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A fundamental solution for the Laplace operator in a doubly connected domain

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Abstract

In this work, we find a harmonic Green function for a doubly connected domain using the theory of conformal mappings and then we define a Poisson kernel. Using these functions an integral representation formula for the solution of the Dirichlet problem for the Poisson equation is derived.

Keywords: Harmonic Green functions; multiply connected domain; conformal invariance.

Resumen

En este trabajo, encontramos una función armónica de Green para un dominio doblemente conexo usando la teoría de las transformaciones conformes y luego mediante ella definimos un núcleo de Poisson. Usando estas funciones se obtiene una fórmula de representación integral para la solución del problema de Dirichlet para la ecuación de Poisson.

Palabras claves: Funciones armónicas de Green; dominio múltiplemente conexo; invarianza conforme.

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1 Introduction

One of the main aim of the theory of boundary value problems for complex partial differential equations is to obtain solutions in analytic or closed form and many results in this direction are known for special types of equations like elliptic equations with constant or analytic coefficients. In most cases, these problems have been considered in simply connected domains, perhaps for the reason that dealing with boundary value problems in multiply connected domains leads to many additional difficulties, which in turn has implied that there are few results known on this subject, despite the fact that they represent a special interest since appear naturally in many problems of mechanics, especially in the filtration theory (see [1–3]) in hydrodynamics and also in the theory of composite materials (see [4, 5]).

It is well known that the Laplace equation is an important differential equation in all of applied mathematics. It arises in a many variety of physical and mathematical systems, like electromagnetism, fluid mechanics, potential theory, heat conduction, among others. That is why it is valuable and important to investigate boundary value problems involving that equation or its corresponding inhomogeneous equation, the Poisson equation, which models many non-steady state phenomena. The classical way to solve boundary value problems for the Poisson equation is by using the representation formulas via the fundamental solutions for the Laplace operator. There are several ways to construct them for a particular domain.

In this work, we find a harmonic Green function for a doubly connected domain using the theory of conformal mappings, which in turn allows us to define a Poisson kernel. With the help of these functions an integral representation formula for the solution of the Dirichlet problem for the Poisson equation is derived.

1.1 Conformal map

In the complex function theory one can say that the conformal mapping is the geometry theory of analytic functions and this can be exploited in solving boundary value problems involving such functions.

We then recall that a complex function $w = f(z)$ defined in a domain $S \subset \mathbb{C}$ is called conformal at a point $z_0 \in S$ if it preserves angles between smooth and oriented curves through z_0 , as well as preserving orientation. If f is conformal for all $z_0 \in S$ we say that f is a conformal map or conformal

transformation (see [6, 7]).

For analytic functions we have the following result:

Theorem 1. [7] *Let f be an analytic function defined in a domain $S \subseteq \mathbb{C}$ and $z_0 \in S$. If $f'(z_0) \neq 0$, then f is conformal at z_0 .*

A well known and important class of conformal mappings are the Möbius Transformations.

Definition 1. [6] *A mapping of the form $w(z) = \frac{az+b}{cz+d}$ is called a linear fractional transformation. If a, b, c, d are constant and also satisfy $ad-bc \neq 0$ then $w(z)$ is called a Möbius Transformation.*

In \mathbb{C} a region G_1 is conformally equivalent to another G_2 if there is an univalent analytic function $f : G_1 \rightarrow \mathbb{C}$ such that $f(G_1) = G_2$. This is an equivalence relation. Therefore by the Liouville's Theorem the finite complex plane \mathbb{C} and the extended complex plane $\mathbb{C} \cup \{\infty\}$ are not equivalents to any bounded region. By Riemann's Mapping Theorem, these regions and the unit disk are the only (except conformal isomorphisms) simply connected domains.

