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Refinement and derivation of statistical resolution limits for circular or rectilinear correlated sources in CES data models

Habti Abeida and Jean-Pierre Delmas

Abstract

This paper analyzes the deterministic (DCRB) and the stochastic (SCRB) Cramér-Rao bound on direction-of-arrival (DOA) estimation for two equi-powered correlated complex circular or rectilinear sources affected by complex circular white noise in different complex elliptically symmetric (CES) data models. Beginning by decomposing these CRBs, into factors depending on signal and noise parameters, and on geometric parameters of the array, some new interpretable closed-form expressions are derived in particular scenarios. These expressions provide useful insight into the behavior of these CRB's dependence on the correlation factor. Approximate closed-form expressions of these CRBs for small DOA separation are also derived. These results lead to new formulas for statistical resolution limit (SRL) based on the Smith criterion at which an unbiased DOA estimation algorithm can resolve two closely-spaced circular or rectilinear sources. These SCRB-derived formulas are much less optimistic than those which have so far been deduced only from the DCRB under the assumption of known sources.

Index Terms

Deterministic Cramér-Rao bound, Stochastic Cramér-Rao bound, Semiparametric Cramér-Rao bound, statistical resolution limit, direction-of-arrival, correlated sources, rectilinear sources, strictly non-circular, complex elliptically symmetric distribution.

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I. Introduction

The ability to resolve two closely-spaced narrowband far-field sources in terms of parameters of interest is an important performance measure of sensor arrays in localizing remote targets. The SRL is an important tool used to quantify estimator performance, which can be defined according to various criteria. There are in the literature four main different approaches to characterize SRL (see e.g., [1, Sec. VI]) The first one is based on the mean null spectrum relating to a specific high-resolution algorithm. (see e.g., [2]–[5]). The second approach rests on hypothesis test using the generalized likelihood ratio test [6], [7], on Rao's test [8] or on Bayesian approach [9]. The third approach is based on the information theory and, more precisely, either on Stein's lemma which relates the false alarm probability resulting from the Neyman-Pearson decision criterion to the relative entropy between two hypothesis [10], on Chernoff upper bound [11], or on the mutual information between DOA, scattering properties and the received signal [12]. The last approach capitalizes on the CRB where two criteria were proposed. One is the Lee criterion [13] for which the SRL is defined by the DOA separation that is equal to half of the maximum value of the two square roots of the CRB of the DOA's. This criterion was derived form the DCRB in [13] and then extended from the SCRB in [14]. The other criterion was proposed in the seminal paper by Smith [15] where the SRL is defined by the DOA separation which is equal to the square root of the CRB of the DOA separation. This criterion has the advantage over Lee's criterion of taking into account the coupling between the interest parameters, and another advantage is that it is closely related to the detection theory approach, as shown in [7], and furthermore, it can be generalized to multiple parameters [16]. Based on the SCRB, this criterion provides the best-case resolution bound for any unbiased algorithm.

The SRL based on the Smith criterion was used in numerous research papers (see e.g., [16]–[25]). We note, however, that all these works (except [17] dedicated to discrete sources and [21] focused on a unifying methodology without given closed-form expressions of the SRL) are based on the DCRB associated with the conditional signal source model. Furthermore, the role of the correlation of the signal sources has not been precisely determined in these contributions, except in [20] which assumes that the signal sources are known. This DCRB has the advantage of being easily derived and making it possible to obtain explicit simple closed-form expressions of the SRL and thus allows to reveal enlightening properties pertinent related to SRL behavior. However, it is well known that the DCRB is not always a tight lower bound on the variance of an unbiased estimator and cannot be attained by the maximum likelihood (ML) estimate. In particular, it was shown in [26] that the difference between the SCRB and the DCRB is very significant for closely spaced sources when the number of sensors is small. Therefore this DCRB would tend to give optimistic values of the SRL and thus makes necessary the derivation and the analysis of the SRL based on the SCRB. Moreover, all these studies

were carried out within the framework of Gaussian distributions. To take into account the effect of impulsive noise encountered in radar clutter [27] [28], made-man noise and interference in indoor and outdoor mobile communications channels [29] [30], the CES distribution has been preferred over the Gaussian distribution in many DOA finding and beamforming processing (see e.g., [31]–[33]) for modeling noise alone or observations.

This paper gives simplified closed-form expressions of the DCRB and SCRB on DOA estimation for two closely-spaced equi-powered narrowband far-field uncorrelated or correlated (including coherent) circular or rectilinear sources affected by complex circular white noise within the framework of CES distributions.

