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# CLIPS OPERATION BETWEEN TYPE II AND TYPE III $O(3)$ -SUBGROUPS WITH APPLICATION TO PIEZOELECTRICITY

P. AZZI AND M. OLIVE

ABSTRACT. Clips is a binary operation between conjugacy classes of closed subgroups of a compact group, which has been introduced to obtain the isotropy classes of a direct sum of representations  $E_1 \oplus E_2$ , if the isotropy classes are known for each factor  $E_1, E_2$ . This allows, in particular, to compute the isotropy classes of any reducible representation, once one knows the decomposition into irreducible factors and the symmetry classes for the irreducible factors. In the specific case of the three dimensional orthogonal group  $O(3)$ , clips between some types of subgroups of  $O(3)$  have already been calculated. However, until now, clips between type II and type III subgroups were missing. Those are encountered in 3D coupled constitutive laws. In this paper, we complete the clips tables by computing the missing clips. As an application, we obtain 25 isotropy classes for the standard  $O(3)$  representation on the full 3D Piezoelectric law, which involves the three Elasticity, Piezoelectricity and Permittivity constitutive tensors.

## CONTENTS

1. Introduction	1
Organization of the paper	2
2. Isotropy classes and Clips operation	2
3. Clips operation between closed $O(3)$ -subgroups	3
4. Proofs for clips operations of type II and III $O(3)$ -subgroups	6
4.1. Clips with $\mathbb{Z}_{2n}^-$	7
4.2. Clips with $\mathbb{D}_n^z$	8
4.3. Clips with $\mathbb{D}_{2n}^d$	10
4.4. Clips with $\mathbb{O}^-$	12
4.5. Clips with $O(2)^-$	13
5. Application to Piezoelectricity	14
Appendix A. Closed $O(3)$ -subgroup	15
References	18

## 1. INTRODUCTION

Given a linear representation  $(V, \rho)$  of a compact group  $G$ , the *symmetry classes* (or isotropy classes) are defined as the sets of conjugacy classes of the symmetry subgroups of  $G$  (see [1], for more details). Finding explicitly the isotropy classes of a given representation has always been a difficult task, but it is known that there exists only a finite number of them. This fact was first conjectured by Montgomery [4, problem 45] and proved by Mostow in the case of the action of a compact Lie group on a compact manifold [17] using the result of Floyd [6].

The explicit calculation of isotropy classes is an interesting problem not only in mathematics but also in mechanics because the symmetry classes are connected to material symmetries. The problem of classifying materials according to their symmetries goes back to the work of Lord Kelvin. Since then, many authors devoted a great effort to formulate the problem especially for  $SO(3)$  and  $O(3)$  tensorial representations (used to model constitutive laws in mechanics). Surprisingly, it was only in 1996 that Forte and Vianello [7] solved definitively the problem for the *Elasticity tensor* (a fourth-order tensor). In that case, eight isotropy classes were found. Based on this method, the problem was solved for other constitutive laws. For instance, 16 isotropy

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classes were obtained for the *Piezoelectricity tensor* (a third-order tensor) [19, 28, 27, 18, 29]. Similar results were obtained for other constitutive tensor spaces [8, 13].

However, the Forte-Vianello approach requires rather fine calculations and reasoning to establish the classification. This complexity makes difficult its application to more involved situations, such as constitutive tensors of order greater than 4 or coupled constitutive laws involving a family of tensors [11, 5]. A systematic way to calculate the isotropy classes was proposed by Chos-sat and Guyard in [2] using a binary operation between conjugacy classes. This operation was named the *clips* operation in [21, 22, 20, 23], where it was generalized and used to determine isotropy classes for reducible representations.

Clips tables for  $SO(3)$ -subgroups were obtained first in [2]. The problem is more complicated for  $O(3)$ -subgroups, since there exist three types of subgroups (see [Appendix A](#)), where type I corresponds to  $SO(3)$ -subgroups. Clips tables for  $O(3)$ -subgroups were calculated in [20], except the ones between type II and type III subgroups. In this paper, we complete these results by providing these missing tables, which have never been obtained before. The clips tables 1 and 2 given in this paper, together with the tables provided in [20], furnish an exhaustive list of clips tables between  $O(3)$ -subgroups. Since the isotropy classes are known for the irreducible representations of  $O(3)$  [9, 3], this allows, in practice, to determine the isotropy classes of any reducible representation of  $O(3)$ . As an original application, we used these results to determine the 25 isotropy classes of the full 3D Piezoelectricity law, in which three constitutive tensors are involved: the Elasticity, the Piezoelectricity and the Permittivity tensors.

**Organization of the paper.** The paper is organized as follows. In [section 2](#), we recall basic material on symmetry classes and we introduce the *clips operation*. In [section 3](#), we provide (in tables 1 and 2) the list of clips between  $O(3)$ -subgroups of type II and III. The proofs and calculations for these clips are given in [section 4](#). The application to the full Piezoelectricity law is detailed in [section 5](#), where 25 symmetry classes are found. To be self-contained, the classification of closed  $O(3)$ -subgroups, up to conjugacy, is recalled in [Appendix A](#).

## 2. ISOTROPY CLASSES AND CLIPS OPERATION

In this section, we recall the notions of isotropy groups and isotropy classes of a group representation and we introduce clips operation for sets of conjugacy classes. We consider a linear representation  $(V, G, \rho)$  of a group  $G$  on a finite dimensional vector space  $V$ , *i.e.*, a group morphism

$$\rho: G \rightarrow GL(V)$$

with  $GL(V)$ , the group of invertible linear mappings of  $V$  into itself.

The *isotropy (or symmetry) group* of  $\mathbf{v} \in V$  is defined as

$$G_{\mathbf{v}} := \{g \in G; \rho(g)\mathbf{v} = \mathbf{v}\},$$

and the *isotropy (or symmetry) class* of  $\mathbf{v}$  is the conjugacy class  $[G_{\mathbf{v}}]$  of its isotropy group

$$[G_{\mathbf{v}}] := \{gG_{\mathbf{v}}g^{-1}; g \in G\}.$$

For a given subgroup  $H \subset G$ , the conjugacy class  $[H]$  is an *isotropy class* of the linear representation  $(V, G, \rho)$  if  $H$  is conjugate to some isotropy group  $G_{\mathbf{v}}$ . Let us then define  $\mathcal{J}(V)$  to be the set of all isotropy classes of the representation  $V$

$$\mathcal{J}(V) := \{[G_{\mathbf{v}}]; \mathbf{v} \in V\}.$$

As mentioned in the introduction, the finiteness of  $\mathcal{J}(V)$  is proven when  $G$  is a compact Lie group and  $V$  is a compact manifold (see, for instance, [17], [1], [14]). The result for a linear representation of a compact Lie group on a vector space follows immediately, since, then, an invariant scalar product exists and the restriction of the representation to the unit sphere, which is a compact manifold, has the same isotropy classes as the full representation.

**Theorem 2.1.** *Let  $(V, G)$  be a linear representation of a compact Lie group  $G$  on a vector space  $V$ . Then there exists a finite number of isotropy classes*

$$\mathcal{J}(V) = \{[H_1], \dots, [H_n]\}.$$

**Remark 2.2.** The set of conjugacy classes  $[H]$  of closed subgroups of a compact group is endowed with a partial order relation (see [1]), given by

$$[H] \preceq [K] \iff \exists g \in G, \quad gHg^{-1} \subset K.$$

The set  $\mathcal{J}(V)$  always has a smallest element, say  $[H_1]$  and a biggest element, say  $[H_n]$ . Hence,

$$[H_1] \preceq [H_k] \preceq [H_n], \quad k = 1, \dots, n$$

Given this finiteness result, it is important to explicitly calculate the isotropy classes of a given representation. In the specific case of SO(3) linear representations, Michel [16] obtained the isotropy classes for the *irreducible representations* and the ones for O(3) irreducible representations were obtained by Ihrig and Golubitsky [10]. Thereafter, Chossat and Guyard [2] get the isotropy classes of a direct sum of two irreducible SO(3) representations. To do so, they introduced a binary operations on conjugate SO(3)-subgroups that allows one to compute the set of isotropy classes  $\mathcal{J}(V)$  of a direct sum  $V = V_1 \oplus V_2$  of linear representations of a group  $G$ , if we know the isotropy classes for each individual irreducible representations. Such an operation is generalized to a binary operation on all conjugacy classes of subgroups of a given group  $G$  and is defined as follows

**Definition 2.3.** For two subgroups  $H_1$  and  $H_2$  of a group  $G$ , we define the *clips operation* of the conjugacy classes  $[H_1]$  and  $[H_2]$  as the subset of conjugacy classes

$$[H_1] \odot [H_2] := \{[H_1 \cap gH_2g^{-1}], \quad g \in G\}.$$

This definition immediately extends to two families (finite or infinite)  $\mathcal{F}_1$  and  $\mathcal{F}_2$  of conjugacy classes

$$\mathcal{F}_1 \odot \mathcal{F}_2 = \bigcup_{[H_i] \in \mathcal{F}_i} [H_1] \odot [H_2].$$

The following lemma states a central result that serves in finding the isotropy classes of a reducible representation once we know the isotropy classes of irreducible ones (see [20] for a proof).

**Lemma 2.4.** *Let  $V_1$  and  $V_2$  be two linear representations of  $G$ . Then the set  $\mathcal{J}(V_1 \oplus V_2)$  of isotropy classes of the diagonal representation of  $G$  on  $V_1 \oplus V_2$  is given by*

$$\mathcal{J}(V_1 \oplus V_2) = \mathcal{J}(V_1) \odot \mathcal{J}(V_2).$$

Using this result, one can find the isotropy classes of any representation  $V$  provided we know

- (1) a stable decomposition  $V = W_1 \oplus \dots \oplus W_r$ ;
- (2) the isotropy classes of each representations  $W_k$ ;
- (3) the clips table of  $[H_1] \odot [H_2]$  for all subgroups of  $G$ .

Such Lemma 2.4 has already been applied to find the isotropy classes of some O(3) reducible representations, which are essentially the standard O(3) representations on *odd order* tensor spaces on  $\mathbb{R}^3$  (see [22, 20]). To extend to all reducible O(3) representations, new clips table of closed O(3)-subgroups have to be established.

### 3. CLIPS OPERATION BETWEEN CLOSED O(3)-SUBGROUPS

As recalled in Appendix A (see also [10] for details), any closed O(3)-subgroup is either of type I, type II or type III. More specifically

- Every type I closed O(3)-subgroup is conjugate to one of the following list
- (1)  $\text{SO}(3)$ ,  $\text{O}(2)$ ,  $\text{SO}(2)$ ,  $\mathbb{D}_n$ ,  $\mathbb{Z}_n$ ,  $\mathbb{T}$ ,  $\mathbb{O}$ ,  $\mathbb{I}$ , or  $\mathbb{1}$ .
  - Every type II closed O(3)-subgroup is conjugate to  $H \oplus \mathbb{Z}_2^c$  with  $H$  a type I subgroup and  $\mathbb{Z}_2^c := \{-\text{Id}, \text{Id}\}$ .
  - Every type III closed O(3)-subgroup is conjugate to one of the following list:
- (2)  $\mathbb{Z}_{2n}^-$ ,  $\mathbb{D}_n^z$ ,  $\mathbb{D}_{2n}^d$ ,  $\mathbb{O}^-$ ,  $\text{O}(2)^-$ .

Each closed subgroups from the lists (1) and (2) is defined in [Appendix A](#).

The clips tables have already been established for two type I subgroups (see [20, Table 1]) and for two type III subgroups (see [20, Table 2]). The clips operation between a type I and a type III subgroup is deduced from [20, Lemma 5.4]. The clips between a type I and a type II subgroup or two type II subgroups are deduced from the clips between two type I subgroups, see remark below.

**Remark 3.1.** Note that

- (1) If  $H_1$  and  $H_2$  are two subgroups of  $\text{SO}(3)$  of type I then

$$[H_1] \odot [H_2 \oplus \mathbb{Z}_2^c] = [H_1] \odot [H_2] \text{ and } [H_1 \oplus \mathbb{Z}_2^c] \odot [H_2 \oplus \mathbb{Z}_2^c] = ([H_1] \odot [H_2]) \oplus \mathbb{Z}_2^c.$$

- (2) For every subgroup  $H$  of  $\text{SO}(3)$  we have

$$[H] \odot [\text{SO}(3) \oplus \mathbb{Z}_2^c] = \{[H]\} \text{ and } [\mathbf{1}] \odot [H] = \{[\mathbf{1}]\}.$$

We propose now clips operations between type II and type III subgroups.