Theorem 2. (Riemann Mapping Theorem) [7] *Let D be a simply connected domain in the z plane, which is neither the z plane or the extended z plane. Then there exists a univalent function $f(z)$, such that $w = f(z)$ maps D onto the disk $|w| < 1$.*

1.2 Harmonic Green Function

Definition 2. *A real-valued function $G(z, z_0)$ in a domain D of \mathbb{C} having the properties*

- i) $G(z, z_0)$ is harmonic in $z \in D \setminus \{z_0\}$
- ii) $\log|z - z_0| + G(z, z_0)$ is harmonic in the neighborhood of z_0
- iii) $\lim_{z \rightarrow \partial D} G(z, z_0) = 0$,

is called the Green function of D , more exactly the Green function of D for the Laplace operator.

Other properties of the Green function that are derived from its definition and the maximum principle are the uniqueness, the positivity $G(z, z_0) > 0$, $z \in D \setminus \{z_0\}$, and the symmetry $G(z, z_0) = G(z_0, z)$, $z_0 \neq z$ (see [6, 8]).

One way of constructing Green's harmonic functions for planar domains is through the method of conformal invariance. More explicitly, let $\Omega \subset \mathbb{C}$ be a domain with Green function $G_{1\Omega}(w, \omega)$ and $w : D \rightarrow \Omega$ a conformal map from a domain $D \subset \mathbb{C}$ onto Ω . Then the Green function for D is given by

$$G_{1D}(z, \zeta) = G_{1\Omega}(w(z), \omega(\zeta)). \quad (1)$$

(see [6, 9]).

2 Harmonic Green Function for a doubly connected domain

By the Riemann mapping theorem any simple connected domain with a boundary of more than one point can be conformally mapped onto the unit disk. However, the picture is different if we are working on multiply connected domains. In this case, we have the following result:

Theorem 3. [10] *Any doubly connected domain can be conformally mapped onto an annulus.*

As an example we consider the following doubly connected domain

$$\mathbb{D}_2 = \{z \in \mathbb{C} : \mathbb{C} \setminus (|z| < 1 \cup |z - s| < r), 0 < r < 1\},$$

where $s = \frac{\delta + \beta}{2}$, $r = \frac{\delta - \beta}{2}$, $\delta, \beta \in \mathbb{R}$ and $1 < \beta < \delta$. As an annulus we take the concentric ring

$$\Omega = \{z \in \mathbb{C} : 0 < \rho < |w| < 1\},$$

where $\rho = \frac{\delta\beta - 1 - \sqrt{(\delta^2 - 1)(\beta^2 - 1)}}{(\delta - \beta)}$. See Fig. 1.

Now we show how is \mathbb{D}_2 conformally mapped onto Ω .

Lemma 1. *The Möbius Transformation*

$$w = w(z) = \frac{z - \alpha}{\alpha z - 1}, \quad (2)$$

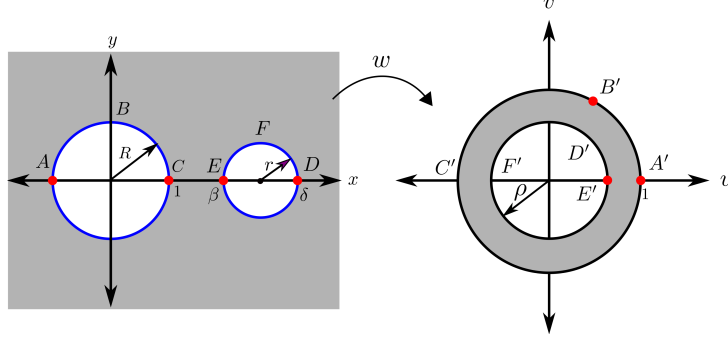


Fig. 1: \mathbb{D}_2 to the left, Ω to the right

where

$$\alpha = \frac{1 + \delta\beta + \sqrt{(\delta^2 - 1)(\beta^2 - 1)}}{(\delta + \beta)}, \quad \beta < \alpha < \delta, \quad (3)$$

conformally maps the doubly connected domain \mathbb{D}_2 onto the concentric ring Ω .

