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# The mathematical physical meaning of the Rarita Schwinger equation factored by a weighted Dirac operator, Part I

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## Abstract

In this article we give a tour through Physics to explain the meaning of the Rarita Schwinger equation and determine its relationship with the Clifford algebras. We also relate the proposed first operators of Rarita Schwinger with the recent operators of that type, given mathematically taking the physical constants as unitary. Finally motivated by the physical applications of considering a weighted Dirac operator, we propose a new Rarita Schwinger type operator and discuss some possible applications.

**Keywords:** Rarita Schwinger equation; Clifford algebras; weighted Dirac operator.

## Resumen

En este artículo damos un recorrido por la Física para explicar el significado de la ecuación de Rarita Schwinger y determinar su relación con las álgebras de Clifford.

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También relacionamos el primer operador de Rarita Schwinger propuesto con los operadores recientes de ese tipo dados matemáticamente, tomando las constantes físicas como unitarias. Finalmente, motivados por las aplicaciones físicas de considerar un operador de Dirac pesado, proponemos un nuevo operador de tipo Rarita Schwinger y discutimos algunas posibles aplicaciones.

**Palabras claves:** Ecuación de Rarita Schwinger; álgebras de Clifford; operador de Dirac pesado.

## 1 Introduction

The discovery of Clifford algebras in the 19th century, gave the bases for the study of geometric phenomena that, even at that time had not been discovered. In the beginning of the 20th century, theories as quantum mechanics and special and general relativity, appeared and dazzled the world; Einstein sentenced that movement is caused by the deformation of space-time geometrics, though it was William K. Clifford the first one to insinuate it, nearly 40 years before. Along with the advances in physics, mathematicians worked making big steps in geometric algebras, extending theories as electromagnetism, framed in vector physics, to superior algebra, which gave it a new point of view. The first one to use geometric algebras in physics problems was Paul Dirac, resolving the electron time evolution equation and obtaining conclusions as the antimatter and the spin existence, just by these algebras' geometric characteristics. As a result of these new edges proposed by this new mathematics, works in partial derivatives in superior dimension began to appear, as the fundamental solution of the Dirac operator[1], the solution of parabolic, hyperbolic and elliptical equations related to physics in this formulation. Other authors [2,3], describe and analyze the Rarita Schwinger operator, finding a fundamental solution for this operator in this particular geometry, which is related with temporal relativistic evolution of fermions with spin  $3/2$ . In this work, motivated by the physics implications of the applications of Clifford algebras in the Rarita Schwinger equation, we give a tour of physics and mathematics to explain the Rarita Schwinger equation, we propose a new Rarita Schwinger type operator and discuss some physical applications.

## 2 Rarita-Schwinger equation from Physics, preliminary theoretical frame in the context of Classical and Relativistic Mechanics

In theoretical physics, the Rarita-Schwinger equation describes the temporal relativistic quantum evolution of fermionic particles with spin  $3/2$ , formulated in 1941 by William Rarita and Julian Schwinger. To understand this approach, we must know terms as temporal evolution equation, quantum and relativistic, all of which will be duly substantiated below.

In Physics, a particle movement equation is the mathematical formulation that relates the particle's position in space with time. This equation connects the temporal derivative of one or several variables, which characterize the particle's physical state, with another physical magnitudes that cause changes in its state.

It is important to note that a fermionic particle, as described in the standard model, is one of two basic types of elementary particles that exist in nature (the other type is the boson). A particularity of fermions is that they present semi-integer spin ( $1/2, 3/2, \dots$ ). In the particular case of  $3/2$  spin fermions, they can be understood as a four-component wave function, which means that these functions, spinors, have 4 degrees of freedom, which determines an irreducible representation of these particles.

### 2.1 First ideas about motion and Newton's equations

One of the first philosophers in attempting to describe the observed phenomena using concepts of movement versus time was Aristoteles, who said that heavier objects fall down before that the lighter objects and this sentence set the guideline for many years. With science development and the introduction of the scientific method, Galileo Galilei, after making experimental measurements, arrived to the conclusion, less logical but right, that objects in vacuum fall at the same velocity and its equations of motions had the mathematical form  $x = v_0t + \frac{1}{2}at^2$ , which corresponds to the movement equations for punctual objects, where we can clearly observe the relation of position in space  $x$ , with time  $t$ . In the same year of Galileo Galilei's death, Isaac Newton was born, known for many as the most famous physicist in history and thanks to the new mathematics that he developed, the differential and integral calculus, he achieved to describe movement in a form not seen before, establishing

three laws of motions as principles to formulate the description of mechanical phenomena in a theoretical frame with the appropriate mathematical tools.