**Theorem 3.2.** *Let  $\Gamma$  be a type III subgroup of  $\text{O}(3)$  and  $H \oplus \mathbb{Z}_2^c$  type II subgroup of  $\text{O}(3)$ . Then the clips operation  $[\Gamma] \odot [H \oplus \mathbb{Z}_2^c]$  always contain the isotropy class  $[\mathbf{1}]$  and all remaining elements in  $[\Gamma] \odot [H \oplus \mathbb{Z}_2^c]$  are given in [Table 1](#) and [Table 2](#), where we have used the notations*

$$d := \gcd(m, n), \quad d_k := \gcd(k, n), \quad d'_k := \gcd(k, m),$$

$$\mathcal{Z}(n) := \begin{cases} [\mathbb{Z}_2] & \text{if } n \text{ even} \\ [\mathbb{Z}_2^-] & \text{else} \end{cases}, \quad \Gamma(m, n) := \begin{cases} [\mathbb{D}_2], [\mathbb{D}_2^z] & \text{if } m \text{ and } n \text{ even} \\ [\mathbb{D}_2^z] & \text{if } m \text{ even and } n \text{ odd} \\ [\mathbb{Z}_2] & \text{if } m \text{ odd and } n \text{ even} \\ [\mathbb{Z}_2^-] & \text{if } m \text{ and } n \text{ odd} \end{cases}$$

and

$$\mathbb{L}_{\mathbb{O}} := [\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], \Gamma(n, 3), [\mathbb{Z}_{d_3}], [\mathbb{D}_{d_3}], [\mathbb{D}_{d_3}^z]$$

**Remark 3.3.** In tables we have used the conventions

$$[\mathbb{Z}_1] := [\mathbf{1}], \quad [\mathbb{D}_1] = [\mathbb{Z}_2], \quad [\mathbb{D}_1^z] = [\mathbb{Z}_2^-], \quad [\mathbb{D}_2^z] = [\mathbb{D}_2^d].$$

**Remark 3.4.** Note that in [20, figure 3] it was stated that  $\mathbb{D}_2^z \subset \mathbb{D}_{2n}^d$  when  $n$  is odd. However, this is true for all  $n$ .

$\odot$	$[\mathbb{Z}_{2n}^-]$	$[\mathbb{D}_n^z]$	$[\mathbb{D}_{2n}^d]$
$[\mathbb{Z}_m \oplus \mathbb{Z}_2^c]$	$\frac{[\mathbb{Z}_{2d}^-] \text{ if } \frac{m}{d} \text{ even}}{[\mathbb{Z}_d] \text{ else}}$	$\frac{[\mathbb{Z}_d] \text{ if } m \text{ odd}}{[\mathbb{Z}_d], [\mathbb{Z}_2^-] \text{ else}}$	$\frac{[\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_{2d}^-] \text{ if } \frac{m}{d} \text{ even}}{[\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_d] \text{ if } m \text{ even and } \frac{m}{d} \text{ odd}}$ $[\mathbb{Z}_d] \text{ else}$
$[\mathbb{D}_m \oplus \mathbb{Z}_2^c]$	$\frac{[\mathbb{Z}_{2d}^-], \mathcal{Z}(n) \text{ if } \frac{m}{d} \text{ even}}{[\mathbb{Z}_d], \mathcal{Z}(n) \text{ else}}$	$\frac{[\mathbb{Z}_d], [\mathbb{Z}_2^-], [\mathbb{D}_{d_2}^z], [\mathbb{D}_d^z] \text{ if } m \text{ even}}{[\mathbb{Z}_d], [\mathbb{Z}_2^-], [\mathbb{Z}_{d_2}], [\mathbb{D}_d^z] \text{ else}}$	$\frac{\Gamma(m, n), [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_{2d}^-], [\mathbb{D}_{2d}^d] \text{ if } \frac{m}{d} \text{ even}}{\Gamma(m, n), [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_d], [\mathbb{D}_d], [\mathbb{D}_d^z] \text{ if } m \text{ even and } \frac{m}{d} \text{ odd}}$ $\Gamma(m, n), [\mathbb{Z}_d], [\mathbb{D}_d], [\mathbb{D}_d^z] \text{ else}$
$[\mathbb{O} \oplus \mathbb{Z}_2^c]$	$\frac{[\mathbb{Z}_2], [\mathbb{Z}_{d_3}], [\mathbb{Z}_4] \text{ if } 4 n}{[\mathbb{Z}_2], [\mathbb{Z}_{d_3}], [\mathbb{Z}_4^-] \text{ if } n \text{ even and } 4 \nmid n}$ $[\mathbb{Z}_2^-], [\mathbb{Z}_{d_3}] \text{ else}$	$[\mathbb{Z}_{d_2}], [\mathbb{Z}_{d_3}], [\mathbb{Z}_{d_4}], [\mathbb{Z}_2^-], [\mathbb{D}_{d_2}^z], [\mathbb{D}_{d_3}^z], [\mathbb{D}_{d_4}^z]$	$\frac{\mathbb{L}_0, [\mathbb{D}_2], [\mathbb{D}_2^z], [\mathbb{Z}_4], [\mathbb{D}_4], [\mathbb{D}_4^z] \text{ if } 4 n}{\mathbb{L}_0, [\mathbb{D}_2], [\mathbb{D}_2^z], [\mathbb{Z}_4^-], [\mathbb{D}_4^d] \text{ if } n \text{ even and } 4 \nmid n}$ $\mathbb{L}_0, [\mathbb{D}_2^z] \text{ else}$
$[\mathbb{T} \oplus \mathbb{Z}_2^c]$	$[\mathbb{Z}_{d_3}], \mathcal{Z}(n)$	$[\mathbb{Z}_2^-], [\mathbb{Z}_{d_2}], [\mathbb{Z}_{d_3}], [\mathbb{D}_{d_2}^z]$	$[\mathbb{Z}_2], [\mathbb{Z}_2^-], \Gamma(2, n), \mathbb{Z}_{d_3}$
$[\mathbb{I} \oplus \mathbb{Z}_2^c]$	$\mathcal{Z}(n), [\mathbb{Z}_{d_3}], [\mathbb{Z}_{d_5}]$	$[\mathbb{Z}_{d_2}], [\mathbb{Z}_{d_3}], [\mathbb{Z}_{d_5}], [\mathbb{Z}_2^-], [\mathbb{D}_{d_2}^z], [\mathbb{D}_{d_3}^z], [\mathbb{D}_{d_5}^z]$	$[\mathbb{Z}_2], [\mathbb{Z}_2^-], \Gamma(2, n), \Gamma(3, n), \Gamma(5, n), [\mathbb{Z}_{d_3}], [\mathbb{D}_{d_3}], [\mathbb{D}_{d_3}^z], [\mathbb{Z}_{d_5}], [\mathbb{D}_{d_5}], [\mathbb{D}_{d_5}^z]$
$[\text{SO}(2) \oplus \mathbb{Z}_2^c]$	$[\mathbb{Z}_{2n}^-]$	$[\mathbb{Z}_2^-], [\mathbb{Z}_n]$	$[\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_{2n}^-]$
$[\text{O}(2) \oplus \mathbb{Z}_2^c]$	$[\mathcal{Z}(n)], [\mathbb{Z}_{2n}^-]$	$[\mathbb{D}_{d_2}^z], [\mathbb{D}_n^z]$	$\mathcal{Z}(n), [\mathbb{D}_{d_2}], [\mathbb{D}_2^z], [\mathbb{D}_{2n}^d]$

TABLE 1. Clips between type II and III O(3)-subgroups (first cases)

$\odot$	$[\mathbb{O}^-]$	$[\mathbb{O}(2)^-]$
$[\mathbb{Z}_m \oplus \mathbb{Z}_2^c]$	$\frac{[\mathbb{Z}_{d'_3}, [\mathbb{Z}_2], [\mathbb{Z}_4^-] \text{ if } 4 m}{[\mathbb{Z}_{d'_2}, [\mathbb{Z}_{d'_3}], [\mathbb{Z}_{d'_2}^-] \text{ else}}$	$[\mathbb{Z}_m], [\mathbb{Z}_{d'_2}^-]$
$[\mathbb{D}_m \oplus \mathbb{Z}_2^c]$	$\frac{[\mathbb{Z}_2], [\mathbb{Z}_{d'_3}], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], [\mathbb{D}_{d'_3}^z], [\mathbb{D}_2^z], [\mathbb{D}_4^d] \text{ if } 4 m}{[\mathbb{Z}_2], [\mathbb{Z}_{d'_3}], [\mathbb{Z}_2^-] \text{ if } m \text{ even and } 4 \nmid m, [\mathbb{D}_2], [\mathbb{D}_{d'_3}^z], [\mathbb{D}_2^z] \text{ else}}$	$\frac{[\mathbb{Z}_m], [\mathbb{Z}_2^-], [\mathbb{D}_2^z], [\mathbb{D}_m^z] \text{ if } m \text{ even}}{[\mathbb{Z}_m], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{D}_m^z] \text{ else}}$
$[\mathbb{O} \oplus \mathbb{Z}_2^c]$	$[\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], [\mathbb{D}_2^z], [\mathbb{D}_3^z], [\mathbb{D}_4^d], [\mathbb{O}^-]$	$[\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_4], [\mathbb{Z}_2^-], [\mathbb{D}_2^z], [\mathbb{D}_3^z], [\mathbb{D}_4^d]$
$[\mathbb{T} \oplus \mathbb{Z}_2^c]$	$[\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_2^-], [\mathbb{D}_2], [\mathbb{D}_2^z], [\mathbb{T}]$	$[\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_2^-], [\mathbb{D}_2^z]$
$[\mathbb{I} \oplus \mathbb{Z}_2^c]$	$[\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{D}_2], [\mathbb{D}_2^z], [\mathbb{Z}_3], [\mathbb{D}_3^z], [\mathbb{T}]$	$[\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_5], [\mathbb{Z}_2^-], [\mathbb{D}_2^z], [\mathbb{D}_3^z], [\mathbb{D}_5^z]$
$[\mathbb{SO}(2) \oplus \mathbb{Z}_2^c]$	$[\mathbb{Z}_3], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-]$	$[\mathbb{Z}_2^-], [\mathbb{SO}(2)]$
$[\mathbb{O}(2) \oplus \mathbb{Z}_2^c]$	$[\mathbb{Z}_2^-], [\mathbb{D}_3^z], [\mathbb{D}_4^d]$	$[\mathbb{D}_2^z], [\mathbb{O}(2)^-]$

TABLE 2. Clips between type II and III  $\mathbb{O}(3)$ -subgroups (second cases)4. PROOFS FOR CLIPS OPERATIONS OF TYPE II AND III  $\mathbb{O}(3)$ -SUBGROUPS

In this section, we provide the details of the proof of Theorem 3.2. The notation  $\mathbf{r}(\mathbf{n}, \theta)$ , with  $\theta \in [0; 2\pi[$  and  $\mathbf{n} = (n_x, n_y, n_z)$ , a unit vector, denotes the Rodrigues formula to represent a rotation by angle  $\theta$  around  $\mathbf{n}$ , which is given by

$$\mathbf{r}(\mathbf{n}, \theta) = \exp(\theta j(\mathbf{n})) = \text{Id} + j(\mathbf{n}) \sin(\theta) + j(\mathbf{n})^2 (1 - \cos(\theta))$$

where  $j(\mathbf{n})$  denotes the antisymmetric matrix with entries

$$j(\mathbf{n}) = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}.$$

To find the clips between subgroups of type II and III we use the following lemma

**Lemma 4.1.** *Let  $\Gamma = \Gamma_+ \cup (-\gamma\Gamma_+)$  be a subgroup of type III where  $\Gamma_+ = \Gamma \cap \mathbb{SO}(3)$  and  $-\gamma \in \Gamma \setminus \Gamma_+$ . Then for every subgroup  $H$  of  $\mathbb{SO}(3)$  we have*

$$\Gamma \cap (H \oplus \mathbb{Z}_2^c) = (\Gamma_+ \cap H) \cup (-((\gamma\Gamma_+) \cap H)).$$

*Proof.* As we have  $H \oplus \mathbb{Z}_2^c = H \cup (-H)$  we deduce that

$$\begin{aligned} \Gamma \cap (H \oplus \mathbb{Z}_2^c) &= (\Gamma_+ \cup (-\gamma\Gamma_+)) \cap (H \cup (-H)) \\ &= (\Gamma_+ \cap H) \cup (\Gamma_+ \cap (-H)) \cup (-(\gamma\Gamma_+) \cap H) \cup (-(\gamma\Gamma_+) \cap (-H)) \\ &= (\Gamma_+ \cap H) \cup (-((\gamma\Gamma_+) \cap H)) \end{aligned}$$

Hence the result. □

In the next subsections, we focus on clips operations  $[\Gamma] \odot [H \oplus \mathbb{Z}_2^c]$ , so from definition 2.3 and lemma 4.1 we have to consider

$$\Gamma \cap (gHg^{-1} \oplus \mathbb{Z}_2^c) = (\Gamma_+ \cap (gHg^{-1})) \cup (-\gamma\Gamma_+ \cap (gHg^{-1})), \quad \Gamma_+ = \Gamma \cap \text{SO}(3), \quad -\gamma \in \Gamma \setminus \Gamma_+$$

with  $g \in \text{SO}(3)$ .