Proof. Substituting $z = Re^{i\theta}$ in (2) we get

$$w = \frac{Re^{i\theta} - \alpha}{\alpha Re^{i\theta} - 1} = \frac{e^{i\theta} - \alpha}{\alpha e^{i\theta} - 1},$$

and

$$|w|^2 = \left(\frac{e^{i\theta} - \alpha}{\alpha e^{i\theta} - 1} \right) \left(\frac{e^{-i\theta} - \alpha}{\alpha e^{-i\theta} - 1} \right) = \frac{1 - 2\alpha \cos(\theta) + \alpha^2}{\alpha^2 - 2\alpha \cos(\theta) + 1} = 1. \quad (4)$$

Therefore the circumference of radius $R = 1$ in the z -plane is mapped onto the circumference of radius 1 of the w -plane.

On the other hand, from $|z - s|^2 = r^2$ it follows

$$|z|^2 = r^2 + s(z + \bar{z}) - s^2. \quad (5)$$

Using (5) in $|w|^2$, we obtain

$$\begin{aligned} |w|^2 &= \left(\frac{z - \alpha}{\alpha z - 1} \right) \left(\frac{\bar{z} - \alpha}{\alpha \bar{z} - 1} \right) = \frac{|z|^2 + \alpha^2 - \alpha(z + \bar{z})}{\alpha^2 |z|^2 + 1 - \alpha(z + \bar{z})} \\ &= \frac{r^2 - s^2 + \alpha^2 + (z + \bar{z})(s - \alpha)}{\alpha^2(r^2 - s^2) + 1 + \alpha(z + \bar{z})(\alpha s - 1)} \end{aligned}$$

$$= \frac{(s - \alpha)}{\alpha(\alpha s - 1)} \left[\frac{\frac{r^2 - s^2 + \alpha^2}{(s - \alpha)} + (z + \bar{z})}{\frac{\alpha^2(r^2 - s^2) + 1}{\alpha(\alpha s - 1)} + (z + \bar{z})} \right]. \quad (6)$$

Substituting α and $s = \frac{\delta + \beta}{2}$ in the factor $\frac{(s - \alpha)}{\alpha(\alpha s - 1)}$ we get

$$\begin{aligned} \frac{(s - \alpha)}{\alpha(\alpha s - 1)} &= \frac{\beta + \delta - 2\alpha}{\alpha^2(\beta + \delta) - 2\alpha} \\ &= \frac{(\delta + \beta)^2 - 2 \left(1 + \delta\beta + \sqrt{(\delta^2 - 1)(\beta^2 - 1)} \right)}{(1 + \delta\beta) + \sqrt{(\delta^2 - 1)(\beta^2 - 1)}} \\ &= \frac{\left(1 + \delta\beta + \sqrt{(\delta^2 - 1)(\beta^2 - 1)} \right) - 2}{\left(1 + \delta\beta + \sqrt{(\delta^2 - 1)(\beta^2 - 1)} \right) - 2} \\ &= \frac{\left(1 + \delta\beta - \sqrt{(\delta^2 - 1)(\beta^2 - 1)} \right) (\delta + \beta)^2 - 2 \left((1 + \delta\beta)^2 - (\delta^2 - 1)(\beta^2 - 1) \right)}{(1 + \delta\beta)^2 - (\delta^2 - 1)(\beta^2 - 1)} \\ &= \frac{\delta\beta - 1 - \sqrt{(\delta^2 - 1)(\beta^2 - 1)}}{\delta\beta - 1 + \sqrt{(\delta^2 - 1)(\beta^2 - 1)}} = \frac{\left(\delta\beta - 1 - \sqrt{(\delta^2 - 1)(\beta^2 - 1)} \right)^2}{(\delta\beta - 1)^2 - (\delta^2 - 1)(\beta^2 - 1)} \\ &= \frac{\left(\delta\beta - 1 - \sqrt{(\delta^2 - 1)(\beta^2 - 1)} \right)^2}{(\delta - \beta)^2} = \rho^2 \end{aligned} \quad (7)$$

