mathbb{T}}_h(\lambda) := \widehat{\Pi} \mathbb{T}_h(\lambda) \widehat{\Pi}$ acting on $\widehat{\Pi} L^2 := \widehat{\Pi}(L^2(\mathbb{T}))$ with domain $D(\widehat{\mathbb{T}}_h(\lambda)) = \{u \in \widehat{\Pi} L^2, u \in H^2(\mathbb{T})\}$ is closed. Moreover, there exist $C, \epsilon_0, h_0 > 0$ such that for all $h \in]0, h_0]$, and for all $\lambda \in D(0, \epsilon_0 h)$, $\widehat{\mathbb{T}}_h(\lambda)$ is invertible, holomorphic with respect to λ , and $\|\widehat{\mathbb{T}}_h(\lambda)^{-1}\| \leq Ch^{-1}$.*

Proof. The proof is very close to the proof of Lemma 2.1 in [20]. We sketch it for reader's convenience. We first observe that since W is bounded, the operator $\mathbb{T}_h(\lambda)$ with domain $D(\mathbb{T}_h(\lambda)) = D(\Delta_{V, h}) = H^2(\mathbb{T})$ is closed and densely defined. Its adjoint $\mathbb{T}_h(\lambda)^*$ satisfies $D(\mathbb{T}_h(\lambda)^*) = H^2(\mathbb{T})$. Suppose that $(u_n, \widehat{\mathbb{T}}_h(\lambda)u_n) \in D(\widehat{\mathbb{T}}_h(\lambda)) \times L^2(\mathbb{T})$ converges to $(u, v) \in L^2(\mathbb{T}) \times L^2(\mathbb{T})$. For any $j = 1, \dots, n_0$, one has $\Psi_j \in D(\Delta_{V, h}) = D(\mathbb{T}_h(\lambda)^*)$ and hence for all $n \in \mathbb{N}$, one has

$$(2.15) \quad \Pi \mathbb{T}_h(\lambda) u_n = \sum_{j=1}^{n_0} \langle \mathbb{T}_h(\lambda) u_n, \Psi_j \rangle \Psi_j = \sum_{j=1}^{n_0} \langle u_n, \mathbb{T}_h(\lambda)^* \Psi_j \rangle \Psi_j.$$

Consequently, the sequence $(\Pi \mathbb{T}_h(\lambda) u_n)_n$ converges and using the identity

$$\mathbb{T}_h(\lambda) u_n = \Pi \mathbb{T}_h(\lambda) u_n + \widehat{\mathbb{T}}_h(\lambda) u_n,$$

it follows that $(\mathbb{T}_h(\lambda) u_n)$ converges. Since $\mathbb{T}_h(\lambda)$ is closed as a bounded perturbation of a closed operator, and $u_n \in H^2(\mathbb{T}) = D(\mathbb{T}_h(\lambda))$, one deduces that $u \in D(\mathbb{T}_h(\lambda))$ and

$\mathbb{T}_h(\lambda)u = \lim \mathbb{T}_h(\lambda)u_n$. Since $\widehat{\Pi} = 1 - \Pi$ is bounded, this implies that $u \in D(\widehat{\mathbb{T}}_h(\lambda))$ and $\widehat{\mathbb{T}}_h(\lambda)u = v$ which proves that $\widehat{\mathbb{T}}_h(\lambda)$ is closed. In addition, the operator $\widehat{\mathbb{T}}_h(\lambda)$ is clearly densely defined on $\widehat{\Pi}L^2$. Let us now study the invertibility of $\widehat{\mathbb{T}}_h(\lambda)$. For any $u \in D(\widehat{\mathbb{T}}_h(\lambda))$, one has by definition

$$\operatorname{Re}\langle \widehat{\mathbb{T}}_h(\lambda)u, u \rangle = \operatorname{Re}\langle (\Delta_{V,h} - \lambda W + \lambda^2)\widehat{\Pi}u, \widehat{\Pi}u \rangle.$$

Using Theorem 5, this implies that for $|\lambda| < \epsilon_0 h$, one has

$$\begin{aligned} \operatorname{Re}\langle \widehat{\mathbb{T}}_h(\lambda)u, u \rangle &\geq \epsilon_* h \|\widehat{\Pi}u\|^2 - \operatorname{Re}(\lambda) \langle W\widehat{\Pi}u, \widehat{\Pi}u \rangle + \operatorname{Re}(\lambda^2) \|\widehat{\Pi}u\|^2 \\ &\geq (\epsilon_* - \epsilon_0 \|W\|_{L^\infty}) h \|\widehat{\Pi}u\|^2 - \epsilon_0^2 h^2 \|\widehat{\Pi}u\|^2 \\ &\geq \frac{\epsilon_*}{2} h \|\widehat{\Pi}u\|^2 \end{aligned}$$

for ϵ_0 small enough. This proves that $\widehat{\mathbb{T}}_h(\lambda)$ is injective when $|\lambda| < \epsilon_0 h$. We observe that the same proof shows that $\widehat{\mathbb{T}}_h(\lambda)^* = \widehat{\Pi}\mathbb{T}_h(\lambda)^*\widehat{\Pi}$ is injective. Moreover, Cauchy-Schwarz inequality implies that for all $u \in D(\widehat{\mathbb{T}}_h(\lambda))$,

$$(2.16) \quad \|\widehat{\mathbb{T}}_h(\lambda)u\| \geq \frac{\epsilon_*}{2} h \|\widehat{\Pi}u\|.$$

Let us now prove that $\widehat{\mathbb{T}}_h(\lambda)$ is surjective if $|\lambda| < \epsilon_0 h$. We first observe that $\operatorname{Ran}(\widehat{\mathbb{T}}_h(\lambda))$ is closed since $\widehat{\mathbb{T}}_h(\lambda)$ is closed and the convergence of any sequence $(\widehat{\mathbb{T}}_h(\lambda)u_n)$ implies the convergence of (u_n) thanks to (2.16). Hence it is sufficient to prove that $\operatorname{Ran}(\widehat{\mathbb{T}}_h(\lambda))^\perp = \{0\}$. Suppose that $v \in L^2(\mathbb{T})$ satisfies $\langle \widehat{\mathbb{T}}_h(\lambda)u, v \rangle = 0$ for all $u \in D(\widehat{\mathbb{T}}_h(\lambda))$. Then

$$\langle \mathbb{T}_h(\lambda)\widehat{\Pi}u, \widehat{\Pi}v \rangle = 0$$

which implies that $\widehat{\Pi}v \in D(\mathbb{T}_h(\lambda)^*)$ and $\langle u, \widehat{\Pi}\mathbb{T}_h(\lambda)^*\widehat{\Pi}v \rangle = 0$ for all $u \in D(\widehat{\mathbb{T}}_h(\lambda))$ which is dense in $\widehat{\Pi}L^2$. Hence $\widehat{\Pi}\mathbb{T}_h(\lambda)^*\widehat{\Pi}v = 0$ and since $\widehat{\mathbb{T}}_h(\lambda)^*$ is injective, this implies that $v = 0$. This ends the proof of the lemma. \square

