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► **To cite this version:**

Walter Briec, Audrey Dumas, Ayman Mekki. Directional distance functions and social welfare: Some axiomatic and dual properties. *Mathematical Social Sciences*, 2021, 113, pp.181-190. 10.1016/j.mathsocsci.2021.07.004 . hal-03550844

**HAL Id: hal-03550844**

**<https://hal.science/hal-03550844v1>**

Submitted on 22 Aug 2023

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# Directional Distance Functions and Social Welfare: some Axiomatic and Dual Properties

Walter Briec, Audrey Dumas and Ayman Mekki\*

March, 2019

## Abstract

The paper introduces in social choice theory a *directional distance function* (DDF) that quantifies the level of inefficiency of a given allocation with respect to the utility possibilities frontier. The paper shows that the DDF has a simple geometric interpretation and involves a complete transitive preference relation. If convexity does not hold true, it is shown that the Rawls welfare function is dual (in a certain sense) to the DDF. Finally a more general class of distance functions is introduced that is shown to be closely related to a large class of welfare functions widely used in the study of economic inequality.

**JEL:** D60, D61.

**Keywords:** social welfare functions, directional distance function, non-convexity duality, utility set, Negishi theorem.

## 1 Introduction

This paper introduces a *directional distance function* that looks for possible increases in consumer satisfaction and quantifies the level of inefficiency of a given allocation with respect to the utility possibilities frontier. The construction of a directional distance function in a social choice context is inspired from a sequence of contributions by Luenberger [18, 19, 20] who proposed a general approach to measure and characterize Pareto efficient allocations. Introducing a function he termed the social benefit function, he showed how to transform the question of Pareto efficiency into an optimality principle.

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The tools introduced by Luenberger have been applied to production theory by Chambers et al. [13, 14] who have extended this concept to measure technical efficiency and productivity. Along this line, Chambers [12] has shown that consumers' surplus can be viewed as an exact and superlative cardinal welfare measure. More recently, Chavas [15] has used the benefit function to explore the connection between benefit functions and fairness. Blackorby C. and D. Donaldson [9] used similar tools in the treatment of indices of absolute inequality. Notice also that Ramos and Silber [25] used efficiency analysis to the study of the dimensions of human development.

Our approach takes into account possible externalities and, like the benefit function due to Luenberger [18], it provides a kind of calculus for Pareto efficiency. However, while Luenberger's framework involves a direct transformation of the bundles consumed by the agents, we focus on the collection of utility functions arranged into a vector function.

The directional distance function translates the evaluation of an allocation at the level of individuals into a ranking of these allocations at the level of the society they compose. The paper shows that the directional distance function satisfies some important properties. However, it fails the axioms of unrestricted domain and indifference of other alternatives proposed by Sen [30, 31] in his definition of welfarism. The approach proposed in the paper involves a complete transitive preorder satisfying some suitable invariance properties. An elementary transformation of the function is proposed to satisfy the criteria of Pareto efficiency and strong monotonicity.

Directional distance functions and social welfare functions are intimately linked. It is established that it is dually linked to the weighted Rawls welfare function and the indirect Rawls welfare function. The weights simply reflect the relative value of marginal consumption which society places upon each individual. Given a weighting scheme of the individual preferences, the indirect Rawls welfare function gives the maximal attainable Rawls welfare function faced with an utility set.

A specific relation to the weighted utilitarian (Bentham) welfare function also exists when convexity holds. The directional distance function appears to be dually linked to the weighted utilitarian welfare function and all the duality results obtained in an utilitarian context can be obtained replacing the summation of individual utility values for the minimum utility value. Under a convexity assumption, the utilitarian social welfare function, arising in an utilitarian context, is dual to the directional distance function. One can then establish the full duality of these concepts and provide an additional characterization of the Negishi theorem.

Lastly, a more general class of distance functions is introduced that includes as a limit case the directional distance function. The construction that is proposed focuses on the criterion of Pareto efficiency. Basically, the traditional way to define inequality measures is a related notion of efficiency. The basic idea is that an inequality measure evaluates some kind of distance between an allocation and an egalitarian allocation. Along this line, a measure

is proposed that quantifies the potential improvement of the value of a large class of functions allowing to retrieve as a special case the Nash approach of the bargaining problem [22] and the Kolm-Pollak form [17].

The paper unfolds as follows. First, we lay down the groundwork and define the notations. Section 3 is devoted to introduce the directional distance function. We show it enjoys some interesting axiomatic properties and, as such, is a tool to characterize Pareto efficiency and compare sub-efficient allocations. The axiomatic of the quasi-order induced by the directional welfare functional are investigated regarding to welfarist theories. Section 4 analyzes the dual properties of the directional distance function and establishes that it can be connected to the weighted utilitarian social welfare functionals. Non-convexities are also analyzed, a duality result derived from the Rawls welfare function is established. Section 5 proposes a generalized notion of directional distance function that establishes some connection to many functional forms used inequality theory.

## 2 Utility, Profiles and Efficiency

### 2.1 Groundwork and Notations

We adopt the formalism proposed by Arrow [3]. First we are given a (finite) set of  $n$  individuals where  $n \in \mathbb{N}$ . For all positive natural numbers  $n$ , let  $[n] = \{1, \dots, n\}$  denotes the set of individuals. Moreover, we consider a subset  $X$  of  $\mathbb{R}^d$  that is *the set of possible social alternatives*.  $X$  is also called *the set of the states of affair*.  $X$  may be as well, for example, a consumption set and in such a case, for each  $i = 1, \dots, n$  there is an identified set of individual allocations  $X_i \subset \mathbb{R}_+^{d_i}$ . An allocation for the whole economy is a vector  $x$  of the form  $x = (x_1, x_2, \dots, x_n) \in X \subset \prod_{i=1}^n X_i \subset \mathbb{R}_+^d$  where each  $x_i \in X_i$  and  $d = \sum_{i \in [n]} d_i$ . For our purpose we shall assume that the number of individuals is at least two and the number of alternatives at least three. A priori, the set of social alternatives is not more specified, though additional restrictions will be useful for some results of the paper.

Each individual has a preference ordering on  $X$  that can be represented by a real valued utility function  $u_i$  defined on  $X$ . The collection of utility functions  $u_i$  are arranged into an  $n$ -dimensional vector function  $U = (u_1, \dots, u_n)$  that defines *a profile*.  $U$  is a real valued vector utility function  $U$  defined on  $X$  as  $U(x) = (u_1(x), \dots, u_n(x)) = \sum_{i=1}^n u_i(x)e_i$ , where the subset  $\{e_i\}_{i=1, \dots, n}$  is the canonical basis of  $\mathbb{R}^n$ . This function summarizes all the utility functions  $u_i$  in the vector  $U$  characterizing the profile of an allocation  $x$ .

We define the utility set  $\mathcal{U}$  as the set of all the real valued functions defined on  $X$ . This represents all the logically possible individual evaluations that are allowed. It follows that  $\mathcal{U}^n$  is the set of all the possible profiles defined for  $n$  individuals.

Let us denote  $x \preceq x'$  if  $u_i(x) \leq u_i(x')$  for all  $i \in [n]$ . We shall write

$x \prec x'$  if  $x \succsim x'$  and  $u_i(x) < u_i(x')$  for some  $i$ . In addition if  $x \succsim x'$  and  $x' \succsim x$  we shall write  $x \sim x'$ .  $\succ$  defines a partial preorder on  $X$ . The notion of *Pareto efficient allocation* was introduced by Vilfredo Pareto [24]. The set of Pareto efficient allocations is defined by:

$$\mathcal{P}(U) = \{x \in X : x \prec x' \implies x' \notin X\}. \quad (2.1)$$

This definition can be weakened by considering the set of weakly Pareto efficient allocations. Thus, the weakly efficiency frontier is the set of all the allocations that are not strictly dominated in the utility set. The set of *weakly Pareto efficient allocations* in  $X$  is defined as:

$$\mathcal{WP}(U) = \{x \in X : u_i(x) < u_i(x'), \forall i \in [n] \implies x' \notin X\}. \quad (2.2)$$

A Pareto efficient allocation is weakly efficient, namely one has:  $\mathcal{P}(U) \subset \mathcal{WP}(U)$ . The converse does not hold true.

The “utility possibilities frontier” was introduced into economics by Maurice Allais [1], who originally called it the “surface de rendement maximum” and, later, renamed it the “surface d’efficacité maximum” (Allais [2]). Independently of Allais, Paul Samuelson drew a utility possibility frontier explicitly in [29]. The utility possibilities frontier is the upper frontier of the utility possibilities set. To simplify the exposition, it is useful to introduce a point to set map  $\mathcal{V}$  defined for all  $U \in \mathcal{U}^n$  as  $\mathcal{V}(U) = U(X) - \mathbb{R}_+^n$ , where  $U(X) = \{U(x) : x \in X\}$ . Given some  $U \in \mathcal{U}$ ,  $\mathcal{V}(U)$  is called the disposal representation set. One can rewrite this disposal representation set  $\mathcal{V}(U)$  as follows:

$$\mathcal{V}(U) = \{V \in \mathbb{R}^n : \exists x \in X, V \leq U(x)\}. \quad (2.3)$$

In the remainder of the paper, we will denote  $V = (v_1, \dots, v_n)$  for all vectors  $V$  in the utility space  $\mathbb{R}^n$ . It is easy to see that  $\mathcal{V}(U)$  is a topologically contractible nonempty set, also having a nonempty interior. Moreover for all  $U$ ,  $\mathcal{V}(U)$  is comprehensive<sup>1</sup>. An analogous construction was discussed in Briec et. al. [10] for portfolios selection problems involving an arbitrary number of statistical moments. For all  $U \in \mathcal{U}^n$ , let  $\partial\mathcal{V}(U)$  denotes the boundary of  $\mathcal{V}(U)$ . By construction,  $\partial\mathcal{V}(U)$  of the disposal representation set encompasses utility possibility frontier in  $U(X)$ .