## 2.2 Development of Theoretical Mechanics and equations of motion

With the development of mathematics and the complexity of the Newtonian formulation to solve problems where particles presented constraint such as the classical example of double pendulum, the Theoretical Mechanics is developed, beginning with D'Alembert Principle which mathematical formulation is:  $\sum_i (\dot{\mathbf{p}}_i - \mathbf{F}_i) \cdot \delta \mathbf{r}_i = 0$ , where  $\mathbf{p}_i$ ,  $\mathbf{F}_i$ ,  $\delta \mathbf{r}_i$  are the linear momentum, the applied external forces and the virtual displacement on the  $i$ -th particle respectively. This principle, that ushered the next great advance in Classical Mechanics, does not contain the forces of constraints taking into account that its virtual work vanishes. From the principle and using generalized coordinates, we arrive to Euler-Lagrange equations:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j, \quad (1)$$

where  $T$  is the kinetic energy of the system and  $Q_j$  are the generalized forces for the  $j$  component or  $Q_j = \sum F_i \frac{\partial r_i}{\partial q_j}$ , the  $q_j$  represent the generalized coordinates of the system and  $\dot{q}_j$  the generalized velocities. Besides if we consider conservative forces, we can say that there exists a potential energy  $V$  and we can reformulate the Euler-Lagrange equations as:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, \quad (2)$$

where  $L = T - V$  is called the Lagrangian of the physical system, in this case of the particle in movement.

The Hamilton's Principle is another of the basis for a variational formulation in order to deduct the Rarita-Schwinger equation. Its basic ingredient is the concept of action functional which is in the spirit of PierreLouis Moreau de Maupertuis' ideas which state: "nature is economic in all of its actions" [4].

The stationary-action principle establishes: the functional  $S[q_i]$  is called

the physical system action and it is determined by:

$$S[q_i] = \int_{t_1}^{t_2} L(q_i; \dot{q}_i(t); t) dt. \quad (3)$$

Then this functional has a minimum value if the function  $L(q_i; \dot{q}_i(t); t)$  describes the system in study; in other words, particles will move following a trajectory that minimizes the Action.

Calculating the functional derivative of action and equaling to zero:

$$\delta S = \delta \int_{t_1}^{t_2} L(q_i(t), \dot{q}_i(t), t) dt = 0 \quad (4)$$

results in Euler Lagrange equations and therefore, the determination of movement equations. The physical mathematical formulation of the Hamilton Principle, is given in the equation (4).

The movement equations that are obtained from the minimum action principle would have a new formulation, cusp of theoretical mechanics, the Hamilton equations. A new physical magnitude was then defined, the system's Hamiltonian  $H$ , just as we had before the Lagrangian; precisely the Hamiltonian comes from applying the Legendre transformation to the Lagrangian:

$$H(q_i; p_i; t) = \dot{q} \frac{\partial L}{\partial \dot{q}} - L, \quad p_i = \frac{\partial L}{\partial \dot{q}_i}. \quad (5)$$

Now let us see the Hamilton movement equations. When equaling the Hamilton generic differential and the Hamilton differential of the formula (5) we arrive to the equations:

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j, \quad \frac{\partial H}{\partial p_j} = \dot{q}_j, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (6)$$

The Hamilton equations summarize all the necessary to understand the temporal evolution of a particle, for example, in the frame of classical mechanics, but there is something more that will help to understand the Rarita Schwinger equation and it is the famous Poisson Brackets, are another way to describe the classical movement of the systems. Those are an useful tool in order to transit of the descriptions of the systems from Classical Mechanics to Quantum Mechanics.

Poisson bracket in canonic coordinates  $(q_i; p_i)$  on the phase space is defined for two functions  $f(q_i, p_i, t)$  and  $g(q_i, p_i, t)$  as:

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}. \quad (7)$$

This bracket has the following properties for certain functions  $f, g, s$ :

1.  $\{f, f\} = 0$
2.  $\{f, g\} = -\{g, f\}$  (antisymmetry property)
3.  $\{\alpha f + \beta g, s\} = \alpha\{f, s\} + \beta\{g, s\}$  (bilinearity)
4.  $[[f, g], s] + [[s, f], g] + [[g, s], f] = 0$  (Jacobi identity)
5.  $\frac{\partial f}{\partial t} = \{f, H\}$ , provided the function  $f$  is a constant of motion.

Therefore the Hamilton equations can be reformulated using the Poisson bracket as

$$\dot{q}_i = \{q_i, H\}, \quad \dot{p}_i = \{p_i, H\}. \quad (8)$$

### 2.3 Special Relativity and its relation with the equations of motion

All these great advances in mathematics propitiated a development of theoretical classic physics. In 1905, a wonderful year for science, 5 articles were published [5–9] which would change the world, from the demonstration of the existence of particles to special relativity, this last article would collapse the absolute time paradigm; in this work Einstein postulates two principles that would later be translated in equations, whose study is fundamental to develop the Rarita Schwinger equation: “The physics laws are the same in any inertial reference system” and “Light’s velocity in vacuum is the same for any inertial observer” [9]. Both postulates gave as result a transformation operator which relates position and time of a physical phenomenon for two observers in two reference systems at different constant velocities. These transformations are known as Lorentz Transformations.

Lorentz Transformations form a group of transformations that include the “boosts” and the rotations in the three dimensional space . The “boosts” in

one position dimension and the time dimension, are nothing more than how position and time change for two observers of the same physical phenomenon, that are moving at different constant velocities [10]:

$$e'_{ct} = \gamma e_{ct} + \gamma\beta e_x, \quad e'_x = \gamma e_x + \gamma\beta e_{ct}, \quad (9)$$

where  $e'_{ct}$  is the coordinate axis for time dimension in the reference system that is moving with constant velocity with respect to the fixed reference system with time coordinate  $e_{ct}$ ; as well as  $e'_x$  is the position coordinate of the system with constant velocity respect to the fixed reference system with position coordinate  $e_x$ . Also  $\gamma = \frac{1}{\sqrt{1-\beta^2}}$  and  $\beta = \frac{v}{c}$  is the relative velocity with respect to the light velocity and  $c$  is the light's velocity in vaccum.

