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# Time Minimal Syntheses in the Abnormal Case using Geometric and Computational Techniques.

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**Abstract:** This article deals with the optimization of chemical network in the frame of geometric optimal control and singularity theory. The objective is to classify the generic syntheses for analytic systems of the form  $\dot{q}(t) = X(q(t)) + u(t)Y(q(t))$ , where the aim is to reach in minimum time a terminal manifold of codimension one. We restrict the study to the three dimensional case considering the McKeithan network for which situations up to codimension 3 have to be investigated. We develop symbolic algorithms to derive semi-normal forms and to compute approximations of the strata of the singular set to obtain classification in the generic cases and to illustrate the role of singularity theory in geometric control, in relation with abnormal geodesics and regularity of the value function.

*Keywords:* Chemical reaction networks · Geometric optimal control · Time minimal syntheses · Regularity of the value function

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## INTRODUCTION

Within the last decades, reactors dynamics has been an important research area [Feinberg, 1972, Horn et al, 1972], having some industrial impact where the control of such processes is a substantial task. One important application in medicine concerns the control of the McKeithan network [McKeithan, 1995] aiming to maximize the production of one chemical species. Assuming that the dependence with respect to the ambient temperature is described by the Arrhenius law, and using the derivative of the temperature as a control, which consists into Goh transformation, the problem can be transformed into a time minimal control problem with a 3d-affine system and where the terminal manifold  $N$  is given by fixing at the final time a fixed amount of the desired species [Bonnard et al, 2019]. Such process can be modeled by a system in  $\mathbb{R}^3$  of the form

$$\dot{q}(t) = X(q(t)) + u(t)Y(q(t)), \quad |u| \leq 1, \quad (1)$$

where  $X, Y$  are analytic vector fields, the control  $u(\cdot)$  is a measurable function on  $[0, T]$  and we require that  $q(T) \in N$ .

In this context, among a series of works, the article [Bonnard et al, 1997] addressed the problem of computing in a neighborhood  $U$  of  $N$  the feedback optimal control  $u^*(q)$ , which reaches  $N$  from  $q \in U$  in minimum time. The techniques combine tools from singularity theory adapted to control theory to determine the time minimal syntheses using the concept of graded semi-normal forms and unfolding [Martinet, 1982]. The aim of this article is to complete the results of [Bonnard et al, 1997, Launay

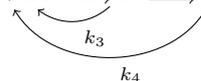
et al, 1997] in the so-called abnormal (or exceptional) case where the geodesics are tangent to the terminal manifold  $N$ .

The singularity theoretical frame was already present in those earliest references: construction of semi-normal form and concept of unfolding but recent progress of formal languages are used to handle complicated computation in particular in relation with the (reversible) McKeithan network.

### 1. TIME MINIMAL EXCEPTIONAL GEODESICS IN OPTIMIZATION OF CHEMICAL NETWORKS

#### 1.1 A brief recap about the optimal control of chemical networks

In this section, we introduce in short the concepts for the optimization of chemical reaction networks. We refer to [Feinberg, 2019] for the modelisation of such process and the underlying differential system. We consider the McKeithan network:  $T + M \xrightarrow{k_1} A \xrightarrow{k_2} B$ . As a deficiency zero



network, the system has a unique (global) asymptotically stable equilibrium point in the positive orthant [Sontag, 2001].

The state space is formed by the concentration vector:

$$c = (c_T, c_M, c_A, c_B)^T$$

of the respective chemical species. We note  $\delta_1 = c_T + c_A + c_B$ ,  $\delta_2 = c_M + c_A + c_B$  the first integrals associated to

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\* The authors thank Bernard Bonnard for the fruitful discussions.

the dynamics and let  $x = c_A$ ,  $y = c_B$ , then the system is described by the equation:

$$\begin{aligned}\dot{x} &= -\beta_2 x v^{\alpha_2} - \beta_3 x v^{\alpha_3} - \delta_3 v(x+y) + \delta_4 v + v(x+y)^2 \\ \dot{y} &= \beta_2 x v^{\alpha_2} - \beta_4 y v^{\alpha_4} \\ \dot{z} &= u, \quad |u| \leq 1,\end{aligned}$$

with

$$0 \leq x \leq \delta_1, \quad 0 \leq y \leq \delta_2, \quad \delta_3 = \delta_1 + \delta_2, \quad \delta_4 = \delta_1 \delta_2,$$

the Arrhenius law gives  $k_i = A_i \exp(-E_i/RT)$ ,  $i = 1, 2, 3, 4$  ( $E_i$  is the activation energy,  $T$  is the temperature,  $A_i, R$  are constant) and

$$v = k_1, \quad k_2 = \beta_2 v^{\alpha_2}, \quad k_3 = \beta_3 v^{\alpha_3}, \quad k_4 = \beta_4 v^{\alpha_4}.$$