Since

$$\alpha(\alpha^2 - \delta\beta)(\alpha s - 1) = (s - \alpha)(1 - \alpha^2\delta\beta)$$

implies

$$\begin{aligned} s(\alpha^4 - 1) - [\alpha^3(1 + \delta\beta) - \alpha(\delta\beta + 1)] &= 0 \\ s(\alpha^2 - 1)(\alpha^2 + 1) - \alpha(1 + \delta\beta)(\alpha^2 - 1) &= 0 \\ \alpha^2(\delta + \beta) - 2\alpha(1 + \beta\delta) + (\delta + \beta) &= 0. \end{aligned}$$

Then the equality

$$\frac{r^2 - s^2 + \alpha^2}{(s - \alpha)} = \frac{\alpha^2(r^2 - s^2) + 1}{\alpha(\alpha s - 1)} \quad (8)$$

is equivalent to

$$\alpha^2(\delta + \beta) - 2\alpha(1 + \beta\delta) + (\delta + \beta) = 0,$$

where the number α defined in (3) is a solution of this equation. Thus taking into account (6) we have $|w| = \rho$ and the circumference of radius r in the z -plane is mapped onto the circumference of radius ρ of the w -plane.

The transformation $w = w(z)$ defined by (2) preserves orientation (see [6]), that is, in the path from A to B and B to C , what is left and right of the disk $|z| < 1$ of this path is mapped to the left and right of the path of A' to B' and B' to C' , the same happens for the disk $|z - s| < r$. \square

Now we are going to find the Green function for \mathbb{D}_2 via conformal invariance.

Using the conformal map $w : \mathbb{D}_2 \rightarrow \Omega$ defined by (2), the equation (1) and the following Green function of the concentric ring Ω (see [11]):

$$G_1(w, \omega) = \frac{\log |w|^2 \log |\omega|^2}{\log \rho^2} - \log \left| \frac{\omega - w}{1 - w\bar{\omega}} \prod_{n=1}^{\infty} \frac{(w - \rho^{2n}\omega)(\omega - \rho^{2n}w)}{(w\bar{\omega} - \rho^{2n})(1 - \rho^{2n}\bar{w}\omega)} \right|^2, \quad (9)$$

we obtain

$$G_1(z, \zeta) = \frac{\log \left| \frac{z-\alpha}{\alpha z-1} \right|^2 \log \left| \frac{\zeta-\alpha}{\alpha \zeta-1} \right|^2}{\log \rho^2} - \log \left| \frac{\frac{\zeta-\alpha}{\alpha \zeta-1} - \frac{z-\alpha}{\alpha z-1}}{1 - \frac{z-\alpha}{\alpha z-1} \frac{\bar{\zeta}-\alpha}{\alpha \bar{\zeta}-1}} \prod_{n=1}^{\infty} \frac{\left[\frac{z-\alpha}{\alpha z-1} - \frac{\zeta-\alpha}{\alpha \zeta-1} \rho^{2n} \right] \left[\frac{\zeta-\alpha}{\alpha \zeta-1} - \frac{z-\alpha}{\alpha z-1} \rho^{2n} \right]}{\left[\frac{z-\alpha}{\alpha z-1} \frac{\bar{\zeta}-\alpha}{\alpha \bar{\zeta}-1} - \rho^{2n} \right] \left[1 - \rho^{2n} \frac{\bar{z}-\alpha}{\alpha \bar{z}-1} \frac{\zeta-\alpha}{\alpha \zeta-1} \right]} \right|^2$$