Rotations in the three dimensional space form a group called  $SO(3)$ . "The orthochronous proper subgroup contains transformations which correspond to rotations in space" [10]. Consider

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & R & & \\ 0 & & & \end{pmatrix}, \quad \text{with } R \in SO(3).$$

Indeed, since  $R^t R = \mathbf{I}_3$ , it is evident that  $L^t \eta L = \eta$ , where  $\eta$  is the metric of the space time, while  $\det L = \det R = 1$ ,  $L_0^0 = 1$ , where  $L_0^0$  denotes entry 11 of  $L$ .

The Lorentz group is that one of linear transformations that leaves invariant the Minkowski space metrics or isometry group of Minkowski space. Mathematically it is formed by any matrix  $\Lambda$  that fulfills the relation

$$\Lambda^T \eta \Lambda = \eta, \quad \eta =: \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (10)$$

Lorentz Transformations are fundamental to understand subsequently the Rarita Schwinger equation, as the system's Lagrangian must fulfill the condition of being a Lorentz invariant. Let us see how a Lorentz transformation can transform a quadrivector in one that fulfills the laws of energy conservation and in consequence, with the movement equations; first of all, let us see

which is the parametrization of a quadrivector that obeys energy conservation and then we will see that we can arrive to the same parametrization by means of Lorentz transformations.

Let us see which is the parametrization of a quadrimomentum that fulfills the equations of energy conservation. Let this be a quadrimomentum, a quadrivector:

$$p^\mu = \gamma^0 p^0 + \gamma^1 p^1 + \gamma^2 p^2 + \gamma^3 p^3, \quad (11)$$

where  $\gamma^\mu$  are the gamma matrices.  $p^\mu$  must comply with the energy conservation equation:

$$(p^\mu)^2 - (mc^2)^2 = (\gamma^0 p^0 + \gamma^1 p^1 + \gamma^2 p^2 + \gamma^3 p^3)^2 - (mc)^2 = 0, \quad (12)$$

Dividing the equation for  $(mc)^2$  and separating  $p^\mu = p^0 + |p^i|$ ,  $p^i = \gamma^1 p^1 + \gamma^2 p^2 + \gamma^3 p^3$ , we obtain:

$$\left(\frac{p^0}{mc}\right)^2 - \left(\frac{|p^i|}{mc}\right)^2 = 1. \quad (13)$$

The equation(13) can be parameterized using the trigonometric formula:

$$\cosh^2(\eta) - \sinh^2(\eta) = 1. \quad (14)$$

This implies that:

$$p^0 = mc \cosh(\eta). \quad (15)$$

On the other hand we have

$$(p^1)^2 + (p^2)^2 + (p^3)^2 = (mc \sinh(\eta))^2. \quad (16)$$

Being the equation (16) a sphere equation and because of this, the parametrization in spherical coordinates of the trimomemtum is:

$$p^1 = mc \sinh(\eta) \sin \theta \cos \varphi$$

$$p^2 = mc \sinh(\eta) \sin \theta \sin \varphi$$

$$p^3 = mc \sinh(\eta) \cos \theta.$$

Adding the equation (15) to the trimomentum, we arrive to a parametrization of the quadrimomentum  $p^\mu$ :

$$\begin{aligned} p^0 &= mc \cosh(\eta) \\ p^1 &= mc \sinh(\eta) \sin \theta \cos \varphi \\ p^2 &= mc \sinh(\eta) \sin \theta \sin \varphi \\ p^3 &= mc \sinh(\eta) \cos \theta. \end{aligned}$$

The quadrimomentum  $p^\mu$  parametrized in this way fulfills the energy conservation:

$$p^\mu = \begin{bmatrix} mc \cosh(\eta) \\ mc \sinh(\eta) \sin \theta \cos \varphi \\ mc \sinh(\eta) \sin \theta \sin \varphi \\ mc \sinh(\eta) \cos \theta \end{bmatrix}. \quad (17)$$

Now let us see how, making Lorentz transformations to a quadrivector, we can arrive to a quadrivector that fulfills the equations of energy conservation, consequently a quadrivector that fulfills the movement equations.