If each individual utility function is continuous, then the set of weakly Pareto efficient allocations corresponds to allocations that generate utility vectors lying in the disposal representation possibility set’s northeast boundary. The Pareto-efficient points correspond to the points on the outer boundary of this region, at such points no single utility can be increased without decreasing another. Points whose utility vectors are interior to the disposal representation set are not weakly Pareto.

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<sup>1</sup>A subset  $\mathcal{D}$  of  $\mathbb{R}^n$  is *comprehensive* if for all  $V \in \mathcal{D}$ ,  $V' \leq V$  implies that  $V' \in \mathcal{D}$ .

## 2.2 Social Welfare Functionals

The Bergson-Samuelson social welfare functions were introduced in the context of utility possibilities frontier developed earlier by Bergson [8] and Samuelson [28]. The attempts of these functions were to make these points comparable, presumably so a society could “rank” points in the utility space according to some acceptable form of social desirability.

Let  $F(U)$  be a real valued function defined on  $X$ . What has become known as the “Bergson-Samuelson” social welfare function takes the following general form:

$$F(u_1(x), \dots, u_n(x)) \quad (2.4)$$

so that “society’s” welfare denoted,  $F$ , is merely a function of the utilities of its constituent members,  $u_i$ ,  $i = 1, 2, \dots, n$ . This general definition includes as a special case two specific social welfare functions. Assuming full measurability and comparability, the utilitarian social welfare function can be constructed as a linear sum of weighted utilities. Let  $\mu \in \mathbb{R}_+^n$ , the function  $W_\mu(U) : X \rightarrow \mathbb{R}$  defined as:

$$W_\mu(U)(x) = \sum_{i=1}^n \mu_i u_i(x) \quad (2.5)$$

is called *the utilitarian welfare function*. The map defined on  $\mathcal{U}^n$  as  $U \mapsto \sum_{i \in [n]} \mu_i u_i$  is called an utilitarian functional. Notice that since the weights  $\mu_i$  may not be identical this functional is asymmetric.

Another popular form is the “Rawlsian” or “maximin” social welfare function which seeks to maximize the utility of society’s least happy member, as argued by the philosopher John Rawls [26, 27]. This yields to a Leontief type of social indifference curves. Let  $\mu \in \mathbb{R}_+^n$ , the function  $R_\mu(U) : X \rightarrow \mathbb{R}$  defined as:

$$R_\mu(U)(x) = \min_{i \in I(\mu)} \{ \mu_i u_i(x) \}, \quad (2.6)$$

where  $I(\mu) = \{i \in [n] : \mu_i > 0\}$ , is called *the Rawls welfare function*. The map defined on  $\mathcal{U}^n$  as  $U \mapsto \max_{i \in [n]} \mu_i u_i$  is called the Rawls functional. When  $\mu$  is the unit  $n$ -dimensional unit vector, the Rawls welfare function is denoted  $R(U)$ .

In studies of inequality, there exists a large class of indices generated from means of order  $r$  (see Atkinson [5]). Suppose  $U(x) > 0$  and  $\mu > 0$ . For all real numbers  $r \neq 0$  the *utilitarian function of order  $r$*  is defined by

$$W_\mu^{(r)}(U)(x) = \left( \sum_{i \in [n]} \mu_i u_i(x)^r \right)^{\frac{1}{r}}. \quad (2.7)$$

The situation where  $r > 1$  reflects the fact that the collective choice is adverse to egalitarian solutions. It is closely related to the inequality index proposed

by Kolm [17] and Atkinson [5] where  $r = 1 - \varepsilon$ . Another function is related to the solution proposed by Nash for the bargaining problem defined as:

$$N_\mu(U)(x) = \prod_{i \in [n]} u_i(x)^{\mu_i}, \quad (2.8)$$

that is the geometric mean of the components of the utility vector  $U(x)$ , with  $U(x) > 0$ . Finally, let us denote  $W_{\max}(U)$  the function defined as

$$W_{\max}(U)(x) = \max_{i \in [n]} u_i(x). \quad (2.9)$$

This functional characterizes defines a lexi-max quasi-order. Suppose that  $\mu > 0$ . The following limit properties are well known:

- (i)  $\lim_{r \rightarrow -\infty} W_\mu^{(r)}(U)(x) = R(U)(x)$  for all  $x \in X$  such that  $U(x) > 0$ ;
- (ii)  $\lim_{r \rightarrow 0} W_\mu^{(r)}(U)(x) = N_\mu(U)(x)$  for all  $x \in X$  such that  $U(x) > 0$ ;
- (iii)  $\lim_{r \rightarrow \infty} W_\mu^{(r)}(U)(x) = W_{\max}(U)(x)$  for all  $x \in X$  such that  $U(x) \geq 0$ .

## 3 Directional Distance Function and Social Choice

### 3.1 Definitions and Properties

Suppose that  $G = (g_1, \dots, g_n) \in \mathbb{R}_+^n \setminus \{0\}$  is a vector defining a preassigned direction. For all utility functions  $U \in \mathcal{U}$ , the function  $D_G(U)$  having values in  $\mathbb{R}_+ \cup \{+\infty\}$  and defined on  $X$  by

$$D_G(U)(x) = \sup \left\{ \delta : U(x) + \delta G \leq U(z), z \in X \right\} \quad (3.1)$$

is called the *directional distance function in the direction of  $G$* . In the following we consider the domain  $\mathcal{U}_G^n$  defined as

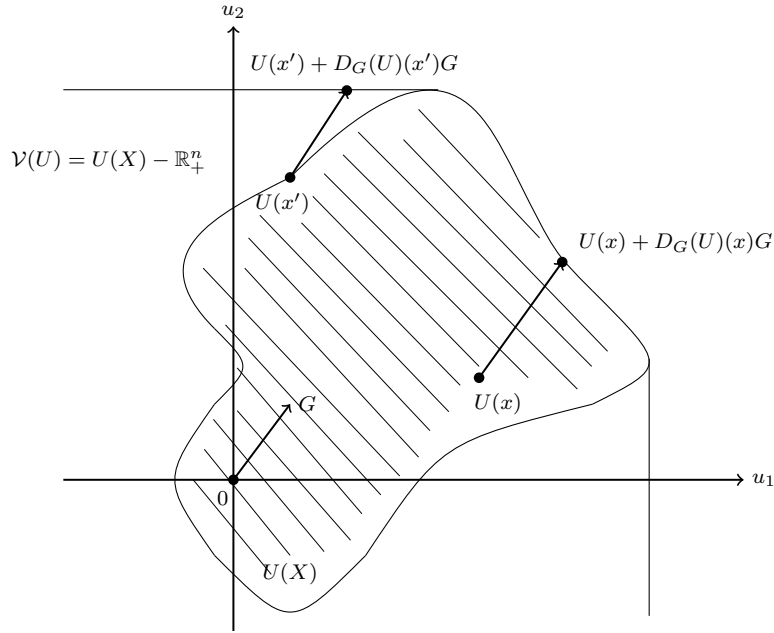
$$\mathcal{U}_G^n = \{U \in \mathcal{U}^n : D_G(U)(x) < \infty, \forall x \in X\}. \quad (3.2)$$

The directional distance function measures the efficiency of an utility vector  $U(x)$  relative to the efficient frontier. More precisely it quantifies the possible improvements of the utility vector  $U(x)$  in the direction of  $G$  while maintaining an allocation in  $X$ .

For each canonical basic vector  $e_i$  of  $\mathbb{R}^n$ , if  $G = e_i$ , then  $D_{e_i}(U)$  is called the *individual directional distance function* for the individual  $i$ .

In the remainder, the map  $U \mapsto D_G(U)$  is termed *directional functional*. The introduction of the disposal representation set in Section 2.2 offers some simplification. Equivalently, we have:

$$D_G(U)(x) = \sup \left\{ \delta : U(x) + \delta G \in \mathcal{V}(U) \right\} \quad (3.3)$$



**Figure 3** Disposal Representation set and Directional Functional.

This directional distance function is a suitable efficiency indicator for each allocation because of its elementary properties. The vector  $G$  is a reference vector defining the measure by which alternative utility points are compared. It might be taken as  $G=(1, 1, \dots, 1)$ . In such a case the transformation of the utility vector is egalitarian.

The directional distance function can be defined as a maxi-min operator and is formally related to the Rawls welfare function. This is stated in the next proposition.

**Proposition 3.1** *Suppose that  $G \in \mathbb{R}_+^n \setminus \{0\}$ . We have for all  $U \in \mathcal{U}_G^n$  and all  $x \in X$*

$$D_G(U)(x) = \sup_{z \in X} \min_{i \in I(G)} \left\{ \frac{u_i(z) - u_i(x)}{g_i} \right\},$$

where  $I(G) = \{i \in [n] : g_i > 0\}$ . Moreover, if  $G \in \mathbb{R}_{++}^n$ , then

$$D_G(U)(x) = \sup_{z \in X} \left\{ R_{G^{-1}}(U - U(x))(z) \right\}.$$

For example if  $G = \mathbb{1}_n$ , and if  $x$  is an egalitarian allocation with  $u_i(x) = \bar{u}$  for all  $i$  then:

$$D_{\mathbb{1}_n}(U)(x) = \sup_{z \in X} \{R(U)(z)\} - \bar{u}. \quad (3.4)$$

The First Welfare Theorem states that a perfect competitive economic equilibrium, with complete information, will be Pareto optimal (see for instance [4] or [16]). In particular, this condition imposes there are no externalities, each actor has perfect information and agents take prices as given. However,



there are many situations where these conditions do not hold and, in such a case, a Paretian optimum solution may not exist in a general equilibrium system. The next result shows that the directional distance function allows us to compare sub-efficient allocations and analyse the situations where a Pareto allocation is not feasible.