**Remark 4.2.** It follows that if  $H = \bigcup_{i=1}^N H_i$  with  $H, H_1, \dots, H_N$  are subgroups of  $\text{SO}(3)$  then for any type III subgroup  $\Gamma = \Gamma_+ \cup (-\gamma\Gamma_+)$  we have

$$(3) \quad \Gamma \cap (gHg^{-1} \oplus \mathbb{Z}_2^c) = \bigcup_{i=1}^N (\Gamma_+ \cap (gH_i g^{-1})) \cup (-\gamma\Gamma_+ \cap (gH_i g^{-1})).$$

**4.1. Clips with  $\mathbb{Z}_{2n}^-$ .** First, let us construct  $\mathbb{Z}_{2n}^-$  from the couple  $(\mathbb{Z}_{2n}, \mathbb{Z}_n)$  as in the equation 11 of appendix A

$$\mathbb{Z}_{2n}^- = \mathbb{Z}_n \cup (-\gamma\mathbb{Z}_n), \quad \gamma = \mathbf{r} \left( \mathbf{e}_3, \frac{\pi}{n} \right)$$

where we note  $\mathbb{Z}_1 := \{Id\}$  for the case  $n = 1$ .

**Lemma 4.3.** Let  $n \geq 1$  and  $m \geq 2$  be two integers and  $d = \text{gcd}(m, n)$ . Then

$$[\mathbb{Z}_{2n}^-] \odot [\mathbb{Z}_m \oplus \mathbb{Z}_2^c] = \begin{cases} \{[\mathbf{1}], [\mathbb{Z}_{2d}^-]\} & \text{if } \frac{m}{d} \text{ even} \\ \{[\mathbf{1}], [\mathbb{Z}_d]\} & \text{else} \end{cases}.$$

*Proof.* We deduce from Lemma 4.1 that

$$\mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_m g^{-1} \oplus \mathbb{Z}_2^c) = (\mathbb{Z}_n \cap g\mathbb{Z}_m g^{-1}) \cup (-\gamma\mathbb{Z}_n \cap g\mathbb{Z}_m g^{-1}), \quad \gamma = \mathbf{r} \left( \mathbf{e}_3, \frac{\pi}{n} \right).$$

In the case when  $g\mathbf{e}_3$  and  $\mathbf{e}_3$  are not colinear, such group reduces to  $\mathbf{1}$ , so we suppose now that  $g\mathbf{e}_3 = \pm\mathbf{e}_3$ . We thus have to consider

$$(\mathbb{Z}_n \cap \mathbb{Z}_m) \cup (-\gamma\mathbb{Z}_n \cap \mathbb{Z}_m)$$

where  $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$  and  $\gamma\mathbb{Z}_n \cap \mathbb{Z}_m$  is obtained by solving the equations of unknown  $k_1, k_2 \in \mathbb{Z}$

$$\frac{2k_1 + 1}{n} = \frac{2k_2}{m} \iff (2k_1 + 1)dm_1 = 2k_2dn_1, \quad \text{gcd}(m_1, n_1) = 1.$$

We get solutions only if  $m_1 = \frac{m}{d}$  is even. By replacing  $m_1$  by  $2p$  we get that  $p$  divides  $k_2$  hence  $k_2 = pk'$  with  $k'$  odd. On one hand,  $\frac{2k_2}{m} = \frac{2pk'}{2pd}$  and on the other hand,  $\frac{2k_1+1}{n} = \frac{k'n_1}{dn_1}$ . We deduce that

$$\gamma\mathbb{Z}_n \cap \mathbb{Z}_m = \left\{ \mathbf{r} \left( \mathbf{e}_3, \frac{(2k+1)\pi}{d} \right) \right\}$$

□

In the following, recall that

$$(4) \quad \mathcal{Z}(n) := \mathbb{Z}_{2n}^- \odot (\mathbb{Z}_2 \oplus \mathbb{Z}_2^c) = \begin{cases} [\mathbb{Z}_2] & \text{if } n \text{ even} \\ [\mathbb{Z}_2^-] & \text{else} \end{cases}.$$

**Lemma 4.4.** Let  $n \geq 1$  and  $m \geq 2$  be two integers and  $d = \text{gcd}(m, n)$ . Then

$$[\mathbb{Z}_{2n}^-] \odot [\mathbb{D}_m \oplus \mathbb{Z}_2^c] = \begin{cases} \{[\mathbf{1}], [\mathbb{Z}_{2d}^-], \mathcal{Z}(n)\} & \text{if } \frac{m}{d} \text{ even} \\ \{[\mathbf{1}], [\mathbb{Z}_d], \mathcal{Z}(n)\} & \text{else} \end{cases}.$$

*Proof.* Recall from [20, Appendix A] that

$$\mathbb{D}_m = \mathbb{Z}_m \cup_{i=1}^m \mathbb{Z}_2^{\mathbf{b}_i}, \quad \mathbb{Z}_2^{\mathbf{b}_i} := \{e, \mathbf{r}(\mathbf{b}_i, \pi)\}$$

where  $\mathbf{b}_i$  are called the secondary axis of the subgroup  $\mathbb{D}_m$ , with

$$\mathbf{b}_1 = \mathbf{e}_1, \quad \mathbf{b}_k = \mathbf{r} \left( \mathbf{e}_3, \frac{\pi}{m} \right) \mathbf{b}_{k-1}, \quad k = 2, \dots, m.$$



We have

$$\mathbb{Z}_{2n}^- \cap (g\mathbb{D}_m g^{-1} \oplus \mathbb{Z}_2^c) = (\mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_m g^{-1} \oplus \mathbb{Z}_2^c)) \cup \left( \mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_2^{\mathbf{b}_i} g^{-1} \oplus \mathbb{Z}_2^c) \right).$$

The left part of the union is deduced from lemma 4.3, and for the right part of the union we get  $\mathcal{Z}(n)$  (if  $g\mathbf{b}_i = \pm\mathbf{e}_3$ ), so we can conclude.  $\square$

**Lemma 4.5.** *Let  $d_3 = \gcd(3, n)$  and  $d_5 = \gcd(5, n)$ . We have*

$$\begin{aligned} [\mathbb{Z}_{2n}^-] \odot [\mathbb{T} \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [\mathbb{Z}_{d_3}], \mathcal{Z}(n)\}, & [\mathbb{Z}_{2n}^-] \odot [\mathbb{I} \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [\mathcal{Z}(n)], [\mathbb{Z}_{d_3}], [\mathbb{Z}_{d_5}]\}. \\ [\mathbb{Z}_{2n}^-] \odot [\mathbb{O} \oplus \mathbb{Z}_2^c] &= \begin{cases} \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3}], [\mathbb{Z}_4]\} & \text{if } n \text{ is even and } 4|n \\ \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3}], [\mathbb{Z}_4^-]\} & \text{if } n \text{ is even but } 4 \nmid n. \\ \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_{d_3}]\} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

*Proof.* Let us consider decomposition (18) of the group  $\mathbb{T}$ . From remark 4.2 we thus have to consider

$$(\mathbb{Z}_n \cap (g\mathbb{D}_2 g^{-1})) \cup (-\gamma\mathbb{Z}_n \cap (g\mathbb{D}_2 g^{-1})) \bigcup_{i=1}^4 (\mathbb{Z}_n \cap (g\mathbb{Z}_3^{\mathbf{st}_i} g^{-1})) \cup (-\gamma\mathbb{Z}_n \cap (g\mathbb{Z}_3^{\mathbf{st}_i} g^{-1})).$$

The only non-trivial cases are obtained for  $g = \text{Id}$  or  $g$  such that  $g\mathbb{Z}_3^{\mathbf{st}_1} g^{-1} = \mathbb{Z}_3$  (for instance). It remains to consider separately

$$(\mathbb{Z}_n \cap \mathbb{D}_2) \cup (-\gamma\mathbb{Z}_n \cap \mathbb{D}_2) \text{ or } (\mathbb{Z}_n \cap \mathbb{Z}_3) \cup (-\gamma\mathbb{Z}_n \cap \mathbb{Z}_3)$$

directly deduced from previous lemmas.

For the cubic case, we deduce from the decomposition (21) and the remark 4.2 that the intersection

$$\mathbb{Z}_{2n}^- \cap (g\mathbb{O}g^{-1} \oplus \mathbb{Z}_2^c)$$

always reduce either to

$$\mathbb{Z}_{2n}^- \cap (g\mathbb{D}_4^{(i)} g^{-1} \oplus \mathbb{Z}_2^c) \text{ or } \mathbb{Z}_{2n}^- \cap (g\mathbb{D}_3^{(j)} g^{-1} \oplus \mathbb{Z}_2^c)$$

for some  $i = 1, 2, 3$  or  $j = 1, 2, 3, 4$ , taking for instance  $g$  such that  $g = \text{Id}$  or  $g\mathbf{st}_1 = \mathbf{e}_3$  (see Appendix A). We thus obtain

$$[\mathbb{Z}_{2n}^-] \odot [\mathbb{O} \oplus \mathbb{Z}_2^c] = ([\mathbb{Z}_{2n}^-] \odot [\mathbb{D}_3 \oplus \mathbb{Z}_2^c]) \cup ([\mathbb{Z}_{2n}^-] \odot [\mathbb{D}_4 \oplus \mathbb{Z}_2^c]).$$

and similarly, from the decomposition (23) of the subgroup  $\mathbb{I}$  we get

$$[\mathbb{Z}_{2n}^-] \odot [\mathbb{I} \oplus \mathbb{Z}_2^c] = ([\mathbb{Z}_{2n}^-] \odot [\mathbb{D}_2 \oplus \mathbb{Z}_2^c]) \cup ([\mathbb{Z}_{2n}^-] \odot [\mathbb{D}_3 \oplus \mathbb{Z}_2^c]) \cup ([\mathbb{Z}_{2n}^-] \odot [\mathbb{D}_5 \oplus \mathbb{Z}_2^c])$$

so the results is directly deduced from previous lemmas.  $\square$

Finally, as a direct consequence of lemma 4.1,

**Lemma 4.6.** *For any integer  $n \geq 1$  we have:*

$$[\mathbb{Z}_{2n}^-] \odot [\text{SO}(2) \oplus \mathbb{Z}_2^c] = \{[\mathbf{1}], [\mathbb{Z}_{2n}^-]\}, \quad [\mathbb{Z}_{2n}^-] \odot [\text{O}(2) \oplus \mathbb{Z}_2^c] = \{[\mathbf{1}], [\mathcal{Z}(n)], [\mathbb{Z}_{2n}^-]\}$$

with  $\mathcal{Z}(n)$  given by (4).

**4.2. Clips with  $\mathbb{D}_n^z$ .** As explained in Appendix A, the subgroup  $\mathbb{D}_n^z$  is obtained for  $-\gamma = -\mathbf{r}(\mathbf{e}_1, \pi)$ , so that

$$\mathbb{D}_n^z = \mathbb{Z}_n \cup -\gamma\mathbb{Z}_n = \mathbb{Z}_n \cup \{-\mathbf{r}(\mathbf{b}_1, \pi), \dots, -\mathbf{r}(\mathbf{b}_n, \pi)\}, \quad \mathbf{b}_1 = \mathbf{e}_1, \quad \mathbf{b}_k = \mathbf{r}\left(\mathbf{e}_3, \frac{\pi}{n}\right) \mathbf{b}_{k-1}$$

where  $\langle \mathbf{b}_i \rangle$  are called secondary axes of the dihedral subgroup  $\mathbb{D}_n$  (see [20, Appendix A]).