The main aim of this paper is twofold. First, it derives a new simplified expression of the SCRB on DOA estimation for two equi-powered correlated complex circular or rectilinear sources affected by complex circular white noise in different CES data models. Such an expression had hitherto been considered uninteresting because it was too complex to analyze. But thanks to our choice of the reference of the phase at the centroid of centro-symmetric arrays, we were able to obtain interpretable closed-form expressions. Rectilinear sources (also called maximally improper or strict-sense noncircular) are frequent in radiocommunications. For example, binary-phase-shift-keying and offset-quadrature-phase-shift-keying after post-rotation are rectilinear. These expressions of SCRB are analyzed and compared to the DCRB w.r.t. array geometric factors, heavy-tailed non-Gaussian noise and observations, and signal sources' correlation coefficients. In particular, they point out the strong effect of the correlation phase, until now only observed by numerical calculations in [34]. Secondly, our paper derives closed-form expressions of the SRL based on the DCRB and SCRB w.r.t. relevant parameters for these different models. This allows us to compare the SRLs between the various models and CRBs.

The paper is organized as follows. Section II describes the deterministic and stochastic data model where the sources are either arbitrary, circular or rectilinear within the framework of CES distributions. Section III gives a review of the DCRB and SCRB as well as the semiparametric DCRB and SCRB (denoted respectively as SDCRB and SSCRB). Section IV focuses on the case of two equi-powered sources, where some new interpretable exact closed-form expressions are derived in particular scenarios with particular attention given to the impact of the correlation phase and correlation magnitude on DCRB and SCRB. This section also derives approximate closed-form expressions of DCRB and SCRB for small DOA separations. Section V gives interpretable closed-form expressions of SRLs deduced from the DCRB and SCRB for known, arbitrary, circular and rectilinear sources. Then, comments to explain how different parameters impact the SRL and how the SRL derived from the SCRB are less optimistic than those that have so far been deduced only from the DCRB, are discussed. Numerical illustrations of the different SRLs are given in Section VI, with particular attention paid to the phase and magnitude of the correlation of the sources. Finally, the paper is concluded in Section VII.

The notations used throughout this paper are the following. Vectors and matrices are denoted by bold-faced lowercase and uppercase letters, respectively. * and ^{H} represent the conjugate and the conjugate transpose

operators, respectively. Re(.) and Im(.) denote the real and imaginary part, respectively, whereas $\odot$ represents the element by element matrix product. $\mathbf{J}$ is the unit antidiagonal matrix and $=_d$ means has the same distribution as.

II. Data model

Consider two narrowband far-field uncorrelated or correlated (including coherent) signals impinging on an arbitrary array of N sensors. The baseband received signal at the time instant t is modeled as

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{n}_t, \qquad t = 1, \ldots, T, \tag{1}$$

where $(\mathbf{y}_t)_{t=1}^T$ are independent. $\mathbf{A} = [\mathbf{a}_1,\ \mathbf{a}_2]$ is the steering matrix where each vector $\mathbf{a}_k = \mathbf{a}(\theta_k)$ is parameterized by the real scalar parameter θ_k with $\|\mathbf{a}_k\|^2 = N$. It is assumed for any $\theta_1 \neq \theta_2$, $\mathbf{A}$ has a full column rank. $\mathbf{x}_t \stackrel{\text{def}}{=} (x_{t,1},\ x_{t,2})^T$ and $\mathbf{n}_t$ are zero mean mutually uncorrelated and respectively model signals transmitted by sources and additive measurement noise which is assumed circular and spatially uncorrelated with $\mathrm{E}(\mathbf{n}_t\mathbf{n}_t^H) = \sigma_n^2\mathbf{I}$. Two different types of data models are currently used for the distribution of $(\mathbf{x}_t, \mathbf{n}_t)$ where $\mathbf{n}_t$ is circular Gaussian distributed [26]. We consider here extensions of these models within the framework of CES distributions.