We now introduce the Grushin operator

$$(2.17) \quad \mathcal{K}_h(\lambda) = \begin{pmatrix} \mathbb{T}_h(\lambda) & R_- \\ R_+ & 0 \end{pmatrix}.$$

Proposition 2.3. *Let U be a Morse function. There exist $\epsilon_0 > 0$ and $h_0 > 0$ such that, for all $h \in]0, h_0]$ and all $\lambda \in D(0, \epsilon_0 h)$, the operator $\mathcal{K}_h(\lambda)$ is invertible. Moreover, its inverse $\mathcal{E}_h(\lambda)$ writes*

$$\mathcal{E}_h(\lambda) = \begin{pmatrix} E(\lambda) & E_+(\lambda) \\ E_-(\lambda) & E_{-+}(\lambda) \end{pmatrix},$$

where E, E_-, E_+, E_{-+} are holomorphic in $D(0, \epsilon_0 h)$ and satisfy the following formulas:

$$(2.18) \quad \begin{aligned} E_+(\lambda) &= R_- - \widehat{\mathbb{T}}_h(\lambda)^{-1} \widehat{\Pi} \mathbb{T}_h(\lambda) R_-, \quad E_-(\lambda) = R_+ - R_+ \mathbb{T}_h(\lambda) \widehat{\mathbb{T}}_h(\lambda)^{-1} \widehat{\Pi}, \\ E_{-+}(\lambda) &= -R_+ \mathbb{T}_h(\lambda) R_- + R_+ \mathbb{T}_h(\lambda) \widehat{\Pi} \widehat{\mathbb{T}}_h(\lambda)^{-1} \widehat{\Pi} \mathbb{T}_h(\lambda) R_- : \mathbb{C}^{n_0} \rightarrow \mathbb{C}^{n_0}, \end{aligned}$$

and

$$(2.19) \quad E(\lambda) = \widehat{\mathbb{T}}_h(\lambda)^{-1} \widehat{\Pi}.$$

Moreover, for $\lambda \in D(0, \epsilon_0 h)$, $\mathsf{T}_h(\lambda)$ is invertible if and only if the matrix $E_{-+}(\lambda)$ is invertible, in which case it holds

$$\mathsf{T}_h(\lambda)^{-1} = E(\lambda) - E_+(\lambda)E_{-+}^{-1}(\lambda)E_-(\lambda).$$

Proof. Thanks to Lemma 2.2, the proof is reduced to an algebraic computation which is completely analogous to the one used in the proof of [20, Lemma 2.2]. \square

We now give some direct consequences of Proposition 2.3 and Lemma 2.2, which will be used in the following. By definition, one has $R_+ \mathsf{T}_h(\lambda) \widehat{\Pi} = -\lambda R_+ W \widehat{\Pi}$. Hence (2.18) becomes

$$E_{-+}(\lambda) = -R_+ \mathsf{T}_h(\lambda) R_- + \lambda^2 R_+ W \widehat{\Pi} \widehat{\mathsf{T}}_h(\lambda)^{-1} \widehat{\Pi} W R_-.$$

Introducing the matrices

$$(2.20) \quad \mathsf{M}_{V,h} = R_+ \Delta_{V,h} R_-, \quad \mathsf{W}_h = R_+ W R_-, \quad \text{and} \quad \mathsf{G}_h(\lambda) = \lambda^2 (\mathbb{I}_{\mathbb{C}^{n_0}} - R_+ W \widehat{\Pi} \widehat{\mathsf{T}}_h(\lambda)^{-1} \widehat{\Pi} W R_-),$$

this rewrites

$$(2.21) \quad -E_{-+}(\lambda) = \mathsf{M}_{V,h} - \lambda \mathsf{W}_h + \mathsf{G}_h(\lambda).$$

Moreover, it follows from Lemma 2.2 that $\lambda \in D(0, \epsilon_0 h) \mapsto \mathsf{G}_h(\lambda)$ is holomorphic and that there exists $C > 0$ such that for all h small enough and all $\lambda \in D(0, \epsilon_0 h)$,

$$(2.22) \quad |\mathsf{G}_h(\lambda)| \leq C |\lambda|^2 h^{-1}.$$

According to (2.8) and to Proposition 2.3,

$$(2.23) \quad \lambda \in D(0, \epsilon_0 h) \cap \sigma(\mathsf{P}_h) \quad \text{if and only if} \quad \lambda \in D(0, \epsilon_0 h) \text{ is a singularity of } E_{-+}^{-1}(\lambda).$$

To prove Theorems 2 and 3, the strategy consists in studying the singularities of $E_{-+}^{-1}(\lambda)$ in $D(0, \epsilon_0 h)$, which first requires to compute asymptotic equivalents of $\mathsf{M}_{V,h}$ and W_h as $h \rightarrow 0$ (see (2.21)).

3. THE DOUBLE WELL CASE

In this section we prove Theorem 2. To this end, we assume throughout this section that $\mathsf{U}^{(0)}$ has exactly two elements \mathbf{m}_1 and \mathbf{m}_2 such that $V(\mathbf{m}_1) < V(\mathbf{m}_2)$. Assume also that the two elements $\mathbf{s}_1, \mathbf{s}_2$ of $\mathsf{U}^{(1)}$ satisfy $V(\mathbf{s}_1) > V(\mathbf{s}_2)$. Recall that in this case, one has by construction of S (see (1.9)) and \mathbf{j} (see (1.8)), $S(\mathbf{m}_2) = V(\mathbf{s}_2) - V(\mathbf{m}_2)$ and $\mathbf{j}(\mathbf{m}_2) = \{\mathbf{s}_2\}$ (see Remark 1.2). Thus, by Theorem 5, it holds

$$(3.1) \quad \mu_{2,h}^\Delta = a_h(\mathbf{m}_2) h e^{-\frac{2(V(\mathbf{s}_2) - V(\mathbf{m}_2))}{h}},$$

with $a_h(\mathbf{m}_2) \sim \sum_{k \geq 0} h^k a_k(\mathbf{m}_2)$ and $a_0(\mathbf{m}_2) = \frac{1}{2\pi} \sqrt{|V''(\mathbf{m}_2) V''(\mathbf{s}_2)|}$.