**Lemma 4.7.** *Let  $m, n \geq 2$  be two integers and  $d = \gcd(m, n)$ . Then*

$$[\mathbb{D}_n^z] \odot [\mathbb{Z}_m \oplus \mathbb{Z}_2^c] = \begin{cases} \{[\mathbf{1}], [\mathbb{Z}_d]\} & \text{if } m \text{ is odd} \\ \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_d]\} & \text{if } m \text{ is even} \end{cases}$$

*Proof.* From lemma 4.1 we have to consider

$$\mathbb{D}_n^z \cap (g\mathbb{Z}_m g^{-1} \oplus \mathbb{Z}_2^c) = (\mathbb{Z}_n \cap g\mathbb{Z}_m g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{Z}_m g^{-1})), \quad \gamma = \mathbf{r}(\mathbf{e}_1, \pi),$$

where  $\gamma\mathbb{Z}_n = \{\mathbf{r}(\mathbf{b}_1, \pi), \dots, \mathbf{r}(\mathbf{b}_n, \pi)\}$ . Here, the only non-trivial cases are obtained when  $g\mathbb{Z}_m g^{-1} = \mathbb{Z}_n$  (see lemma 4.3) or  $\mathbb{Z}_2^{\mathbf{b}_i}$ , which leads directly to the result.  $\square$

**Lemma 4.8.** *Let  $m, n \geq 2$  be two integers and  $d = \gcd(m, n)$  and  $d_2 = \gcd(2, n)$ . Then*

$$[\mathbb{D}_n^z] \odot [\mathbb{D}_m \oplus \mathbb{Z}_2^c] = \begin{cases} \{[\mathbf{1}], [\mathbb{Z}_d], [\mathbb{Z}_2^-], [\mathbb{D}_{d_2}^z], [\mathbb{D}_d^z]\} & \text{if } m \text{ is even} \\ \{[\mathbf{1}], [\mathbb{Z}_d], [\mathbb{Z}_2^-], [\mathbb{Z}_{d_2}], [\mathbb{D}_d^z]\} & \text{if } m \text{ is odd} \end{cases}.$$

*Proof.* By lemma 4.1, the result is deduced from the intersection

$$(\mathbb{Z}_n \cap g\mathbb{D}_m g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{D}_m g^{-1})), \quad \gamma\mathbb{Z}_n = \{\mathbf{r}(\mathbf{b}_i, \pi), i = 1, \dots, n\}.$$

By considering the decomposition  $\mathbb{D}_m = \mathbb{Z}_m \cup_{i=1}^m \mathbb{Z}_2^{\mathbf{b}_i}$ , where  $\mathbb{Z}_2^{\mathbf{b}_i} := \{e, \mathbf{r}(\mathbf{b}_i, \pi)\}$ , this intersection becomes

$$(\mathbb{Z}_n \cap g\mathbb{D}_m g^{-1}) \cup (-(\gamma\mathbb{Z}_n \cap g\mathbb{Z}_m g^{-1})) \cup (-(\gamma\mathbb{Z}_n \cap g \cup_{i=1}^m \mathbb{Z}_2^{\mathbf{b}_i} g^{-1}))$$

The first part of the union gives  $\mathbf{1}$ ,  $\mathbb{Z}_{d_2}$  or  $\mathbb{Z}_d$  by [20, lemma A.3]. The second and the third part of the union give  $\mathbb{Z}_{d_2}'$  only if  $g\mathbf{e}_3 = \mathbf{b}_i$  ( $i = 1, \dots, n$ ) and  $\mathbb{Z}_2$  only if  $g\mathbf{b}_i = \mathbf{b}_j$  ( $i = 1, \dots, n$  and  $j = 1, \dots, m$ ). We deduce the result by regrouping these subgroups according to the corresponding  $g$ .  $\square$

As for the proof of the lemma 4.5, we can use group decompositions (18)–(21)–(23) and intersections (3) so that previous lemmas 4.7 and 4.8 lead to

**Lemma 4.9.** *For any integer  $n \geq 2$  we have*

$$\begin{aligned} [\mathbb{D}_n^z] \odot [\mathbb{T} \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_{d_2}], [\mathbb{Z}_{d_3}], [\mathbb{D}_{d_2}^z]\} \\ [\mathbb{D}_n^z] \odot [\mathbb{O} \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [\mathbb{Z}_{d_2}], [\mathbb{Z}_{d_3}], [\mathbb{Z}_{d_4}], [\mathbb{Z}_2^-], [\mathbb{D}_{d_2}^z], [\mathbb{D}_{d_3}^z], [\mathbb{D}_{d_4}^z]\} \\ [\mathbb{D}_n^z] \odot [\mathbb{I} \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [\mathbb{Z}_{d_2}], [\mathbb{Z}_{d_3}], [\mathbb{Z}_{d_5}], [\mathbb{Z}_2^-], [\mathbb{D}_{d_2}^z], [\mathbb{D}_{d_3}^z], [\mathbb{D}_{d_5}^z]\} \end{aligned}$$

**Lemma 4.10.** *For any integer  $n \geq 2$ , we have*

$$\begin{aligned} [\mathbb{D}_n^z] \odot [\text{SO}(2) \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_n]\} \\ [\mathbb{D}_n^z] \odot [\text{O}(2) \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [\mathbb{D}_{d_2}^z], [\mathbb{D}_n^z]\} \end{aligned}$$

*Proof.* We propose here the proof for the clips operation  $[\mathbb{D}_n^z] \odot [\text{O}(2) \oplus \mathbb{Z}_2^c]$  as the other one is obtained in the same way. First, using lemma 4.1, we have to consider intersection

$$(5) \quad (\mathbb{Z}_n \cap (g\text{O}(2)g^{-1})) \cup (-(\gamma\mathbb{Z}_n \cap g\text{O}(2)g^{-1})), \quad \gamma = \mathbf{r}(\mathbf{e}_1, \pi).$$

For any  $g \in \text{SO}(3)$  we can write

$$(6) \quad g\text{O}(2)g^{-1} = \{\mathbf{r}(\mathbf{u}, \theta), \quad \mathbf{r}(\mathbf{v}, \pi), \quad \theta \in [0; 2\pi[, \quad \mathbf{u} \cdot \mathbf{v} = 0, \quad \|\mathbf{v}\| = 1\}$$

where  $\mathbf{u}$  is some unit vector and  $\mathbf{u} \cdot \mathbf{v}$  is the scalar product between  $\mathbf{u}$  and  $\mathbf{v}$ . Only two non-trivial cases can occur:

- $\mathbf{u} = \pm\mathbf{e}_3$  in which case intersection (5) reduces to  $\mathbb{D}_n^z$ .
- $\mathbf{u} = \pm\mathbf{e}_1$  in which case we have

$$\mathbb{Z}_n \cap (g\text{O}(2)g^{-1}) = \mathbb{Z}_{d_2}, \quad \gamma\mathbb{Z}_n \cap g\text{O}(2)g^{-1} = \begin{cases} \{\mathbf{r}(\mathbf{e}_1, \pi), \mathbf{r}(\mathbf{e}_2, \pi)\} & \text{if } n \text{ even} \\ \{\mathbf{r}(\mathbf{e}_1, \pi)\} & \text{otherwise} \end{cases}$$

so we can conclude the proof.  $\square$

4.3. **Clips with  $\mathbb{D}_{2n}^d$ .** First we have (see [Appendix A](#))

$$\mathbb{D}_{2n}^d = \mathbb{D}_n \cup (-\gamma\mathbb{D}_n), \quad \gamma = \mathbf{r}\left(\mathbf{e}_3, \frac{\pi}{n}\right)$$

where we can write

$$\begin{aligned} \mathbb{D}_n &= \left\{ \mathbf{r}\left(\mathbf{e}_3, \frac{2k_1\pi}{n}\right), \mathbf{r}(\mathbf{b}_{2k_1+1}, \pi); \quad k_1 = 0, \dots, n-1 \right\} \\ -\gamma\mathbb{D}_n &= \left\{ -\mathbf{r}\left(\mathbf{e}_3, \frac{(2k_2+1)\pi}{n}\right), -\mathbf{r}(\mathbf{b}_{2(k_2+1)}, \pi); \quad k_2 = 0, \dots, n-1 \right\} \end{aligned}$$

with

$$(7) \quad \mathbf{b}_1 = \mathbf{e}_1 \text{ and } \mathbf{b}_k = \mathbf{r}\left(\mathbf{e}_3, \frac{\pi}{2n}\right) \mathbf{b}_{k-1}.$$

**Lemma 4.11.** *Let  $m, n \geq 2$  be two integers and  $d = \gcd(m, n)$ . Then*

$$[\mathbb{D}_{2n}^d] \odot [\mathbb{Z}_m \oplus \mathbb{Z}_2^c] = \begin{cases} \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_{2d}^-]\} & \text{if } \frac{m}{d} \text{ even} \\ \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_d]\} & \text{if } m \text{ even and } \frac{m}{d} \text{ odd.} \\ \{[\mathbf{1}], [\mathbb{Z}_d]\} & \text{else} \end{cases}$$

*Proof.* From lemma 4.1 we have to consider intersection

$$\mathbb{D}_{2n}^d \cap (g\mathbb{Z}_m g^{-1} \oplus \mathbb{Z}_2^c) = (\mathbb{D}_n \cap g\mathbb{Z}_m g^{-1}) \cup (-\gamma\mathbb{D}_n \cap g\mathbb{Z}_m g^{-1}) \quad \gamma = \mathbf{r}\left(\mathbf{e}_3, \frac{\pi}{n}\right)$$

which can always reduce to  $\mathbf{1}$ . Otherwise we have only to consider three cases:

- $g\mathbf{e}_3 = \mathbf{e}_3$  and we deduce intersection from lemma 4.3;
- $g\mathbf{e}_3 = \mathbf{b}_{2k}$  for some  $k$  and intersection reduces to  $\mathbb{Z}_2^-$  for  $m$  even;
- $g\mathbf{e}_3 = \mathbf{b}_{2k+1}$  for some  $k$  and intersection reduces to  $\mathbb{Z}_2$  for  $m$  even, and we can conclude the proof. □

For the following, we recall the notation introduced in theorem 3.2

$$(8) \quad \Gamma(m, n) := \begin{cases} [\mathbb{D}_2], [\mathbb{D}_2^z] & \text{if } m \text{ and } n \text{ even} \\ [\mathbb{D}_2^z] & \text{if } m \text{ even and } n \text{ odd} \\ [\mathbb{Z}_2] & \text{if } m \text{ odd and } n \text{ even} \\ [\mathbb{Z}_2^-] & \text{if } m \text{ and } n \text{ odd} \end{cases}$$

**Lemma 4.12.** *Let  $m, n \geq 2$  be two integers,  $d = \gcd(m, n)$  and  $\Gamma(m, n)$  defined by (8). Then*

$$[\mathbb{D}_{2n}^d] \odot [\mathbb{D}_m \oplus \mathbb{Z}_2^c] = \begin{cases} \{[\mathbf{1}], \Gamma(m, n), [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_{2d}^-], [\mathbb{D}_{2d}^d]\} & \text{if } \frac{m}{d} \text{ is even} \\ \{[\mathbf{1}], \Gamma(m, n), [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_d], [\mathbb{D}_d], [\mathbb{D}_d^z]\} & \text{if } \frac{m}{d} \text{ is odd and } m \text{ is even.} \\ \{[\mathbf{1}], \Gamma(m, n), [\mathbb{Z}_d], [\mathbb{D}_d], [\mathbb{D}_d^z]\} & \text{else} \end{cases}$$

*Proof.* We have to consider intersection

$$(\mathbb{D}_n \cap (g\mathbb{D}_m g^{-1})) \cup (-\gamma(\mathbb{D}_n \cap (g\mathbb{D}_m g^{-1}))), \quad \gamma = \mathbf{r}\left(\mathbf{e}_3, \frac{\pi}{n}\right)$$

where (see (16))

$$g\mathbb{D}_m g^{-1} = \left\langle \mathbf{r}\left(\mathbf{u}, \frac{2\pi}{m}\right), \mathbf{r}(\mathbf{v}, \pi) \right\rangle.$$

Only seven non-trivial cases can occur:

- $\mathbf{u} = \pm\mathbf{e}_3$  and  $\mathbf{v} = \pm\mathbf{b}_{2k+1}$  for some  $k$  (see (7)), and in the same way of the proof of lemma 4.3 we obtain  $[\mathbb{D}_{2d}^d]$  if  $m/d$  is even, and  $[\mathbb{D}_d]$  otherwise.
- $\mathbf{u} = \pm\mathbf{e}_3$  and  $\mathbf{v} = \pm\mathbf{b}_{2k}$  for some  $k$  (see (7)), and in the same way of the proof of lemma 4.3 we obtain  $[\mathbb{D}_{2d}^d]$  if  $m/d$  is even, and  $[\mathbb{D}_d^z]$  otherwise.
- $\mathbf{u} = \pm\mathbf{e}_3$  only, and using again the proof of lemma 4.3 we obtain  $[\mathbb{Z}_{2d}^-]$  if  $m/d$  is even and  $[\mathbb{Z}_d]$  otherwise.

