

# Oleinik's inequality for conservation laws with a Lipschitz convex flux

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UNIVERSITÉ  
CÔTE D'AZUR



*Inria*  
inventeurs du monde numérique

- 1 Generalized  $BV$  spaces:  $BV^s$  and  $\Phi$ - $BV$
- 2 Smoothing effect for nonlinear convex  $C^1$  flux
  - Oleinik's inequality revisited
- 3 Lipschitz convex flux
  - Oleinik's inequality lost & recovered
  - Generalized inverse for a non decreasing function
  - Generalized Lax-Oleinik formula
  - Generalized characteristics
- 4 Nonconvex flux

# BV space for conservation laws

$$\partial_t u + \partial_x f(u) = 0, \quad u(0, x) = u_0(x), \quad t > 0, \quad x \in \mathbb{R}$$

- propagation:  $TV u(t, \cdot) \leq TV u_0$
- smoothing for **uniform convex  $C^2$  flux** [Lax & Oleinik, 1957]  $u_0 \in L^\infty \Rightarrow u(t, \cdot) \in BV_{loc}$
- Compact embedding from  $BV$  to  $L^1_{loc}$
- Regulated functions: Left and right **traces** everywhere.
- **Not all solutions in  $BV$**  [K. S. Cheng JMAA 1983], . . . , [Ghoshal-Jana SIMA 2020]
- Compensated compactness for nonlinear flux [Tartar 1979]

## Question ( $BV$ is not enough)

*How to to quantify the compactness (regularity) of the solutions?*

## Definition (generalised $BV$ spaces, Muzielak-Orlicz 1959)

For a convex function  $\Phi(0) = 0, r > 0 \Rightarrow \Phi(r) > 0$

$$\Phi\text{-TV } u \stackrel{\text{defi}}{=} \sup_{\sigma} \sum_n \Phi(|u(x_{n+1}) - u(x_n)|),$$

$BV^s = p\text{-}BV = \Phi\text{-}BV$ , with  $\Phi(u) = |u|^p, p = \frac{1}{s}, 0 < s \leq 1$

$BV = BV^1 \subset BV^s \subset L^\infty (= BV^0)$ .

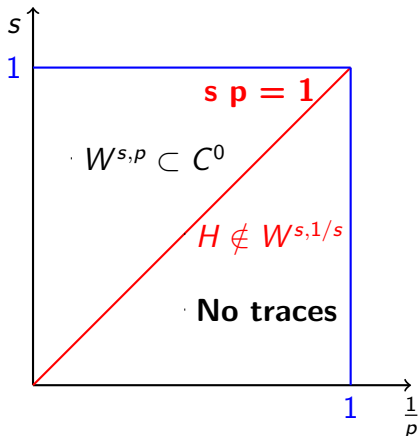
### **$BV$ -like properties.**

- $\Phi\text{-}BV$  compactly embedded in  $L^1_{loc}$ .
- If  $u \in \Phi\text{-}BV$ , then  $u$  has traces everywhere.
- Sobolev regularity,  $\forall \varepsilon > 0, BV^s \subset W^{s-\varepsilon, 1/s}_{loc}$

$BV^s \neq W^{s,1/s}$  on a bounded interval

Heaviside  
step  
function,

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$



## Theorem ( Propagation of $\Phi$ -BV regularity )

*Lipschitz flux,*

$$\Phi\text{-TV } u \leq \Phi\text{-TV } u_0$$

- $BV^s$  Bourdarias-Gisclon-J., JHDE 2014
- $\Phi$ -BV Jenssen-Ridder, JHDE 2020
- Numerical schemes:
  - Godunov scheme 1959
  - Glimm scheme 1969
  - [Wave Front Tracking algorithm](#) Dafermos 1972

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# Nonlinear Flux

The most general: **Nowhere linear** on any non-empty open interval Tartar 1972

**Definition** Power law nonlinearity  $p$  for  $C^1$  flux

The flux has a  $p$ -power law nonlinearity on  $[-M, M]$  if

$$\exists p \geq 1, \exists C > 0, \forall (u, v) \in [-M, M]^2 \quad |f'(u) - f'(v)| \geq C|u - v|^p$$