Let us begin with a particle at rest and let us do some Lorentz transformations:

$$p^\nu = \begin{bmatrix} mc \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (18)$$

If we make a Boost in  $z$  axis, we can pass from a particle at rest to one that is moving at relativistic velocities in this axis:

$$\begin{aligned} \begin{bmatrix} mc \cosh(\eta) \\ 0 \\ 0 \\ mc \sinh(\eta) \end{bmatrix} &= (\Lambda_{(z)})^{-1} \begin{bmatrix} mc \\ 0 \\ 0 \\ 0 \end{bmatrix} = \\ &= \begin{bmatrix} \cosh(\eta) & 0 & 0 & \sinh(\eta) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh(\eta) & 0 & 0 & \cosh(\eta) \end{bmatrix} \begin{bmatrix} mc \\ 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \quad (19)$$

Now, we can rotate the particle moving in the  $z$  axis with respect to  $y$  axis by the Lorentz transformation:

$$\begin{aligned} \begin{bmatrix} mc \cosh(\eta) \\ mc \sinh(\eta) \sin \theta \\ 0 \\ mc \sinh(\eta) \cos \theta \end{bmatrix} &= R_y \begin{bmatrix} mc \cosh(\eta) \\ 0 \\ 0 \\ mc \sinh(\eta) \end{bmatrix} = \\ & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & 0 & \sin \theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} mc \cosh(\eta) \\ 0 \\ 0 \\ mc \sinh(\eta) \end{bmatrix}. \end{aligned} \quad (20)$$

Now we can rotate the particle with respect the  $z$  axis by means of the Lorentz transformation:

$$\begin{aligned} \begin{bmatrix} mc \cosh(\eta) \\ mc \sinh(\eta) \sin \theta \cos \varphi \\ mc \sinh(\eta) \sin \theta \sin \varphi \\ mc \sinh(\eta) \cos \theta \end{bmatrix} &= R_z \begin{bmatrix} mc \cosh(\eta) \\ mc \sinh(\eta) \sin(\theta) \\ 0 \\ mc \sinh(\eta) \cos \theta \end{bmatrix} = \\ & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi & 0 \\ 0 & \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} mc \cosh(\eta) \\ mc \sinh(\eta) \sin \theta \\ 0 \\ mc \sinh(\eta) \cos \theta \end{bmatrix}. \end{aligned} \quad (21)$$

We can observe that, by means of a Lorentz  $\Lambda$  transformation we can pass from a quadrimomentum or quadrivector on the  $z$  axis, to a quadrivector with components that depend on the parameters  $\eta, \varphi, \theta$  and satisfies the energy conservation equations:

$$\begin{bmatrix} mc \cosh(\eta) \\ mc \sinh(\eta) \sin \theta \cos \varphi \\ mc \sinh(\eta) \sin \theta \sin \varphi \\ mc \sinh(\eta) \cos \theta \end{bmatrix} = R_z R_y (\Lambda_{(z)})^{-1} \begin{bmatrix} mc \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (22)$$

$$R_z R_y (\Lambda_{(z)})^{-1} = \Lambda. \quad (23)$$

### 3 Quantum Mechanics, Quantum Field Theory and Rarita-Schwinger equation

In the first half of the 20th century new experiments as Young's double slit, the development of the emission theory and the measurement of the black body radiation, the photoelectric effect and the experiment of Stern and Gerlach, gave the basis to accept that these particles did not move nor interact following the laws of classical mechanics and in this way, quantum mechanics emerged, fundamental to understand the movement of very small particles as fermions of spin <sup>1</sup>  $\frac{3}{2}$ .

The knowledge that scientists had by that time, had to be applied in some way to this new physics that was arising, so, fundamental principles as the Hamilton Principle, the Euler Lagrange equations and Einstein's Postulates, were used to describe quantum phenomena; the classic theories were quantized and managed explaining the movement of quantum particles, as the electron or the fermions with spin  $\frac{3}{2}$ .

How to quantize a theory? Basically and without studying very deeply the formulation, the postulate that goes from classic to quantum world was proposed by Paul Dirac [11], it consists in equaling Poisson's Bracket (7) to the commutator:

$$[a, b] = i\hbar\{a, b\} = ab - ba, \quad (24)$$

where in this quantum case, a and b are hermitic operators,  $[a, b]$  is the commutator between a and b.

The most important quantum operators are:

$$\vec{v} = \frac{-i\hbar}{m}\vec{\nabla} \quad (25)$$

$$\vec{p} = -i\hbar\vec{\nabla} \quad (26)$$

$$E = \hat{H} \quad (27)$$

$$\vec{l} = \vec{r} \times (-i\hbar\vec{\nabla}), \quad (28)$$

where  $\vec{v}$  represents the particle's velocity,  $\vec{p}$  is the linear momentum,  $E$  is the system's energy,  $H$  is the Hamiltonian,  $\vec{l}$  is the angular momentum and  $\hbar$  is the reduced Planck's constant,  $\hbar = \frac{h}{2\pi}$ , where  $h$  is Planck's constant.

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<sup>1</sup> spin is a physical property of elementary particles. The spin of a particle is always an integer multiple of  $\frac{\hbar}{2}$ .

### 3.1 A fields theory

In the case of Copenhagen's interpretation of quantum physics, particles are described by a wave function, which expressed that a particle's punctual position could be described with functions, as we observed in Young's double slit experiment [12] that determined the wave nature of light and subsequently the quantum nature of light would be verified with the photoelectric effect [9].

Let us see how to construct a quantum field theory. Imagine we have discrete particles upon which there are forces acting and they can move, for this non relativistic case it is enough to obtain the Lagrangian and with the Euler Lagrange equations we can obtain the movement equations; but if we make the distance between particles tend to zero, then, in each space point a force would be experienced by a particle that would make it move, consequently in this limit we can change the position by a field, which will give an scalar value in each point, a detail formulation of the quantum field theory can be seen in [13].