- $\mathbf{u} = \mathbf{b}_{2k+1}$  for some  $k$  and  $\mathbf{v} = \mathbf{e}_3$ , in which case we obtain

$$\begin{cases} [\mathbb{D}_2] & \text{if } m \text{ and } n \text{ even} \\ [\mathbb{D}_2^z] & \text{if } m \text{ even and } n \text{ odd} \\ [\mathbb{Z}_2] & \text{if } m \text{ odd and } n \text{ even} \\ [\mathbb{Z}_2^-] & \text{if } m \text{ and } n \text{ odds} \end{cases}$$

- $\mathbf{u} = \mathbf{b}_{2k+1}$  for some  $k$  only, and we obtain  $[\mathbb{Z}_2]$ .
- $\mathbf{u} = \mathbf{b}_{2k+2}$  for some  $k$  and  $\mathbf{v} = \mathbf{e}_3$ , in which case we obtain

$$\begin{cases} [\mathbb{D}_2^z] & \text{if } m \text{ even} \\ [\mathbb{Z}_2] & \text{if } m \text{ odd and } n \text{ even} \\ [\mathbb{Z}_2^-] & \text{if } m \text{ and } n \text{ odds} \end{cases}$$

- $\mathbf{u} = \mathbf{b}_{2k+2}$  for some  $k$  only, where we obtain  $[\mathbb{Z}_2^-]$  if  $m$  is even, and  $[\mathbf{1}]$  otherwise, so we can conclude the proof. □

From groups decompositions (18)–(21)–(23) and remark 4.2, using the same proof as for the proof of the Lemma 4.5 we get

$$\begin{aligned} [\mathbb{D}_{2n}^d] \odot [\mathbb{T} \oplus \mathbb{Z}_2^c] &= \left( [\mathbb{D}_{2n}^d] \odot [\mathbb{D}_2 \oplus \mathbb{Z}_2^c] \right) \cup \left( [\mathbb{D}_{2n}^d] \odot [\mathbb{Z}_3 \oplus \mathbb{Z}_2^c] \right) \\ [\mathbb{D}_{2n}^d] \odot [\mathbb{O} \oplus \mathbb{Z}_2^c] &= \left( [\mathbb{D}_{2n}^d] \odot [\mathbb{D}_3 \oplus \mathbb{Z}_2^c] \right) \cup \left( [\mathbb{D}_{2n}^d] \odot [\mathbb{D}_4 \oplus \mathbb{Z}_2^c] \right) \\ [\mathbb{D}_{2n}^d] \odot [\mathbb{I} \oplus \mathbb{Z}_2^c] &= \left( [\mathbb{D}_{2n}^d] \odot [\mathbb{D}_5 \oplus \mathbb{Z}_2^c] \right) \cup \left( [\mathbb{D}_{2n}^d] \odot [\mathbb{D}_3 \oplus \mathbb{Z}_2^c] \right) \cup \left( [\mathbb{D}_{2n}^d] \odot [\mathbb{D}_2 \oplus \mathbb{Z}_2^c] \right). \end{aligned}$$

As a direct consequence of previous lemmas, we thus get

**Lemma 4.13.** *For any integer  $n \geq 2$  we have*

$$\begin{aligned} [\mathbb{D}_{2n}^d] \odot [\mathbb{T} \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], \Gamma(2, n), \mathbb{Z}_{d_3}\} \\ [\mathbb{D}_{2n}^d] \odot [\mathbb{O} \oplus \mathbb{Z}_2^c] &= \begin{cases} \{\mathbb{L}_0, [\mathbb{D}_2], [\mathbb{D}_2^z], [\mathbb{Z}_4], [\mathbb{D}_4], [\mathbb{D}_4^z]\} & \text{if } 4|n \\ \{\mathbb{L}_0, [\mathbb{D}_2], [\mathbb{D}_2^z], [\mathbb{Z}_4^-], [\mathbb{D}_4^d]\} & \text{if } n \text{ is even and } 4 \nmid n \\ \{\mathbb{L}_0, [\mathbb{D}_2^z]\} & \text{if } n \text{ is odd} \end{cases} \\ \mathbb{L}_0 &:= [\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], \Gamma(n, 3), [\mathbb{Z}_{d_3}], [\mathbb{D}_{d_3}], [\mathbb{D}_{d_3}^z] \\ [\mathbb{D}_{2n}^d] \odot [\mathbb{I} \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], \Gamma(2, n), \Gamma(3, n), \Gamma(5, n), [\mathbb{Z}_{d_3}], [\mathbb{D}_{d_3}], [\mathbb{D}_{d_3}^z], \\ &\quad [\mathbb{Z}_{d_5}], [\mathbb{D}_{d_5}], [\mathbb{D}_{d_5}^z]\} \end{aligned}$$

where  $\Gamma(m, n)$  is given by (3.2) and  $d_k = \gcd(n, k)$  for  $k = 3, 5$ .

Finally, we have

**Lemma 4.14.** *For any integer  $n \geq 2$  we have*

$$\begin{aligned} [\mathbb{D}_{2n}^d] \odot [\text{SO}(2) \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_{2n}^-]\} \\ [\mathbb{D}_{2n}^d] \odot [\text{O}(2) \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], \mathcal{Z}(n), [\mathbb{D}_{d_2}], [\mathbb{D}_2^z], [\mathbb{D}_{2n}^d]\}. \end{aligned}$$

*Proof.* The first clips operation is obtained in the same way as the one given by lemma 4.11. We now have to consider intersection

$$(\mathbb{D}_n \cap (g\text{O}(2)g^{-1})) \cup (-(\gamma\mathbb{D}_n \cap (g\text{O}(2)g^{-1}))), \quad \gamma = \mathbf{r} \left( \mathbf{e}_3, \frac{\pi}{n} \right)$$

where  $g\text{O}(2)g^{-1}$  is given by (6). Three non-trivial cases can occur:

- $\mathbf{u} = \pm \mathbf{e}_3$  and we obtain  $[\mathbb{D}_{2n}^d]$ ;

- $\mathbf{u} = \mathbf{b}_{2k+1}$  for some  $k$ , say  $\mathbf{u} = \mathbf{b}_1 = \mathbf{e}_1$  (see (7)). Then, if  $n$  is even, there exists  $k'$  such that  $\mathbf{b}_{2k'+1} = \mathbf{e}_2$ , otherwise there exists  $k'$  such that  $\mathbf{b}_{2k'} = \mathbf{e}_2$ . We deduce that

$$\mathbb{D}_n \cap (g\mathbb{O}(2)g^{-1}) = \begin{cases} \mathbb{D}_2 & \text{if } n \text{ is even} \\ \text{Id}, \mathbf{r}(\mathbf{e}_1, \pi) & \text{otherwise} \end{cases}$$

and

$$(\gamma\mathbb{D}_n) \cap (g\mathbb{O}(2)g^{-1}) = \begin{cases} \emptyset & \text{if } n \text{ is even} \\ \mathbf{r}(\mathbf{e}_3, \pi), \mathbf{r}(\mathbf{e}_2, \pi) & \text{otherwise} \end{cases}$$

and we thus obtain  $[\mathbb{D}_2]$  if  $n$  is even, or  $[\mathbb{D}_2^z]$  if  $n$  is odd.

- $\mathbf{u} = \mathbf{b}_{2k}$  for some  $k$ , say  $\mathbf{u} = \mathbf{b}_2$  (see (7)). In such a case, whatever  $n$ 's parity, we always obtain  $[\mathbb{D}_2^z]$ .
- $\mathbf{v} = \pm\mathbf{e}_3$ , since  $\mathbf{v} = g\mathbf{b}_i$  then we have two cases:
  - if  $\mathbf{b}_i$  is within the secondary axis of  $\mathbb{D}_{2n}$  then the discussion is similar to the previous case.
  - if  $\mathbf{b}_i$  is not within the secondary axis of  $\mathbb{D}_{2n}$  then we obtain  $\mathcal{Z}(n)$ .

□

**4.4. Clips with  $\mathbb{O}^-$ .** First, we introduce a useful decomposition of the subgroup  $\mathbb{O}^-$ , deduced from (19):

$$(9) \quad \mathbb{O}^- = \bigcup_{i=1}^3 (\mathbb{Z}_4^{\mathbf{e}_i})^- \bigcup_{j=1}^3 \mathbb{Z}_3^{\mathbf{st}_j} \bigcup_{k=1}^3 (\mathbb{Z}_2^{\mathbf{a}_{c_k}})^-$$

where

$$(\mathbb{Z}_4^{\mathbf{e}_i})^- = \left\{ \text{Id}, \mathbf{r}(\mathbf{e}_i, \pi), -\mathbf{r}\left(\mathbf{e}_i, \frac{\pi}{2}\right), -\mathbf{r}\left(\mathbf{e}_i, \frac{3\pi}{2}\right) \right\}, \quad (\mathbb{Z}_2^{\mathbf{a}_{c_k}})^- = \{\text{Id}, -\mathbf{r}(\mathbf{a}_{c_k}, \pi)\}.$$

**Lemma 4.15.** *Let  $m \geq 2$  be an integer and  $d'_k = \gcd(k, m)$  ( $k = 2, 3$ ). Then we have*

$$[\mathbb{O}^-] \odot [\mathbb{Z}_m \oplus \mathbb{Z}_2^c] = \begin{cases} \left\{ [\mathbf{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_{d'_3}], [\mathbb{Z}_4^-] \right\} & \text{if } 4|m \\ \left\{ [\mathbf{1}], [\mathbb{Z}_{d'_2}], [\mathbb{Z}_{d'_2}^-], [\mathbb{Z}_{d'_3}] \right\} & \text{otherwise} \end{cases}.$$

*Proof.* For any  $g \in \text{SO}(3)$ , let us write  $g\mathbb{Z}_m g^{-1} = \mathbb{Z}_m^{\mathbf{u}}$  (see (9)). When considering intersection

$$\mathbb{O}^- \cap (\mathbb{Z}_m^{\mathbf{u}} \oplus \mathbb{Z}_2^c)$$

the only non-trivial cases are when  $\mathbf{u} = \pm\mathbf{e}_i$ ,  $\mathbf{u} = \pm\mathbf{st}_j$  or  $\mathbf{u} = \pm\mathbf{a}_{c_k}$ , and the problem then reduces to clips operations  $[\mathbb{Z}_{2n}^-] \odot [\mathbb{Z}_m \oplus \mathbb{Z}_2^c]$  for  $n = 1, 2$  or  $[\mathbb{Z}_3] \odot [\mathbb{Z}_m \oplus \mathbb{Z}_2^c]$ , which are already known (see lemma 4.3 and [20]), so we can conclude. □

For the following, instead of using decomposition (9), we will write  $\mathbb{O}^-$  using  $\mathbb{T}$  (Appendix A)

$$\mathbb{O}^- = \mathbb{T} \cup (-\gamma\mathbb{T}), \quad \gamma = \mathbf{r}\left(\mathbf{e}_3, \frac{\pi}{2}\right).$$

**Lemma 4.16.** *Let  $m \geq 2$  be an integer. We have*

$$[\mathbb{O}^-] \odot [\mathbb{D}_m \oplus \mathbb{Z}_2^c] = \begin{cases} \left\{ [\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_{d'_3}], [\mathbb{D}_{d'_3}^z], [\mathbb{Z}_4^-], [\mathbb{D}_2^z], [\mathbb{D}_4^d] \right\} & \text{if } 4|m \\ \left\{ [\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_{d'_3}], [\mathbb{D}_{d'_3}^z], [\mathbb{D}_2], [\mathbb{D}_2^z] \right\} & \text{if } m \text{ is even and } 4 \nmid m \\ \left\{ [\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_{d'_3}], [\mathbb{D}_{d'_3}^z] \right\} & \text{if } m \text{ is odd} \end{cases}$$

*Proof.* For any  $g \in \text{SO}(3)$ , let us write  $g\mathbb{D}_m g^{-1} = \mathbb{D}_m^{\mathbf{u}, \mathbf{v}}$  with  $\mathbf{u}, \mathbf{v}$  two non-zero orthogonal vectors (see (16)). From lemma 4.1 We then have to consider intersection

$$(\mathbb{T} \cap \mathbb{D}_m^{\mathbf{u}, \mathbf{v}}) \cup (-\gamma\mathbb{T} \cap \mathbb{D}_m^{\mathbf{u}, \mathbf{v}})$$

and 8 non-trivial cases can occur:

- $\mathbf{u} = \pm\mathbf{e}_k$ , says  $\pm\mathbf{e}_3$  and  $\mathbf{v} = \pm\mathbf{e}_j$  with  $j \neq k$ , say  $\mathbf{v} = \pm\mathbf{e}_1$ . If  $4 | m$ , then intersection reduces to  $\mathbb{D}_4^d$ , while if  $m$  is even and  $4 \nmid m$  we obtain  $\mathbb{D}_2$ , and we get  $\mathbf{1}$  otherwise.

- $\mathbf{u} = \pm \mathbf{e}_3$  (for instance) and  $\mathbf{v} = \pm \mathbf{a}_{c_k}$ , say  $\pm \mathbf{a}_{c_1}$ . Once again, if  $4 \mid m$ , then intersection reduces to  $\mathbb{D}_4^d$ , but if  $m$  is even and  $4 \nmid m$  we obtain a subgroup conjugate to  $\mathbb{D}_2^z$ , otherwise we get  $\mathbb{Z}_2^-$ .
- $\mathbf{u} = \pm \mathbf{e}_3$  only, in which case we obtain a subgroup conjugate to  $\mathbb{Z}_4^-$  if  $4 \mid m$ ,  $\mathbb{Z}_2$  if  $m$  is even and  $4 \nmid m$ , and  $\mathbb{1}$  otherwise.
- $\mathbf{u} = \pm \mathbf{a}_{c_1}$  (for instance) and  $\mathbf{v} = \pm \mathbf{e}_3$ , leading to a subgroup conjugate to  $\mathbb{D}_2^z$  if  $m$  is even, or  $\mathbb{Z}_2$  if  $m$  is odd.
- $\mathbf{v} = \pm \mathbf{e}_3$  only, so we obtain a subgroup conjugate to  $\mathbb{Z}_2$ .
- $\mathbf{v} = \mathbf{a}_{c_1}$  only and we obtain a subgroup conjugate to  $\mathbb{Z}_2^-$ .
- $\mathbf{u} = \pm \mathbf{s}_{t_1}$  (for instance) and  $\mathbf{v} = \mathbf{a}_{c_2}$ , so we get a subgroup conjugate to  $\mathbb{D}_3^z$  if  $3 \mid m$ , otherwise we get a subgroup conjugate to  $\mathbb{Z}_2^-$ .
- $\mathbf{u} = \mathbf{s}_{t_1}$  only, and we finally get  $\mathbb{Z}_{d_3}^z$ .