*Example: Burgers*  $p = 1$ ,  $f(u) = \frac{|u|^{p+1}}{p+1}$

The flux is necessary convex



## Theorem (Smoothing for $C^1$ flux with $p$ -power law nonlinearity)

If  $u_0 \in L^\infty(\mathbb{R})$  then,  $\forall t > 0$ ,  $u(t, \cdot) \in BV_{loc}^s(\mathbb{R})$ ,  $s = \frac{1}{p}$

- Nonlinear effect:  $\partial_t u + c \partial_x u = u$ ,  
 $u(t, x) = u_0(x - c t)$  no smoothing effect
- $f \in C^2$ ,  $f'' > 0 \rightarrow p = 1$ , Lax & Oleinik 1957
- $f \in C^1$ ,  $p > 1$ , Bourdarias, Gisclon, J. 2014
- Optimality: Castelli, J. 2014,  
Ghoshal, Guelmame, Jana, J. 2020
- shortest (best) proof: one-sided Oleinik inequality

# one-sided Lipschitz Oleinik's inequality

Oleinik 1957 rewritten by Hoff 1983

$$f'(u(t, x)) - f'(u(t, y)) \leq \frac{x - y}{t} \quad \text{a.e. } x > y.$$

- 1 It implies the Lax entropy inequality  
no increasing velocity through a jump
- 2 Oleinik's uniqueness proof
- 3 bound the expansion of wave.
- 4 Optimality: "=" for the Rarefaction Waves
- 5  $f'(u(t, \cdot)) \in BV_{loc}$

$$f'(u(t, \cdot)) \in BV_{loc} \implies u(t, \cdot) \in BV_{loc}^s$$

$$(f'(u(t, x)) - f'(u(t, y)))^+ \leq \frac{x - y}{t}, \quad v^+ = \max(0, v)$$

①  $f'(u(t, \cdot)) \in BV_{loc}$

$$TV^+ v[a, b] = \sup_{n \in \mathbb{N}, a \leq x_0 < x_1 < \dots < x_n \leq b} \sum_{i < n} (v(x_{i+1}) - v(x_i))^+.$$

$$TV^+ f'(u)[a, b] \leq \frac{b - a}{t}.$$

$f'(u) \in L^\infty$  by the maximum principle  $|u| \leq \|u_0\|_\infty$

If  $v \in L^\infty$  and  $TV^+ v < \infty$  then  $v \in BV$ .

②  $|f'(u_1) - f'(u_2)| \geq C|u_1 - u_2|^p$

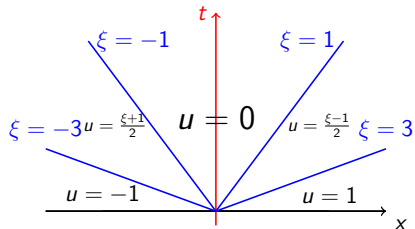
$\implies$  for  $t > 0$   $u(t, \cdot) \in p\text{-}BV_{loc} = BV_{loc}^s$

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# Oleinik inequality lost for Lipschitz flux

$$\xi \stackrel{\text{defi}}{=} x/t,$$

$$f(u) = u^2 + |u|$$
$$u_0(x) = \text{sign}(x).$$
$$f'(0) = ?$$



$$\bar{f}'(u) \stackrel{\text{defi}}{=} \frac{1}{2}(f'(u-0) + f'(u+0))$$

$$\bar{f}'(u(t, x)) - \bar{f}'(u(t, y)) > (x - y)/t$$

# Recovering an Oleinik inequality

For the Riemann problem,  $f'$  discontinuous at  $u_{\#}$ ,

$$u(t, x) = u_{\#}, \quad \xi_- \leq \xi \stackrel{\text{defi}}{=} \frac{x - x_{\#}}{t} \leq \xi_+$$