3.1. Asymptotic equivalent of W_h , as $h \rightarrow 0$. To compute an asymptotic equivalent of W_h in the limit $h \rightarrow 0$, we first need to define (Ψ_1, Ψ_2) with the help of so-called quasi-modes, where we recall that (Ψ_1, Ψ_2) is an orthonormal basis of eigenvectors of $\Delta_{V,h}$ associated respectively with the eigenvalues 0 and $\mu_h(\mathbf{m}_2)$. First of all, we choose for $h > 0$,

$$\Psi_1 = \frac{1}{Z_{1,h}} e^{-(V-V(\mathbf{m}_1))/h} \text{ with } Z_{1,h} = \|e^{-(V-V(\mathbf{m}_1))/h}\|_{L^2(\mathbb{T})},$$

where by Laplace's method: $Z_{1,h} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_1))^{1/4})$ (since \mathbf{m}_1 is the unique global minimum of V on \mathbb{T}). Because $\Delta_{V,h}\Psi_1 = 0$, $\Psi_1 \in \mathbf{E}_0$. We then construct Ψ_2 as follows. Define

$$(3.2) \quad \varphi_2 = \frac{1}{Z_{2,h}} \chi_2 e^{-(V-V(\mathbf{m}_2))/h} \text{ with } Z_{2,h} = \|e^{-(V-V(\mathbf{m}_2))/h}\|_{L^2(\mathbb{T})},$$

and where $\chi_2 \in \mathcal{C}_c^\infty(\mathbb{T}, [0, 1])$ satisfies $\mathbb{1}_{B(\mathbf{m}_2, r)} \leq \chi_2 \leq \mathbb{1}_{B(\mathbf{m}_2, 2r)}$, and where $r > 0$. If $r > 0$ is small enough, \mathbf{m}_2 is the unique global minimum of V on the closure of $B(\mathbf{m}_2, 2r)$ (because V is a Morse function). Thus, by Laplace's method: $Z_{2,h} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_2))^{1/4})$. In addition, for such fixed $r > 0$, $V > V(\mathbf{m}_2)$ on $\text{supp}(\nabla \chi_2)$, and therefore, there exists $C > 0$ such that for h small enough:

$$(3.3) \quad \Delta_{V,h}\varphi_2 = O(e^{-C/h}) \text{ in } L^2(\mathbb{T}).$$

On the other hand, since $\Delta_{V,h}$ is self-adjoint, it follows from the localisation of the spectrum in Theorem 5 (with $n_0 = 2$ there) that

$$(3.4) \quad \forall z \in \partial D(0, \epsilon_* h/2), (\Delta_{V,h} - z)^{-1} = O(h^{-1}).$$

By definition of Π and by Theorem 5,

$$\Pi = \frac{1}{2i\pi} \int_{\partial D(0, \frac{\epsilon_*}{2} h)} (z - \Delta_{V,h})^{-1} dz$$

and using (3.3) and (3.4), it follows that $\Pi\varphi_2$ satisfies

$$(3.5) \quad \Pi\varphi_2 - \varphi_2 = \frac{1}{2i\pi} \int_{\partial D(0, \frac{\epsilon_*}{2} h)} z^{-1} (z - \Delta_{V,h})^{-1} \Delta_{V,h}\varphi_2 dz = O(e^{-C/h}).$$

Moreover, since $V(\mathbf{m}_2) > V(\mathbf{m}_1)$ and $\Pi\Psi_1 = \Psi_1$, one has for h small enough:

$$(3.6) \quad \langle \Pi\varphi_2, \Psi_1 \rangle_{L^2(\mathbb{T})} = \langle \varphi_2, \Psi_1 \rangle_{L^2(\mathbb{T})} = O(e^{-C/h}).$$

We finally set

$$\Psi_2 = \frac{\Pi\varphi_2 - \langle \Pi\varphi_2, \Psi_1 \rangle_{L^2(\mathbb{T})} \Psi_1}{\|\Pi\varphi_2 - \langle \Pi\varphi_2, \Psi_1 \rangle_{L^2(\mathbb{T})} \Psi_1\|_{L^2(\mathbb{T})}}.$$

The function Ψ_2 belongs to \mathbf{E}_0 , is orthogonal to Ψ_1 , and $\|\Psi_2\|_{L^2(\mathbb{T})} = 1$. From now on, we consider (Ψ_1, Ψ_2) constructed as above, as a orthonormal basis of \mathbf{E}_0 . Notice that, using (3.5) and (3.6), one has

$$(3.7) \quad \Psi_2 = \varphi_2 + O(e^{-C/h}) \text{ in } L^2(\mathbb{T}).$$

Lemma 3.1. *With the above choice of Ψ_1, Ψ_2 , there exists $C > 0$ such that for all h small enough,*

$$W_h = \begin{pmatrix} \gamma_{1,h} & 0 \\ 0 & \gamma_{2,h} \end{pmatrix} + O(e^{-C/h}),$$

where we recall that W_h is defined in (2.20) and where, for $i \in \{1, 2\}$, $\gamma_{i,h}$ satisfies as $h \rightarrow 0$: $\gamma_{i,h} \sim \sum_{k \geq 0} h^k \gamma_{\alpha,k}(\mathbf{m}_i) + \sqrt{h} \sum_{k \geq 0} h^k \gamma_{V,k}(\mathbf{m}_i)$, with

$$\gamma_{\alpha,0}(\mathbf{m}_i) = \alpha(\mathbf{m}_i) \text{ and } \gamma_{V,0}(\mathbf{m}_i) = 2\sqrt{\frac{V''(\mathbf{m}_i)}{\pi}}.$$

Proof. Since (Ψ_1, Ψ_2) is an orthonormal family, one has $W_h = (\langle W\Psi_i, \Psi_j \rangle_{L^2(\mathbb{T})})_{i,j=1,2}$. In addition, $\Psi_1 = O(e^{-C/h})$ in $L^2(\text{supp}(\varphi_2))$. Hence, using (3.7), for all $i \neq j$, $\langle W\Psi_i, \Psi_j \rangle_{L^2(\mathbb{T})} = O(e^{-C/h})$. Suppose now that $i \in \{1, 2\}$ is fixed. By definition of W , one has $\langle W\Psi_i, \Psi_i \rangle_{L^2(\mathbb{T})} = \langle \alpha\Psi_i, \Psi_i \rangle_{L^2(\mathbb{T})} + 2\langle |\partial_x V| \Psi_i, \Psi_i \rangle_{L^2(\mathbb{T})}$. By definition of Ψ_1 and Ψ_2 above, using Laplace's method, one has as $h \rightarrow 0$: $\langle \alpha\Psi_i, \Psi_i \rangle_{L^2(\mathbb{T})} \sim \sum_{k \geq 0} h^k \gamma_{\alpha,k}(\mathbf{m}_i)$ with $\gamma_{\alpha,0}(\mathbf{m}_i) = \alpha(\mathbf{m}_i)$. On the other hand, for $i \in \{1, 2\}$, one has for $\delta > 0$ small enough,

$$\begin{aligned} \langle |\partial_x V| \Psi_i, \Psi_i \rangle_{L^2(\mathbb{T})} &= Z_{i,h}^{-2} \int_{|x-\mathbf{m}_i| < \delta} |\partial_x V(x)| e^{-2(V(x)-V(\mathbf{m}_i))/h} dx + O(e^{-c/h}) \\ &= Z_{i,h}^{-2} \left(\int_{\mathbf{m}_i}^{\mathbf{m}_i+\delta} \partial_x V(x) e^{-2(V(x)-V(\mathbf{m}_i))/h} dx \right. \\ &\quad \left. - \int_{\mathbf{m}_i-\delta}^{\mathbf{m}_i} \partial_x V(x) e^{-2(V(x)-V(\mathbf{m}_i))/h} dx \right) + O(e^{-c/h}) \\ &= h Z_{i,h}^{-2} + O(e^{-c/h}). \end{aligned}$$

Using the fact that $Z_{i,h} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_i))^{1/4})$, this ends the proof of the lemma. \square

We are now in position to prove Theorem 2.