Maximizing the product of the  $A$  species leads to a time minimal control problem with a terminal manifold  $N : x = d$ ,  $d$  being the desired production of  $A$ , see [Bonnard et al, 2019] for more details.

### 1.2 Time minimal control for 3d-affine systems and Maximum Principle

We consider a three dimensional single-input control system of the form

$$\frac{dq}{dt}(t) = X(q(t)) + u(t)Y(q(t))$$

with  $q = (x, y, z)$ ,  $(X, Y)$  being smooth ( $C^\infty$  or  $C^\omega$ ) vector fields and the control  $u(\cdot)$  is a measurable mapping on  $J = [0, t_f(u)]$  valued in the interval  $[-1, +1]$ . We denote by  $q(\cdot, q_0, u)$  the response to  $u(\cdot)$  defined on a subinterval of  $J$ , with  $q(0) = q_0$ , written in short as  $q(\cdot)$ . Together with this controlled dynamics, our aim is to minimize the time to reach a terminal manifold  $N$  of codimension 1.

**Maximum principle [Pontryagin et al, 1964].** Consider the problem of optimizing the transfer time from  $q_0$  to a smooth submanifold  $N$  of  $\mathbb{R}^3$ . Then, if the pair  $(u(\cdot), q(\cdot))$  is optimal on  $[0, t_f]$  there exists an absolutely continuous function  $p(\cdot) \in \mathbb{R}^3 \setminus \{0\}$  such that the following conditions are satisfied. Denoting  $H(q, p, u) = p \cdot (X + uY)$ , with  $\cdot$  the scalar product of  $\mathbb{R}^3$ , one has:

- (1) the triplet  $(q, p, u)$  is solution a.e. on  $[0, t_f]$  of

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \\ H(q, p, u) &= \max_{|v| \leq 1} H(q, p, v).\end{aligned}\tag{2}$$

- (2)  $M(q, p) = \max_{|v| \leq 1} H(q, p, v)$  is constant and equals to  $-p_0$  and  $-p_0$  is nonnegative in the time minimal case and nonpositive in the time maximal case.  
(3) The vector  $p(\cdot)$  satisfies at the final time  $t_f$  the transversality conditions

$$p(t_f) \perp T_{q(t_f)}N.$$

*Definition 1.* A triplet  $(q, p, u)$  solutions of (2) is called an extremal and the  $q$ -projection is called a geodesic. If moreover it satisfies the transversality condition it is called a BC-extremal. An extremal is called regular if the control is given by  $u(t) = \text{sign } p(t) \cdot Y(q(t))$  a.e.. It is called bang-bang if the number of switches is finite. An extremal is called singular if  $p(t) \cdot Y(q(t)) = 0$  on  $[0, t_f]$ . An extremal is called abnormal (or exceptional) if  $p_0 = 0$ , so that it is candidate to minimize or maximize the transfer time.

**Computation of the singular controls.** If  $Z_1, Z_2$  denote two (smooth) vector fields, the Lie bracket is computed with the convention:

$$[Z_1, Z_2](q) = \frac{\partial Z_1}{\partial q}(q)Z_2(q) - \frac{\partial Z_2}{\partial q}(q)Z_1(q).$$

In the singular case, deriving twice the equation  $t \mapsto p(t) \cdot Y(q(t)) = 0$ , one gets:

$$\begin{aligned}p(t) \cdot Y(q(t)) &= p(t) \cdot [Y, X](q(t)) = 0, \\ p(t) \cdot ([Y, X], X](q(t)) + u(t)[Y, X], Y](q(t)) &= 0.\end{aligned}$$

Introducing the determinants:

$$\begin{aligned}D &= \det(Y, [Y, X], [[Y, X], Y]), \\ D' &= \det(Y, [Y, X], [[Y, X], X]), \\ D'' &= \det(Y, [Y, X], X),\end{aligned}$$

one has the following.