A. Deterministic CES data model

In the conditional or deterministic model, the signal sources sequence $(\mathbf{x}_t)_{t=1,..,T}$ are conditioned from an independent zero-mean process (as it was explained in [26]), either of arbitrary circularity with $\mathbf{R}_{x,T} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t^H$ such that $\lim_{T\to\infty} \mathbf{R}_{x,T} = \mathbf{R}_{x,\infty}$, or rectlinear, i.e. satisfying the condition

$$x_{t,k} = r_{t,k}e^{i\phi_k}, k = 1, 2 \text{ where } r_{t,k} \text{ are real-valued with } \Delta\phi \stackrel{\text{def}}{=} \phi_1 - \phi_2 \in [0, \pi], \tag{2}$$

with $\mathbf{R}_{r,T} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^T \mathbf{r}_t\mathbf{r}_t^T$ where $\mathbf{r}_t \stackrel{\text{def}}{=} (r_{t,1}, r_{t,2})^T$ such that $\lim_{T\to\infty} \mathbf{R}_{r,T} = \mathbf{R}_{r,\infty}$. The phases ϕ_k associated with different propagation delays are assumed fixed, but unknown during the array observation. In this model $(\mathbf{x}_t)_{t=1,..,T}$ or $(\mathbf{r}_t)_{t=1,..,T}$ and (ϕ_1, ϕ_2) are unknown deterministic parameters. Noise $\mathbf{n}_t$ is assumed in this model circular CES (C-CES) distributed.

B. Stochastic Gaussian data model

In the unconditional or stochastic model, both $\mathbf{x}_t$ and $\mathbf{n}_t$ are usually assumed Gaussian distributed and independent of each other. $\mathbf{x}_t$ is here either circular or strictly non-circular (also called rectilinear). In the circular case, the distribution of $\mathbf{y}_t$ is characterized by the covariance given by

$$\mathbf{R}_y \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{y}_t\mathbf{y}_t^H) = \mathbf{A}\mathbf{R}_x\mathbf{A}^H + \sigma_n^2\mathbf{I}, \tag{3}$$

where

$$\mathbf{R}_x \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}_t\mathbf{x}_t^H) = \begin{pmatrix} \sigma_{x_1}^2 & \rho\sigma_{x_1}\sigma_{x_2} \\ \rho^*\sigma_{x_1}\sigma_{x_2} & \sigma_{x_2}^2 \end{pmatrix} \text{ with } \rho = |\rho|e^{i\phi} \in \mathbb{C} \text{ and } |\rho| \leq 1. \tag{4}$$

In the rectilinear case, the sources $x_{t,k}$, $k = 1, 2$, satisfy the constraint (2). In this case, the distribution of $\mathbf{y}_t$ is characterized by the covariance of the extended observation $\widetilde{\mathbf{y}}_t \stackrel{\text{def}}{=} [\mathbf{y}_t^T, \mathbf{y}_t^H]^T$ given by

$$\mathbf{R}_{\tilde{y}} \stackrel{\text{def}}{=} \mathrm{E}(\widetilde{\mathbf{y}}_t\widetilde{\mathbf{y}}_t^H) = \widetilde{\mathbf{A}}\mathbf{R}_r\widetilde{\mathbf{A}}^H + \sigma_n^2\mathbf{I}, \tag{5}$$

where $\widetilde{\mathbf{A}} \stackrel{\text{def}}{=} [\widetilde{\mathbf{a}}_1, \widetilde{\mathbf{a}}_2]$ with $\widetilde{\mathbf{a}}_k \stackrel{\text{def}}{=} [\mathbf{a}_k^T e^{i\phi_k}, \mathbf{a}_k^H e^{-i\phi_k}]^T$ and $\mathbf{R}_r \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{r}_t\mathbf{r}_t^T)$ given by

$$\mathbf{R}_r = \begin{pmatrix} \sigma_{x_1}^2 & \rho'\sigma_{x_1}\sigma_{x_2} \\ \rho'\sigma_{x_1}\sigma_{x_2} & \sigma_{x_2}^2 \end{pmatrix} \quad \text{with} \quad \rho' \in [-1, +1]. \tag{6}$$

Thus, $\mathbf{R}_x$ in (4) is written in the following form:

$$\mathbf{R}_x = \begin{pmatrix} \sigma_{x_1}^2 & \rho' e^{i\Delta\phi}\sigma_{x_1}\sigma_{x_2} \\ \rho'\sigma_{x_1}\sigma_{x_2}e^{-i\Delta\phi} & \sigma_{x_2}^2 \end{pmatrix}. \tag{7}$$

Consequently, for rectilinear sources, the phase separation $\Delta\phi$ associated with the sign of ρ' corresponds to the phase ϕ of the correlation of the sources if $\rho' \neq 0$.