3.2. End of the proof of Theorem 2. Recall that the strategy consists in localizing the singularities of $E_{-+}^{-1}(\lambda)$ in $D(0, \epsilon_0 h)$ (see (2.23)). First of all, $\lambda = 0$ is always a singularity of $\lambda \in D(0, \epsilon_0 h) \mapsto E_{-+}^{-1}(\lambda)$, since $\mathbb{T}_h(0)\Psi_1 = 0$ (see Proposition 2.3). Let us now look for the other singularities of $\lambda \in D(0, \epsilon_0 h) \mapsto E_{-+}^{-1}(\lambda)$. Since $\dim \mathbf{E}_0 = 2$, $\Delta_{V,h}\Psi_1 = 0$, and $\Psi_2 \in \mathbf{E}_0$ is orthogonal to Ψ_1 , it holds: $\Delta_{V,h}\Psi_2 = \mu_{2,h}^\Delta \Psi_2$ and therefore, $\mathbf{M}_{V,h} = \text{diag}(0, \mu_{2,h}^\Delta)$. By Lemma 3.1, (2.21), and (2.22), one then has

$$-E_{-+}(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & \mu_{2,h}^\Delta \end{pmatrix} - \lambda \begin{pmatrix} \gamma_{1,h} & 0 \\ 0 & \gamma_{2,h} \end{pmatrix} + r_h(\lambda),$$

where $\lambda \in D(0, \epsilon_0 h) \mapsto r_h(\lambda)$ is holomorphic and $r_h(\lambda) = O(\lambda^2 h^{-1} + \lambda e^{-C/h})$ for all $\lambda \in D(0, \epsilon_0 h)$ and h small enough. Set

$$\Gamma_h = \text{diag}(\gamma_{1,h}, \gamma_{2,h}).$$

Then, by Lemma 3.1:

$$(3.8) \quad \Gamma_h^{-1} = O(h^{-1/2}).$$

Therefore, one deduces that

$$(3.9) \quad -E_{-+}(\lambda) = \Gamma_h(F_h(\lambda) + R_h(\lambda)) \text{ with } F_h(\lambda) = \begin{pmatrix} -\lambda & 0 \\ 0 & \mu_{2,h}^\Delta/\gamma_{2,h} - \lambda \end{pmatrix},$$

where $R_h(\lambda)$ is holomorphic with respect to $\lambda \in D(0, \epsilon_0 h)$ and $R_h(\lambda) = O(\lambda^2 h^{-\frac{3}{2}} + \lambda h^{-\frac{1}{2}} e^{-C/h})$ for all $\lambda \in D(0, \epsilon_0 h)$ and h small enough. Set

$$\eta_{2,h} := \mu_{2,h}^\Delta/\gamma_{2,h}.$$

According to Lemma 3.1, to (3.1), and to the relation $\alpha(\mathbf{m}_2) > 0$ implied by (1.4) (see indeed (1.5)), one has in the limit $h \rightarrow 0$:

$$(3.10) \quad \eta_{2,h} = \zeta_h(\mathbf{m}_2) h e^{-\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))},$$

where $\zeta_h(\mathbf{m}_2) \sim \sum_{k \geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_2)$ and $\zeta_0(\mathbf{m}_2) = \frac{1}{2\pi} \sqrt{|V''(\mathbf{m}_2)V''(\mathbf{s}_2)|}/\alpha(\mathbf{m}_2)$. Let $K \geq 2$ be fixed in what follows. Set

$$\mathcal{D}_K = \left\{ \lambda \in \mathbb{C}, |\lambda - \eta_{2,h}| < h^K e^{-\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))} \right\},$$

whose closure is by (3.10) included in $D(0, \epsilon_0 h)$ for h small enough. Hence, for any $\lambda \in \overline{\mathcal{D}_K}$, since $K \geq 2$, one has, for all h small enough,

$$(3.11) \quad C^{-1} h e^{-\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))} \leq |\lambda| \leq C h e^{-\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))}.$$

Consequently, for any $\lambda \in \partial \mathcal{D}_K$, the matrix $F_h(\lambda)$ is invertible and

$$(3.12) \quad F_h(\lambda)^{-1} = O(h^{-K} e^{\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))}) \text{ on } \partial \mathcal{D}_K.$$

On the other hand, it follows from the estimate on R_h below (3.9) that for any $\lambda \in \partial \mathcal{D}_K$,

$$R_h(\lambda) = O(\sqrt{h} e^{-4(V(\mathbf{s}_2) - V(\mathbf{m}_2))/h} + \sqrt{h} e^{-(2(V(\mathbf{s}_2) - V(\mathbf{m}_2)) + C)/h}) = O(e^{-(2(V(\mathbf{s}_2) - V(\mathbf{m}_2)) + C)/h}).$$

Hence, by (3.9), $E_{-+}(\lambda)$ is invertible on $\partial \mathcal{D}_K$ for all h small enough and

$$-E_{-+}^{-1}(\lambda) = (1 + O(e^{-\frac{C}{2h}})) F_h(\lambda)^{-1} \Gamma_h^{-1} \text{ on } \partial \mathcal{D}_K,$$

where we have used (3.12) and the invertibility of Γ_h . Using in addition $\|F_h(\lambda)^{-1}\| |\partial \mathcal{D}_K| = O(1)$ and (3.8), this implies that for all h small enough:

$$(3.13) \quad \begin{aligned} \frac{1}{2i\pi} \int_{\partial \mathcal{D}_K} E_{-+}^{-1}(\lambda) d\lambda &= - \left[\frac{1}{2i\pi} \int_{\partial \mathcal{D}_K} \begin{pmatrix} -\lambda & 0 \\ 0 & \eta_{2,h} - \lambda \end{pmatrix}^{-1} d\lambda \right] \Gamma_h^{-1} + \|\Gamma_h^{-1}\| O(e^{-\frac{C}{2h}}) \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{2,h}^{-1} \end{pmatrix} + O(e^{-\frac{C}{4h}}) \text{ is non trivial.} \end{aligned}$$

Hence, for all $K \geq 2$, $\lambda \mapsto E_{-+}^{-1}(\lambda)$ admits at least a singularity $\alpha_{h,K}$ in the disk \mathcal{D}_K . In particular, one has for all h small enough,

$$\alpha_{h,K} = \eta_{2,h} + O(h^K e^{-\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))}), \text{ where } \eta_{2,h} \text{ satisfies (3.10).}$$

But, since (1.4) holds, Theorem 1 implies that for all h small enough, $\sigma(\mathbf{P}_h) \cap \{\operatorname{Re}(z) \leq \epsilon_0 h^2\}$ is composed of two elements 0 and $\lambda_{2,h}$. According to (2.23), the nonzero eigenvalue $\lambda_{2,h}$ then necessarily satisfies $\lambda_{2,h} = \alpha_{h,K}$ for all $K \geq 2$. This ends the proof of Theorem 2.