□

For the subgroups  $\mathbb{O}$ ,  $\mathbb{T}$  and  $\mathbb{I}$ , we use, as before, the decompositions (18)–(21)–(23).

**Lemma 4.17.** *We have*

$$\begin{aligned} [\mathbb{O}^-] \odot [\mathbb{O} \oplus \mathbb{Z}_2^c] &= \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_3], [\mathbb{Z}_4], [\mathbb{D}_2^z], [\mathbb{D}_3^z], [\mathbb{D}_4^d], [\mathbb{O}^-]\}. \\ [\mathbb{O}^-] \odot [\mathbb{T} \oplus \mathbb{Z}_2^c] &= \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{D}_2], [\mathbb{D}_2^z], [\mathbb{Z}_3], [\mathbb{T}]\}. \\ [\mathbb{O}^-] \odot [\mathbb{I} \oplus \mathbb{Z}_2^c] &= \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{Z}_3], [\mathbb{D}_2], [\mathbb{D}_2^z], [\mathbb{D}_3^z], [\mathbb{T}]\}. \end{aligned}$$

**Lemma 4.18.** *We have*

$$\begin{aligned} [\mathbb{O}^-] \odot [\text{SO}(2) \oplus \mathbb{Z}_2^c] &= \{[\mathbb{1}], [\mathbb{Z}_3], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-]\}. \\ [\mathbb{O}^-] \odot [\text{O}(2) \oplus \mathbb{Z}_2^c] &= \{[\mathbb{1}], [\mathbb{Z}_2^-], [\mathbb{D}_3^z], [\mathbb{D}_4^d]\}. \end{aligned}$$

*Proof.* The first clips operation is obtained in the same way as the one given by lemma 4.15. We now have to consider intersection

$$(\mathbb{T} \cap (g\text{O}(2)g^{-1})) \cup (-(\gamma\mathbb{T} \cap (g\text{O}(2)g^{-1}))), \quad \gamma = \mathbf{r}\left(\mathbf{e}_3, \frac{\pi}{n}\right)$$

where  $g\text{O}(2)g^{-1}$  is given by (6),  $\mathbb{T}$  by (17) and  $\gamma\mathbb{T}$  by

$$\gamma\mathbb{T} = \left\{ \mathbf{r}(\mathbf{a}_{c_k}, \pi), \mathbf{r}\left(\mathbf{e}_i, \frac{\pi}{2}\right), \mathbf{r}\left(\mathbf{e}_i, \frac{3\pi}{2}\right) \right\}$$

Three non-trivial cases can occur:

- $\mathbf{u} = \pm \mathbf{s}_{t_i}$  and we obtain  $\mathbb{D}_3^z$ .
- $\mathbf{u} = \pm \mathbf{e}_i$ , say  $\mathbf{u} = \mathbf{e}_3$ , then we obtain  $\mathbb{D}_4^d$ .
- $\mathbf{u} = \pm \mathbf{a}_{c_k}$  for some  $k$ , then we obtain  $\mathbb{Z}_2^-$ .

□

**4.5. Clips with  $\text{O}(2)^-$ .** We construct  $\text{O}(2)^-$  as follows

$$\begin{aligned} \text{O}(2)^- &= \text{SO}(2) \cup -(\gamma\text{SO}(2)) \quad \text{where } \gamma = \mathbf{r}(\mathbf{e}_1, \pi) \\ &= \{\mathbf{r}(\mathbf{e}_3, \theta), \theta \in [0, 2\pi], -\mathbf{r}(\mathbf{b}, \pi)\}. \end{aligned}$$

where  $\mathbf{r}(\mathbf{b}, \pi)$  represent the symmetry with respect to all the axes in the  $xy$  plane.

The argumentation for the calculation of the clips with  $\text{O}(2)^-$  is very similar to the ones for  $\mathbb{D}_n^z$  exposed in subsection 4.2.

**Lemma 4.19.** *Let  $m \geq 2$  be an integer and  $d_2' = \text{gcd}(2, m)$ . We have*

$$[\text{O}(2)^-] \odot [\mathbb{Z}_m \oplus \mathbb{Z}_2^c] = \{[\mathbb{1}], [\mathbb{Z}_m], [\mathbb{Z}_{d_2'}^-]\}$$

*Proof.* We get the result by considering the following intersection (from lemma 4.1)

$$\text{O}(2)^- \cap (g\mathbb{Z}_m g^{-1} \oplus \mathbb{Z}_2^c) = (\text{SO}(2) \cap g\mathbb{Z}_m g^{-1}) \cup (-(\gamma\text{SO}(2) \cap g\mathbb{Z}_m g^{-1}))$$

where  $\gamma = \mathbf{r}(\mathbf{e}_1, \pi)$  and  $\gamma\text{SO}(2) = \{\mathbf{r}(\mathbf{b}, \pi)\}$ .

□

The proof of the following lemma is similar to the proof of lemma 4.8.

**Lemma 4.20.** *For any integer  $m \geq 2$ , we have*

$$[\mathrm{O}(2)^-] \odot [\mathbb{D}_m \oplus \mathbb{Z}_2^c] = \begin{cases} \{[\mathbf{1}], [Z_m], [Z_2^-], [\mathbb{D}_2^z], [\mathbb{D}_m^z]\} & \text{if } m \text{ is even} \\ \{[\mathbf{1}], [Z_m], [Z_2], [Z_2^-], [\mathbb{D}_m^z]\} & \text{if } m \text{ is odd} \end{cases}.$$

Using the decompositions (18)–(21)–(23) of the subgroups  $\mathbb{T}$ ,  $\mathbb{O}$  and  $\mathbb{I}$ , we get the following

**Lemma 4.21.** *We have*

$$\begin{aligned} [\mathrm{O}(2)^-] \odot [\mathbb{T} \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [Z_2], [Z_3], [Z_2^-], [\mathbb{D}_2^z]\}. \\ [\mathrm{O}(2)^-] \odot [\mathbb{O} \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [Z_2], [Z_3], [Z_4], [Z_2^-], [\mathbb{D}_2^z], [\mathbb{D}_3^z], [\mathbb{D}_4^z]\}. \\ [\mathrm{O}(2)^-] \odot [\mathbb{I} \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [Z_2], [Z_3], [Z_5], [Z_2^-], [\mathbb{D}_2^z], [\mathbb{D}_3^z], [\mathbb{D}_5^z]\}. \end{aligned}$$

And finally we deduce the clips with  $\mathrm{SO}(2) \oplus \mathbb{Z}_2^c$  and  $\mathrm{O}(2) \oplus \mathbb{Z}_2^c$  in the same way as in lemma 4.10.

**Lemma 4.22.** *We have*

$$\begin{aligned} [\mathrm{O}(2)^-] \odot [\mathrm{SO}(2) \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [Z_2^-], [\mathrm{SO}(2)]\}. \\ [\mathrm{O}(2)^-] \odot [\mathrm{O}(2) \oplus \mathbb{Z}_2^c] &= \{[\mathbf{1}], [\mathbb{D}_2^z], [\mathrm{O}(2)^-]\}. \end{aligned}$$

## 5. APPLICATION TO PIEZOELECTRICITY

We propose here to apply clips operation to the specific case of the *Piezoelectricity law*. We introduce the space of constitutive tensors occurring in the mechanical description of Piezoelectricity, which describes the electrical behavior of a material subject to mechanical stress. It is defined by a triplet of tensors given by an elasticity tensor, a Piezoelectricity tensor and a permittivity tensor. Such a space is naturally endowed with an  $\mathrm{O}(3)$  representation, and the finite set of isotropy classes is obtained in theorem 5.1 below.

We recall now the Piezoelectricity law, while details can be found in [26, 12, 25, 15]. First, the mechanical state of a material is characterized by two fields of symmetric second order tensors: the stress tensor  $\boldsymbol{\sigma}$  and the strain tensor  $\boldsymbol{\varepsilon}$ . The relation between these two fields forms the constitutive law that describes the mechanical behavior of a specific material. In linear elasticity, the relation is linear, given by

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon}$$

which is known as the generalized Hook's law. Such fourth order elasticity tensor  $\mathbf{E}$  have the index symmetries

$$\mathbf{E}_{ijkl} = \mathbf{E}_{jikl} = \mathbf{E}_{ijlk} = \mathbf{E}_{klij}$$

and we define the associated space of elasticity tensors  $\mathbb{E}la$ , which is a 21 dimensional vector space.

Similarly to the mechanical state, the electrical state of a material is described by two vector fields: the electric displacement field  $\mathbf{d}$  and the electric field  $\mathbf{e}$ . These two fields are related and the relation between them forms the constitutive law that describes the electrical behavior of a material. In the linear case, it is given by

$$\mathbf{d} = \mathbf{S} \cdot \mathbf{e}$$

where the second order symmetric tensor  $\mathbf{S}$  is the *permittivity* tensor. We define  $\mathbb{S}$  to be the vector space of the permittivity tensors, which is of dimension 6.

Finally, the Piezoelectricity law is given by the coupled law

$$\begin{cases} \boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon} - \mathbf{e} \cdot \mathbf{P} \\ \mathbf{d} = \mathbf{P} : \boldsymbol{\sigma} + \mathbf{S} \cdot \mathbf{e} \end{cases}$$

which involves a third order tensor  $\mathbf{P}$  called the *Piezoelectricity* tensor, satisfying the index symmetry

$$\mathbf{P}_{ijk} = \mathbf{P}_{ikj}$$

The vector space of Piezoelectricity tensors is an 18 dimensional vector space, noted  $\mathbb{P}iez$ .

As a consequence, the linear electromechanical behavior of any homogeneous material is defined by a triplet  $\mathcal{P}$  of constitutive tensors

$$\mathcal{P} := (\mathbf{E}, \mathbf{P}, \mathbf{S}) \in \mathbb{E}la \oplus \mathbb{P}iez \oplus \mathbb{S}$$

and we define  $\mathbb{P}iez$  to be the space of Piezoelectricity constitutive tensors:

$$\mathbb{P}iez = \mathbb{E}la \oplus \mathbb{P}iez \oplus \mathbb{S}.$$

The natural O(3) representation on  $(\mathbf{E}, \mathbf{P}, \mathbf{S}) \in \mathbb{P}iez$  is given in any orthonormal basis by

$$(10) \quad (\rho(g)\mathbf{E})_{ijkl} := g_{ip}g_{jq}g_{kr}g_{ls}E_{pqrs}, \quad (\rho(g)\mathbf{P})_{ijk} := g_{ip}g_{jq}g_{kr}P_{pqr}, \quad (\rho(g)\mathbf{S})_{ij} := g_{ip}g_{jq}S_{pq}$$

where  $g \in O(3)$ .