C. Stochastic CES data model

To take into account the effect of heavy tailed impulse noise, the CES distribution has been preferred over the Gaussian distribution to model the observations $\mathbf{y}_t$ in many DOA finding and beamforming processing (see e.g., [31]–[33]). In this case the distributions of $\mathbf{x}_t$ and $\mathbf{n}_t$ are not specified, but only their second-order statistics are imposed. More specifically, in the case of circular to the second-order [resp., rectilinear] sources $\mathbf{x}_t$, the associated $(\mathbf{y}_t)_{t=1,\ldots,T}$ are assumed independent zero-mean C-CES [resp., noncircular CES (NC-CES)] identically distributed. The p.d.f. of $\mathbf{y}_t$ is

$$p(\mathbf{y}_t) = c_{N,g}|\mathbf{R}_y|^{-1}g(\mathbf{y}_t^H\mathbf{R}_y^{-1}\mathbf{y}_t), \text{ [resp., } c_{N,g}|\mathbf{R}_{\tilde{y}}|^{-1/2}g\left(\frac{1}{2}\tilde{\mathbf{y}}_t^H\mathbf{R}_{\tilde{y}}^{-1}\tilde{\mathbf{y}}_t\right)], \tag{8}$$

where $\mathbf{R}_y$ and $\mathbf{R}_{\tilde{y}}$ are the structured covariance in (3) and extended covariance in (5), respectively. The density generator $g(.)$: $\mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfies $\delta_{N,g} \stackrel{\text{def}}{=} \int_0^\infty t^{N-1}g(t)dt < \infty$ to ensure the integrability of $p(\mathbf{y}_t)$ and $c_{N,g}$ is a normalizing constant given by $c_{N,g} \stackrel{\text{def}}{=} 2(s_N\delta_{N,g})^{-1}$ where $s_N \stackrel{\text{def}}{=} 2\pi^N/\Gamma(N)$ is the surface area of the unit complex N-sphere. We note that the so-called scale ambiguity usually present in the p.d.f. of $\mathbf{y}_t$ with the scatter and pseudo-scatter matrices, is here removed thanks to the constraint on g: $\delta_{N+1,g}/\delta_{N,g} = N$ [33] which ensures that the scatter matrices are equal to the covariance matrices.

The r.v. $\mathbf{y}_t$ admits the following stochastic representation [35]:

$$\mathbf{y}_t =_d \sqrt{\mathcal{Q}_t}\mathbf{R}_y^{1/2}\mathbf{u}_t, \quad \text{circular source case} \tag{9}$$

$$\mathbf{y}_t =_d \sqrt{\mathcal{Q}_t}[\mathbf{I},\mathbf{0}]\mathbf{R}_{\tilde{y}}^{1/2}\tilde{\mathbf{u}}_t, \quad \text{rectilinear source case} \tag{10}$$

where $\tilde{\mathbf{u}}_t \stackrel{\text{def}}{=} (\mathbf{u}_t^T, \mathbf{u}_t^H)^T$. $\mathcal{Q}_t$ and $\mathbf{u}_t$ are independent, $\mathbf{u}_t$ is uniformly distributed on the unit complex N-sphere and $\mathcal{Q}_t$ has the p.d.f.

$$p(\mathcal{Q}_t) = \delta_{N,g}^{-1}\mathcal{Q}_t^{N-1}g(\mathcal{Q}_t), \tag{11}$$

with $\mathrm{E}(\mathcal{Q}_t) = N$. Note that this CES distribution model includes the standard Gaussian model, for which $g(x) = e^{-x}$, $c_{N,g} = \pi^{-N}$ and $\mathcal{Q}_t$ is $1/2\chi^2_{2N}$ distributed in the circular and rectilinear source cases.

III. Review of deterministic and stochastic Cramér-Rao bounds derivation

We consider here the general framework of K deterministic or circular stochastic [resp., rectilinear] sources, where the range of the steering matrix $\mathbf{A}$ [resp., $\widetilde{\mathbf{A}}$] characterizes the DOA[1]. To derive the DCRB and SCRB for DOA estimation, we have to carefully specify all the unknown parameters associated with the distribution of $\mathbf{y}_t$.