Let us see how the physics equations change following this method. Given an Action, using Hamilton's Principle, we can obtain the movement equations: In the classic case, the Action would be given by (3) but in classic field theory we would change position for a field. Finally, we arrive to a quantum theory resulting from integrating classical mechanics, relativity and quantum mechanics; this is made using fields that are invariant relativistic and that follow the equations of minimum action, using the commutator in place of Poisson Bracket and in this way we pass from the classic action (3) to the quantum action:

$$S[\phi] = \int_{t_1}^{t_2} \int_a^b L_\rho \left( \phi, \frac{\partial \phi}{\partial \mu} \right) d^4x dt, \quad (29)$$

where  $L_\rho \left( \phi, \frac{\partial \phi}{\partial \mu} \right)$ , would be a Lagrangian density Poincare invariant.

An example of how to obtain a relativistic quantum equation, can be seen when we apply the Hamilton Principle (4) to the Klein Gordon action:

$$S[\phi] = \int_{t_1}^{t_2} \int_a^b \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \left( \frac{mc^2}{\hbar} \right)^2 \phi \right) d^4x dt, \quad (30)$$

where  $\phi$  is the Klein Gordon real field and  $m$  its mass, we obtain the Klein

Gordon equations of movement:

$$\left( \partial_\mu \partial^\mu - \left( \frac{mc^2}{\hbar} \right)^2 \right) \phi = 0. \quad (31)$$

### 3.2 Temporal evolution equations of relativistic quantum particles

Having a temporal evolution relativistic quantum theory, one of the first successes in explaining quantum particle's movement was the Schrödinger equation [14].

Schrödinger equation dependent on time for a non relativistic electron is:

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left[ \frac{-\hbar^2}{2\mu} \nabla^2 + V(\mathbf{r}, t) \right] \Psi(\mathbf{r}, t), \quad (32)$$

where  $\Psi(\mathbf{r}, t)$  is a wave function, the parameter  $\mu$  is the relative mass and  $V$  is the potential.

Knowing that, for the theory to be complete it should fulfill the relativity laws, the next step was made by Klein Gordon, with his relativistic quantum theory of the electron's movement [15] and its equation (31):

$$\left( \partial_\mu \partial^\mu - \left( \frac{mc}{\hbar} \right)^2 \right) \phi = 0.$$

Although this theory came from a Lorentz invariant Lagrangian that fulfills the laws of special relativity, it had inconsistencies with Copenhagen's interpretation of quantum mechanics, it predicted negative probabilities and it was unable to treat particles as wave functions.

The next step was made by Paul Dirac with his theory of temporal evolution of relativistic electrons [16] and its equation:

$$(i\gamma^\mu \partial_\mu - m) \psi = 0. \quad (33)$$

Here  $\psi$  is a Dirac spinor, a wave function with four components, totally compatible with Copenhagen's interpretation of quantum mechanics and Special Relativity, coming from a Lorentz invariant Lagrangian.

### 3.3 Development of the Rarita Schwinger Equation in the quantum Field Theory

In 1941 William Rarita and Julian Schwinger [17], had the physical bases to develop a theory of the temporal evolution of fermions of spin  $\frac{3}{2}$ , that consisted in obtaining a Lagrangian density of the field of fermionic particles, based on the studies of the theory of Proca and Dirac [16,18], and also in the work of Fierz and Pauli [19]. The Lagrangian density that they proposed in modern notation is:

$$\mathcal{L} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma^5\gamma_\nu\partial_\rho\psi_\sigma - m\bar{\psi}_\mu\psi^\mu, \quad (34)$$

where the coefficients will be defined in more detail in the following lines.

Hamilton Principle for the Action of the Rarita Schwinger field.

$$\delta \left[ \int_{t_1}^{t_2} \int_a^b \left( \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma^5\gamma_\nu\partial_\rho\psi_\sigma - m\bar{\psi}_\mu\psi^\mu \right) d^4x dt \right] = 0. \quad (35)$$

Developing the equation (35) we obtain the Rarita Schwinger equation:

$$\epsilon^{\mu\nu\rho\sigma}\gamma^5\gamma_\nu\partial_\rho\psi_\sigma + m\psi^\mu = 0. \quad (36)$$

Let us see the significate of each part of this equation:

The Levi-Civita symbol  $\epsilon^{\mu\nu\rho\sigma}$ :

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 \\ -1 \\ 0 \end{cases}$$

The positive value 1 corresponds to all the even permutations of  $\mu\nu\rho\sigma$  (it is even value if it can be reduced to the initial ordered series by a even number of positions exchanges) and in the opposite case, when no term is repeated, it is odd and then it has the value  $-1$ . For the case that the terms are repeated, the corresponding value is 0.

Let us see the meaning of the term  $\gamma_\nu$ .

$\gamma_\nu$  is by definition:

$$g_{\mu\nu}\gamma^\nu = \{-\gamma_0, +\gamma_1, +\gamma_2, +\gamma_3\},$$

where  $\gamma^\nu$  are Dirac's matrixes. In Dirac's representation, the four gamma contravariants matrixes are:

$$\begin{aligned} \gamma^0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} & \gamma^1 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \\ \gamma^2 &= \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} & \gamma^3 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (37)$$

Let us see the significance of the term  $\gamma^5$ .