*Proposition 2.* If  $D$  is nonzero, the singular controls are defined by the feedback  $u_s(q) = -D'(q)/D(q)$  so that the corresponding geodesics are solutions of the vector field :  $X_s(q) = X(q) + u_s(q)Y(q)$ . Abnormal (exceptional) geodesics are contained in the set  $D'' = 0$ .

**High-order Maximum Principle in the singular case.** To distinguish between time minimizing or maximizing geodesics, one needs to use the high-order Maximum Principle, see [Krener, 1977].

*Proposition 3.* Candidates to time minimizing are contained in  $DD'' \geq 0$  and candidates to time maximizing are contained in the set  $DD'' \leq 0$ . If the corresponding inequalities are strict, they are respectively called hyperbolic or elliptic.

*Proof.* It is based on the Legendre-Clebsch condition, which takes the form  $p(t) \cdot [[Y, X], Y](q(t)) \geq 0$ , see [Kupka, 1987] for details.  $\square$

## 2. TIME MINIMAL SYNTHESIS IN THE EXCEPTIONAL CASES

### 2.1 General concepts and notation

We consider a local neighborhood  $U$  of  $q_0 \in N$ , which can be identified to 0. One wants to describe the time minimal syntheses in a small neighborhood  $U$  of 0. We denote respectively by  $\sigma_\pm^0$ ,  $\sigma_s^0$  bang and singular arcs terminating at 0. Our aim is to analyze the situation where the arc is tangent to  $N$  at 0. This is related to the bang exceptional case and the singular exceptional case.

In  $U$ , each optimal solution is a concatenation by arcs  $\sigma_+, \sigma_-$ , where the control is  $u = \pm 1$  and singular arcs  $\sigma_s$ . Our method consists of the construction of *semi-normal forms* for the action of the pseudo-group  $\mathcal{G}$  of local diffeomorphisms  $\varphi$  such that  $\varphi(0) = 0$ ,  $\varphi * Y = Y$  and feedback transformation  $u \rightarrow -u$  (so that  $\sigma_+$  and  $\sigma_-$  can be exchanged).

If the optimal control  $u^*(v) \in [-1, 1]$  exists and is unique, it is regular on an open subset of  $U$ , union of  $U_+$  where  $u^*(v) = 1$  and  $U_-$  where  $u^*(v) = -1$ . The surface  $S$  which separates  $U_+$  from  $U_-$  is subanalytic and can be stratified into [Boltyanskii, 1966]

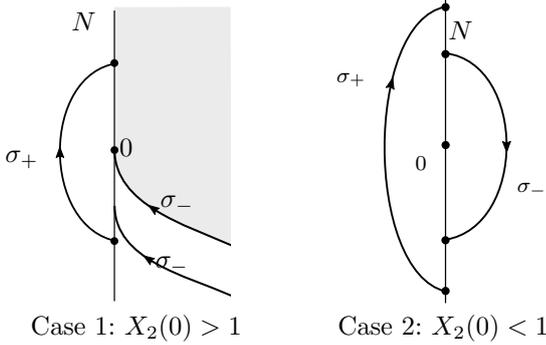


Fig. 1. Local synthesis near  $\mathcal{E}$  in the generic case.

- a switching locus  $W$ : closure of the set of points where  $u^*$  is regular and not continuous. We denote by  $W_+$  (resp.  $W_-$ ) the points of  $W$  where the optimal control is  $+1$  (resp.  $-1$ ) on  $N$ ,
- a cut-locus  $C$ : closure of the set of points where a trajectory loses its (global) optimality,
- singular locus  $\Gamma_s$ : union of optimal singular trajectories,

and these strata can be approximated by semialgebraic sets using seminormal forms.

The syntheses are described in details in [Bonnard et al, 1997, Launay et al, 1997] up to the codimension two situations and we recall below the main points to be applied to the McKeithan network.

## 2.2 Local syntheses in the exceptional cases

*The bang exceptional case.* In the bang exceptional situation, we have to consider the two following cases.