Since $\rho^{2n} = \left(\frac{\alpha s-1}{r}\right)^{2n}$ and denoting $\tau = \alpha s - 1$, we have

$$G_1(z, \zeta) = \frac{\log \left| \frac{z-\alpha}{\alpha z-1} \right|^2 \log \left| \frac{\zeta-\alpha}{\alpha \zeta-1} \right|^2}{\log \left(\frac{\alpha s-1}{r}\right)^2} - \log \left| \frac{\frac{(\zeta-\alpha)(\alpha z-1) - (z-\alpha)(\alpha \zeta-1)}{(\alpha \zeta-1)(\alpha z-1)}}{\frac{(\alpha z-1)(\alpha \bar{\zeta}-1) - (z-\alpha)(\bar{\zeta}-1)}{(\alpha z-1)(\alpha \bar{\zeta}-1)}} \right|$$

$$\times \prod_{n=1}^{\infty} \left[\frac{\left(\frac{r^{2n}(z-\alpha)(\alpha \zeta-1) - (\zeta-\alpha)(\alpha z-1)\tau^{2n}}{r^{2n}(\alpha z-1)(\alpha \zeta-1)} \right)}{\left(\frac{r^{2n}(z-\alpha)(\bar{\zeta}-\alpha) - (\alpha z-1)(\alpha \bar{\zeta}-1)\tau^{2n}}{r^{2n}(\alpha z-1)(\alpha \bar{\zeta}-1)} \right)} \right]$$

$$\begin{aligned}
& \times \left[\frac{\left(\frac{r^{2n}(\zeta-\alpha)(\alpha z-1)-(z-\alpha)(\alpha\zeta-1)\tau^{2n}}{r^{2n}(\alpha\zeta-1)(\alpha z-1)} \right)}{\left(\frac{r^{2n}(\alpha\bar{z}-1)(\alpha\zeta-1)-(\bar{z}-\alpha)(\zeta-z)\tau^{2n}}{r^{2n}(\alpha\bar{z}-1)(\alpha\zeta-1)} \right)} \right]^2 \\
& = \frac{\log \left| \frac{z-\alpha}{\alpha z-1} \right|^2 \log \left| \frac{\zeta-\alpha}{\alpha\zeta-1} \right|^2}{\log \left(\frac{\alpha s-1}{r} \right)^2} \\
& \quad - \log \left| \frac{(\alpha\bar{\zeta}-1) [(\zeta-\alpha)(\alpha z-1) - (z-\alpha)(\alpha\zeta-1)]}{(\alpha\zeta-1) [(\alpha z-1)(\alpha\bar{\zeta}-1) - (z-\alpha)(\bar{\zeta}-1)]} \right| \\
& \quad \times \prod_{n=1}^{\infty} \left[\frac{(\alpha\bar{z}-1)(\alpha\bar{\zeta}-1)(r^{2n}(z-\alpha)(\alpha\zeta-1) - \tau^{2n}(\zeta-\alpha)(\alpha z-1))}{(\alpha z-1)(\alpha\zeta-1)(r^{2n}(z-\alpha)(\bar{\zeta}-\alpha) - \tau^{2n}(\alpha\bar{\zeta}-1)(\alpha z-1))} \right] \\
& \quad \times \left[\frac{(r^{2n}(\zeta-\alpha)(\alpha z-1) - \tau^{2n}(z-\alpha)(\alpha\zeta-1))}{(r^{2n}(\alpha\bar{z}-1)(\alpha\zeta-1) - \tau^{2n}(\bar{z}-\alpha)(\zeta-\alpha))} \right]^2 \\
& = \frac{\log \left| \frac{z-\alpha}{\alpha z-1} \right|^2 \log \left| \frac{\zeta-\alpha}{\alpha\zeta-1} \right|^2}{\log \left(\frac{\alpha s-1}{r} \right)^2} - \log \left| \frac{(\alpha\bar{\zeta}-1)(\zeta-z)}{(\alpha\zeta-1)(1-z\bar{\zeta})} \right| \\
& \quad \times \prod_{n=1}^{\infty} \left[\frac{(\alpha\bar{z}-1)(\alpha\bar{\zeta}-1)(r^{2n}(z-\alpha)(\alpha\zeta-1) - \tau^{2n}(\zeta-\alpha)(\alpha z-1))}{(\alpha z-1)(\alpha\zeta-1)(r^{2n}(z-\alpha)(\bar{\zeta}-\alpha) - \tau^{2n}(\alpha\bar{\zeta}-1)(\alpha z-1))} \right] \\
& \quad \times \left[\frac{r^{2n}(\zeta-\alpha)(\alpha z-1) - \tau^{2n}(z-\alpha)(\alpha\zeta-1)}{r^{2n}(\alpha\bar{z}-1)(\alpha\zeta-1) - \tau^{2n}(\bar{z}-\alpha)(\zeta-\alpha)} \right]^2 \\
& = \frac{\log \left| \frac{z-\alpha}{\alpha z-1} \right|^2 \log \left| \frac{\zeta-\alpha}{\alpha\zeta-1} \right|^2}{\log \left(\frac{\alpha s-1}{r} \right)^2} - \log \left| \frac{(\alpha\bar{\zeta}-1)}{(\alpha\zeta-1)} \right|^2 \left| \frac{(z-\alpha)}{(1-z\bar{\zeta})} \right|^2 \\
& \quad - \log \prod_{n=1}^{\infty} \left| \frac{r^{2n}(z-\alpha)(\alpha\zeta-1) - \tau^{2n}(\zeta-\alpha)(\alpha z-1)}{r^{2n}(z-\alpha)(\bar{\zeta}-\alpha) - \tau^{2n}(\alpha\bar{\zeta}-1)(\alpha z-1)} \right|^2 \\
& \quad - \log \prod_{n=1}^{\infty} \left| \frac{r^{2n}(\zeta-\alpha)(\alpha z-1) - \tau^{2n}(z-\alpha)(\alpha\zeta-1)}{r^{2n}(\alpha\bar{z}-1)(\alpha\zeta-1) - \tau^{2n}(\bar{z}-\alpha)(\zeta-\alpha)} \right|^2
\end{aligned}$$