In the deterministic model where the density generator g(.) of the C-CES distributed noise is known, $\mathbf{y}_t$ in (1) is parameterized by the parameter $\boldsymbol{\alpha} = (\theta_1, .., \theta_K, \boldsymbol{\rho}^T, \sigma_n^2)^T$ where $\boldsymbol{\rho} \stackrel{\text{def}}{=} ((\mathrm{Re}^T(\mathbf{x}_t), \mathrm{Im}^T(\mathbf{x}_t))_{t=1,..,T})^T$ (with $K < N$) in arbitrary circularity case and $\boldsymbol{\rho} \stackrel{\text{def}}{=} (\phi_1, .., \phi_K, (r_{t,1}..., r_{t,K})_{t=1,..,T})^T$ (with $K < 2N$) in rectilinear case. For both stochastic Gaussian and stochastic CES data model where the density generator g(.) is known, $\mathbf{y}_t$ in (1) is parameterized by the parameter $\boldsymbol{\alpha} = (\theta_1, .., \theta_K, \boldsymbol{\rho}^T, \sigma_n^2)^T$ where $\boldsymbol{\rho} \stackrel{\text{def}}{=} ([\mathbf{R}_x]_{i,i}, \mathrm{Re}([\mathbf{R}_x]_{i,j}), \mathrm{Im}([\mathbf{R}_x]_{i,j}))^T$, $1 \leq i < j \leq K$ (with $K < N$) in circular case and $\boldsymbol{\rho} \stackrel{\text{def}}{=} (\phi_1, .., \phi_K, [\mathbf{R}_r]_{i,j})^T$, $1 \leq i \leq j \leq K$ (with $K < 2N$) in rectilinear case. Under these parametrizations, the components of the Fisher Information matrix (FIM) associated with DOA parameters and nuisance parameters can be computed. Simple general closed-form expressions of these components called Slepian-Bangs formula were given for the real Gaussian distribution in [37] and [38], then extended to the circular and noncircular complex Gaussian distribution in [39] and [40], respectively. This formula has been extended to the C-CES distribution in [41] and [42], and then in [43] to the NC-CES distribution. In contrast, if the density generator $g(.)$ of the C-CES distribution of the noise in the deterministic model or of the C-CES ot NC-CES distribution of the data $\mathbf{y}_t$ in the stochastic model, is unknown, an additional infinite dimensional nuisance parameter must

[1]This excludes the SCRB derived under the prior knowledge of uncorrelated sources applied to sparse linear arrays with $K > N$ [36].

be considered. To handle this parameterization, a semiparametric FIM associated with the finite dimensional parameter was derived in [44].

In the deterministic model with circular Gaussian noise, with no prior knowledge is introduced on the sources, the DCRB for DOA was first derived in [45] by picking the DOA-block of the inverse of the FIM. Then for rectilinear sources, a closed-form expression of this DCRB was given in [46] and [47] and then in [48] under more general rectilinear models. Noting that the parameter σ_n^2 is decoupled from the other parameters in the FIM of the C-CES distributions where the density generator $g(.)$ is known, and that the expectation terms of this FIM is proportional to those of the Gaussian noise case, the DCRB for DOA is directly derived. Furthermore, when the density generator $g(.)$ is unknown, σ_n^2 is also decoupled from the other parameters in the semiparametric FIM from [49, rel. (46)] and the expectation term of this semiparametric FIM is proportional to the expectation term of the FIM associated with circular Gaussian distributed data with the coefficient of proportionality $\xi_1 = \frac{\mathrm{E}[\phi^2(\mathcal{Q}_t)\mathcal{Q}_t]}{N}$ where $\phi(x) \stackrel{\text{def}}{=} -\frac{1}{g(x)}\frac{dg(x)}{dx}$. Comparing these DCRB and SDCRB, we get the following theorem:

Theorem 1: The DCRB and SDCRB for DOA estimation in the general scenario of K sources are given by the following expressions:

$$\mathrm{DCRB}(\boldsymbol{\theta}) = \mathrm{SDCRB}(\boldsymbol{\theta}) = \frac{1}{T}\frac{\sigma_n^2}{2\xi_1}\left\{\mathrm{Re}\left((\mathbf{D}_\theta^H \boldsymbol{\Pi}_{\mathbf{A}}^{\perp}\mathbf{D}_\theta)\odot\mathbf{R}_{x,T}^T\right)\right\}^{-1}, \tag{12}$$

$$\begin{aligned}\mathrm{DCRB}_{\mathrm{Rec}}(\boldsymbol{\theta}) &= \mathrm{SDCRB}_{\mathrm{Rec}}(\boldsymbol{\theta}) \\ &= \frac{1}{T}\frac{\sigma_n^2}{\xi_1}\left\{\left((\widetilde{\mathbf{D}}_\theta^H \boldsymbol{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\widetilde{\mathbf{D}}_\theta)\odot\mathbf{R}_{r,T}\right)-\left((\widetilde{\mathbf{D}}_\theta^H \boldsymbol{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\widetilde{\mathbf{D}}_\phi)\odot\mathbf{R}_{r,T}\right)\left((\widetilde{\mathbf{D}}_\phi^H \boldsymbol{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\widetilde{\mathbf{D}}_\phi)\odot\mathbf{R}_{r,T}\right)^{-1}\left((\widetilde{\mathbf{D}}_\phi^H \boldsymbol{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\widetilde{\mathbf{D}}_\theta)\odot\mathbf{R}_{r,T}\right)\right\}^{-1}\end{aligned} \tag{13}$$