**Generic case (codimension one).** In this case both arcs  $\sigma_+^0$  and  $\sigma_-^0$  arc tangent to  $N$  but with a contact of order 2. Using the concept of unfolding, one can define a  $C^0$ -foliation of  $U$  by invariant planes so that in each plane the system takes the semi-normal form:

$$\begin{cases} \dot{x} = by + o(|x|, |y|) \\ \dot{y} = X_2(x, y) + u, & |u| \leq 1, \end{cases} \quad (3)$$

where  $b = n \cdot [Y, X] \neq 0$ , which can be normalized to 1,  $n = (1, 0)$  being the normal to  $N$  identified to  $x = 0$ . Moreover one can assume that  $1 + X_2(0) > 0$  and we have two cases.

*Proposition 4.* Using the previous normalization we have two cases described in Fig.1.

The difference between the two cases is related to different accessibility properties of the system. In the case  $X_2(0) > 1$ , the target  $N$  is not accessible from the points in  $x > 0$  above the arc  $\sigma_-^0$  terminating at 0. In the case  $X_2(0) < 1$ , each point of  $U$  can be steered in minimum time to  $N$ , the domain  $U_+$  where the optimal control is  $+1$  being  $x < 0$  and the domain  $U_-$  with optimal control  $-1$  being  $x > 0$ .

**Codimension two case.** A more complex situation occurs assuming that the arc  $\sigma_-^0$  has a contact of order

three with  $N$ , while  $\sigma_+^0$  has a contact of order two. The optimal syntheses cannot be described by foliating  $N$  by  $2d$ -planes as in the previous cases.

One needs to introduce the following normalization. We assume that  $Y = \frac{\partial}{\partial z}$ ,  $N$  is the plane  $x = 0$  parameterized as the image of:  $(w, s) \mapsto (0, w, s)$  and, along the  $y$ -axis,  $n \cdot X = 0$  and  $[X, Y] = \frac{\partial}{\partial x}$ . Denoting  $n = (1, 0, 0)$ , the normal to  $N$  at 0, we assume

- bang exceptional case:  $n \cdot X(0) = 0, n \cdot [Y, X](0) \neq 0,$
- $\det(X, Y, [Y, X]) \neq 0$  at 0,
- $\{n \cdot X = 0\} \cap N$  is a curve which is neither tangent to  $X$  nor to  $Y$  at 0.

Using the concept of semi-normal form the optimal syntheses can be described by the following model:

$$\begin{cases} \dot{x} = z \\ \dot{y} = b, & N : (w, s) \mapsto (0, w, s), \\ \dot{z} = 1 + u + y \end{cases} \quad (4)$$

and we have two types of time minimal syntheses.

*Proposition 5.* Assume  $b < 0$ . Then each point of  $U$  can be steered to  $N$ . Moreover

- (1)  $U^+ \setminus N \subset \{x < 0\}$  and  $U^- \setminus N \subset \{x > 0\}$
- (2) Optimal trajectories  $\sigma_-$  arrive at any point  $(0, w, s < 0)$  or  $(0, w \geq 0, s)$  of  $N \cap U$ .

The optimal synthesis is described by Fig.2.

*Proposition 6.* Assume  $b > 0$ . In this case the system is not locally controllable at 0. We represent on Fig.3 the synthesis in this case.

We shall refer to [Launay et al, 1997] for the full details of the computation and description of the syntheses.

*The singular exceptional case.* In this case we can assume  $Y = \frac{\partial}{\partial z}$ ,  $N$  is the plane  $x = 0$ , the normal to  $N$  at 0 is  $n = (1, 0, 0)$  and moreover:

$$\begin{cases} n \cdot X(0) = 0 \\ n \cdot [Y, X](0) = 0 \end{cases}$$

and we assume the following generic conditions

- $X$  and  $Y$  are independent at 0,
- $\det(Y, [Y, X], \text{ad}^2 Y \cdot X) \neq 0$  at 0,
- $\{n \cdot X = 0\} \cap N$  is a curve which is not tangent to  $X$  at 0.

Using the concept of semi-normal form the optimal syntheses can be described by the following

$$\begin{cases} \dot{x} = y + z^2 \\ \dot{y} = b + b_1 z. \\ \dot{z} = c + u \end{cases} \quad (5)$$

In this model, the exceptional locus  $\mathcal{E} \cap N$  is approximated by the parabola:  $w + s^2 = 0$  and we denote by  $\mathcal{E}_- : \{n \cdot X(q) < 0\} \cap N$  and  $\mathcal{E}_+ = \{n \cdot X(q) > 0\} \cap N$ .