4. THE MULTIPLE WELL CASE

In this section, we prove Theorem 3. We assume that U is a Morse function and that (1.10) is satisfied. Recall that the local minima $\mathbf{m}_1, \dots, \mathbf{m}_{n_0}$ of V are labeled such that $(S(\mathbf{m}_j))_{j \in \{1, \dots, n_0\}}$ is decreasing (see (1.11)). We then label $\{\Psi_1, \dots, \Psi_{n_0}\}$, the basis of \mathbf{E}_0 , accordingly.

4.1. An adapted basis of quasimodes. In order to compute the matrices $\mathbf{M}_{V,h}$ and \mathbf{W}_h accurately, we will build the basis $(\Psi_j)_{j=1, \dots, n_0}$ of \mathbf{E}_0 from a family of quasi-modes $(\varphi_j)_{j=1, \dots, n_0}$ constructed in [6]. First, as in the previous section, set, for $h > 0$,

$$\varphi_1 = \frac{1}{Z_{1,h}} e^{-(V-V(\mathbf{m}_1))/h} \text{ with } Z_{1,h} = \|e^{-(V-V(\mathbf{m}_1))/h}\|_{L^2(\mathbb{T})} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_1))^{1/4}),$$

and $\Psi_1 = \varphi_1$. For any $j = 2, \dots, n_0$, let φ_j be the $L^2(\mathbb{T})$ -normalized quasi-mode associated with \mathbf{m}_j given in [6, Definition 4.3] applied to the case of the Witten Laplacian $\Delta_{V,h}$. Since φ_i is by definition supported in a neighborhood of $\mathbf{C}(\mathbf{m}_i)$ (see [6, Section 4]), it follows from the second item of Assumption (1.10) that, for all $i \in \{1, \dots, n_0\}$ and $r > 0$ small enough, one has for all h small enough:

$$(4.1) \quad \varphi_i = \frac{\mathbf{1}_{D(\mathbf{m}_i,r)}}{Z_{i,h}} e^{-(V-V(\mathbf{m}_i))/h} + O(e^{-c/h}) \text{ in } L^2(\mathbb{T}),$$

where $Z_{i,h} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_i))^{1/4})$. For $j \in \{1, \dots, n_0\}$, we set

$$\kappa_{j,h} := \langle \Delta_{V,h} \varphi_j, \varphi_j \rangle.$$

We have obviously $\kappa_{1,h} = 0$ and, from [6, Proposition 5.1], we have for all $j \in \{2, \dots, n_0\}$,

$$(4.2) \quad \kappa_{j,h} \in \mathcal{E}_{cl}\left(h e^{-2S(\mathbf{m}_j)/h} \sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m}_j)} \frac{|V''(\mathbf{m}_j)V''(\mathbf{s})|^{1/2}}{2\pi}\right).$$

The basis $(\Psi_j)_{j=1, \dots, n_0}$ of $\mathbf{E}_0 = \text{Ran } \Pi$ is then constructed from the family $(\varphi_j)_{j=1, \dots, n_0}$ by the following procedure. Set, for $j = 1, \dots, n_0$: $v_j = \Pi \varphi_j$. For h small enough, the family $(v_j)_{j=1, \dots, n_0}$ is then a basis of \mathbf{E}_0 (since, according to [6, Proposition 5.3], $\langle v_j, v_i \rangle_{L^2(\mathbb{T})} = \delta_{i,j} + O(e^{-c/h})$, for all $i, j = 1, \dots, n_0$). We then consider the family $(\Psi_j)_{j=1, \dots, n_0}$ obtained from $(v_j)_{j=1, \dots, n_0}$ by a Gram-Schmidt procedure as in [6, page 30]. According to [6, Proposition 5.3 and Eq. (5.15)], this orthonormal basis of \mathbf{E}_0 satisfies the following properties:

$$(4.3) \quad \text{for all } j, k = 1, \dots, n_0, \quad \langle \Delta_{V,h} \Psi_j, \Psi_k \rangle = \delta_{j,k} \kappa_{j,h} + O(h^\infty \sqrt{\kappa_j(h) \kappa_k(h)})$$

and

$$(4.4) \quad \text{for all } j = 1, \dots, n_0, \quad \Psi_j = \varphi_j + O(e^{-c/h}) \text{ in } L^2(\mathbb{T}).$$

Working in the basis $(\Psi_j)_{j=1, \dots, n_0}$ of \mathbf{E}_0 , we obtain the following asymptotic equivalent of \mathbf{W}_h (see (2.20)).

Lemma 4.1. *There exists $C > 0$ such that the matrix \mathbf{W}_h satisfies, for h small enough,*

$$\mathbf{W}_h = \Gamma_h + O(e^{-C/h}),$$

where $\Gamma_h = \text{diag}(\gamma_{1,h}, \dots, \gamma_{n_0,h})$ and, for $j = 1, \dots, n_0$:

$$\gamma_{j,h} \sim \sum_{k \geq 0} h^k \gamma_{\alpha,k}(\mathbf{m}_j) + \sqrt{h} \sum_{k \geq 0} h^k \gamma_{V,k}(\mathbf{m}_j)$$

with $\gamma_{\alpha,0}(\mathbf{m}_j) = \alpha(\mathbf{m}_j)$, and $\gamma_{V,0}(\mathbf{m}_j) = 2\sqrt{\frac{V''(\mathbf{m}_j)}{\pi}}$.

Proof. The asymptotic computation of $\langle W\Psi_i, \Psi_i \rangle$ as $h \rightarrow 0$ is exactly the same as in Lemma 3.1, using (4.1) and (4.4). The only point to be checked is that for any $i \neq j$, one has $\langle W\Psi_i, \Psi_j \rangle = O(e^{-C/h})$. Thanks to (4.4), this is equivalent to say that for all $i \neq j$, one has $\langle W\varphi_i, \varphi_j \rangle = O(e^{-C/h})$, which follows directly from (4.1). \square

4.2. Proof of Theorem 3. Let Γ_h be defined by Lemma 4.1 and let $\mathbf{M}_{V,h}$ be given by (2.20). Notice that in the case $n_0 \geq 3$, the matrix $\mathbf{M}_{V,h}$ has not to be diagonal and we will use the following result.