for sources of arbitrary circularity and rectilinear sources, respectively, where $\mathbf{D}_\theta \stackrel{\text{def}}{=} [\mathbf{d}_1,..,\mathbf{d}_K]$, $\widetilde{\mathbf{D}}_\theta \stackrel{\text{def}}{=} [\tilde{\mathbf{d}}_1,..,\tilde{\mathbf{d}}_K]$, $\widetilde{\mathbf{D}}_\phi \stackrel{\text{def}}{=} [\tilde{\mathbf{d}}_{\phi_1},..,\tilde{\mathbf{d}}_{\phi_K}]$ with $\mathbf{d}_k \stackrel{\text{def}}{=} \frac{\partial \mathbf{a}_k}{\partial \theta_k}$, $\tilde{\mathbf{d}}_k \stackrel{\text{def}}{=} \frac{\partial \tilde{\mathbf{a}}_k}{\partial \theta_k}$, $\tilde{\mathbf{d}}_{\phi_k} \stackrel{\text{def}}{=} \frac{\partial \tilde{\mathbf{a}}_k}{\partial \phi_k}$, $\boldsymbol{\Pi}_{\mathbf{A}}^{\perp}$ [resp. $\boldsymbol{\Pi}_{\tilde{\mathbf{A}}}^{\perp}$] denote the orthogonal projector on the columns of $\mathbf{A}$ [resp. $\widetilde{\mathbf{A}}$]. Note that if the signal sources $x_{t,k}$ or $r_{t,k}$ are known, the DCRB are more simply derived from the associated FIM. They are also given by (12) and (13) where $\boldsymbol{\Pi}_{\mathbf{A}}^{\perp}$ and $\boldsymbol{\Pi}_{\tilde{\mathbf{A}}}^{\perp}$ is replaced by the identity matrix.

In the stochastic circular Gaussian model, the SCRB for DOA was first derived, indirectly, as the asymptotic covariance matrix of the maximum likelihood estimator [26], then directly by picking the DOA block of the inverse of the FIM [50]. Following this approach, the SCRB for DOA was derived in the stochastic rectilinear Gaussian model [51] and then extended to the stochastic C-CES data model where the density generator $g(.)$ is known in [43] to C-CES and NC-CES distributions. Similarly, using the semiparametric FIM [44], the SSCRB on the DOA only was given in [49, rel. (62)] for C-CES distributions and extended to NC-CES distributions in [35]. Comparing these SCRB given in [43] and SSCRB, we get the following theorem:

Theorem 2: The SCRB and SSCRB for DOA estimation in the general scenario of correlated or uncorrelated (including coherent) K sources are given for both conventional Gaussian and robust distribution models by the following common expressions:

$$\mathrm{SCRB}_{\mathrm{Cir}}(\boldsymbol{\theta}) = \mathrm{SSCRB}_{\mathrm{Cir}}(\boldsymbol{\theta}) = \frac{1}{T}\frac{\sigma_n^2}{2\xi_2}\left\{\mathrm{Re}\left((\mathbf{D}_\theta^H \boldsymbol{\Pi}_{\mathbf{A}}^{\perp}\mathbf{D}_\theta)\odot\mathbf{H}^T\right)\right\}^{-1}, \tag{14}$$

$$\begin{aligned}\mathrm{SCRB}_{\mathrm{Rec}}(\boldsymbol{\theta}) &= \mathrm{SSCRB}_{\mathrm{Rec}}(\boldsymbol{\theta}) \\ &= \frac{1}{T}\frac{\sigma_n^2}{\xi_2}\left\{\left((\widetilde{\mathbf{D}}_\theta^H \boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^{\perp}\widetilde{\mathbf{D}}_\theta)\odot\widetilde{\mathbf{H}}\right) - \left((\widetilde{\mathbf{D}}_\theta^H \boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^{\perp}\widetilde{\mathbf{D}}_\phi)\odot\widetilde{\mathbf{H}}\right)\left((\widetilde{\mathbf{D}}_\phi^H \boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^{\perp}\widetilde{\mathbf{D}}_\phi)\odot\widetilde{\mathbf{H}}\right)^{-1}\left((\widetilde{\mathbf{D}}_\phi^H \boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^{\perp}\widetilde{\mathbf{D}}_\theta)\odot\widetilde{\mathbf{H}}\right)\right\}^{-1}\end{aligned} \tag{15}$$