Green's harmonic function is invariant under conformal mappings (see [6]),

so Green's harmonic function of \mathbb{D}_2 is

$$\begin{aligned}
 G_1(z, \zeta) &= \frac{\log \left| \frac{z-\alpha}{\alpha z-1} \right|^2 \log \left| \frac{\zeta-\alpha}{\alpha \zeta-1} \right|^2}{\log \left(\frac{\alpha s-1}{r} \right)^2} - \log \left| \frac{(\zeta-z)}{(1-z\bar{\zeta})} \right|^2 \\
 &\quad - \log \prod_{n=1}^{\infty} \left| \frac{(\alpha s-1)^{2n} (z-\alpha)(\alpha \zeta-1) - r^{2n} (\zeta-\alpha)(\alpha z-1)}{(\alpha s-1)^{2n} (z-\alpha)(\bar{\zeta}-\alpha) - r^{2n} (\alpha \bar{\zeta}-1)(\alpha z-1)} \right|^2 \\
 &\quad - \log \prod_{n=1}^{\infty} \left| \frac{(\alpha s-1)^{2n} (\zeta-\alpha)(\alpha z-1) - r^{2n} (z-\alpha)(\alpha \zeta-1)}{(\alpha s-1)^{2n} (\alpha \bar{z}-1)(\alpha \zeta-1) - r^{2n} (\bar{z}-\alpha)(\zeta-\alpha)} \right|^2
 \end{aligned}$$

3 Boundary value problems

As we know the harmonic Green function, is a fundamental solution to the Laplace operator. Then it is used to solve the Dirichlet boundary value problem for the Poisson equation via corresponding integral representation formulas for solutions. To do that, first we define the Poisson kernel $K(z, \zeta)$ for our doubly connected domain \mathbb{D}_2 in the form:

$$K(z, \zeta) = -\frac{1}{2} \partial_{\nu_\zeta} G_1(z, \zeta), \quad z \in \mathbb{D}_2, \quad \zeta \in \partial \mathbb{D}_2. \quad (10)$$

This Poisson kernel has the following property

$$\lim_{z \rightarrow z_0} \frac{1}{2\pi i} \int_{\partial \mathbb{D}_2} \gamma(\zeta) K(z, \zeta) \frac{d\zeta}{\zeta} = \gamma(z_0), \quad z \in \mathbb{D}_2, \quad z_0 \in \partial \mathbb{D}_2, \quad \gamma \in C(\partial \mathbb{D}_2, \mathbb{C}). \quad (11)$$

A way to prove this property is through the conformal mapping $w : \mathbb{D}_2 \rightarrow \Omega$ defined by (2), see also Fig. 1.