Lemma 4.2. *Introduce the symmetric positive semi-definite matrix*

$$\mathbf{M}_{V,h}^\Gamma = \Gamma_h^{-\frac{1}{2}} \mathbf{M}_{V,h} \Gamma_h^{-\frac{1}{2}}.$$

Then, there exists $\epsilon_0 > 0$ such that, for all h small enough, the n_0 eigenvalues $\beta_{1,h} \leq \dots \leq \beta_{n_0,h}$ of $\mathbf{M}_{V,h}^\Gamma$ satisfy: $\beta_{1,h} = 0$ and, for all $k = 2, \dots, n_0$, $\beta_{k,h} \in \mathcal{E}_{cl}(\mu_{k,h}^\Delta/\gamma_{k,h})$, where $\mu_{k,h}^\Delta$ is given by (2.10).

Proof. First, observe that $\mathbf{M}_{V,h}$ admits 0 as a simple eigenvalue (since 0 is a simple eigenvalue of $\Delta_{V,h}$), so it is also a simple eigenvalue of $\mathbf{M}_{V,h}^\Gamma$. For $A = (a_{i,j})_{1 \leq i, j \leq n_0}$ a matrix, we define $\tilde{A} := (a_{i,j})_{2 \leq i, j \leq n_0}$. We then have:

$$\tilde{\mathbf{M}}_{V,h}^\Gamma = (\tilde{\Gamma}_h)^{-\frac{1}{2}} \tilde{\mathbf{M}}_{V,h} (\tilde{\Gamma}_h)^{-\frac{1}{2}}.$$

Moreover, by (4.2) and (4.3), the matrix $\tilde{\mathbf{M}}_{V,h}$ writes

$$\tilde{\mathbf{M}}_{V,h} = \Omega_h (\mathbf{D}_{V,h} + O(h^\infty)) \Omega_h,$$

with $\Omega_h = \text{diag}(e^{-S(\mathbf{m}_2)/h}, \dots, e^{-S(\mathbf{m}_{n_0})/h})$ and $\mathbf{D}_{V,h} = \text{diag}(a_h(\mathbf{m}_2), \dots, a_h(\mathbf{m}_{n_0}))$. Since $\tilde{\Gamma}_h$ is diagonal, one has $\Omega_h \tilde{\Gamma}_h = \tilde{\Gamma}_h \Omega_h$ and it follows that $\tilde{\mathbf{M}}_{V,h}^\Gamma$ writes

$$\tilde{\mathbf{M}}_{V,h}^\Gamma = \Omega_h (\mathbf{D}_{V,h}^\Gamma + O(h^\infty)) \Omega_h$$

with $\mathbf{D}_{V,h}^\Gamma = \text{diag}(\mu_{j,h}^\Delta/\gamma_{j,h}, j = 2, \dots, n_0)$. Hence $\tilde{\mathbf{M}}_{V,h}^\Gamma$ admits a *graded structure* in the sense of [20, Definition A.1]. We can thus apply [20, Theorem A.4] which yields the result. \square

We are now in position to prove Theorem 3. Let U be a Morse function and assume that (1.10) is satisfied. Recall that we look for the singularities of $E_{-+}^{-1}(\lambda)$ and that $\lambda = 0$ is always a singularity of $E_{-+}^{-1}(\lambda)$ (since $\dim \text{Ker}(\mathbf{P}_h) = 1$). Let us now look for the remaining singularities of $E_{-+}^{-1}(\lambda)$ in $D(0, \epsilon_0 h)$. From Lemma 4.2, there exists a unitary change of basis B_h such that $\mathbf{M}_{V,h}^\Gamma = B_h^* \text{diag}(\beta_{1,h}, \dots, \beta_{n_0,h}) B_h$ (since $\mathbf{M}_{V,h}^\Gamma$ is symmetric and thus

diagonalizable in an orthonormal basis). Combined with (2.21), (2.22), and Lemma 4.1, this yields

$$(4.5) \quad -E_{-+}(\lambda) = \Gamma_h^{1/2} B_h^* \left[\text{diag}(\beta_{1,h}, \dots, \beta_{n_0,h}) - \lambda I_{\mathbb{C}^{n_0}} - R_h(\lambda) \right] B_h \Gamma_h^{1/2},$$

where, for h small enough, $R_h(\lambda)$ is holomorphic with respect to $\lambda \in D(0, \epsilon_0 h)$ and

$$(4.6) \quad R_h(\lambda) = O(\lambda^2 h^{-\frac{3}{2}} + \lambda h^{-\frac{1}{2}} e^{-C/h}),$$

and where we have used that $\Gamma_h^{-1} = O(h^{-1/2})$ by Lemma 4.1. Hence, the singularities of $E_{-+}^{-1}(\lambda)$ in $D(0, \epsilon_0 h)$ are exactly those of L_h^{-1} , where

$$(4.7) \quad L_h(\lambda) = F_h(\lambda) - R_h(\lambda), \text{ with } F_h(\lambda) = \text{diag}(\beta_{1,h} - \lambda, \dots, \beta_{n_0,h} - \lambda).$$

Recall that $\beta_{1,h} = 0$ and $\beta_{j,h} \in \mathcal{E}_{cl}(\mu_{j,h}^\Delta / \gamma_{j,h})$ for all $j \geq 2$. For all $j = 2, \dots, n_0$, using the asymptotic equivalents of $\mu_{j,h}^\Delta$ and $\gamma_{j,h}$ given in Theorem 5 and in Lemma 4.1, one has when $h \rightarrow 0$, using (1.4) which implies $\alpha(\mathbf{m}_j) > 0$:

$$(4.8) \quad \beta_{j,h} = \zeta_h(\mathbf{m}_j) h e^{-\frac{2}{h} S(\mathbf{m}_j)},$$

where $\zeta_h(\mathbf{m}_j) \sim \sum_{k \geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_j)$ and $\zeta_0(\mathbf{m}_j)$ is given by (1.13). Let us now consider $j \in \{2, \dots, n_0\}$ and $K \geq 2$. Denote $\mathcal{D}_{j,K} = D(\beta_{j,h}, h^K e^{-\frac{2}{h} S(\mathbf{m}_j)})$ and let $\lambda \in \partial \mathcal{D}_{j,K}$. Since $S(\mathbf{m}_\ell) < S(\mathbf{m}_k)$ when $\ell \geq k$, for $h > 0$ small enough, the $\mathcal{D}_{j,K}$ are pairwise disjoint, their closures are included in $D(0, \epsilon_0 h)$, and for all $i \in \{2, \dots, n_0\}$:

$$(4.9) \quad \begin{aligned} \forall i > j, |\lambda - \beta_{i,h}| &\geq |\beta_{j,h} - \beta_{i,h}| - |\lambda - \beta_{j,h}| \geq C h e^{-\frac{2}{h} S(\mathbf{m}_i)} \geq C h e^{-\frac{2}{h} S(\mathbf{m}_j)} \\ \forall i < j, |\lambda - \beta_{i,h}| &\geq |\beta_{j,h} - \beta_{i,h}| - |\lambda - \beta_{j,h}| \geq C h e^{-\frac{2}{h} S(\mathbf{m}_j)}. \end{aligned}$$