In the following the Pareto efficiency criterion is relaxed and the set of the *Pareto efficient allocations in the direction of  $G$*  is introduced.

$$\mathcal{P}_G(U) = \{x \in X : x' \succsim x \text{ and } u_i(x') > u_i(x), \forall i \in I(G) \Rightarrow x' \notin X\}. \quad (3.5)$$

Clearly, if  $G > 0$  then  $\mathcal{P}_G(U) = \mathcal{WP}(U)$ . An elementary algebraic manipulation shows that:

$$\mathcal{P}(U) = \bigcap_{i \in [n]} \mathcal{P}_{e_i}(U) \quad \text{and} \quad \mathcal{WP}(U) = \bigcup_{i \in [n]} \mathcal{P}_{e_i}(U). \quad (3.6)$$

It follows that for all  $i \in [n]$ ,  $\mathcal{P}(U) \subset \mathcal{P}_{e_i}(U) \subset \mathcal{WP}(U)$ . In addition let us introduce the relation  $\succ_G$  defined as  $x' \succ_G x$  if  $x' \succsim x$  and  $u_i(x') > u_i(x)$  for all  $i \in I(G)$ . A real valued map  $f$  defined on  $X$  is strongly monotonic in the direction of  $G$  if  $x' \succ_G x$  implies that  $f(x') > f(x)$ .

**Proposition 3.2** *Suppose that  $G \in \mathbb{R}_+^n \setminus \{0\}$ . For all  $U \in \mathcal{U}_G^n$ ,  $D_G(U)$  satisfies the following properties on  $X$ :*

- (a) *If  $G \in \mathbb{R}_{++}^n$  then, for all  $x \in X$ ,  $D_G(U)(x)=0$  if and only if  $x \in \mathcal{WP}(U)$  (weak Pareto efficiency).*
- (a') *If  $G \in \mathbb{R}_+^n \setminus \{0\}$  then, for all  $x \in X$ ,  $D_G(U)(x)=0$  if and only if  $x \in \mathcal{P}_G(U)$  (Pareto efficiency in the direction of  $G$ ).*
- (a'')  *$x^* \in X$  is a Pareto efficient allocation if and only if  $D_{e_i}(U)(x^*) = 0$  for all  $i \in [n]$ .*
- (b)  *$D_G(U)$  is weakly monotonic, i.e., for all  $x, x' \in X$ , we have  $x' \succsim x$  implies that  $0 \leq D_G(U)(x') \leq D_G(U)(x)$ .*
- (b')  *$D_G(U)$  is strongly monotonic in the direction of  $G$ , i.e., for all  $x, x' \in X$ , we have  $x' \succ_G x$  implies that  $0 \leq D_G(U)(x') < D_G(U)(x)$ .*
- (c) *Suppose that  $G \in \mathbb{R}_{++}^n$ . If  $X$  is a closed subset of  $\mathbb{R}^d$  and if all the utility functions are continuous, then  $D_G(U)$  is continuous on  $X$ .*
- (d) *Suppose that  $X$  is convex and that for all  $i \in [n]$ ,  $u_i$  is concave. Then  $D_G(U)$  is convex.*
- (e) *For all real numbers  $\alpha > 0$  and for all  $b \in \mathbb{R}^n$ , we have*

$$D_G(\alpha U + b) = \alpha D_G(U).$$

- (f) *For all definite positive diagonal matrices  $L$ , we have*

$$D_{LG}(LU) = D_G(U).$$

The value of the directional distance function is zero if and only if the allocation belongs to the Pareto efficient frontier in the direction of  $G$ . A

positive value indicates possible inefficiency and an optimal resource reallocation within the aggregated entity. Moreover, an allocation is Pareto-optimal if and only if, for any vector of the canonical basis  $\{e_i\}_{i \in [n]}$ , the individual directional distance function is zero. Secondly, a weakly dominated allocation is less efficient as the directional distance function is large. Finally, if the direction vector is positive, then the directional distance function is continuous.

Notice that in Figure 3.1 there may exist an allocation that achieves the projection of an utility vector onto the boundary of the disposal representation set. However, if the utility functions are continuous and  $X$  is closed there always exists some  $x^* \in X$  such that  $U(x) + D_G(U)(x) \leq U(z^*)$ .

## 3.2 Axiomatic Properties

In the following it is shown that the directional distance function can be used to construct a [preference relation over  \$X\$](#)  according to traditional welfare theory. Let us consider the binary relation  $f_G(U)$  defined for a given  $G \in \mathbb{R}_+^n \setminus \{0\}$  by

$$x f_G(U) x' \iff D_G(U)(x) \leq D_G(U)(x'). \quad (3.7)$$

This relation satisfies the following properties that can immediately be derived from Proposition 3.2.

**Proposition 3.3** *For all  $U \in \mathcal{U}_G^n$ ,  $f_G(U)$  satisfies the following properties.*

- (a)  $f_G(U)$  is complete and transitive.
- (b)  $f_G(U)$  is a weakly monotonic relation.
- (c) If  $G \in \mathbb{R}_{++}^n$ ,  $X$  is a closed subset of  $\mathbb{R}^d$  and  $U$  is continuous, then  $f_G(U)$  is continuous.
- (d) If  $X$  is convex, and if for all  $i \in [n]$   $u_i$  is quasi-concave, then  $f_G(U)$  is convex.

Sen [30, 31] provided an axiomatic characterization of welfarism from the axioms that a social welfare functional satisfies. The three axiomatic components of welfarism are the following. First, the *axiom of unrestricted domain* specifies if there are constraints imposed on the individual evaluations of states of affairs. In this instance, all the logically possible individual evaluations are allowed. This implies that a social welfare functional must be defined for every conceivable profile of preferences. The *Pareto indifference axiom* states that if all the individuals assess that  $x$  is indifferent to  $x'$ , then  $x$  will be considered indifferent to  $x'$  at the social level. Formally, it states that for all  $x, x'$  and for all vector utility functions  $U$  defined on  $X$ , if  $U(x) = U(x')$ , then  $x \sim x'$ .

The *indifference of other alternatives* means that ranking two states of affairs, all information about any other state of affairs than these two ones is excluded, i.e. considered as irrelevant. More formally, this axiom states

that: for all  $x, x' \in X$ , and for all  $U, U' \in \mathcal{U}^n$ , we have  $U(x) = U'(x)$  and  $U(x') = U'(x')$  implies that  $xf(U)x' \iff xf(U')x'$ .

The *non dictatorial rule* means that there is no  $i \in [n]$  such that, for all  $x, x' \in X$ ,  $u_i(x) > u_i(x') \implies xf(U)x'$  and  $x \approx x'$ . In the following we say that a binary relation  $f(U)$  satisfies a strong Pareto axiom in the direction of  $G$  if, for all  $x, x' \in X$ ,  $x \succ_G x'$  implies that  $xf(U)x'$  and  $x \approx x'$ .

It is further established that this binary relation is reflexive and defines a quasi-order on  $X$  having some suitable properties.

**Proposition 3.4** *For all  $G \in \mathbb{R}^n \setminus \{0\}$ , and all  $U \in \mathcal{U}_G^n$ , the quasi-order  $f_G(U)$  satisfies the following relations:*

- (a) *If  $X$  is finite then,  $\mathcal{U}_G^n = \mathcal{U}^n$  and  $f_G(U)$  satisfies the axiom of unrestricted domain.*
- (b)  *$f_G(U)$  satisfies the axiom of Pareto indifference.*
- (c)  *$f_G(U)$  satisfies the weak Pareto axiom.*
- (c')  *$f_G(U)$  satisfies the strong Pareto axiom in the direction of  $G$ .*
- (d)  *$f_G(U)$  satisfies the non-dictatorial rule.*
- (e) *For all  $a \in \mathbb{R}_{++}$  and all  $b \in \mathbb{R}_+^n$ , if  $U' = aU + b$ , then the quasi-order  $f_G(U)$  and  $f_G(U')$  are equivalent.*
- (f) *For all positive  $n \times n$  diagonal matrices  $L$ , the relations  $f_G(U)$  and  $f_{LG}(LU)$  are equivalent.*

**Remark 3.5** *In general the axiom of unrestricted domain does not hold. Suppose that for any individual  $i$ ,  $X_i = \mathbb{R}^{d_i}$  where  $d_i$  is a positive integer. Set  $d = \sum_{i \in [n]} d_i$ . Suppose that for all  $i$ ,  $u_i$  is a linear map defined by  $u_i(x_i) = \sum_{j \in [d_i]} a_{i,j} x_{i,j}$ , where  $a_i \in \mathbb{R}_{++}^{d_i}$ . Then  $U(X) = \mathbb{R}^n$  and  $D_G(U)(x) = \infty$  for all  $x \in \mathbb{R}_{++}^d$ , which implies that  $D_G(U)$  cannot be evaluated of  $x$ . It is easy to provide some example showing that the strong Pareto axiom fails. Also the axiom of indifference of other alternative does not hold. This comes from the fact that the utility set depends on  $U$ . However, note that Sen [30, 31] clearly criticized the informational basis of welfarism he considered exceedingly narrow.*

## 4 Weighted Welfare Functionals and Duality

### 4.1 On Utilitarian Welfare Functions and Duality

In economics, duality is the principle that optimization problems the individuals are facing may be viewed from either of two perspectives, the primal problem or the dual problem. The solution to the primal and dual problems are intimately linked when some suitable convexity assumptions are satisfied.

To provide a dual interpretation of the directional distance function, it is convenient to define a corresponding general indirect welfare function. This implies that the welfare measures we consider should have a linear structure.

Along this line, we focus on the utilitarian welfare function  $W_\mu(U)$ , that is a special case of the Bergson-Samuelson welfare function introduced in [5] and [28]. The next results will establish that, under a convexity assumption of the disposal representation set  $\mathcal{V}(U)$ , this function can be connected to the directional distance function. In this section it is shown that, if the disposal representation set is convex, then the utilitarian welfare function is a relevant measure because it is dual to the directional distance function and can be used to characterize any Pareto efficient allocation.