We revise the six cases presented in [Launay et al, 1997], where we compute the optimal policies using symbolic computation (obtaining a different result for the Case 3). One noticeable difference with the non exceptional case (see [Bonnard et al, 1997]) is the presence of trajectories intersecting  $N$  twice and leads to an other component of  $S$ .

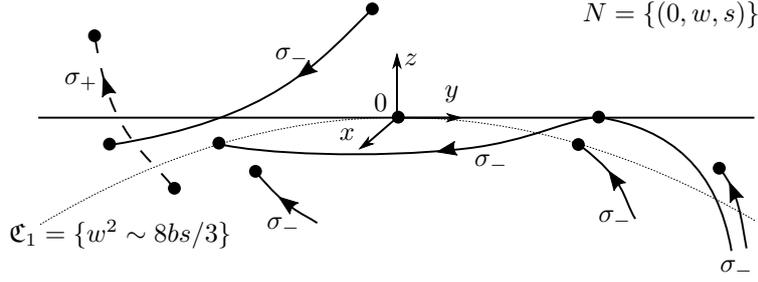


Fig. 2. Local synthesis near  $\mathcal{E}$  in the codimension 2 case for  $b < 0$ . The dashed curves are in the region  $x \leq 0$ .

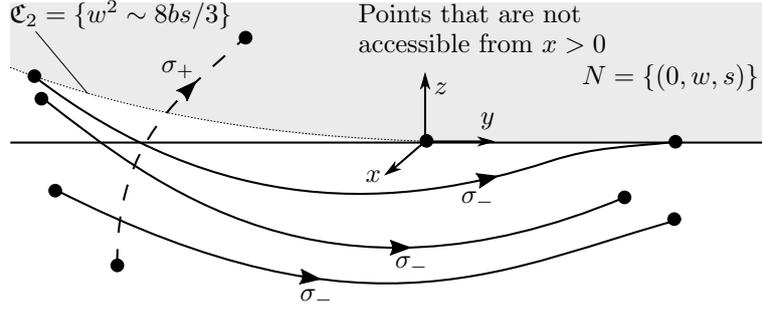


Fig. 3. Local synthesis near  $\mathcal{E}$  in the codimension 2 case for  $b > 0$ .

- *Case 1:*  $b > 0$ ,  $u_s(0) > 3$ . The surface  $S$  is a union of the switching locus and a subset of  $\mathcal{E}_-$ .
- *Case 2:*  $b > 0$ ,  $1 < u_s(0) < 3$ . The surface  $S$  is the union of a splitting locus, a switching locus and a subset of  $\mathcal{E}_-$ .
- *Case 3:*  $b > 0$ ,  $0 \leq u_s(0) < 1$ . The surface  $S$  is the union of a singular locus filled by admissible singular arcs, a splitting locus, a switching locus and a subset of  $\mathcal{E}_-$ .
- *Case 4:*  $b < 0$ ,  $u_s(0) > 3$ . In this situation the set  $S$  is the union of a switching locus and a subset of  $\mathcal{E}_+$ .
- *Case 5:*  $b < 0$ ,  $1 < u_s(0) < 3$ . In this case  $S$  is the union of a splitting locus, a switching locus  $W_+$  and a subset of  $\mathcal{E}_+$ .
- *Case 6:*  $b < 0$ ,  $0 \leq u_s(0) < 1$ . In this case  $S$  is a union of a singular locus, a splitting locus and  $\mathcal{E}_+$ .

This gives the complete classification under generic assumptions. Again the stratification of  $S$  can be computed in the original coordinates and applied to the McKeithan network.

**Illustration of the method based on symbolic computation.** We present an algorithm to compute an approximation of the surface  $S$  in the codimension 2 case, more specifically we treat the Case 3 described above and given by the model (5) with  $b > 0$  and such that the singular trajectory arriving at 0 is not saturating, that is  $0 \leq b_1/2 - c < 1$ .

*Method.* The following steps involve symbolic computation to obtain the optimal policy based on [Kupka, 1987].

1. Take  $q(0) = (0, w, s) \in N \cap U$ . We first determine the stratification of the surface  $N$ . Since  $Y$  is tangent to  $N$ ,

$q(0)$  is a switching point. If  $q(0)$  is an ordinary switching point, the optimal control is regular:  $u(0) = \text{sign}(p(0) \cdot [Y, X](0))$ . If it is a fold point, the optimal control may be singular and the optimal policy is determined using [Bonnard et al., 1993]. Note that since  $u_s(0) = b_1/2 - c < 1$ , the singular trajectories cannot be parabolic but either hyperbolic or elliptic, which corresponds respectively to time minimizing or time maximizing trajectories.