Moreover, for h small enough,

$$(4.10) \quad |\lambda| \geq C h e^{-\frac{2}{h} S(\mathbf{m}_j)}.$$

Consequently, for $\lambda \in \partial \mathcal{D}_{j,K}$, the matrix $F_h(\lambda)$ is invertible and

$$F_h(\lambda)^{-1} = O(h^{-K} e^{\frac{2}{h} S(\mathbf{m}_j)}) \text{ on } \mathcal{D}_{j,K}.$$

Combining this estimate with (4.6) and reasoning as around (3.13), we prove that for all $j = 2, \dots, n_0$, $K \geq 2$, and h small enough,

$$(4.11) \quad L_h^{-1}(\lambda) \text{ (and thus } E_{-+}^{-1}(\lambda)) \text{ admits a singularity } \alpha_{j,h,K} \text{ in } \mathcal{D}_{j,K},$$

so in particular $\alpha_{j,h,K} \neq \alpha_{i,h,K}$ when $i \neq j \in \{2, \dots, n_0\}$ (the $\mathcal{D}_{j,K}$ being pairwise disjoint).

In addition, since (1.4) holds, Theorem 1 implies that for all h small enough, $\sigma(\mathbf{P}_h) \cap \{\text{Re}(z) \leq \epsilon_0 h^2\} \setminus \{0\}$ is made of $n_0 - 1$ real eigenvalues $0 < \lambda_{2,h} \leq \dots \leq \lambda_{n_0,h}$ (counted with algebraic multiplicity). It then follows from (2.23) that, for each $j = 2, \dots, n_0$, the eigenvalue $\lambda_{j,h}$ satisfies $\lambda_{j,h} = \alpha_{j,h,K}$ for all $K \geq 2$. Since $\alpha_{j,h,K} = \beta_{j,h} + O(h^K e^{-\frac{2}{h} S(\mathbf{m}_j)})$ for all $K \geq 2$ and $\beta_{j,h} \in \mathcal{E}_{cl}(\mu_{j,h}^\Delta / \gamma_{j,h})$, this completes the proof of Theorem 3, using the asymptotic equivalent of $\mu_{j,h}^\Delta / \gamma_{j,h}$ as $h \rightarrow 0$ (see Theorem 5 and Lemma 4.1).

4.3. The case when (1.4) is not satisfied. In this section, we prove Theorem 4, where, compared to Theorem 3, we do no longer assume that (1.4) holds but that (1.14) holds. We can thus no longer use Theorem 1 as we did at the very end of the proof of Theorem 3 to say that P_h admits $n_0 - 1$ nonzero eigenvalues (counted with multiplicity) in $\{\operatorname{Re}(z) \leq \epsilon_0 h^2\}$. We can however make use of all the intermediate results in the proof of Theorem 3 until (4.11) (included), except that one has the following minor scaling changes:

(1) Equation (4.8) must be changed into

$$(4.12) \quad \beta_{j,h} = \zeta_h(\mathbf{m}_j) \sqrt{h} e^{-\frac{2}{h}S(\mathbf{m}_j)},$$

where $\zeta_h(\mathbf{m}_j) \sim \sum_{k \geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_j)$ with $\zeta_0(\mathbf{m}_j)$ satisfying (1.16).

(2) In Equations (4.9) and (4.10), all the Ch must be replaced by $C\sqrt{h}$.

Let $j \in \{2, \dots, n_0\}$ and note that, for h small enough, the sequence $(\alpha_{j,h,K})_{K \geq 2}$ defined in the proof of Theorem 3 is stationary by analyticity of the nontrivial map $\lambda \mapsto E_{-+}(\lambda)$. We denote by $\alpha_{j,h}$ its limit and recall that, according to (2.23), $\{0, \alpha_{2,h}, \dots, \alpha_{n_0,h}\} \subset \sigma(P_h)$. In addition, the $\alpha_{j,h}$'s are exponentially small and thus belong to $D(0, ch^2)$ for any $c > 0$ and all h small enough. Let us now prove that, for h small enough, 0 and $\alpha_{j,h}$, $j = 2, \dots, n_0$, are the only singularities of $\lambda \mapsto E_{-+}^{-1}(\lambda)$ in $D(0, ch^2)$, that they are all real, and have geometric multiplicity 1 as eigenvalues of P_h . Recall that by (4.5), $\lambda \in D(0, \epsilon_0 h)$ is a singularity of $E_{-+}^{-1}(\lambda)$ if and only if it is a singularity of $L_h(\lambda)^{-1}$ (see (4.7)). Let us denote, for $\lambda \in \mathbb{C}$,

$$d_h(\lambda) = \det L_h(\lambda) = \det(F_h(\lambda) - R_h(\lambda)),$$

which is holomorphic on $D(0, \epsilon_0 h)$. For all $\lambda \in \partial D(0, ch^2)$, using (4.6), we get $R_h(\lambda) = O(h^4 h^{-3/2}) = O(h^{5/2})$, and by (4.12),

$$|\det F_h(\lambda)| = |\lambda \times \prod_{j=2}^{n_0} (\lambda - \beta_{j,h})| = c^{n_0} h^{2n_0} (1 + o_h(1)).$$

Thus, for h small enough, it holds uniformly on $\partial D(0, ch^2)$:

$$(4.13) \quad d_h(\lambda) = \lambda \times \prod_{j=2}^{n_0} (\lambda - \beta_{j,h}) + O(h^{5/2} h^{2(n_0-1)}) = \det F_h(\lambda) (1 + o_h(1)).$$

In particular, for any $\lambda \in \partial D(0, ch^2)$, one has

$$|d_h(\lambda) - \det F_h(\lambda)| < |\det F_h(\lambda)|,$$

which implies by Rouché's theorem that $d_h(\lambda)$ and $\det F_h(\lambda)$ have the same number n_0 of zeros (counted with multiplicity) in $D(0, ch^2)$. These n_0 zeros are thus 0 and $\alpha_{j,h}$, $j = 2, \dots, n_0$. Let us recall that the $\alpha_{j,h}$ are pairwise distinct and satisfy $\alpha_{j,h} = \beta_{j,h} + O(h^K e^{-\frac{2}{h}S(\mathbf{m}_j)})$ for all $K \geq 2$ and $j = 2, \dots, n_0$. They are then all simple zeros of $d_h(\lambda)$ (and thus $\dim \operatorname{Ker} E_{-+}(\alpha_{j,h}) = 1$ for $j = 2, \dots, n_0$). Using also (2.9) and [25, Equation (2.7)], $\dim \operatorname{Ker}(P_h - \alpha_{j,h}) = \dim \operatorname{Ker} T_h(\alpha_{j,h}) = \dim \operatorname{Ker} E_{-+}(\alpha_{j,h}) = 1$, for all $j = \{2, \dots, n_0\}$. Lastly, since the operator P_h has real coefficients, its spectrum is stable by complex conjugation. Hence, since moreover 0 and the $\alpha_{j,h}$, $j = 2, \dots, n_0$, have different asymptotic equivalents as $h \rightarrow 0$, one has $\overline{\alpha_{j,h}} = \alpha_{j,h}$ for every $j = 2, \dots, n_0$. To conclude the proof of Theorem 4, it then just remains to use that, for every $j = 2, \dots, n_0$, $\beta_{j,h} \in \mathcal{E}_{cl}(\mu_{j,h}^\Delta / \gamma_{j,h})$, and the asymptotic equivalents of $\mu_{j,h}^\Delta / \gamma_{j,h}$ as $h \rightarrow 0$ given by Theorem 5 and Lemma 4.1.