Before to achieve these goals, an indirect utilitarian welfare function  $W^*(U)$  is introduced. For all  $\mu \in \mathbb{R}_+^n$  and all utilitarian social welfare functions  $W_\mu(U)$ , let  $W^*(U)$  be a map valued in  $\mathbb{R} \cup \{+\infty\}$  and defined on  $\mathbb{R}_+^m$  as:

$$W_U^*(\mu) = \sup \{ W_\mu(U)(x) : x \in X \} \quad (4.1)$$

is called *the indirect utilitarian welfare function*.

The map  $U \mapsto W_U^*$  is called the indirect utilitarian functional. This indirect welfare function represents the maximization of the weighted sum of the utilities of each individual subject to the feasibility of an allocation.

We come now to our duality result showing that the directional distance function is primal to the indirect utilitarian welfare function that is dual to the directional distance function. Clearly the function  $V \mapsto \mu \cdot V = \sum_{i=1}^n \mu_i v_i$  is linear on  $\mathbb{R}_+^n$ . Moreover,  $W_U^*(\mu) = \sup \{ \mu \cdot V : V \in \mathcal{V}(U) \}$ . Therefore, some earlier results due to Luenberger [18] and Chambers et. al [14] can be transposed in this context. To that end, let us consider the map  $\tau_{G,U}$  valued in  $\mathbb{R} \cup \{-\infty, +\infty\}$  and defined on  $\mathbb{R}^n$  as  $\tau_{G,U}(V) = \sup \{ \delta : V + \delta G \in \mathcal{V}(U) \}$ . We have the immediate relation:

$$D_G(U)(x) = \tau_{G,U} \circ U(x) \quad (4.2)$$

for all  $x$  in  $X$ . The map  $\tau_{G,U}$  is *inverse translation homothetic in the direction of  $G$* , that is for all  $\alpha \in \mathbb{R}$   $\tau_{G,U}(V + \alpha G) = \tau_{G,U}(V) - \alpha$ .

In the following, the Negishi theorem [23] is used, which asserts that, under a convexity assumption, any Pareto efficient allocation maximizes a suitable utilitarian function.

The next duality result establishes a relation between the directional distance function and the indirect social welfare function.

**Proposition 4.1** *For all  $U \in \mathcal{U}_G^n$ , if  $\mathcal{V}(U)$  is a closed and non-empty convex subset of  $\mathbb{R}^n$ , then:*

$$D_G(U)(x) = \inf_{\mu \geq 0} \{ W_U^*(\mu) - W_\mu(U)(x) : \mu \cdot G = 1 \}$$

and

$$W_U^*(\mu) = \sup_{x \in X} \{ W_\mu(U)(x) - D_G(U)(x) \mu \cdot G \}.$$

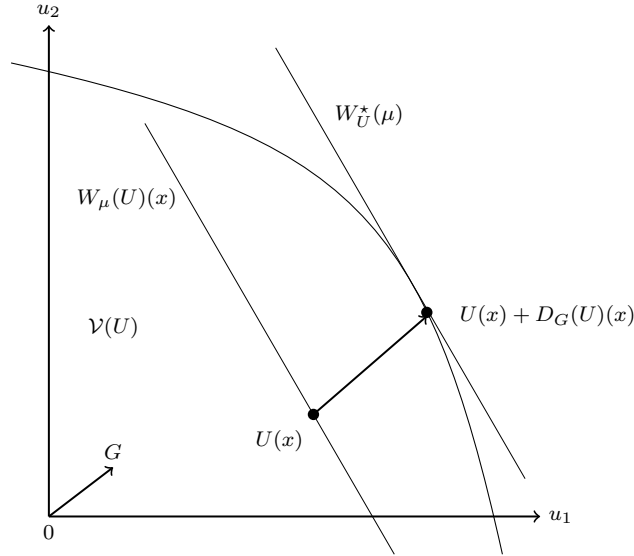
Remark that an elementary calculus shows that

$$W_U^*(\mu) - W_\mu(U)(x) = W_{U-U(x)}^*(\mu). \quad (4.3)$$

Hence, we have:

$$D_G(U)(x) = \inf_{\mu \geq 0} \{W_{U-U(x)}^*(\mu) : \mu \cdot G = 1\}. \quad (4.4)$$

Therefore, the directional distance function is primal to the indirect welfare function which is dual to the directional distance function. Figure 4.1 illustrates this duality result. The social welfare maximizing utility vector is given by the point labelled  $U^*$ , which is determined by the tangency between the frontier of  $\mathcal{V}(U)$  and the line segment whose slope is given by the normalized weighting scheme. Hence, the dual problem can be interpreted as one of finding the vector parallel to  $G$  which includes the efficient point  $U^*$ . It follows that calculating the directional distance function boils down finding a normalized dual weighting scheme that minimizes the difference between the indirect utilitarian welfare function and the utilitarian welfare of an allocation  $x$ . The reference utility vector  $U^*$  is then associated to the maximisation of the utilitarian welfare function subject to the choice of a suitable weighting scheme  $\mu^*$ .



**Figure 4.1** : Utilitarian Welfare Function and Duality.

In the following a connection to the Negishi theorem is established. For this purpose, the so-called adjusted weight function is introduced. Along this line, an adjusted social welfare function is defined. In a consumer context, Luenberger introduced a set valued map called the adjusted price function that plays a crucial role to measure the willingness to pay of consumers.

The set valued map  $\mu^*(U)$  defined on  $X$  by

$$\mu^*(U)(x) = \arg \min \{W_U^*(\mu) - W_\mu(U)(x) : \mu \cdot G = 1, \mu \in \mathbb{R}_+^n\}$$

is called *the adjusted weight function relative to  $G$* . The corresponding social welfare function  $W_{\mu^*(x)}(U)$  is called *the adjusted utilitarian welfare function*. Let us consider the set-valued map  $\mu^*$  defined on  $\mathbb{R}^n$  by  $\mu^*(V) = \arg \min \{W_U^*(\mu) - \mu \cdot V : \mu \cdot G = 1, \mu \in \mathbb{R}_+^n\}$  that is derived from the dual program of  $\tau_{G,U}$  defined in equation (4.2). We have the relation  $\mu^*(U) = \mu^* \circ U$ .

Note that the minimum value of  $W_U^*(\mu) - W_\mu(U)(x)$  is the directional distance function under a convexity assumption of the disposal representation set. Hence,  $\mu^*(U)(x)$  yields the welfare function that gives the directional distance function value. In general, of course, if this minimum is achieved, it may not be unique, in which case the adjusted weight function is a set valued map.

In fact, under some regularity conditions, the adjusted weight function can be obtained from the Kuhn-Tucker multipliers of the mathematical program computing the directional distance function.

Now, we come to the following result that yields an extension of the Negishi theorem.

**Proposition 4.2** *Assume that  $G \in \mathbb{R}_{++}^n$  and  $U \in \mathcal{U}_G^n$ . Suppose moreover that  $\mathcal{V}(U)$  is closed and convex. Let  $x^*$  be a Pareto efficient allocation. There is some  $\mu^* \in \mathbb{R}_+^n$  such that:*

$$W_U^*(\mu^*) = \sup \{W_{\mu^*}(U)(x) : x \in X\} = W_{\mu^*}(U)(x^*).$$

*Moreover, if  $U(x^*)$  is not a kink point of  $\mathcal{V}(U)$ , and if  $\tau_{G,U}$  is differentiable at  $U(x^*)$ , then there is a uniqueness element  $\mu^*$  such that for each  $i=1, 2, \dots, n$ :*

$$\mu_i^* = - \left. \frac{\partial \tau_{G,U}(V)}{\partial v_i} \right|_{v_i = u_i(x^*)}.$$

This result establishes that the weight  $\mu^*$  for which  $x^*$  maximizes the social welfare function  $W_{\mu^*}$  is obtained from the derivatives of the function  $\tau_G$ .

## 4.2 Non-Convexities and Rawlsian Duality

The analysis of economic efficiency relies extensively on convexity assumptions: convexity of production sets and convexity of preferences (e.g, Debreu [16]). If convexity is relaxed, then many of the properties of competitive markets do not hold. Thus, non-convexity is sometimes associated with market failures, where market equilibria can be inefficient. **However**, in many situations the structure of preferences may be non-convex. For example, convexity is less than essential when one takes into account some possible anti-complementarity in the structure of preferences. In the case where preferences are not convex, the utility set may not be convex. Another situation arises when the goods are indivisible. In such a case the consumption set may be discrete and therefore the utility set is a finite number of points and

also has discrete structure. If the disposal representation set is not convex, then there is no dual relation between the directional distance function and the indirect utilitarian welfare function.

First of all, note that, by construction,  $\mathcal{V}(U)$  is a downward set involving some interesting properties. The properties of downward sets have been analyzed in [21]. First, let us define the indirect Rawls welfare function as follows. For all  $\mu \in \mathbb{R}_+^n$  and all Rawls welfare function  $R_\mu(U)$ , the function  $R_U^* : \mathbb{R}_+^n \rightarrow \mathbb{R}$  defined by

$$R_U^*(\mu) = \sup\{R_\mu(U)(x) : x \in X\}, \quad (4.5)$$

is called the *indirect Rawls welfare function*. The next statement is an immediate consequence of Proposition 3.1.