$v(t, x) = f'(u(t, x))$  when is well defined else:

$$v(t, x) = \xi, \quad \xi_- \leq \xi \leq \xi_+, \quad t \leq t_*$$

Oleinik inequality for  $v$ :  $x < y$ , a.e.

$$v(t, y) - v(t, x) \leq \frac{y - x}{t}$$

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$$u(t, x) = (f')^{-1}(v(t, x)) ?$$

# Generalized inverse for a non decreasing function

$\varphi : [a, b] \mapsto [c, d]$  non decreasing

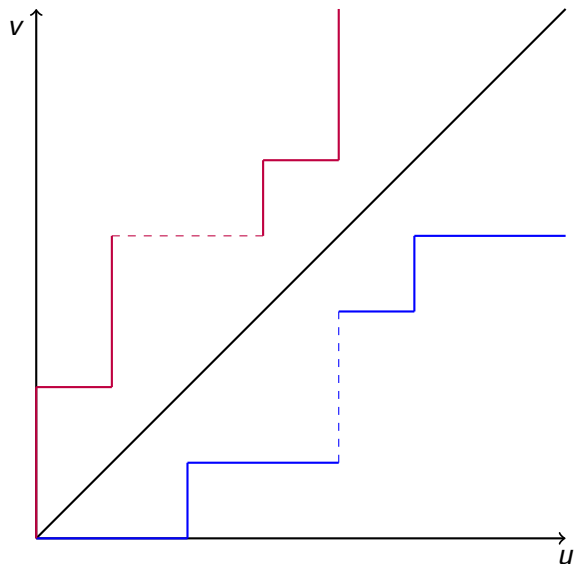
generalized inverse,  $y \in [c, d]$ ,

$$\varphi^{-1}(x) \stackrel{\text{defi}}{=} \inf \{x \in [a, b], y \leq \varphi(x)\}$$

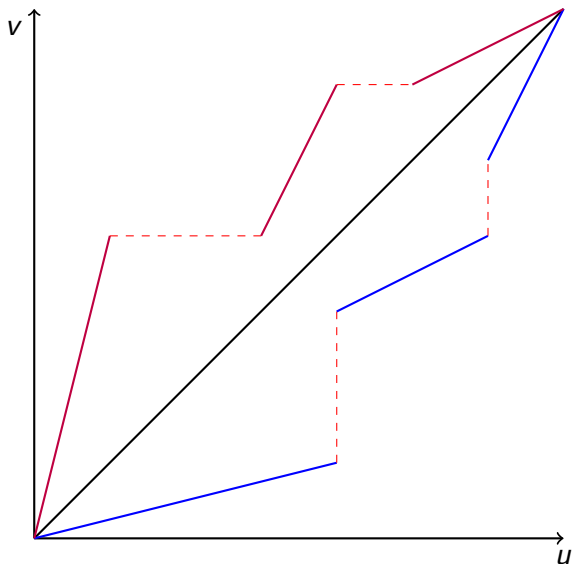
- $\varphi \in C^0$  & increasing  $\Rightarrow \varphi^{-1}$  is the usual inverse function
- $\varphi$  increasing  $\Leftrightarrow \varphi^{-1}$  continuous
- $\varphi$  is constant on an interval  $\Leftrightarrow \varphi^{-1}$  has a jump
- $(\varphi^{-1} \circ \varphi)(x) \leq x$
- $\varphi^{-1}$  is right continuous