The  $\gamma^5$  matrix is defined as the multiplication of the four Dirac's gamma matrixes and  $i$  (37):

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (38)$$

The  $\gamma^5$  matrix is related to the particle's chirality because of its properties.

Finally let us see which is the significance of  $\psi^\mu$ .

$\psi^\mu$  is a space time spinor, it is a wave function of four components and it belongs to an intern space of the Minkowsky real space. The wave function can be explicitly written as a column matrix:

$$\psi^\mu(\mathbf{x}, t) \equiv \begin{bmatrix} \psi^0(\mathbf{x}, t) \\ \psi^1(\mathbf{x}, t) \\ \psi^2(\mathbf{x}, t) \\ \psi^3(\mathbf{x}, t) \end{bmatrix}. \quad (39)$$

The dual wave function can be written as a simple matrix:

$$\psi^\dagger(\mathbf{x}, t) \equiv [\psi_1^*(\mathbf{x}, t) \quad \psi_2^*(\mathbf{x}, t) \quad \psi_3^*(\mathbf{x}, t) \quad \psi_4^*(\mathbf{x}, t)], \quad (40)$$

where the Dirac adjoint is defined by  $\bar{\psi} \equiv \psi^\dagger\gamma^0$ ,  $\psi^\dagger$  denotes the adjoint hermitic spinor (conjugated transposed) of de  $\psi$ , and  $\gamma^0$  is the time gamma matrix.

Finally,  $\partial_\rho\psi_\sigma$ , are the spinor  $\psi_\sigma$  partial derivatives following the Einstein additive agreement, where  $\rho, \sigma$  varie between 0 and 3.

The Rarita Schwinger equation (36) manages to explain the movement of quantum particles with spin  $\frac{3}{2}$ , in total agreement with the interpretation of quantum mechanics and the Special Theory of Relativity, with spinor functions (39) with which quantum variables as position and movement, can be probabilistically determined.

## 4 Clifford Algebras and Rarita Schwinger Equation

Clifford algebras were developed initially by Hermann Grassmann and William K. Clifford. They consist in a unified language for physicists and mathematicians, that gathers and unifies concepts that allow the expression and description of some geometric relations with applications well defined in physics, as the complex numbers system, matrixes algebra, quaternions, vector algebras, tensors, spinors and differential forms.

Let us see some generalities and characteristics of these algebras, taken from the book [20].

Universal Clifford Algebra:

Given  $V_{n,s}(0 < s < n)$ , being a  $n$ -dimensional ( $n \geq 1$ ) linear real space with base

$$\{e_1, e_2, \dots, e_n\},$$

provided by a bilinear form

$$(v | w), \quad v, w \in V_{n,s}$$

such that

$$\begin{aligned} (e_i | e_j) &= 0 & , & \quad i \neq j \\ (e_i | e_i) &= 1 & , & \quad i = 1, \dots, s \\ (e_i | e_i) &= -1 & , & \quad i = s + 1, \dots, n \end{aligned}$$

and  $v = \sum_{i=1}^n v_i e_i \in V_{n,s}$ . Then the associated quadratic form is

$$(v | v) = \sum_{i=1}^s v_i^2 - \sum_{i=s+1}^n v_i^2.$$

Consider the linear real space of dimension  $2^n$   $C(V_{n,s})$  with basis

$$\{e_A = e_{h_1 \dots h_r} : A = (h_1, \dots, h_r) \in PN, 1 \leq h_1 < \dots < h_r \leq n\},$$

where  $N$  is the set  $\{1, \dots, n\}$  and  $e_\phi$  is  $e_0$ . Now a product on  $C(V_{n,s})$  is defined by:

$$e_A e_B = (-1)^{n((A \cap B) \setminus S)} (-1)^{p(A,B)} e_{A \Delta B}, \quad (41)$$

where  $S$  represents the set  $\{1, \dots, s\}$ ,  $n(A) = \#A$ ,

$$p(A, B) = \sum_{j \in B} p(A, j), p(A, j) = \#\{i \in A : i > j\}, \quad (42)$$

and the sets  $A, B$  and  $A \Delta B$  are ordered in the prescribed manner. Following the multiplications rules (41), with  $e_0$ , the identity element, we have

$$\begin{aligned} e_i^2 &= 1 & , & \quad i = 1, \dots, s \\ e_i^2 &= -1 & , & \quad i = s + 1, \dots, n \\ e_i e_j &= -e_j e_i, & i &\neq j \end{aligned}$$

and for  $1 < h_1 < \dots < h_r < n$ ,

$$e_{h_1} \cdot e_{h_2} \cdots e_{h_r} = e_{h_1 \dots h_r}.$$

Moreover for all  $v, w \in V_{n,s}$  vector:

$$vw + wv = 2(v | w).$$

“It is easy to verify that, in this way,  $C(V_{n,s})$  becomes in a linear algebra, associative but not commutative on  $\mathbb{R}$ . It is called universal Clifford algebra on  $V_{n,s}$ ” (see [20]).