**Proposition 4.3** *For all  $U \in \mathcal{U}_G^n$ , if  $G \in \mathbb{R}_{++}^n$  we have for all  $x \in X$*

$$D_G(U)(x) = R_{U-U(x)}^*(G^{-1}).$$

*Moreover if  $x$  is an egalitarian allocation with  $u_i(x) = \bar{u}$  for all  $i$  then:*

$$D_{\mathbf{1}_n}(U)(x) = R_U^*(\mathbf{1}_n) - \bar{u}. \quad (4.6)$$

This indirect Rawls welfare function is a kind of support function replacing the usual operation of addition “+” with the operation of minimum “min”. Suppose that  $U(X) \subset \mathbb{R}_+^n$ . In such a case, Martínez, Rubinov and Singer [21] have established that, given  $V \in \mathbb{R}_+^n \setminus \mathcal{V}(U)$ , it is always possible to find some  $\mu \in \mathbb{R}_+^n$  such that  $\min_{i \in I(\mu)} \{\mu_i v_i\} > R_U^*(\mu)$ . Consequently, the restriction to the nonnegative Euclidean orthant of the disposal representation set  $\mathcal{V}(U)$  is the intersection of all its supporting halfspaces. Namely, we have:

$$\mathcal{V}(U) \cap \mathbb{R}_+^n = \bigcap_{\mu \in \mathbb{R}_+^n} \{V \in \mathbb{R}_+^n : \min_{i \in I(\mu)} \mu_i v_i \leq R_U^*(\mu)\}. \quad (4.7)$$

Notice that one can also derive the properties above from the separation properties established in [11]. Among the consequences of this result, it is possible to give an analogue to the Negishi theorem expressed in terms of the Rawls welfare function.

**Proposition 4.4** *Suppose that  $U(X)$  is a nonempty closed subset of  $\mathbb{R}_+^n$ . For each Pareto efficient allocation  $x^*$  there exists some  $\mu^* \in \mathbb{R}_+^n$  such that  $R_U^*(\mu^*) = R_{\mu^*}(U)(x^*)$ .*

The following result establishes that, though the utilities are not concave, there is a dual relationship between the directional distance function and the Rawls welfare function. Notice that we should assume that the range of  $U$  is a subset of  $\mathbb{R}_+^n$ .

**Proposition 4.5** *Suppose that  $U(X)$  is a nonempty closed subset of  $\mathbb{R}_+^n$ . Then, for all  $x \in X$  and all  $G \in \mathbb{R}_+^n \setminus \{0\}$ , we have*

$$D_G(U)(x) = \inf_{\mu \geq 0} \min_{i \in I(\mu) \cap I(G)} \frac{R_U^*(\mu)/\mu_i - u_i(x)}{g_i}$$

and

$$R_U^*(\mu) = \sup_{x \in X} \min_{i \in I(\mu) \cap I(G)} \mu_i (u_i(x) + D_G(U)(x)g_i).$$

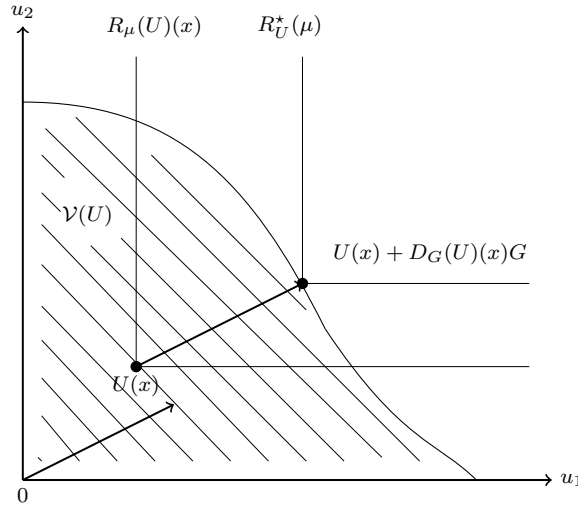


Figure 4.2 Duality with Rawls Welfare Functions.

## 5 Generalized Means and Extended Distance Functions

### 5.1 $\phi$ -Generalized Distance Functions

In the earlier section it was established that the directional distance function characterizes all the weak Pareto efficient allocations. In this subsection the directional distance function is slightly modified to get an encompassing reformulation that permits to characterize Pareto efficient allocations. More importantly, a relation to the social welfare functions introduced by Atkinson [5] is established. First let us define  $\mathcal{U}_B^n$  and the set of all the vector functions in  $\mathcal{U}$  that are bounded above, i.e.

$$\mathcal{U}_B^n = \{U \in \mathcal{U}_B^n : \exists \bar{V} \in \mathbb{R}^n \text{ such that } U(z) \leq \bar{V}, \forall z \in X\}.$$

Notice that we have  $\mathcal{U}_B^n \subset \mathcal{U}_G^n$ . Suppose that  $\phi$  is a bijective map defined from  $\mathbb{R}_+$  to an arbitrary subset  $K$  of  $\mathbb{R} \cup \{+\infty\}$ , that is  $\phi$  may have an infinite value.



In the following, for all  $(\delta_1, \dots, \delta_n) \in \mathbb{R}^n$  we adopt the notation  $\Delta = (\delta_1, \dots, \delta_n)$ . Let  $\sum^\phi$  denotes the generalized sum defined as

$$\sum_{i \in [n]}^\phi \delta_i = \phi^{-1} \left( \sum_{i \in [n]} \phi(\delta_i) \right). \quad (5.1)$$

The map  $\Delta \mapsto \phi^{-1}(\frac{1}{n}) \sum_{i \in [n]}^\phi \delta_i$  yields an encompassing formulation of the usual notion of generalized mean. In particular for all  $r > 0$ , let  $\phi_r$  be the map defined on  $\mathbb{R}_+$  by  $\phi_r(\delta_i) = \delta_i^r$ . We have  $\phi_r^{-1}(\frac{1}{n}) \sum_{i \in [n]}^{\phi_r} \delta_i = (\frac{1}{n} \sum_{i \in [n]} \delta_i^r)^{\frac{1}{r}}$  and we retrieve as a special case the generalized mean of order  $r$ . Suppose moreover, that: (i) If  $\Delta = 0$  then  $\sum_{i \in [n]}^\phi \delta_i = 0$ ; (ii) The map  $\Delta \mapsto \sum_{i \in [n]}^\phi \delta_i$  is nondecreasing on  $\mathbb{R}_+^n$ . For all utility functions  $U \in \mathcal{U}_B^n$ , the nonnegative real valued map  $SD_\phi(U)$  defined on  $X$  by

$$SD_\phi(U)(x) = \sup_{\Delta} \left\{ \phi^{-1} \left( \frac{1}{n} \sum_{i \in [n]}^\phi \delta_i : U(x) + \Delta \leq U(z), \Delta \in \mathbb{R}_+^n, z \in X \right) \right\} \quad (5.2)$$

is called the  $\phi$ -generalized distance function. Such a construction will be useful in the next section to consider a class of distance functions that takes into account the aversion to inequalities in the case of a generalised mean of order  $r$  that is specifically treated in the next section.

The  $\phi$ -generalized distance function quantifies the possible individual improvements of the utility vector  $U(X)$ , while maintaining an allocation  $x$  in  $X$ . For the sake of simplicity, this definition does not refer to any either weighting scheme or directions that can be endogenously incorporated to the map  $\phi$ .

Along this line, for all  $x \in X$  such that  $U(x) \geq 0$ , it is natural to consider a general form of the utilitarian social welfare function defined as  $W^\phi(U)(x) = \sum_{i \in [n]}^\phi u_i(x)$ . One can then provide an equivalent formulation with:

$$SD_\phi(U)(x) = \sup_{\Delta} \left\{ \phi^{-1} \left( \frac{1}{n} W^\phi(\Delta) : U(x) + \Delta \leq U(z), \Delta \in \mathbb{R}_+^n, z \in X \right) \right\}. \quad (5.3)$$

In the next statement, it is shown that a useful reformulation allows to establish some connections to the generalized utilitarian function presented in the earlier sections.

**Proposition 5.1** *Suppose that  $\phi$  is a bijective map defined from  $\mathbb{R}_+$  to an arbitrary subset  $K$  of  $\mathbb{R} \cup \{+\infty\}$ . Let  $SD_\phi(U) : X \rightarrow \mathbb{R}_+$  be a  $\phi$ -generalized distance function. Then we have for all  $x \in X$*

$$SD_\phi(U)(x) = \sup_{z \in X} \left\{ \phi^{-1} \left( \frac{1}{n} W^\phi(U - U(x))(z), z \succeq x \right) \right\}.$$

## 5.2 $\phi_r$ -Generalized Distance Functions and Limit Properties

For all  $\Delta = (\delta_1, \dots, \delta_n) \in \mathbb{R}_{++}^n$ , and all  $r \in \mathbb{R} \setminus \{0\}$ , let us consider the map  $(\delta_1, \dots, \delta_n) \mapsto \left( \sum_{i \in [n]} \delta_i^r \right)^{\frac{1}{r}}$ . In the case where  $r < 0$ , it may not be properly defined, especially when some components of  $\Delta$  are either null or negative. To overcome this problem, a slightly more general formulation is introduced.

For all  $r \in ]0, +\infty[$ , let  $\phi_r$  be the map defined by  $\phi_r(\lambda) = \lambda^r$ . For all  $r \neq 0$ , the reciprocal map is  $\phi_r^{-1} := \phi_{\frac{1}{r}}$ . First, it is quite straightforward to state that: (i)  $\phi_r$  is defined over  $\mathbb{R}_+$ ; (ii)  $\phi_r$  is continuous over  $\mathbb{R}_+$ ; and (iii)  $\phi_r$  is bijective over  $\mathbb{R}_+$ . Second, let us focus on the case  $r \in ]-\infty, 0[$ . The map  $\lambda \mapsto \lambda^r$  is not defined at point  $\lambda = 0$ . Thus, it is not possible to construct a bijective endomorphism on  $\mathbb{R}_+$ .

For all  $r \in ]-\infty, 0[$  we consider the functions  $\phi_r$  and its reciprocal  $\phi_{\frac{1}{r}}$  respectively defined on  $\mathbb{R}_+$  and  $\mathbb{R}_{++} \cup \{+\infty\}$  as:

$$\phi_r(\lambda) = \begin{cases} \lambda^r & \text{if } \lambda > 0 \\ +\infty & \text{if } \lambda = 0 \end{cases} \quad \text{and} \quad \phi_{\frac{1}{r}}(\lambda) = \begin{cases} \lambda^{\frac{1}{r}} & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = \infty. \end{cases} \quad (5.4)$$

Let us consider the  $\phi_r$  generalized sum analyzed by Ben-Tal [7]. For all  $(\delta_1, \dots, \delta_n) \in \mathbb{R}_+^n$ , and for all  $r > 0$  the  $\phi_r$ -generalized sum is given by:  $\sum_{i \in [n]}^{\phi_r} \delta_i := \phi_r^{-1} \left( \sum_{k \in [n]} \phi_r(\delta_k) \right) = \left( \sum_{i \in [n]} \delta_i^r \right)^{\frac{1}{r}}$ .