Acknowledgement

This work was supported by the ANR-19-CE40-0010, Analyse Quantitative de Processus Métastables (QuAMProcs). B.N. is supported by the grant IA20Nectoux from the Projet I-SITE Clermont CAP 20-25. The authors are grateful to Pierre Monmarché for fruitful discussions.

REFERENCES

1. C. Andrieu, A. Durmus, N. Nüsken, and J. Roussel, *Hypercoercivity of Piecewise Deterministic Markov Process-Monte Carlo*, To appear in *Annals of Applied Probability* (2018).
2. M. Benaim, S. Le Borgne, F. Malrieu, and P-A. Zitt, *Qualitative properties of certain piecewise deterministic Markov processes*, *Annales de l'IHP Probabilités et Statistiques* **51** (2015), no. 3, 1040–1075.
3. J. Bierkens, P. Fearnhead, and G. Roberts, *The zig-zag process and super-efficient sampling for Bayesian analysis of big data*, *The Annals of Statistics* **47** (2019), no. 3, 1288–1320.
4. J. Bierkens, G. O Roberts, and P-A. Zitt, *Ergodicity of the zigzag process*, *The Annals of Applied Probability* **29** (2019), no. 4, 2266–2301.
5. J. Bierkens and S. M. Verduyn Lunel, *Spectral analysis of the zigzag process*, *Annales de l'Institut Henri Poincaré (B) Probabilités et Statistiques*, to appear, arXiv:1905.01691.
6. J.-F. Bony, D. Le Peutrec, and L. Michel, *Eyring-Kramers law for Fokker-Planck type differential operators*, arXiv:2201.01660.
7. A. Bouchard-Côté, S. J. Vollmer, and A. Doucet, *The bouncy particle sampler: A nonreversible rejection-free markov chain Monte Carlo method*, *Journal of the American Statistical Association* **113** (2018), no. 522, 855–867.
8. A. Bovier, V. Gayrard, and M. Klein, *Metastability in reversible diffusion processes. II. Precise asymptotics for small eigenvalues*, *J. Eur. Math. Soc.* **7** (2005), no. 1, 69–99.
9. M. H. A Davis, *Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models*, *Journal of the Royal Statistical Society: Series B (Methodological)* **46** (1984), no. 3, 353–376.
10. G. Deligiannidis, A. Bouchard-Côté, and A. Doucet, *Exponential Ergodicity of the Bouncy Particle Sampler*, *The Annals of Statistics* **47** (2019), no. 3, 1268–1287. MR 3911112
11. G. Di Gesù, T. Lelièvre, D. Le Peutrec, and B. Nectoux, *Jump markov models and transition state theory: the quasi-stationary distribution approach*, *Faraday Discussions* **195** (2017), 469–495.
12. J. Dolbeault, C. Mouhot, and C. Schmeiser, *Hypocoercivity for linear kinetic equations conserving mass*, *Transactions of the American Mathematical Society* **367** (2015), no. 6, 3807–3828.
13. A. Durmus, A. Guillin, and P. Monmarché, *Piecewise Deterministic Markov Processes and their invariant measure*, To appear in *Annales de l'IHP Probabilités et Statistiques* (2018).
14. ———, *Geometric ergodicity of the Bouncy Particle Sampler*, To appear in *Annals of Applied Probability* (2019).
15. A. Faggionato, D. Gabrielli, and M. Ribezzi-Crivellari, *Non-equilibrium thermodynamics of piecewise deterministic Markov processes*, *Journal of Statistical Physics* **137** (2009), no. 2, 259.
16. A. Guillin and B. Nectoux, *Low-Lying Eigenvalues and Convergence to the Equilibrium of Some Piecewise Deterministic Markov Processes Generators in the Small Temperature Regime*, *Ann. Henri Poincaré* **21** (2020), no. 11, 3575–3608.
17. B. Helffer, M. Klein, and F. Nier, *Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach*, *Mat. Contemp.* **26** (2004), 41–85.
18. B. Helffer and J. Sjöstrand, *Puits multiples en mécanique semi-classique. IV. Étude du complexe de Witten*, *Comm. Partial Differential Equations* **10** (1985), no. 3, 245–340.
19. F. Hérau, M. Hitrik, and J. Sjöstrand, *Tunnel effect and symmetries for Kramers-Fokker-Planck type operators*, *J. Inst. Math. Jussieu* **10** (2011), no. 3, 567–634.
20. D. Le Peutrec and L. Michel, *Sharp asymptotics for non-reversible diffusion processes*, *Probability and Mathematical Physics* **1** (2020), no. 1, 3–53.
21. L. Michel, *About small eigenvalues of the Witten Laplacian*, *Pure Appl. Anal.* **1** (2019), no. 2, 149–206.

22. L. Miclo and P. Monmarché, *Étude spectrale minutieuse de processus moins indécis que les autres*, Séminaire de Probabilités XLV, Springer, 2013, pp. 459–481.
23. P. Monmarché, *Piecewise deterministic simulated annealing*, ALEA, Latin American Journal of Probability and Mathematical Statistics **13** (2014), no. 1, 357–398.
24. E. A. J. F. Peters and G. de With, *Rejection-free Monte Carlo sampling for general potentials*, Physical Review E **85** (2012), no. 2, 026703.
25. J. Sjöstrand and M. Zworski, *Elementary linear algebra for advanced spectral problems*, Annales de l’institut Fourier, vol. 57, 2007, pp. 2095–2141.
26. E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), no. 4, 661–692 (1983).

D. LE PEUTREC, INSTITUT DENIS POISSON, UNIVERSITÉ D’ORLÉANS

Email address: dorian.le-peutrec@univ-orleans.fr

L. MICHEL, INSTITUT MATHÉMATIQUE DE BORDEAUX, UNIVERSITÉ DE BORDEAUX

Email address: laurent.michel@math.u-bordeaux.fr

BORIS NECTOUX, LABORATOIRE DE MATHÉMATIQUES BLAISE PASCAL, UNIVERSITÉ CLERMONT AUVERGNE

Email address: boris.nectoux@uca.fr