# Generalized inverse for a step function

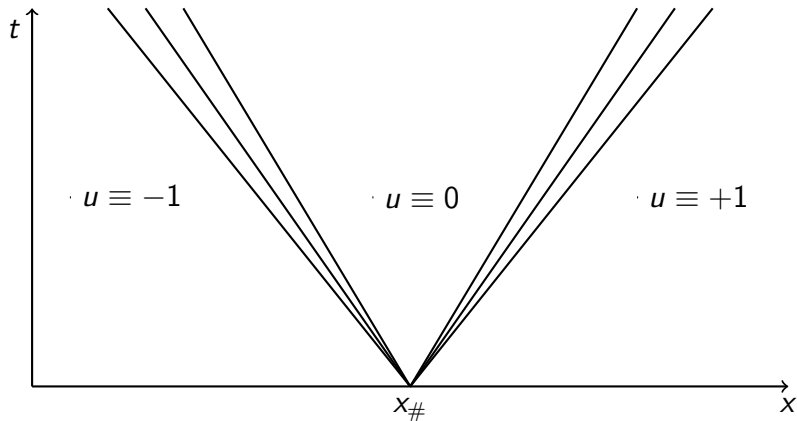


# $C^0$ generalised inverse for increasing fct



$$u(t, x) = \mathfrak{I}(v(t, x)) \quad ,$$

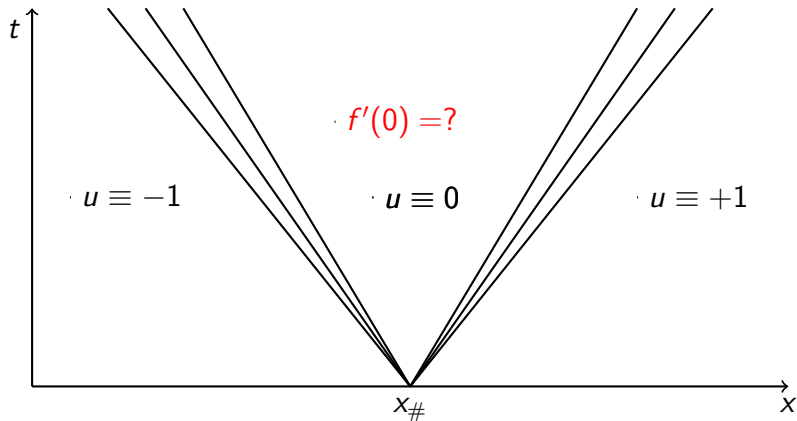
$$\mathfrak{I} = (f')^{-1}$$



Riemann problem,  $f(u) = |u| + u^2$ ,  $u_0(x) = \text{sign}(x)$

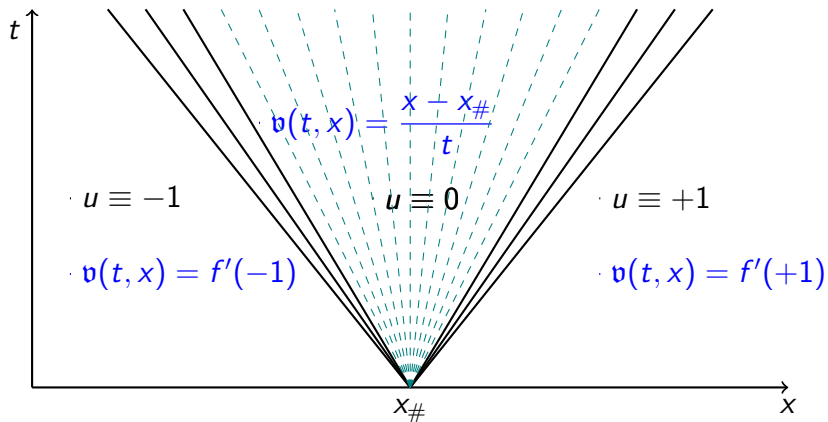
$$u(t, x) = \mathfrak{I}(v(t, x)) \quad ,$$

$$\mathfrak{I} = (f')^{-1}$$



Riemann problem,  $f(u) = |u| + u^2$ ,  $u_0(x) = \text{sign}(x)$

$$u(t, x) = \mathfrak{I}(v(t, x)) \quad , \quad \mathfrak{I} = (f')^{-1}$$



Riemann problem,  $f(u) = |u| + u^2$ ,  $u_0(x) = \text{sign}(x)$

## Plan:

- use a wave front tracking algorithm  $u \simeq u_\nu$ ,  $\nu > 1$
- Get Oleinik inequality for  $\mathfrak{v}_\nu$
- $\mathfrak{v}_\nu \in BV_{loc}$
- $u \in \Phi\text{-}BV$  since  $u = \mathfrak{J}(\mathfrak{v})$  as a generalized
  - Bruneau's decomposition of  $\Phi\text{-}BV$  function 1974
  - Lax-Oleinik formula 1957