2. We then integrate the system backward in time from  $q(0)$  and compute the equations characterizing the switching surface, the splitting locus  $C_s$  and the singular locus  $\Gamma_s$ .

A trajectory  $\sigma_\varepsilon, \varepsilon \in \{-1, 1\}$  can switch at time  $t_1^\varepsilon < 0$ , can intersect the surface  $N$  at time  $t_2^\varepsilon < 0$  or there may exist a time  $t_3 < 0$  and a point  $q_3 \in V$  such that  $\exp(-t_3(X + Y))(q_3) = q(0)$ ,  $\exp(-t_3(X - Y))(q_3) \in N$  and  $t_3 < 0$ .

The weights of the variable  $t, s, w$  is respectively 1, 2, 1. We develop the regular flow using Taylor expansion up to order 3 in  $t$ , we obtain :

$$\begin{aligned}
 p_3(t_1^\varepsilon) = 0 &\Rightarrow t_1^\varepsilon = \frac{2s}{b_1/2 - c - \varepsilon} + \dots, \\
 e^{t_2^\varepsilon(X + \varepsilon Y)}(q(0)) \in N &\Rightarrow t_2^\varepsilon = \frac{-2}{b}(w + s^2) + \dots, \\
 \exists (t_3, q) \in \mathbb{R}_*^- \times U, &\begin{cases} e^{-t_3(X+Y)}(q) \in N \\ e^{-t_3(X-Y)}(q) \in N \end{cases} \\
 &\Rightarrow t_3 = \frac{3s}{b_1/2 - c - 3} + \dots
 \end{aligned} \tag{6}$$

and a parameterization of

- the switching surface  $W_- = (x(w, s), y(w, s), z(w, s))$  is

### 3. CONCLUSION

$$\begin{aligned}
 x(w, s) &= \frac{s^2 (9b_1^2 - 4b_1(c + \varepsilon) + 4(c + \varepsilon)^2) + 3w(b_1 - 2(c + \varepsilon))^2}{3(b_1 - 2(c + \varepsilon))^3/4s} + \\
 &\frac{6bs(b_1 - 2(c + \varepsilon))}{3(b_1 - 2(c + \varepsilon))^3/4s} + \dots, \\
 y(w, s) &= \frac{4s (b(b_1 - 2(c + \varepsilon)) + b_1^2 s)}{(b_1 - 2(c + \varepsilon))^2} + w + \dots, \\
 z(w, s) &= \frac{s(b_1 + 2(c + \varepsilon))}{b_1 - 2(c + \varepsilon)} + \dots,
 \end{aligned} \tag{7}$$

- the singular surface  $\Gamma_s := \Gamma_s(t, w)$  is

$$\Gamma_s = \left( \frac{bt^2}{2} + \frac{b_1^2 t^3}{6} + tw, bt + \frac{b_1^2 t^2}{4} + w, \frac{b_1 t}{2} \right) + \dots, \tag{8}$$

- and the splitting locus  $C_s = (x(w, s), y(w, s), z(w, s))$  is

$$\begin{aligned}
 x(w, s) &= \frac{2s^2 (b_1 c + b_1(2b_1 - 9) + 2c^2 + 6) + w(b_1 - 2c - 6)^2}{(b_1 - 2(c + 3))^3/6s} \\
 &+ \frac{3bs(b_1 - 2(c + 3))}{(b_1 - 2(c + 3))^3/6s} + \dots, \\
 y(w, s) &= \frac{6s(b(b_1 - 2(c + 3)) + b_1 s(b_1 + c - 3))}{(b_1 - 2(c + 3))^2} + w + \dots, \\
 z(w, s) &= \frac{s(b_1 + 4c)}{b_1 - 2(c + 3)} + \dots
 \end{aligned} \tag{9}$$