If  $r < 0$ , using the symbolism  $\frac{1}{0} = +\infty$  we have, by construction:

$$\sum_{i \in [n]}^{\phi_r} \delta_i := \phi_r^{-1} \left( \sum_{i \in [n]} \phi_r(\delta_i) \right) = \begin{cases} \left( \sum_{i \in [n]} (\delta_i)^r \right)^{\frac{1}{r}} & \text{if } \min_i \delta_i > 0 \\ 0 & \text{if } \min_i \delta_i = 0. \end{cases} \quad (5.5)$$

It is then easy to see that, when  $r < 0$ , this generalized sum is a continuous extension of the generalized sum  $(\delta_1, \dots, \delta_n) \mapsto \left( \sum_{i \in [n]} \delta_i^r \right)^{\frac{1}{r}}$  from  $\mathbb{R}_{++}^n$  to  $\mathbb{R}_+^n$ . It follows that we have:  $\lim_{r \rightarrow \infty} \sum_{i \in [n]}^{\phi_r} \delta_i = \min_{i \in [n]} \delta_i$ . Moreover, for all  $r \neq 0$ ,  $\Delta = 0$  implies  $\sum_{i \in [n]}^{\phi_r} \delta_i = 0$  and if  $\Delta \geq 0$  then  $\sum_{i \in [n]}^{\phi_r} \delta_i \geq 0$ . In addition, the map  $\Delta \mapsto \sum_{i \in [n]}^{\phi_r} \delta_i$  is nondecreasing on  $\mathbb{R}_+^n$ . Therefore, the map  $\phi_r$  satisfies the conditions of a  $\phi_r$ -generalized distance function as defined in Section 5.1. For all real numbers  $r \neq 0$ , the  $\phi_r$ -generalized distance function is

$$SD_{\phi_r}(U)(x) = \sup_{\Delta} \left\{ \frac{1}{n^{\frac{1}{r}}} \sum_{i \in [n]}^{\phi_r} \delta_i : U(x) + \Delta \leq U(z), \Delta \in \mathbb{R}_+^n, z \in X \right\}. \quad (5.6)$$

One can also define, for all  $x \in X$  such that  $U(x) \geq 0$ , a social welfare function  $W^{\phi_r}(U) = \phi_r^{-1} \left( \sum_{i \in [n]} \phi_r(u_i) \right)$ . In the remainder, for the sake of

simplicity  $W^{\phi_r}$  is denoted  $W^{(r)}$  and we retrieve the notations used in Section 2.2. Therefore:  $SD_{\phi_r}(U)(x) = \sup_{z \in X} \left\{ n^{-\frac{1}{r}} W^{(r)}(U - U(x))(z), z \succsim x \right\}$ .

Notice that setting  $\psi_r = \phi_r(\exp)$ , we have  $KP^{(r)}(U) = W^{\psi_r}(U)$ . Setting  $SD_{\psi_r} = KP^{(r)}$ , we retrieve a *Kolm-Pollak* formulation with

$$KP^{(r)}(U)(x) = \sup_{\Delta} \left\{ \frac{1}{r} \ln \left( \sum_{i \in [n]} \exp(r\delta_i) \right) : U(x) + \Delta \leq U(z), \Delta \geq 0, z \in X \right\} \quad (5.7)$$

In addition, we introduce a *multiplicative* distance function defined as

$$ND(U)(x) = \sup_{\Delta} \left\{ \prod_{i \in [n]} \delta_i^{\frac{1}{n}} : U(x) + \Delta \leq U(z), \Delta \geq 0, z \in X \right\}. \quad (5.8)$$

From Proposition 5.1:

$$ND(U)(x) = \sup_{z \in X} \left\{ N(U - U(x))(z), z \succsim x \right\}. \quad (5.9)$$

Also, let us denote  $AD_{\max}(U) = \max_{i \in [n]} \{D_{e_i}(U)\}$  the *asymmetric* directional distance function because it involves an asymmetric transformation of each individual utility function.

Atkinson [5] proposed a normative measure of income inequality by imposing a coefficient  $\varepsilon = 1 - r \in [0, 1[$  to weight incomes. If  $\varepsilon = 1$  the generalized mean of exponent 0 is interpreted as the geometric mean. The Atkinson index can be derived from the utilitarian function of order  $W^{(r)}$  that is the  $\phi_r$ -generalized mean of the individual utilities. Atkinson showed that  $\varepsilon$ , represents the level of *inequality aversion* since it quantifies the amount of social utility that is assumed to be gained from complete redistribution of resources. For  $\varepsilon = 0$  ( $r = 1$ ), there is no aversion to inequality and it is assumed that no social utility is gained by complete redistribution. Considering a broader set of parameters for inequality aversion, if  $\varepsilon = \infty$  ( $r = -\infty$ ) there is an infinite aversion to inequality and social utility is gained by complete redistribution. The next proposition yields some parallel results. If  $r = -\infty$  ( $\varepsilon = \infty$ ) then one retrieves as a limit case the directional distance function that is linked to the Rawls welfare function.

**Proposition 5.2** *Suppose that  $U \in \mathcal{U}_{\mathbb{B}}^n$ . If for all  $i \in [n]$   $u_i$  is upper-semicontinuous, then, for all  $x \in X$ , we have the following properties:*

- (a)  $\lim_{r \rightarrow -\infty} SD_{\phi_r}(U)(x) = D_{\mathbb{1}_n}(U)(x)$ ;
- (b)  $\lim_{r \rightarrow 0} SD_{\phi_r}(U)(x) = ND(U)(x)$ ;
- (c)  $\lim_{r \rightarrow \infty} SD_{\phi_r}(U)(x) = AD_{\max}(U)(x)$ .

**Lemma 5.3** *Suppose that  $U \in \mathcal{U}_{\mathbb{B}}^n$  and that for all  $i \in [n]$   $u_i$  is upper-semicontinuous. For all  $r, s \in [-\infty, \infty]$ , if  $r \leq s$ , then*

$$SD_{\phi_r}(U) \leq SD_{\phi_s}(U).$$

Notice that if  $r > 0$ , the map  $\Delta \mapsto \left(\sum_{i \in [n]} \delta_i^r\right)^{\frac{1}{r}}$  is increasing. The next result immediately follows.

**Proposition 5.4** *For all  $U \in \mathcal{U}_B^n$  and all  $r \in \mathbb{R}_+ \setminus \{0\}$ ,  $SD^{(r)}(U)$  is strongly monotonic, i.e. for all  $x, x' \in x$  if  $x \succsim x'$  and  $x \approx x'$  then  $SD_{\phi_r}(U)(x) < SD_{\phi_r}(U)(x')$ . Moreover,  $x^*$  is a Pareto efficient allocation if and only if*

$$SD_{\phi_r}(U)(x^*) = 0.$$

## 6 Conclusion

We have introduced [in social choice theory a measure to compare sub-efficient allocations](#): the directional distance function. This paper has outlined that the directional distance function is [dually](#) linked to the Rawls welfare function. Considering a convexity assumption, it is also dual to the indirect utilitarian social welfare function introduced in this paper. Thereafter, it has been shown that the directional distance function also enjoys many interesting axiomatic properties from a welfare viewpoint. Finally, in line with Atkinson [5], the paper has proposed an extension of the directional distance function depending on an inequality aversion parameter. It includes as a limit case (infinite inequality aversion) the directional distance function.

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## Appendix

**Proof of Proposition 3.1:** By definition we have

$$\begin{aligned}
 D_G(U)(x) &= \sup \left\{ \delta : U(x) + \delta G \leq U(z), z \in X \right\} \\
 &= \sup \left\{ \delta : \delta G \leq U(z) - U(x), z \in X \right\} \\
 &= \sup \left\{ \delta : \delta \leq \frac{u_i(z) - u_i(x)}{g_i}, z \in X, i \in I(G) \right\}. \quad \square
 \end{aligned}$$

The relation to the Rawls welfare function is then immediate.  $\square$

**Proof of Proposition 3.2:** The proof of (a) is immediate. (a') Since  $U \in \mathcal{U}_G^n$ ,  $D_G(U)(x) < +\infty$ . If  $x \notin \mathcal{P}_G(U)$ , then there is some  $x' \in X$  such that  $u_i(x') \geq u_i(x)$  for  $i = 1, \dots, n$  and  $u_i(x') > u_i(x)$  for  $i \in I(G)$ . Therefore, it is immediate to see that  $D_G(U)(x) > 0$ . Conversely, if  $D_G(U)(x) > 0$ ,