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## Question

$\Phi = ?$

$$\Phi \stackrel{\text{defi}}{=} \text{upper convex envelope of } (\omega[(f')^{-1}])^{-1}$$

①  $\mathfrak{J} = (f')^{-1}$  continuous since  $f$  strictly convex

②  $0 \leq \omega[\mathfrak{J}]$  modulus of continuity of  $\mathfrak{J}$

③  $0 \leq \phi \stackrel{\text{defi}}{=} (\omega[\mathfrak{J}])^{-1} \notin C^0$

④  $\Phi \stackrel{\text{defi}}{=} \sup_{\text{convex } \psi, 0 \leq \psi \leq \phi} \psi \leq \phi$



# Dafermos' wave front tracking algorithm 1972

- Uniform mesh  $u_\nu \in \nu^{-1}\mathbb{Z}$ ,  $\nu \geq 1$
- $f \leftrightarrow f_\nu$  a piecewise linear flux.

$$f'_\nu = s_k \stackrel{\text{defi}}{=} \nu \left( f \left( \frac{k}{\nu} + \frac{1}{\nu} \right) - f \left( \frac{k}{\nu} \right) \right)$$

on  $(k/\nu, (k+1)/\nu)$  non-decreasing piece-wise constant

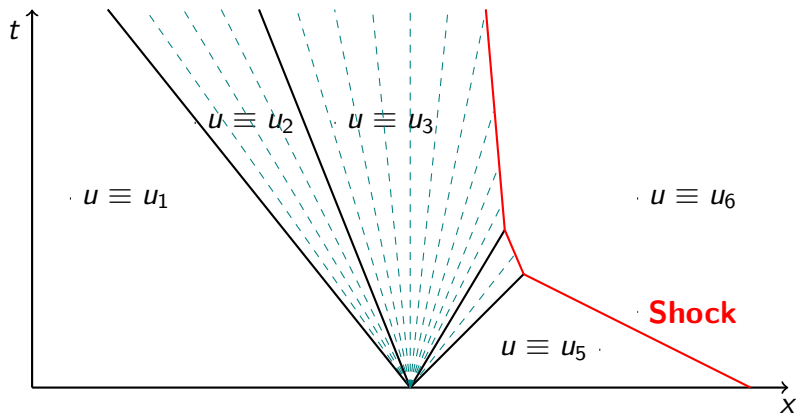
- $u_0 \leftrightarrow u_{0,\nu}$  a piecewise constant approximation
- Riemann problems: Contact Discontinuities & Shocks
- $\partial_t u_\nu + \partial_x f_\nu(x) = 0$ ,  $u_\nu(0, x) = u_{0,\nu}(x)$

# The velocity $v_\nu$ satisfies the Oleinik's inequality

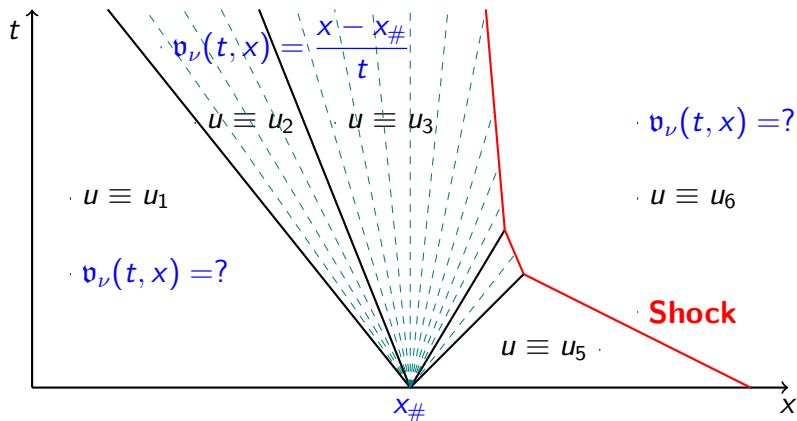
Initially,  $v_\nu$  is defined for contact discontinuities to fill the discontinuity of  $f'_\nu$  at  $(k/\nu)$  between the two fronts with velocity  $s_{k-1}$  and  $s_k$ .