In Ω the solution of the Dirichlet problem for the Laplace equation

$$\Delta u = 0 \text{ in } \Omega, \quad u = \gamma \text{ on } \partial \Omega,$$

is given by the Poisson integral formula:

$$u(\cdot) = \int_{\partial \Omega} \gamma(\zeta) K(\cdot, \zeta) \frac{d\zeta}{\zeta} \quad (\text{see [11]}).$$

Using the conformal mapping w , the function $u_1(z) = u(w(z))$, $z \in \mathbb{D}_2$ is a solution of Dirichlet problem:

$$\Delta u_1 = 0 \text{ in } \mathbb{D}_2, \quad u_1 = \gamma_1 \text{ on } \partial\mathbb{D}_2,$$

since it is clear that the Poisson integral formula

$$u_1(z) = \int_{\partial\mathbb{D}_2} \gamma_1(\zeta) K(z, \zeta) \frac{d\zeta}{\zeta},$$

where $K(z, \zeta)$ is given by (10), is a harmonic function. On the other hand, for $z \in \mathbb{D}_2$ and $z_0 \in \partial\mathbb{D}_2$, we obtain:

$$\lim_{z \rightarrow z_0} u_1(z) = \lim_{z \rightarrow z_0} u(w(z)) = u(w(z_0)) = \gamma(w(z_0)) = \gamma_1(z_0),$$

because $w(z_0) \in \partial\Omega$.

Next, we have the following result:

Theorem 4. *The Dirichlet problem for the Poisson equation in \mathbb{D}_2*

$$u_{z\bar{z}} = f \text{ in } \mathbb{D}_2 \quad u = \gamma \text{ on } \partial\mathbb{D}_2 \quad (12)$$

for $f \in C(\mathbb{D}_2, \mathbb{C}) \cap L_2(\mathbb{D}_2, \mathbb{C})$, $\gamma \in C(\partial\mathbb{D}_2, \mathbb{C})$ given has a unique solution belonging to $C^2(\mathbb{D}_2, \mathbb{C}) \cap C(\overline{\mathbb{D}_2}, \mathbb{C})$. The solution is expressed by

$$u(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}_2} \gamma(\zeta) |\zeta| K(z, \zeta) \frac{d\zeta}{\zeta} - \frac{1}{\pi} \int_{\mathbb{D}_2} f(\zeta) G_1(z, \zeta) d\xi d\eta. \quad (13)$$

Proof. (see [12])

Since the Poisson kernel is a harmonic function, and the Pompeiu operator

$$T_\Omega f(z) = -\frac{1}{\pi} \int_{\Omega} f(\zeta) G_1(z, \zeta) d\xi d\eta$$

satisfies that $\partial_{z\bar{z}} T_\Omega f(z) = f(z)$ in the distributional sense, we get:

$$u_{z\bar{z}} = f.$$

On the other hand, by the kernel Poisson property (11) and the behavior of the Green function $G_1(z, \zeta)$ at the boundary $\partial\mathbb{D}_2$, we have:

$$u = \gamma \text{ in } \partial\mathbb{D}_2$$

□

4 Concluding remarks

The method of obtaining harmonic Green functions for doubly connected domains via conformal transformations can also be used to obtain other special kernel functions such as harmonic Neumann functions and harmonic Robin functions in such domains. With such functions we can then solve other boundary value problems different from the one posed in this work which can serve as models for different physical phenomena. The study of boundary value problems in multiply connected domains and from the numerical point of view is still in progress.

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