As a consequence of lemma 2.4, the isotropy classes  $\mathcal{J}(\mathbb{P}iez)$  can be deduced from isotropy classes  $\mathbb{E}la$ ,  $\mathbb{P}iez$  and  $\mathbb{S}$ :

$$\mathcal{J}(\mathbb{P}iez) = (\mathcal{J}(\mathbb{E}la) \odot \mathcal{J}(\mathbb{P}iez)) \odot \mathcal{J}(\mathbb{S}).$$

Recall from [23] the isotropy classes of  $\mathbb{P}iez$  (the notations and definitions of O(3)-subgroups have been moved to Appendix A)

$$\mathcal{J}(\mathbb{P}iez) = \left\{ [1], [Z_2], [Z_3], [\mathbb{D}_2^z], [\mathbb{D}_3^z], [Z_2^-], [Z_4^-], [\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4^d], [\mathbb{D}_6^d], \right. \\ \left. [\text{SO}(2)], [\text{O}(2)], [\text{O}(2)^-], [\mathbb{O}^-], [\text{O}(3)] \right\}.$$

The symmetry classes for O(3) representation on  $\mathbb{E}la$  and  $\mathbb{S}$  are the same as the symmetry classes for SO(3) representation which can be found in [8] and [24], except that each type I subgroup occurring in the list of isotropy classes has to be replaced by the corresponding type II subgroup (see Appendix A). Indeed,  $-\text{Id}$  acts trivially on  $\mathbb{E}la$  and  $\mathbb{S}$ , and we have

$$\mathcal{J}(\mathbb{E}la) = \{ [1], [Z_2 \oplus Z_2^c], [\mathbb{D}_2 \oplus Z_2^c], [\mathbb{D}_3 \oplus Z_2^c], [\mathbb{D}_4 \oplus Z_2^c], [\mathbb{O} \oplus Z_2^c], [\text{O}(2) \oplus Z_2^c], [\text{SO}(3) \oplus Z_2^c] \},$$

and

$$\mathcal{J}(\mathbb{S}) = \{ [\mathbb{D}_2 \oplus Z_2^c], [\text{O}(2) \oplus Z_2^c], [\text{SO}(3) \oplus Z_2^c] \}.$$

We deduce the isotropy classes of the Piezoelectricity law  $\mathbb{P}iez = \mathbb{E}la \oplus \mathbb{P}iez \oplus \mathbb{S}$  from lemma 2.4 by calculating the clips operations between the isotropy classes of  $\mathbb{E}la$ ,  $\mathbb{P}iez$  and  $\mathbb{S}$  (see Table 1 and Table 2 for clips between type II and III O(3)-subgroups and [20, table 1] for clips between two type I and remark 3.1 (1) for type I with type II and two type II).

**Theorem 5.1.** *There exists 25 isotropy classes for the Piezoelectricity law  $\mathbb{E}la \oplus \mathbb{P}iez \oplus \mathbb{S}$  given by*

$$\mathcal{J}(\mathbb{P}iez) = \{ [1], [Z_2], [Z_3], [Z_4], [\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4], [\text{SO}(2)], [\text{O}(2)], [\text{O}(3)], \\ [Z_2 \oplus Z_2^c], [\mathbb{D}_2 \oplus Z_2^c], [\mathbb{D}_3 \oplus Z_2^c], [\mathbb{D}_4 \oplus Z_2^c], [\mathbb{O} \oplus Z_2^c], [\text{O}(2) \oplus Z_2^c], \\ [Z_2^-], [Z_4^-], [\mathbb{D}_2^z], [\mathbb{D}_3^z], [\mathbb{D}_4^z], [\mathbb{D}_2^d], [\mathbb{D}_4^d], [\mathbb{D}_6^d], [\mathbb{O}^-], [\text{O}(2)^-] \}.$$

## APPENDIX A. CLOSED O(3)-SUBGROUP

Given a subgroup  $\Gamma$  of O(3), set

$$\Gamma_+ = \{g \in \Gamma; \det g = 1\} \text{ and } \Gamma_- = \{g \in \Gamma; \det g = -1\}.$$

Then  $\Gamma = \Gamma_+ \cup \Gamma_-$  and we have the following classification.

**Type I:**  $\Gamma$  is of type I if  $\Gamma_- = \emptyset$ , in which case  $\Gamma$  is a subgroup of SO(3). Hence, a subgroup of type I is conjugate to one of the following list

$$\text{SO}(3), \text{O}(2), \text{SO}(2), \mathbb{D}_n, \mathbb{Z}_n, \mathbb{T}, \mathbb{O}, \mathbb{I}, \text{ or } \mathbb{1}$$

**Type II:**  $\Gamma$  is of type II if  $-\text{Id} \in \Gamma$ , in which case  $-\Gamma_- = \Gamma_+$  and  $\Gamma = \Gamma_+ \cup -\Gamma_+$ . Hence, a subgroup of type II is a subgroup of type I to which we add the group  $Z_2^c = \{\pm \text{Id}\}$

$$\text{SO}(3) \oplus Z_2^c, \text{O}(2) \oplus Z_2^c, \text{SO}(2) \oplus Z_2^c, \mathbb{Z}_m \oplus Z_2^c, \mathbb{D}_m \oplus Z_2^c, \mathbb{T} \oplus Z_2^c, \mathbb{O} \oplus Z_2^c, \mathbb{I} \oplus Z_2^c$$



**Type III:**  $\Gamma$  is of type III if  $\Gamma_- \neq \emptyset$  but  $-\text{Id} \notin \Gamma$ . Then,  $\Gamma_+$  is a subgroup of index 2 in  $\tilde{\Gamma} = \Gamma_+ \cup (-\Gamma_-) \subset \text{SO}(3)$ . In that case, there exists  $\gamma \in \tilde{\Gamma} \setminus \Gamma_+$  such that  $-\Gamma_- = \gamma\Gamma_+$  and  $\Gamma = \Gamma_+ \cup -\gamma\Gamma_+$ . Hence, five subgroups of type III can be deduced

$$(11) \quad \mathbb{Z}_{2n}^- = \mathbb{Z}_n \cup (-\mathbf{r}(\mathbf{e}_3, \frac{\pi}{n}))\mathbb{Z}_n \quad \forall n \geq 1.$$

$$(12) \quad \mathbb{D}_n^z = \mathbb{Z}_n \cup (-\mathbf{r}(\mathbf{e}_1, \pi))\mathbb{Z}_n \quad \forall n \geq 2.$$

$$(13) \quad \mathbb{D}_{2n}^d = \mathbb{D}_n \cup (-\mathbf{r}(\mathbf{e}_3, \frac{\pi}{n}))\mathbb{D}_n \quad \forall n \geq 1.$$

$$(14) \quad \mathbb{O}^- = \mathbb{T} \cup (-\mathbf{r}(\mathbf{e}_3, \frac{\pi}{2}))\mathbb{T}.$$

$$(15) \quad \text{O}(2)^- = \text{SO}(2) \cup (-\mathbf{r}(\mathbf{e}_1, \pi))\text{SO}(2).$$

**Example A.1.**  $\mathbb{Z}_4^- = \mathbb{Z}_2 \cup (-\mathbf{r}(\mathbf{e}_3, \pi/2))\mathbb{Z}_2 = \{\text{Id}, \mathbf{r}(\mathbf{e}_3, \pi), -\mathbf{r}(\mathbf{e}_3, \pi/2), -\mathbf{r}(\mathbf{e}_3, 3\pi/2)\}$

We propose in Table 3 generators of type I and type III closed  $\text{O}(3)$ -subgroups.

Type I subgroup	Order	Generators
$\mathbb{Z}_n, n \geq 2$	$n$	$\mathbf{r}(\mathbf{e}_3, 2\pi/n)$
$\mathbb{D}_n, n \geq 2$	$2n$	$\mathbf{r}(\mathbf{e}_3, 2\pi/n), \mathbf{r}(\mathbf{e}_1, \pi)$
$\mathbb{T}$	12	$\mathbf{r}(\mathbf{e}_3, \pi), \mathbf{r}(\mathbf{e}_1, \pi), \mathbf{r}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, 2\pi/3)$
$\mathbb{O}$	24	$\mathbf{r}(\mathbf{e}_3, \pi/2), \mathbf{r}(\mathbf{e}_1, \pi), \mathbf{r}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, 2\pi/3)$
$\mathbb{I}$	60	$\mathbf{r}(\mathbf{e}_3, \pi), \mathbf{r}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, 2\pi/3), \mathbf{r}(\mathbf{e}_1 + \phi\mathbf{e}_3, 2\pi/5) \quad \phi := (1 + \sqrt{5})/2$
$\text{SO}(2)$	$\infty$	$\mathbf{r}(\mathbf{e}_3, \theta), \quad \theta \in [0, 2\pi]$
$\text{O}(2)$	$\infty$	$\text{SO}(2), \mathbf{r}(\mathbf{e}_1, \pi)$
Type III subgroup		
$\mathbb{Z}_2^-$	2	$-\mathbf{r}(\mathbf{e}_3, \pi)$
$\mathbb{Z}_{2n}^-, n \geq 2$	$2n$	$-\mathbf{r}(\mathbf{e}_3, \pi/n)$
$\mathbb{D}_{2n}^d, n \geq 2$	$4n$	$-\mathbf{r}(\mathbf{e}_3, \pi/n), \mathbf{r}(\mathbf{e}_1, \pi)$
$\mathbb{D}_n^z, n \geq 2$	$2n$	$\mathbf{r}(\mathbf{e}_3, 2\pi/n), -\mathbf{r}(\mathbf{e}_1, \pi)$
$\mathbb{O}^-$	24	$-\mathbf{r}(\mathbf{e}_3, \pi/2), -\mathbf{r}(\mathbf{e}_2 - \mathbf{e}_3, \pi)$
$\text{O}(2)^-$	$\infty$	$\text{SO}(2), -\mathbf{r}(\mathbf{e}_1, \pi)$

TABLE 3. Generators of closed  $\text{O}(3)$ -subgroups

We give now some useful decompositions of subgroups  $\mathbb{T}, \mathbb{O}$  and  $\mathbb{I}$ . To do so, first introduce subgroups

$$(16) \quad \mathbb{Z}_n^{\mathbf{u}} := \left\langle \mathbf{r}\left(\mathbf{u}, \frac{2\pi}{n}\right) \right\rangle, \quad \mathbb{D}_n^{\mathbf{u}, \mathbf{v}} := \left\langle \mathbf{r}\left(\mathbf{u}, \frac{2\pi}{n}\right), \mathbf{r}(\mathbf{v}, \pi) \right\rangle$$

where  $\mathbf{u}, \mathbf{v}$  are a non-zero orthogonal vectors. In such notations, the axis  $\langle \mathbf{u} \rangle$  generated by  $\mathbf{u}$  is said to be the primary axis of  $\mathbb{Z}_n^{\mathbf{u}}$  and  $\mathbb{D}_n^{\mathbf{u}, \mathbf{v}}$ , while  $\langle \mathbf{v} \rangle$  is said to be a secondary axis of  $\mathbb{D}_n^{\mathbf{u}, \mathbf{v}}$ . In fact we have

$$\mathbb{Z}_n = \mathbb{Z}_n^{\mathbf{e}_3}, \quad \mathbb{D}_n = \mathbb{D}_n^{\mathbf{e}_3, \mathbf{e}_1}$$

and for any  $g \in \text{SO}(3)$

$$g\mathbb{Z}_n g^{-1} = \mathbb{Z}_n^{g\mathbf{e}_3}, \quad g\mathbb{D}_n g^{-1} = \mathbb{D}_n^{g\mathbf{e}_3, g\mathbf{e}_1}.$$

We propose now details on compositions of the subgroups  $\mathbb{T}$ ,  $\mathbb{O}$  and  $\mathbb{I}$ , with explicit axes for all cyclic and dihedral subgroups they contain.

First we have (see [10] for instance)

$$(17) \quad \mathbb{T} = \bigcup_{i=1}^3 \mathbb{Z}_2^{\mathbf{e}_i} \bigcup_{j=1}^4 \mathbb{Z}_3^{\mathbf{s}_{t_j}}$$

with

$$\mathbf{s}_{t_1} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \quad \mathbf{s}_{t_2} = \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \mathbf{s}_{t_3} = -\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3, \quad \mathbf{s}_{t_4} = -\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3.$$

Such decomposition (17) can also write

$$(18) \quad \mathbb{T} = \mathbb{D}_2 \bigcup_{j=1}^4 \mathbb{Z}_3^{\mathbf{s}_{t_j}}.$$

For the cubic group we have

$$(19) \quad \mathbb{O} = \bigcup_{i=1}^3 \mathbb{Z}_4^{\mathbf{e}_i} \bigcup_{j=1}^4 \mathbb{Z}_3^{\mathbf{s}_{t_j}} \bigcup_{k=1}^6 \mathbb{Z}_2^{\mathbf{a}_{c_k}}$$

with vectors  $\mathbf{a}_k$  given by

$$(20) \quad \begin{aligned} \mathbf{a}_{c_1} &= \mathbf{e}_1 + \mathbf{e}_2, & \mathbf{a}_{c_2} &= \mathbf{e}_1 - \mathbf{e}_2, & \mathbf{a}_{c_3} &= \mathbf{e}_1 + \mathbf{e}_3 \\ \mathbf{a}_{c_4} &= \mathbf{e}_1 - \mathbf{e}_3, & \mathbf{a}_{c_5} &= \mathbf{e}_2 + \mathbf{e}_3, & \mathbf{a}_{c_6} &= \mathbf{e}_2 - \mathbf{e}_3. \end{aligned}$$