for circular and rectilinear sources, respectively, where $\mathbf{D}_\theta$, $\widetilde{\mathbf{D}}_\theta$, $\widetilde{\mathbf{D}}_\phi$, $\boldsymbol{\Pi}_{\mathbf{A}}^{\perp}$ and $\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^{\perp}$ are defined in Theorem 1, $\mathbf{H} \stackrel{\mathrm{def}}{=} \mathbf{R}_x\mathbf{A}^H\mathbf{R}_y^{-1}\mathbf{A}\mathbf{R}_x$, $\widetilde{\mathbf{H}} \stackrel{\mathrm{def}}{=} \mathbf{R}_r\widetilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1}\widetilde{\mathbf{A}}\mathbf{R}_r$, and where $\xi_2 \stackrel{\mathrm{def}}{=} \frac{\mathrm{E}[\phi^2(\mathcal{Q}_t)\mathcal{Q}_t^2]}{N(N+1)}$.

We note that in the standard Gaussian model, $\phi(x) = 1$ gives $\xi_1 = \frac{\mathrm{E}(\chi_{2N}^2)}{N} = 1$ and $\xi_2 = \frac{\mathrm{E}[(\chi_{2N}^2)^2]}{4N(N+1)} = 1$ and thus the DCRB and SDCRB given for arbitrary C-CES distributions of the noise, and the SCRB and SSCRB given for arbitrary C-CES or NC-CES distributions of the observation are scaled expressions of the associated CRB given in the conventional Gaussian model. Furthermore, we notice that $\xi_1 \geq 1$, while $\xi_2 \leq 1$ (sub-Gaussian case) or $\xi_2 \geq 1$ (super-Gaussian case) [43].

Note also that from this theorem, the DCRB and SCRB derived under the full knowledge of g, and the SDCRB and SSCRB that assume g as infinite-dimensional nuisance parameter are equal. But we cannot conclude that knowing or not knowing g can lead to the same DOA performance. When g is known, the ML estimate of the DOA is asymptotically (w.r.t. the number of snapshots T) efficient, and therefore, its covariance reaches the CRB for the stochastic model. But when g is unknown, some estimates have been proposed in [32] by exploiting the MUSIC algorithm together with the Tyler's or Hubert's M estimate of the covariance with better performance than for the conventional MUSIC algorithm. But none of these estimators is efficient w.r.t. the SSCRB. Finding such an estimator appears to be an open problem to the best of our knowledge.

IV. DCRB AND SCRB FOR TWO EQUI-POWERED SOURCES

This subsection derives exact closed-form expressions of the DCRB and SCRB given for two equi-powered sources deduced from (12), (13), (14) and (15) in particular scenarios and analyzes the significant role played by the magnitude and the phase of the correlation. It gives also approximate closed-form expressions of this SCRB for small DOA separation. Unfortunately, without special conditions on arrays, the expressions of these different CRB's are too complicated (see (71)-(80) in Appendix A) to provide useful insights into the behavior of the CRB's dependence on the different parameters. To simplify these expressions, we impose the assumption that the steering vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are defined up to a multiplicative phase depending on the DOAs and the

origin of the coordinate system. Consequently, without loss of generality, we can suppose that $\beta \stackrel{\text{def}}{=} \mathbf{a}_1^H \mathbf{a}_2$ is real-valued. But as this condition is not yet sufficient to obtain simple expressions, it is reinforced by supposing that the array is centro-symmetric, i.e., satisfying $\mathbf{a}_k^* = \mathbf{J}\mathbf{a}_k$[2]

This latter condition ensures not only that β is real-valued, but that it is also the case for the geometric terms $\alpha_{1,2}$, η_k' and η_2'' defined below, which allows us to obtain simplified expressions of SCRBs.