As we can observe in the Rarita Schwinger equation (36):

$$\epsilon^{\mu\nu\rho\sigma} \gamma^5 \gamma_\nu \partial_\rho \psi_\sigma + m \psi^\mu = 0.$$

The terms  $\epsilon^{\mu\nu\rho\sigma} \gamma^5 \partial_\rho$  are multiplying the term  $\gamma_\nu$  which by definition is  $g_{\mu\nu} \gamma^\nu$ .  $\gamma^\nu$  are the gamma Dirac matrixes (37), these matrixes fulfill an algebra structure that is very particular in the Clifford algebras:

Let  $V_{4,1}$  be a 4-dimensional linear real space with basis:

$$\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$$

provided with a bilinear form:

$$(v | w), \quad v, w \in V_{4,1}$$

such that

$$\begin{aligned} (\gamma_i | \gamma_j) &= 0 & , & \quad i \neq j \\ (\gamma_i | \gamma_i) &= 1 & , & \quad i = 0 \\ (\gamma_i | \gamma_i) &= -1 & , & \quad i = 1, 2, 3. \end{aligned}$$

The  $\gamma_\nu$  factors, fulfill the particular Clifford algebra  $V_{4,1}$ , so that the equation's right part (36) can be understood as a function in the particular Clifford algebra.

Understand as a Clifford algebra valued function,  $f = \sum e_A f_A$  according [20], where  $e_A$  are the elements of the basis that generate Clifford algebra and  $f_A$  represents functions with real values.

The gamma matrixes  $\gamma_\nu$  were introduced for the first time by Paul Dirac in his attempt to solve the physics problems that resulted from solving (31). To eliminate physical problems in the equation (31), Dirac contrary to Klein-Gordon did not quantize the energy  $E^2 = p^2c^2 + m^2c^4$ . Instead, he eliminated the root of the equation by finding terms  $e_\mu$  such that  $(e_\gamma pc + e_\epsilon mc^2)^2 = p^2c^2 + m^2c^4$ , these  $e_\mu$  terms comply with the structure relations of algebras proposed by William K. Clifford many years earlier. Finding the  $e_\mu$  terms allowed Dirac to find his equation (33) electron motion from quantize  $e_\gamma pc + e_\epsilon mc^2 = E$ , which is a linear equation with respect to the energy and moment operators much simpler to solve than the equation (31).

The terms  $e_\mu$  resulted in 4x4 matrixes (37), fact that gave a new twist to the understanding of physics at the moment, introducing concepts as spinors  $\psi(\mathbf{x}, t)$  (39) and giving a theoretical demonstration of the spin.

## 5 Rarita Schwinger operator seen from mathematics

From Rarita Schwinger equation (36), there were developed operators in partial derivatives. Ones of the firsts in dealing with the problem of the movement of fermions of spin 3/2 in this way, were Deser, S. Kay, J. H. and Stelle, K. in their work of 1977 "Hamiltonian formulation of supergravity" [21], where the Rarita Schwinger operator was defined through

$$R^\nu \equiv \gamma_\mu * f^{\mu\nu} = 0, \tag{43}$$

where

$$*f^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} f_{\alpha\beta}, f_{\mu\nu} \equiv D_\mu\psi_\nu - D_\nu\psi_\mu$$

and it must be fulfilled that

$$\begin{aligned}\gamma_\mu f^{\mu\nu} &= 0 \\ f_{\mu\nu} + \gamma_5 * f_{\mu\nu} &= 0.\end{aligned}$$

Many mathematicians kept dealing with the Rarita Schwinger equation by using operators, but in [22], the authors defined a family of Rarita Schwinger type operators, where a case coincides with the operator of the same name proposed by [21] and enunciate it as: “A recent study of operators conformably invariants has left clear that there is a sequence of similar elliptic operators of first order, conformably invariants that act on more complex functions with values in spinor representations. The most simple of these operators is an analog of Rarita Schwinger operator introduced by physicists in dimension 4” [22].

In the cited work [22], then it is defined a Rarita Schwinger operator that belongs to a family of elliptic first order operators conformably invariant that act on functions with values in spinor representations, where the authors textually express that: “it is a generalization of the classic Rarita-Schwinger equation for spin 3/2 fields to the case of functions taking values in irreducible representation spaces with weight  $k + 1/2$ . These fields may be realised as functions taking values in spaces of spherical monogenics”.

In this work [22], the authors define the Rarita Schwinger operator in the following way:

Let the Dirac operator be

$$\partial_x = \sum_{j=1}^m e_j \partial_{x_j}. \quad (44)$$

A monogenic decomposition of this operator is

$$\partial_x = D_{0x} + D_{1x}, \quad (45)$$

where  $D_{0x}$  and  $D_{1x}$  are the total Rarita Schwinger operator and the Twistor operator, respectively. They are defined as

$$D_{0x} = -(\Gamma_u \partial_x + \partial_x (\Gamma_u - m + 2)) (m - 2 - 2\Gamma_u)^{-1} \quad (46)$$

$$D_{1x} = (\Gamma_u \partial_x - \partial_x \Gamma_u) (m - 2 - 2\Gamma_u)^{-1}, \quad (47)$$

where

$$\Gamma_u = -u \wedge \partial_u = -\sum_{i < j} e_{ij} L_{ij,u} \quad (48)$$

is the Gamma operator and  $L_{ij,u} = u_i \partial_{u_j} - u_j \partial_{u_i}$  is the angular momentum operator.