As for the tetragonal case, such decomposition (19) also writes

$$(21) \quad \mathbb{O} = \bigcup_{i=1}^3 \mathbb{D}_4^{(i)} \bigcup_{j=1}^4 \mathbb{D}_3^{(j)}$$

with

$$(22) \quad \begin{aligned} \mathbb{D}_4^{(1)} &:= \mathbb{D}_4, & \mathbb{D}_4^{(2)} &:= \mathbb{D}_4^{\mathbf{e}_2, \mathbf{e}_3}, & \mathbb{D}_4^{(3)} &:= \mathbb{D}_4^{\mathbf{e}_1, \mathbf{e}_3} \\ \mathbb{D}_3^{(1)} &:= \mathbb{D}_3^{\mathbf{s}_{t_1}, \mathbf{a}_{c_2}}, & \mathbb{D}_3^{(2)} &:= \mathbb{D}_3^{\mathbf{s}_{t_2}, \mathbf{a}_{c_1}}, & \mathbb{D}_3^{(3)} &:= \mathbb{D}_3^{\mathbf{s}_{t_3}, \mathbf{a}_{c_1}}, & \mathbb{D}_3^{(4)} &:= \mathbb{D}_3^{\mathbf{s}_{t_4}, \mathbf{a}_{c_2}}. \end{aligned}$$

Finally we have

$$(23) \quad \mathbb{I} = \bigcup_{i=1}^6 \mathbb{Z}_5^{\mathbf{u}_i} \bigcup_{j=1}^{10} \mathbb{Z}_3^{\mathbf{v}_j} \bigcup_{k=1}^{15} \mathbb{Z}_2^{\mathbf{w}_k} = \bigcup_{i=1}^6 \mathbb{D}_5^{(i)} \bigcup_{j=1}^{10} \mathbb{D}_3^{(j)} \bigcup_{l=1}^5 \mathbb{D}_2^{(l)}.$$

Taking  $\phi := (1 + \sqrt{5})/5$  to be the golden ratio, vectors  $\mathbf{u}_i$  are obtained as centers of icosahedron faces ([20, Figure 11]):

$$\begin{aligned} \mathbf{u}_1 &:= (1 + 3\phi)\mathbf{e}_1 + (2 + \phi)\mathbf{e}_3, & \mathbf{u}_2 &:= (2 + \phi)\mathbf{e}_1 + (1 + 3\phi)\mathbf{e}_2, & \mathbf{u}_3 &:= (2 + \phi)\mathbf{e}_2 - (1 + 3\phi)\mathbf{e}_3 \\ \mathbf{u}_4 &:= -(2 + \phi)\mathbf{e}_2 - (1 + 3\phi)\mathbf{e}_3, & \mathbf{u}_5 &:= (1 + 3\phi)\mathbf{e}_1 - (2 + \phi)\mathbf{e}_3, & \mathbf{u}_6 &:= (2 + \phi)\mathbf{e}_1 - (1 + 3\phi)\mathbf{e}_2. \end{aligned}$$

Then vectors  $\mathbf{v}_j$  are obtained from vertices of icosahedron:

$$\begin{aligned}\mathbf{v}_1 &:= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, & \mathbf{v}_2 &:= \phi \mathbf{e}_1 + \frac{1}{\phi} \mathbf{e}_2, & \mathbf{v}_3 &:= \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3, & \mathbf{v}_4 &:= \phi \mathbf{e}_2 - \frac{1}{\phi} \mathbf{e}_3 \\ \mathbf{v}_5 &:= \phi \mathbf{e}_2 + \frac{1}{\phi} \mathbf{e}_3, & \mathbf{v}_6 &:= -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, & \mathbf{v}_7 &:= -\phi \mathbf{e}_1 + \frac{1}{\phi} \mathbf{e}_2, & \mathbf{v}_8 &:= -\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 \\ \mathbf{v}_9 &:= -\frac{1}{\phi} \mathbf{e}_1 - \phi \mathbf{e}_3, & \mathbf{v}_{10} &:= \frac{1}{\phi} \mathbf{e}_1 - \phi \mathbf{e}_3\end{aligned}$$

while vectors  $\mathbf{w}_k$  are obtained from its edges:

$$\begin{aligned}\mathbf{w}_1 &:= \mathbf{e}_1 + (\phi + 1)\mathbf{e}_2 + \left(1 + \frac{1}{\phi}\right)\mathbf{e}_3, & \mathbf{w}_2 &:= \left(1 + \frac{1}{\phi}\right)\mathbf{e}_1 + \mathbf{e}_2 + (\phi + 1)\mathbf{e}_3 \\ \mathbf{w}_3 &:= (\phi + 1)\mathbf{e}_1 + \left(1 + \frac{1}{\phi}\right)\mathbf{e}_2 + \mathbf{e}_3, & \mathbf{w}_4 &:= \phi \mathbf{e}_1, & \mathbf{w}_5 &:= (\phi + 1)\mathbf{e}_1 + \left(1 + \frac{1}{\phi}\right)\mathbf{e}_2 - \mathbf{e}_3 \\ \mathbf{w}_6 &:= (\phi + 1)\mathbf{e}_1 - \left(1 + \frac{1}{\phi}\right)\mathbf{e}_2 + \mathbf{e}_3, & \mathbf{w}_7 &:= \left(1 + \frac{1}{\phi}\right)\mathbf{e}_1 - \mathbf{e}_2 + (\phi + 1)\mathbf{e}_3, \\ \mathbf{w}_8 &:= (\phi + 1)\mathbf{e}_1 - \left(1 + \frac{1}{\phi}\right)\mathbf{e}_2 - \mathbf{e}_3, & \mathbf{w}_9 &:= \mathbf{e}_1 + (\phi + 1)\mathbf{e}_2 - \left(1 + \frac{1}{\phi}\right)\mathbf{e}_3 \\ \mathbf{w}_{10} &:= \phi \mathbf{e}_2, & \mathbf{w}_{11} &:= -\mathbf{e}_1 + (\phi + 1)\mathbf{e}_2 + \left(1 + \frac{1}{\phi}\right)\mathbf{e}_3, & \mathbf{w}_{12} &:= \phi \mathbf{e}_3 \\ \mathbf{w}_{13} &:= -\left(1 + \frac{1}{\phi}\right)\mathbf{e}_1 + \mathbf{e}_2 + (\phi + 1)\mathbf{e}_3, & \mathbf{w}_{14} &:= -\mathbf{e}_1 + (\phi + 1)\mathbf{e}_2 - \left(1 + \frac{1}{\phi}\right)\mathbf{e}_3, \\ \mathbf{w}_{15} &:= -\left(1 + \frac{1}{\phi}\right)\mathbf{e}_1 - \mathbf{e}_2 + (\phi + 1)\mathbf{e}_3.\end{aligned}$$

All dihedral subgroups from (23) are given by:

$$(24) \quad \begin{aligned}\mathbb{D}_5^{\mathbf{u}_1, \mathbf{w}_{10}}, & \quad \mathbb{D}_5^{\mathbf{u}_2, \mathbf{w}_6}, & \mathbb{D}_5^{\mathbf{u}_3, \mathbf{w}_1}, & \mathbb{D}_5^{\mathbf{u}_4, \mathbf{w}_4}, & \mathbb{D}_5^{\mathbf{u}_5, \mathbf{w}_1}, & \mathbb{D}_5^{\mathbf{u}_6, \mathbf{w}_2}, \\ \mathbb{D}_3^{\mathbf{v}_1, \mathbf{w}_8}, & \mathbb{D}_3^{\mathbf{v}_2, \mathbf{w}_{11}}, & \mathbb{D}_3^{\mathbf{v}_3, \mathbf{w}_2}, & \mathbb{D}_3^{\mathbf{v}_4, \mathbf{w}_2}, & \mathbb{D}_3^{\mathbf{v}_5, \mathbf{w}_4}, & \mathbb{D}_3^{\mathbf{v}_6, \mathbf{w}_3}, & \mathbb{D}_3^{\mathbf{v}_7, \mathbf{w}_1}, \\ \mathbb{D}_3^{\mathbf{v}_8, \mathbf{w}_1}, & \mathbb{D}_3^{\mathbf{v}_9, \mathbf{w}_5}, & \mathbb{D}_3^{\mathbf{v}_{10}, \mathbf{w}_3}, & & & & \\ \mathbb{D}_2^{\mathbf{w}_1, \mathbf{w}_6}, & \mathbb{D}_2^{\mathbf{w}_2, \mathbf{w}_8}, & \mathbb{D}_2^{\mathbf{w}_3, \mathbf{w}_{13}}, & \mathbb{D}_2^{\mathbf{w}_4, \mathbf{w}_{10}}, & \mathbb{D}_2^{\mathbf{w}_5, \mathbf{w}_7}.\end{aligned}$$

## REFERENCES

- [1] G. Bredon. Finiteness of number of orbit types. In *A. Borel, Seminar on transformation groups. With contributions by G. Bredon, EE Floyd, D. Montgomery, R. Palais. Annals of Mathematics Studies*, number 46, 1960.
- [2] P. Chossat and F. Guyard. A classification of 2-modes interactions with so (3) symmetry and applications. In *Dynamics, Bifurcation and Symmetry*, pages 79–95. Springer, 1994.
- [3] P. Chossat, R. Lauterbach, and I. Melbourne. Steady-State bifurcation with 0(3)-Symmetry. *Arch. Rational Mech. Anal.*, 113(3):313–376, 1990.
- [4] S. Eilenberg. On the problems of topology. *Annals of Mathematics*, pages 247–260, 1949.
- [5] A. Eringen and G. Maugin. *Electrodynamics of Continua, tomes I et II*. Springer-Verlag, 1990.
- [6] E. E. Floyd. Orbits of torus groups operating on manifolds. *Annals of Mathematics*, pages 505–512, 1957.
- [7] S. Forte and M. Vianello. Symmetry classes for elasticity tensors. *Journal of Elasticity*, 43(2):81–108, 1996.
- [8] S. Forte and M. Vianello. Symmetry classes and harmonic decomposition for photoelasticity tensors. *International Journal of Engineering Science*, 35(14):1317–1326, 1997.
- [9] M. Golubitsky, I. Stewart, and D. G. Schaeffer. *Singularities and groups in bifurcation theory. Vol. II*, volume 69 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1988.
- [10] E. Ihrig and M. Golubitsky. Pattern selection with O(3) symmetry. *Physica D: Nonlinear Phenomena*, 13(1-2):1–33, 1984.
- [11] H. Juretschke. *Crystal physics; macroscopic physics of anisotropic solids*. W.A. Benjamin, Advanced Book Program, Reading, Mass, 1974.
- [12] L. D. Landau, J. S. Bell, M. J. Kearsley, L. P. Pitaevskii, and E. M. Lifshitz. *Electrodynamics of Continuous Media*. Elsevier Science, Oct. 2013.

- [13] H. Le Quang and Q.-C. He. The number and types of all possible rotational symmetries for flexoelectric tensors. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 467(2132):2369–2386, 2011.
- [14] L. Mann et al. Finite orbit structure on locally compact manifolds. *The Michigan Mathematical Journal*, 9(1):87–92, 1962.
- [15] A. Meitzler, H. Tiersten, A. Warner, D. Berlincourt, G. Couqin, and F. Welsh III. IEEE standard on piezoelectricity, 1988.
- [16] L. Michel. Symmetry defects and broken symmetry. configurations hidden symmetry. *Reviews of Modern Physics*, 52(3):617, 1980.
- [17] G. D. Mostow and G. P. Hochschild. *On a conjecture of Montgomery*. Mathematics Division, Air Force Office of Scientific Research, 1957.
- [18] R. E. Newnham. *Properties of Materials: Anisotropy, Symmetry, Structure*. OxfordUniversity Press, Jan. 2005.
- [19] J. F. Nye. *Physical Properties of Crystals*. Oxford University Press, May 1985.
- [20] M. Olive. Effective computation of  $so(3)$  and  $o(3)$  linear representation symmetry classes. *Mathematics and Mechanics of Complex Systems*, 7(3):203–237, 2019.
- [21] M. Olive and N. Auffray. Symmetry classes for even-order tensors. *Mathematics and Mechanics of Complex Systems*, 1(2):177–210, 2013.
- [22] M. Olive and N. Auffray. Symmetry classes for odd-order tensors. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 94(5):421–447, 2014.
- [23] M. Olive and N. Auffray. Symmetry classes in piezoelectricity from second-order symmetries. *Mathematics and Mechanics of Complex Systems*, 9(1):77–105, mar 2021.
- [24] M. Olive, B. Kolev, R. Desmorat, and B. Desmorat. Characterization of the symmetry class of an elasticity tensor using polynomial covariants, 2018.
- [25] D. Royer and E. Dieulesaint. *Elastic waves in solids I: Free and guided propagation*. Springer Science & Business Media, 1999.
- [26] J. A. Schouten. *Tensor Analysis for Physicists*. Oxford, At the Clarendon Press, 1951.
- [27] T. Weller. *Etude des symétries et modèles de plaques en piézoélectricité linéarisée*. PhD thesis, Université de Montpellier Sciences et Techniques du Languedoc, 2004.
- [28] Q. S. Zheng and J. P. Boehler. The description, classification, and reality of material and physical symmetries. *Acta Mechanica*, 102(1-4):73–89, mar 1994.
- [29] W.-N. Zou, C.-X. Tang, and E. Pan. Symmetry types of the piezoelectric tensor and their identification. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 469(2155):20120755, 2013.

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