3. The optimal policy is deduced by computing  $t^* = \max(t_1^\varepsilon, t_2^\varepsilon, t_3)$  with  $\varepsilon = \text{sign}(s(n \cdot X(0, w, s)))$  and we represent in Fig.5 the region of  $N$  where  $t^*(w, s)$  corresponds to a switching time, a splitting time or a time at which the trajectory intersects  $N$ . The surface  $S$  separating  $U_+$  and  $U_-$  is the union of the switching surface  $W_-$ , the singular surface  $\Gamma_s$  foliated by singular arcs, the splitting locus  $C_s$  and a subset  $\tilde{\mathcal{E}}_- \subset \mathcal{E}_-$  from which  $\sigma_\pm$  intersects  $N$  in the green region of Fig. 5. The set  $S$  is represented in Fig. 4 together with some trajectories emanating from  $N$  to the components of  $S$ . The green trajectories starting from  $\mathcal{E}_+$  intersect  $N$  in  $\tilde{\mathcal{E}}_-$ .

#### 2.3 Application to the McKeithan network

The McKeithan network exhibits situations of codimension 2 described in Section 2.2. Another codimension two case occurs at the collinearity locus  $\mathcal{C} = \{q \in N, X \text{ and } Y \text{ are collinear}\}$ . This corresponds to singular points of the 2d-vector  $X$  and it occurs in the McKeithan network.

$\mathcal{C} \cap N$  leads to isolated singular points and in the generic case such a point is not a singular trajectory. A semi-normal form at such point identified to 0 can be derived using the following normalization

- $X(0)$  and  $Y(0)$  are collinear,
- $X$  at 0 is not tangent to the curve  $\mathcal{E} \cap N$ ,
- There is no singular trajectory arriving at 0,
- The collinearity locus  $\mathcal{C} \cap N$  is transverse to  $\mathcal{E} \cap N$  at 0 and it not tangent to  $X$  at 0.

Thus we obtain a model of the form  $\dot{q} = Aq + ub$  and the local synthesis can be easily described with the method described above.

The stratification of the terminal manifold  $N$  is given in Fig. 6 and describes the codimension 2 situations of this network.

We presented a method to determine efficiently the local time minimal synthesis for optimal control problems and complete the work [Launay et al, 1997] using geometric analysis and formal computation in the exceptional situations of codimension 2.

This is applied to the control of the McKeithan network where complex situations appear. In the continuity of the study of this reversible network, we can extend the proposed techniques to treat an important problem related to the existence of equilibria corresponding to codimension 3 situations.

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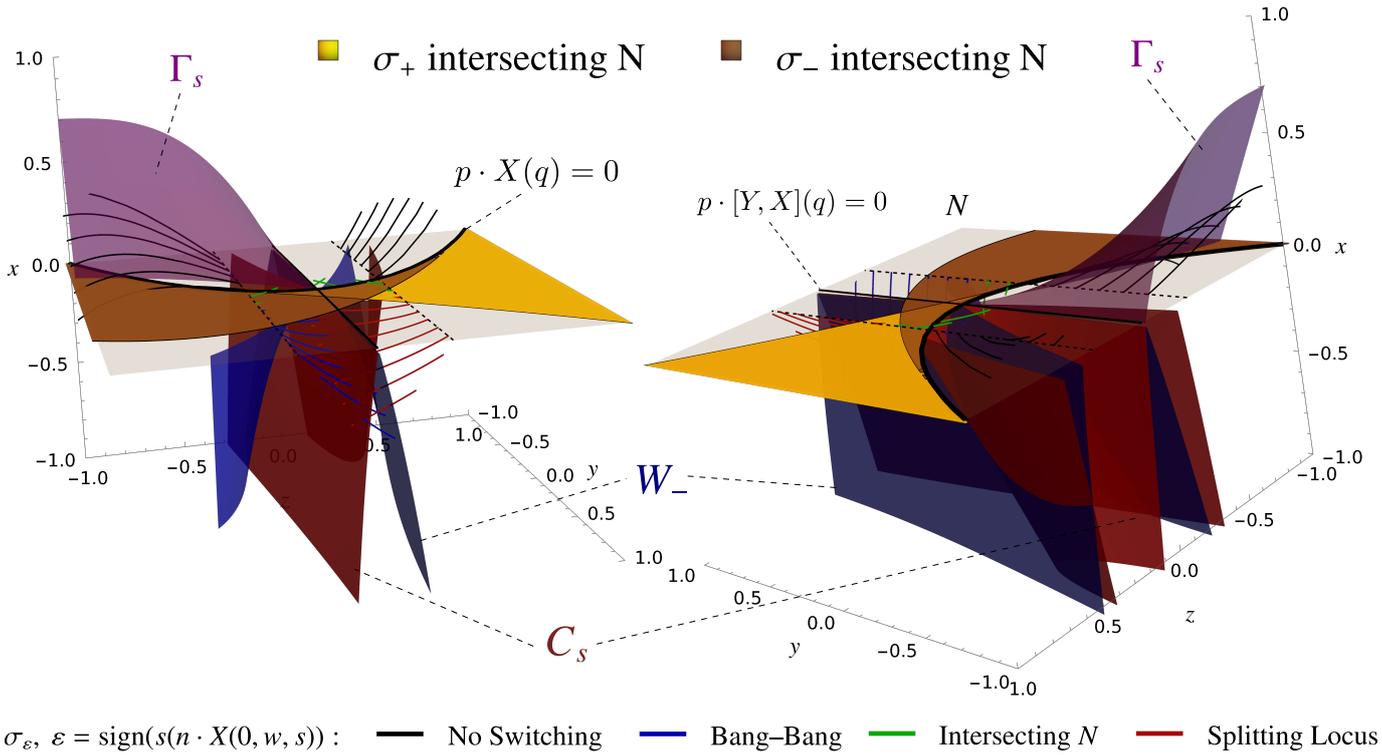