then  $U(x) + D_G(U)(x)g \in \mathcal{V}(U)$ . Since  $g_i > 0$  for all  $i \in I(G)$  we deduce that  $u_i(x) < u_i(x) + D_G(U)(x)G$ . Therefore,  $x \notin \mathcal{P}_G(U)$ . (a'') Suppose that there is some  $i \in [n]$  such that  $D_{e_i}(U)(x^*) > 0$ . In such a case, one can find some  $V = (v_1, \dots, v_n) \in \mathcal{V}(U)$  such that  $v_i > u_i(x^*)$ . Since  $V \in \mathcal{V}(U) = U(X) - \mathbb{R}_+^n$ , there is some  $z \in X$  such that  $u_i(z) \geq v_i > u_i(x^*)$  and  $x$  cannot be Pareto efficient. Consequently, if  $x^*$  is Pareto efficient, then  $D_{e_i}(U)(x^*) = 0$  for  $i = 1, \dots, n$ . Let us prove the converse and suppose that  $x^*$  is not Pareto efficient. In such a case there is some  $i \in [n]$  and some  $z \in X$  such that  $u_i(z) > u_i(x^*)$ . Since  $U(z) \in U(X) \subset \mathcal{V}(U)$ , we deduce that  $D_{e_i}(U)(z) > 0$ . Consequently, if  $D_{e_i}(U)(x^*) = 0$  for  $i = 1, \dots, n$  then  $x^*$  is Pareto efficient, which ends the proof. (b) is an immediate consequence of the definition of  $\mathcal{V}(U)$ . Assume that  $U(x) \leq U(x')$ . Since  $\mathcal{V}(U) = \mathcal{V}(U) - \mathbb{R}_+^n$ , we have  $\{\delta \geq 0 : U(x') + \delta G \in \mathcal{V}(U)\} \subset \{\delta \geq 0 : U(x) + \delta G \in \mathcal{V}(U)\}$ . Then, by definition, we deduce that  $D_G(U)(x') \leq D_G(U)(x)$ . The proof of (b') is similar. (c) Let us consider the map  $\tau_{G,U} : \mathcal{V}(U) \rightarrow \mathbb{R}_+$  defined by  $\tau_{G,U}(V) = \sup\{\delta : V + \delta G \in \mathcal{V}(U)\}$ . Since  $\mathcal{V}(U) = \mathcal{V}(U) - \mathbb{R}_+^n$  it follows, from Luenberger [18], that  $\tau_{G,U}$  is continuous. However, by construction  $D_G(U)(x) = \tau_{G,U}(U(x))$ . Since preferences are continuous,  $U$  is continuous vector function defined over  $\mathcal{V}(U)$ . It follows that  $D_G(U)$  is continuous on  $X$ . (d) For all  $i$ , if  $u_i$  is concave then it is quasi-concave. Therefore,  $\mathcal{V}(U)$  is convex. This implies that  $D_G$  is concave on  $\mathcal{V}(U)$ . Consequently, for all  $x, x' \in X$ , and all  $t, t' \in [0, 1]$  with  $t + t' = 1$ , we have  $D_G(tU(x) + t'U(x')) \geq tD_G(U)(x) + t'D_G(U)(x')$ . However, if for all  $i$   $u_i$  is concave, then we have  $U(tx + t'x') \geq tU(x) + t'U(x')$ . Since  $D_G$  is weakly monotonic on  $\mathcal{V}(U)$ , we deduce that  $D_G(U)(tx + t'x') \leq tD_G(U)(x) + t'D_G(U)(x')$  which proves (d). (e) and (f) are immediate from Proposition 3.1.  $\square$

**Proof of Proposition 3.3:** (a) Since  $U \in \mathcal{U}_G^n$ , for all  $x$ ,  $D_G(U)(x) < \infty$ . Therefore  $f_G(U)$  is complete and transitive. (b) From Proposition 3.2.(c),  $D_G(U)$  is weakly monotonic which implies (b). (c) From Proposition 3.2.(c),  $D_G(U)$  is continuous. It follows that  $f_G(U)(y) = \{x \in X : xf_G(U)y\}$  is closed. Moreover,  $f_G(U)^{-1}(x) = \{y \in X : xf_G(U)y\}$  also is closed. (d) Let  $x, x' \in X$ , with  $x \succsim x'$ . This implies that for all  $i$ , and all  $t, t' \in [0, 1]$ , with  $t + t' = 1$ , we have from the quasi-concavity of  $u_i(tx + t'x') \geq \min\{u_i(x), u_i(x')\} \geq u_i(x')$ . Therefore  $D_G(U)(tx + t'x') \geq D_G(x')$ , which implies that  $tx + t'x' f_G(U)x'$ .  $\square$

**Proof of Proposition 3.4:** (a) Suppose that  $X$  is finite. If for all  $i$ ,  $u_i$  is a real valued function, then  $U(X)$  is a finite set. It follows that  $\mathcal{V}(U)$  is bounded above and that  $D_G(U)(x)$  takes real values over  $X$ . Therefore the functional  $D_G$  is well defined. (b) is immediate since for all  $x, z \in X$ ,  $x \sim z$  implies that  $U(x) = U(z)$  and consequently  $D_G(U)(x) = D_G(U)(z)$ . Moreover, since  $U \in \mathcal{U}_G^n$ ,  $D_G(U)$  is well defined over  $X$ . (c) Suppose that  $u_i(x') > u_i(x)$  for all  $i \in [n]$ . This implies that one can find some  $\epsilon > 0$  such

that for all  $i$ ,  $u_i(x') > u_i(x) + \epsilon g_i$  for all  $i \in I(G)$ . We have

$$D_G(U)(x) = \sup_{z \in X} \min_{i \in I(G)} \left\{ \frac{u_i(z) - u_i(x)}{g_i} \right\},$$

where  $I(G) = \{i \in [n] : g_i > 0\}$  and

$$\begin{aligned} D_G(U)(x') &= \sup_{z \in X} \min_{i \in I(G)} \left\{ \frac{u_i(z) - u_i(x')}{g_i} \right\} \\ &\leq \sup_{z \in X} \min_{i \in I(G)} \left\{ \frac{u_i(z) - u_i(x) + \epsilon g_i}{g_i} \right\} \\ &\leq \sup_{z \in X} \min_{i \in I(G)} \left\{ \frac{u_i(z) - u_i(x)}{g_i} \right\} + \epsilon. \end{aligned}$$

Hence  $D_G(U)(x') < D_G(U)(x)$ . The proof of (c') uses similar arguments. (d) To prove this result it is just sufficient to find an utility set that contradicts the dictatorial rule. in any case. Assume for example that  $n = 2$  and  $U(X) = \{(u_1, u_2) \in \mathbb{R}_+^2 : u_1 + u_2 \leq 1\}$ . Suppose some  $G = (g_1, g_2) \in \mathbb{R}_+^2 \setminus \{0\}$ . Suppose that  $g_1 > 0$ . Fix  $(u_1, u_2) \in U(X)$  such that  $D_G(u_1, u_2) > 0$ . Hence, one can find some  $\epsilon > 0$  such that  $D_G(u_1 + \epsilon g_1, u_2 + \epsilon g_2) > 0$ . This implies that there is some  $x, x' \in X$  such that  $u_1(x') = u_1 + \epsilon g_1 > u_1 = u_1(x)$ . Now take  $G' = (0, g_2)$  with  $g_2 > 0$ .  $(u_1, u_2 + D_{G'}(u))$  lies in the hyperplane define by  $u_1 + u_2 = 1$ , and there is some  $x'' \in U(X)$  such that  $U(x'') = (u_1, u_2 + D_{G'}(u))$ . This implies that  $D_G(U)(x'') = 0$  and  $x'' \succ x'$ . However  $u_1(x') = u_1 + \epsilon > u_1 = u_1(x'')$ . It follows that in such a case, for all  $G = (g_1, g_2) \in \mathbb{R}_+^2 \setminus \{0\}$ , the dictator rule does not apply which ends the proof. (e) Suppose that  $x, x' \in X$ ,  $x' f_G(U)x$  and  $D_G(U)(x) \neq D_G(U)$ . We have for all  $i \in I(G)$  and all  $z \in X$ :

$$\frac{u_i(z) - u_i(x)}{g_i} = \frac{au_i(z) + b_i - (au_i(x) + b_i)}{g_i}$$

and

$$\frac{u_i(z) - u_i(x')}{g_i} = \frac{au_i(z) + b_i - (au_i(x') + b_i)}{g_i}.$$

The result immediately follows. (f) is established following a similar procedure.  $\square$

**Proof of Proposition 4.1:** Let us consider the function  $\tau_{G,U} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$  defined by  $\tau_{G,U}(V) = \sup\{\delta : V + \delta G \in \mathcal{V}(U)\}$  if the line spanned from  $V$  in the direction of  $G$  meets  $\mathcal{V}(U)$  and taking the value  $-\infty$  otherwise. Moreover, let  $h : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be the functional support of  $\mathcal{V}(U)$  defined by  $h(\mu) = \sup\{\mu \cdot V : V \in \mathcal{V}(U)\}$ . From Luenberger [18] we have:

$$\tau_{G,U}(V) = \inf_{\mu \geq 0} \{h(\mu) - \mu \cdot V : \mu \cdot G = 1\} \text{ and } h(\mu) = \sup_V \{\mu \cdot V - \tau_{G,U}(V) \mu \cdot G\}.$$

Clearly, for all  $\mu \geq 0$ , we have  $h(\mu) = \sup\{\mu \cdot V : V \in U(X) - \mathbb{R}_+^n\} = \sup\{\mu \cdot V : V \in \mathcal{V}(U)\} = \sup\{\mu \cdot U(x) : x \in X\} = W_U^*(\mu)$ . Setting  $V = U(x)$ ,



we obtain  $\mu \cdot V = W_\mu(U)(x)$  and  $\tau_{G,U}(V) = D_G(U)(x)$ . Substituting into the equations above, yields the result.  $\square$ .