The Euler operator is defined as:

$$E_u = \sum_{j=1}^m u_j \partial_{u_j} \quad (49)$$

The Rarita-Schwinger operator  $R_k : C^\infty(\mathbb{R}^m, M_k) \rightarrow C^\infty(\mathbb{R}^m, M_k)$  given by

$$R_k f(x, u) = D_{0x} f(x, u), \quad f(x, \cdot) \in M_k \quad (50)$$

Given this, we have:

$$\Gamma_u f(u) = (-u \wedge \partial_u) f(u) = (-u \partial_u - E_u) f(u) = (-u \partial_u - k) f(u).$$

For all  $f \in C^\infty(\mathbb{R}^m, M_k)$

$$\begin{aligned} R_k f(x, u) &= -(\Gamma_u \partial_x + \partial_x (\Gamma_u - m + 2)) (m - 2 - 2\Gamma_u)^{-1} f(x, u) \\ &= \frac{-1}{2k+m-2} ((-u \partial_u - k) \partial_x + \partial_x (-k - m + 2)) f(x, u) \\ &= \left( \frac{u \partial_u}{2k+m-2} + 1 \right) \partial_x f(x, u), \end{aligned}$$

defining then the Rarita Schwinger operator as

$$R_k = \left( \frac{u \partial_u}{2k + m - 2} + 1 \right) \partial_x. \quad (51)$$

## 6 Weighted Dirac and Rarita Schwinger operators

Recent works about the Dirac operator (44) yield interesting applications in physics; for example, in the work [23] the authors define the weighted Dirac Operator and find its fundamental solution and in [24], the same authors calculate the fundamental solution for the weighted Dirac operator for anisotropic media.

Let us see some important definitions taken from this work.

“We consider that in Clifford algebra  $\mathcal{A}_n$ , is a more general Dirac operator defined as” [23]:

$$D_w = \sum_{i=1}^n \gamma_i e_i \partial_i. \quad (52)$$

The conjugated operator to (52) is also defined:

$$\bar{D}_w = - \sum_{i=1}^n e_i \bar{\gamma}_i \partial_i \quad (53)$$

With the purpose of factorizing the Laplacian  $D_w \bar{D}_w = \Delta_n = \bar{D}_w D_w$ , it must also be fulfilled that:

$$\psi_{ii} \bar{\psi}_{jj} + \psi_{jj} \bar{\psi}_{ii} = 2\delta_{ij}, \quad i, j = 1, 2, \dots, n \quad (54)$$

$$\bar{\psi}_{ii} \psi_{jj} + \bar{\psi}_{jj} \psi_{ii} = 2\delta_{ij}, \quad i, j = 1, 2, \dots, n, \quad (55)$$

where  $\psi_{ii} = \gamma_i e_i$ .

We can see an example of a Weighted Dirac operator in the following

$$D_w = \frac{1}{\sqrt{2}} (-e_3 + e_{123}) e_1 \partial_1 + \frac{1}{\sqrt{2}} (-e_3 + e_{123}) e_2 \partial_2 + \frac{1}{\sqrt{2}} (-e_3 - e_{123}) e_3 \partial_3, \quad (56)$$

where

$$\gamma_1 = \frac{1}{\sqrt{2}} (-e_3 + e_{123}), \gamma_2 = \frac{1}{\sqrt{2}} (-e_3 + e_{123}), \gamma_3 = \frac{1}{\sqrt{2}} (-e_3 - e_{123}) \quad (57)$$

The weighted Dirac operator can be understood as

$$D_w = w \cdot D, \quad w = (\gamma_1, \gamma_2, \gamma_3), \quad (58)$$

where  $w$  can be interpreted as an operator that transforms  $D$ .

Analogously, from a physical point of view, remembering the Lorentz  $\Lambda$  transformations that transform quadrivectors in a similar way (see the equation (22)) and if we remember that the quadrimomentum  $p^\mu$  from a quantum point of view, making unitary the physical constants, transforms in the operator  $p^\mu = e_\mu \partial_\mu$ , we can appreciate a similarity in:

$$D_w = w \cdot D \quad (59)$$

and

$$p^\mu = \Lambda p_\mu. \quad (60)$$

Motivated by the applications in physics proposed by [24], and considering that Rarita Schwinger operator (51) is factorized by a Dirac operator, we can propose a new Rarita Schwinger type operator in the form:

$$R_k^w = \left( \frac{u \partial_u}{2k + m - 2} + 1 \right) w \partial_x, \quad (61)$$

where:

$$w\partial_x = D_w = \sum_{i=1}^n \gamma_i e_i \partial_i. \quad (62)$$

Due to the similarities between (59) and (60), we can suppose that a Rarita Schwinger type operator solution would imply obtaining a solution for different quadrivectors dependent on weights, which would result in a solution for a more general problem than the one formulated by the Rarita Schwinger equation, where varying instantly the weights we would obtain a solution for a different problem, for a quadrivector transformed by a Lorentz special transformation, where  $\Lambda p_\eta$  is proportional to  $w \cdot D$ , i.e.,  $\Lambda p_\eta \propto w \cdot D$ .

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