Fig. 4. Strata of the surface  $S$  separating the regions of  $U$  where the control is  $\pm 1$  for the model (5) with  $b = b_1 = 1, c = 0$ . We also represent the regions where  $\sigma_\pm$  intersect  $N$  for  $t < 0$  and several trajectories which generate a switching locus and a splitting locus.

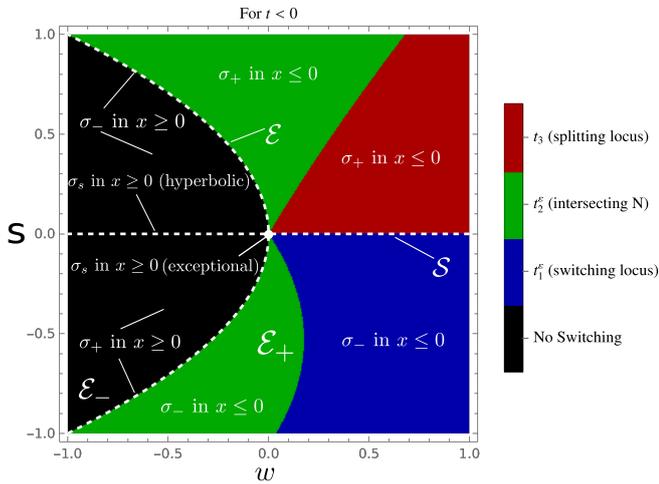


Fig. 5. Minimum time  $t^* = \max(t_1^\varepsilon, t_2^\varepsilon, t_3)$  to reach  $(0, w, s) \in N$  from a neighbourhood  $U$  of 0 for the model (5) with  $b = b_1 = 1, c = 0$ . The exceptional locus is  $\mathcal{E} : y = -z^2$  and the singular locus is  $\mathcal{S} : n \cdot [Y, X](q) = 0 : z = 0$ .

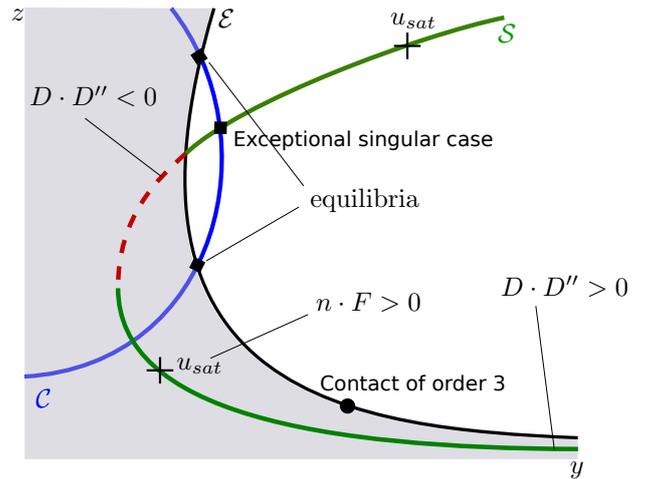


Fig. 6. Description of the codimension 2 cases analyzed in our section for the McKeithan network.