When a shock interacts with one of the two contact discontinuity front, the front disappears.  $v_\nu$  is prolonged between the shock and the remaining contact discontinuity front.

# Define $v_\nu$ , contact Discontinuity, Shock



# Define $v_\nu$ , contact Discontinuity, Shock



## $v_\nu$ outside waves

outside waves  $u_\nu = cst$  constant

$\mathfrak{J}(s_k) = k/\nu$  by right continuity

- between two contact discontinuities (CD) ,  
 $v_\nu$  is the speed of CD
- between one contact discontinuity and one shock,  
 $v_\nu$  is the speed of CD
- between two shocks ,  
 $v_\nu = f'(cst)$

$$u_\nu(t, x) = \mathfrak{J}_\nu(v_\nu(t, x)) + \mathcal{O}(1/\nu)$$

# Regularity and compactness for $\mathbf{v}_\nu$

A priori, no uniqueness for  $\mathbf{v}$

- space
  - Uniform  $BV$  estimates : Oleinik's inequality
  - Compactness in  $L^1_{loc}$  and pointwise convergence
- Time regularity
  - Finite speed and  $\sup_\nu TV\mathbf{v}_\nu < \infty$   
yields  $\mathbf{v} \in \text{Lip}_{t,loc}([0, +\infty], L^1_{x,loc}(\mathbb{R}))$
- Compactness in  $L^1_{loc}$  and pointwise convergence in space and time
- From  $u_\nu(t, x) = \mathfrak{J}_\nu(\mathbf{v}_\nu(t, x)) + \mathcal{O}(1/\nu)$

$$u(t, x) = \mathfrak{J}(\mathbf{v}(t, x))$$

# Generalized Lax-Oleinik formula

$$u(t, x) = \mathfrak{I}(v(t, x))$$

$\mathfrak{I}_\nu = (f'_\nu)^{-1} = k/\nu$  on  $(s_{k-1}, s_k)$  piecewise constant

$$u_\nu = \mathfrak{I}_\nu(v_\nu) + \mathcal{O}(1/\nu)$$

Locally  $\|\mathfrak{I} - \mathfrak{I}_\nu\|_\infty \rightarrow 0$  when  $\nu \rightarrow +\infty$

Dini Lemma for non decreasing functions

# The classical Lax-Oleinik formula 1957

for smooth flux, historically  $f \in C^2$ ,  $f'' > 0$

$$u(t, x) = (f')^{-1} \left( \frac{x - y(t, x)}{t} \right),$$

$y(t, x)$  non decreasing w.r.t.  $x$  defined by:

$$G(y(t, x)) := \min_{y \in \mathbb{R}} G(y),$$

$$G(y) := t f \left( \frac{x - y}{t} \right) + U_0(y),$$

$$f^*(v) := \max_{u \in \mathbb{R}} (v u - f(u)), \quad (f^*)' = (f')^{-1}$$

$$U_0(x) := \int^x u_0(y) dy$$

to compare with  $u(t, x) = \mathfrak{J}(v(t, x))$



$$\Phi(\Delta u) \leq |\Delta v|$$

$$u_i = u(t_i, x), \quad v_i = v(t_i, x), \quad u_i = \mathcal{I}(v_i)$$

$$\begin{aligned} \Phi(|u_1 - u_2|) &\leq \Phi(|\mathcal{I}(v_1) - \mathcal{I}(v_2)|) \\ &\leq \phi(|\mathcal{I}(v_1) - \mathcal{I}(v_2)|) \\ &\leq \phi\left(\omega[\mathcal{I}] (|v_1 - v_2|)\right) \\ &\leq \left(\omega[\mathcal{I}]^{-1} \circ \omega[\mathcal{I}]\right) (|v_1 - v_2|) \\ &\leq |v_1 - v_2|. \end{aligned}$$