**Proof of Proposition 4.2:** Since  $G \in \mathbb{R}_{++}^n$ , the map  $\mu \mapsto W_U^*(\mu) - W_\mu(x)$  achieves its minimum on  $\Sigma_G$ . Hence, from Proposition 4.1, there is some  $\mu^* \geq 0$  such that

$$D_G(U)(x^*) = \min_{\mu \geq 0} \{W_U^*(\mu) - W_\mu(U)(x^*) : \mu \cdot G = 1\} = W_U^*(\mu^*) - W_{\mu^*}(U)(x^*).$$

Since  $x^*$  is Pareto efficient, it follows that  $D_G(U)(x^*) = 0$ . Hence,  $W_U^*(\mu^*) = W_{\mu^*}(U)(x^*)$ . If  $U(x^*)$  is not a kink point, there is a unique  $\mu^* \in \Sigma_G$  satisfying the equality above and the map  $\mu^*$  is single valued. The standard envelope theorem then gives the derivative of the left side as equal to the derivative inside the minimum evaluated at the minimum point. Thus  $\partial \tau_{G,U}(V) / \partial V|_{V=U(x^*)} = -\mu^*$  where  $\mu^*$  achieves the minimum. Therefore:

$$\left. \frac{\partial \tau_{G,U}(V)}{\partial v_i} \right|_{v_i=u_i(x^*)} = -\mu_i^*(x^*). \quad \square$$

**Proof of Proposition 4.4:** Suppose that  $x^*$  is a Pareto efficient allocation. If  $U(x^*) = 0$ , then  $U(X) = \{0\}$  and any Rawls welfare function with  $\mu \neq 0$  achieves its maximum at  $x^*$ . Suppose that  $U(x^*) \neq 0$  and fix  $\mu^* \in \mathbb{R}_+^n$  defined by:

$$\mu_i^* = \begin{cases} 1/u_i(x^*) & \text{if } u_i(x^*) > 0 \\ 0 & \text{if } u_i(x^*) = 0. \end{cases}$$

By construction, we have  $R_{\mu^*}(U)(x^*) = 1$ . Since  $x^*$  is a Pareto efficient allocation one equivalently has  $(\mathbb{R}_+^n \setminus \{U(x^*)\}) \cap \mathcal{V}(U) = \emptyset$ . Suppose that  $x' \in \mathbb{R}_+^d$  and  $R_{\mu^*}(U)(x') > 1$ . This implies that  $U(x') \in \mathbb{R}_+^n \setminus \{U(x^*)\}$ . Hence  $U(x') \notin \mathcal{V}(U)$  and we deduce that  $x' \notin X$ . Consequently  $R_U^*(\mu^*) = R_{\mu^*}(U)(x^*) = 1$ , which ends the proof.  $\square$

**Proof of Proposition 4.5:** Since  $U(X)$  is a subset of  $\mathbb{R}_+^n$ , we have for all  $x \in X$

$$D_G(U)(x) = \sup\{\delta : U(x) + \delta G \in \mathcal{V}(U) \cap \mathbb{R}_+^n\}.$$

One equivalently has

$$D_G(U)(x) = \inf\{\delta : U(x) + \delta G \notin \mathcal{V}(U) \cap \mathbb{R}_+^n\}.$$

Since  $\mathcal{V}(U) = \bigcap_{\mu \in \mathbb{R}_+^n} \{V \in \mathbb{R}_+^n : \min_{i \in I(\mu)} \mu_i v_i \leq R_U^*(\mu)\}$ , it follows that

$$\begin{aligned} \mathbb{R}_+^n \setminus \mathcal{V}(U) &= \bigcup_{\mu \in \mathbb{R}_+^n} \mathbb{R}_+^n \setminus \{V \in \mathbb{R}_+^n : \min_{i \in I(\mu)} \mu_i v_i \leq R_U^*(\mu)\} \\ &= \bigcup_{\mu \in \mathbb{R}_+^n} \{V \in \mathbb{R}_+^n : \min_{i \in I(\mu)} \mu_i v_i > R_U^*(\mu)\}. \end{aligned}$$

Consequently,

$$\begin{aligned}
D_G(U)(x) &= \inf \left\{ \delta : U(x) + \delta G \in \bigcup_{\mu \in \mathbb{R}_+^n} \{V \in \mathbb{R}_+^n : \min_{i \in I(\mu)} \mu_i v_i > R_U^*(\mu)\} \right\} \\
&= \inf_{\mu \in \mathbb{R}_+^n} \inf \left\{ \delta : U(x) + \delta G \in \{V \in \mathbb{R}_+^n : \min_{i \in I(\mu)} \mu_i v_i > R_U^*(\mu)\} \right\} \\
&= \inf_{\mu \in \mathbb{R}_+^n} \inf \left\{ \delta : \min_{i \in I(\mu)} \mu_i (u_i(x) + \delta g_i) > R_U^*(\mu) \right\} \\
&= \inf_{\mu \geq 0} \min_{i \in I(\mu) \cap I(G)} \frac{R_U^*(\mu) / \mu_i - u_i(x)}{g_i}.
\end{aligned}$$

To prove the second part of the statement, we note that the map  $V \mapsto \min_{i \in [n]} \mu_i v_i$  is nondecreasing over  $\mathbb{R}_+^n$ . Hence,  $R_U^*(\mu) = \sup\{\min_{i \in [n]} \mu_i v_i : V \in \mathcal{V}(U)\}$  and

$$\min_{i \in I(\mu) \cap I(G)} \mu_i (u_i(x) + D_G(U)(x)g_i) \geq \min_{i \in I(\mu) \cap I(G)} \mu_i u_i(x) = R_\mu(U)(x).$$

Moreover, for all  $x \in X$ ,  $U(x) + D_G(U)(x)G \in \mathcal{V}(U)$ . It follows that

$$R_U^*(\mu) \geq \min_{i \in [n]} \{\mu_i (u_i(x) + D_G(U)(x)g_i)\} \geq R_\mu(U)(x),$$

which ends the proof.  $\square$

**Proof of Proposition 5.1:** First note that the map  $\delta \mapsto \sum_{i \in [n]}^\phi \delta_i$  is increasing on  $\mathbb{R}^n$ . By definition we have

$$\begin{aligned}
SD_\phi(U)(x) &= \sup_{\Delta} \left\{ \phi^{-1}\left(\frac{1}{n}\right) \sum_{i \in [n]}^\phi \delta_i : U(x) + \Delta \leq U(z), \Delta \geq 0, z \in X \right\} \\
&= \sup_{\Delta} \left\{ \phi^{-1}\left(\frac{1}{n}\right) \sum_{i \in [n]}^\phi \delta_i : \delta_i \leq u_i(z) - u_i(x), i \in [n], \Delta \geq 0, z \in X \right\} \\
&= \sup \left\{ \phi^{-1}\left(\frac{1}{n}\right) \sum_{i \in [n]}^\phi (u_i(z) - u_i(x)) : U(z) \geq U(x), z \in X \right\}. \quad \square
\end{aligned}$$

**Proof of Proposition 5.2:** Let  $D$  be a compact subset of  $\mathbb{R}^n$ . Suppose that  $r_0 \in [-\infty, \infty]$ . Let  $\{f_r\}_{r \in \mathbb{N}}$  a sequence of continuous functions that is pointwise convergent to some continuous map  $f$  on  $D$ . Since  $f_{r_0}$  is continuous over  $D$ , we deduce that the sequence  $\{f_r\}_{r \in \mathbb{N}}$  converges uniformly to  $f_{r_0}$ . Since  $D$  is compact, there is some  $\delta_r \in D$  that achieves the maximum for each  $r$ , i.e  $f_r(\delta_r) = \sup_{\delta \in B} f_r(\delta)$ . Moreover, there is some  $\delta_{r_0} \in B$  which achieves the maximum of  $f_{r_0}$  over  $D$ . From the continuity of  $f_{r_0}$  and  $f_r$  for each  $r$ , we deduce that  $\lim_{r \rightarrow \infty} \sup_{\delta \in D} f_r(\delta) = \sup_{\beta \in B} f(\delta)$ . Now, note that the maps  $\delta \mapsto \min_{i \in [n]} \delta_i$ ,  $\delta \mapsto \max_{i \in [n]} \delta_i$ , and  $\delta \mapsto \prod_{i \in [n]} \delta_i^{\frac{1}{n}}$  are continuous.

Moreover, since these utility functions are upper-semicontinuous and since  $\mathcal{V}(U)$  is bounded above, it follows that  $\mathcal{D}(U) = \mathbb{R}_+^n \cap (\mathcal{V}(U) - U(x))$  is closed and bounded, therefore it is a compact subset of  $\mathbb{R}^n$ . However, by construction

$$SD_{\phi_r}(U)(x) = \sup \left\{ \sum_{i \in [n]}^{\phi_r} \frac{1}{n^{\frac{1}{r}}} \delta_i : \delta \in \mathcal{V}(U) - U(x), \Delta \geq 0 \right\}$$

Hence, it is then immediate to deduce the following properties

$$\begin{aligned} \lim_{r \rightarrow -\infty} SD_{\phi_r}(U)(x) &= \sup \left\{ \min_{i \in [n]} \delta_i : U(x) + \delta \leq U(z), \Delta \geq 0, z \in X \right\} \\ &= \sup \left\{ \min_{i \in [n]} \{u_i(z) - u_i(x)\} : z \in X \right\} = D_{\mathbb{1}_n}(U)(x); \end{aligned}$$

$$\begin{aligned} \lim_{r \rightarrow 0} SD_{\phi_r}(U)(x) &= \sup \left\{ \prod_{i \in [n]} \delta_i^{\frac{1}{n}} : U(x) + \delta \leq U(z), \Delta \geq 0, z \in X \right\} \\ &= \sup \left\{ \prod_{i \in [n]} (u_i(z) - u_i(x))^{\frac{1}{n}} : z \succsim x, z \in X \right\} = ND(U)(x); \end{aligned}$$

$$\begin{aligned} \lim_{r \rightarrow \infty} SD_{\phi_r}(U)(x) &= \sup \left\{ \max_{i \in [n]} \delta_i : U(x) + \delta \leq U(z), \Delta \geq 0, z \in X \right\} \\ &= \sup \left\{ \max_{i \in [n]} \{u_i(z) - u_i(x)\} : z \in X \right\} \\ &= \max_{i \in [n]} \sup \left\{ u_i(z) - u_i(x) : z \in X \right\} \\ &= \max_{i \in [n]} D_{e_i}(U)(x) = AD_{\max}(U)(x). \quad \square \end{aligned}$$

**Proof of Lemma 5.3:** For all  $\delta \in \mathbb{R}_+^n$ , the map  $r \mapsto \sum_{i \in [n]}^{\phi_r} \delta_i$  is increasing with respect to  $r$ . Therefore, the result immediately follows.  $\square$