# Regularity for $u$ from $u(t, x) = \mathfrak{I}(v(t, x))$

$$v \in L_{t,loc}^\infty([0, +\infty], BV_{x,loc}(\mathbb{R})) \cap \text{Lip}_{t,loc}([0, +\infty], L_{x,loc}^1(\mathbb{R}))$$

- In space:  $\Phi\text{-TV}u \leq \text{TV}v$
- Time regularity in Orlicz space:  
 $u \in \text{Lip}([0, +\infty], L_{loc}^\Phi(\mathbb{R})), 0 < t_1 < t_2,$

$$\int_{\mathbb{R}} \Phi(|u(t_1, x) - u(t_2, x)|) dx \leq \mathcal{O}\left(1 + \frac{1}{t_1}\right) |t_1 - t_2|$$

- Guelmame, J., Clamond, CMS 2019
- Jenssen, Ridder, JHDE 2020

# Theorem (Smoothing for strictly convex flux

Guelmame, J., Clamond. *CMS* 2019, revisited 2022)

$f$  strictly convex,  $\mathfrak{J} := (f')^{-1} \in C^0$ ,  $u$  entropy solution of,

$$\partial_t u + \partial_x f(u) = 0, \quad u_0 \in L^\infty(\mathbb{R})$$

$\Rightarrow u \in L^\infty_{loc}([0, +\infty], \Phi\text{-}BV_{loc}(\mathbb{R})) \cap Lip_{loc}([0, +\infty], L^\Phi_{loc}(\mathbb{R}))$

$\Phi \stackrel{\text{defi}}{=} \text{upper convex envelope of } (\omega[\mathfrak{J}])^{-1}$

Generalized Lax-Oleinik formula:

$u(t, x) = \mathfrak{J}(v(t, x))$  with  $v \in L^\infty_{loc}([0, +\infty], BV_{loc}(\mathbb{R}))$

$\Phi\text{-}TVu \leq TVv$ , Oleinik's inequality:  $\Delta v \leq \frac{\Delta x}{t}$

# Generalized characteristics for Lipschitz flux

$$\frac{dX(t;x)}{dt} = (f' \circ u)(X(t;x)) \text{ not defined}$$

- $v$  represents the waves speed but not unique
- **generalized characteristics**  
usually,  $BV$  framework & smooth flux Dafermos 1977

$$\frac{dX(t;x)}{dt} = v(t, X(t,x)), \quad X(0;x) = x$$

**defined** A. F. Fillppov 1988 (in Russian 1985)  
new qualitative results in  $\Phi$ - $BV$ ?

- Hamilton Jacobi approach
- to compare with Lagrangian representations  
smooth flux, Bianchini, Marconi, ARMA 2017

# Nonconvex flux

- $f'(u) \in BV$  [ K.S. Cheng, JDE 1986]

- $f'(u) \notin BV$  [Marconi, JHDE 2018]

**Theorem.** [Marconi. 2018]  $C^3$  non convex flux,  $u \in \Phi\text{-}BV$ .

*Example.*  $f(u) = u^3$ ,  $u \in BV_{loc}^{1/2}$  for  $t > 0$ .

# Nonconvex flux

- $f'(u) \in BV$  [ K.S. Cheng, JDE 1986]

Hidden assumptions:

$f \in C^3$  with polynomial expansion:

$$f(u) = c_0 + c_1(u - u_0) + \cdots + c_d(u - u_0)^d + \mathcal{O}(u - u_0)^{d+1}$$

- $f'(u) \notin BV$  [Marconi, JHDE 2018]

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2 counter-examples,  $f \in C^2$

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