We obtain from (12) the following closed-form expressions:

$$\mathrm{DCRB}(\theta_k, \theta_k) = \frac{1}{2T\xi_1}\frac{1}{r}\frac{\alpha_{3-k}}{(\alpha_1\alpha_2 - \alpha_{1,2}^2|\rho|^2\cos^2\phi)}, \quad k = 1, 2 \tag{16}$$

$$\mathrm{DCRB}(\theta_1, \theta_2) = -\frac{1}{2T\xi_1}\frac{1}{r}\frac{\alpha_{1,2}|\rho|\cos\phi}{(\alpha_1\alpha_2 - \alpha_{1,2}^2|\rho|^2\cos^2\phi)}, \tag{17}$$

where $\alpha_k \stackrel{\text{def}}{=} \mathbf{d}_k^H \mathbf{\Pi}_{\mathbf{A}}^{\perp} \mathbf{d}_k$ and $\alpha_{1,2} \stackrel{\text{def}}{=} \mathbf{d}_1^H \mathbf{\Pi}_{\mathbf{A}}^{\perp} \mathbf{d}_2$, $r \stackrel{\text{def}}{=} \sigma_x^2/\sigma_n^2$ with $\sigma_x^2 \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^T |x_{t,1}|^2 = \frac{1}{T}\sum_{t=1}^T |x_{t,2}|^2$ and $\rho \stackrel{\text{def}}{=} \frac{1}{\sigma_x^2}\frac{1}{T}\sum_{t=1}^T x_{t,1}x_{t,2}^*$. We see from (16) that the DCRB of the DOA is an increasing function of the real part of the correlation. This DCRB reaches its minimum for uncorrelated sources or for $\phi = \pi/2 \bmod \pi$ for which

$$\mathrm{DCRB}(\theta_k, \theta_k) = \frac{1}{2T\xi_1}\frac{1}{r}\frac{1}{\alpha_k}, \quad k = 1, 2. \tag{18}$$

This expression is similar to that of a single source k where α_k is replaced by $\mathbf{d}_k^H \mathbf{\Pi}_{\mathbf{a}_k}^{\perp} \mathbf{d}_k = \|\mathbf{d}_k\|^2$ (for centro-symmetric arrays). It also follows that the minimum bound (18) remains greater than that for a single source. We see from (17) that the DOA are decoupled in the DCRB if and only if the real part of the correlation is zero. It is also clear that the DCRB is inversely proportional to SNR. We note that (16) and (17) are not given in [13] which focused on approximate DCRB expressions for closely-spaced sources impinging on a ULA array. We also point out that (16) and (17) are also valid in the context of known signal sources where $\mathbf{\Pi}_{\mathbf{A}}^{\perp}$ is replaced by the identity matrix in the definition of α_k and $\alpha_{1,2}$. These expressions have also been derived in [20] for the linear array in a more complicated form due to the choice of the first sensor as the origin of the phases.

A. Exact closed-form expressions of the DCRB and SCRB in particular scenarios

In contrast to (12), the application of (13), (14) and (15) for two equi-powered sources gives intricate expressions of $\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k)$ and $\mathrm{SCRB}_{\mathrm{Cir}}(\theta_1, \theta_2)$ [resp., $\mathrm{DCRB}_{\mathrm{Rec}}(\theta_k, \theta_k)$, $\mathrm{DCRB}_{\mathrm{Rec}}(\theta_1, \theta_2)$, $\mathrm{SCRB}_{\mathrm{Rec}}(\theta_k, \theta_k)$, $\mathrm{SCRB}_{\mathrm{Rec}}(\theta_1, \theta_2)$], $k = 1, 2$ in which geometric terms through (β, α_k and $\alpha_{1,2}$) and signal terms through (r and ρ), [resp., geometric and phase terms ($\tilde{\beta} = \tilde{\mathbf{a}}_1^H \tilde{\mathbf{a}}_2$, $\tilde{\alpha}_k = \tilde{\mathbf{d}}_k^H \mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp} \tilde{\mathbf{d}}_k$ and $\tilde{\alpha}_{1,2} = \tilde{\mathbf{d}}_1^H \mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp} \tilde{\mathbf{d}}_2$) and signal

[2]For example, uniform linear arrays, uniform circular arrays with an even number of sensors, regular hexagonal shaped arrays, cross-based centro-symmetric arrays, square-based centro-symmetric array, for which the array centroid is chosen as the reference of the phases are centro-symmetric arrays [52].

terms (r, ρ')] are mixed (see Appendix A). However, in the particular scenarios of orthogonal steering vectors, uncorrelated sources or coherent sources, closed-form interpretable expressions are available.

From (13), only the following scenarios make it possible to obtain such expressions:

- Case $\rho' = 0$ (uncorrelated sources)

  In this case, the two DOAs are decoupled in the DCRB with expressions given by

$$\mathrm{DCRB}_{\mathrm{Rec}}(\theta_k, \theta_k) = \frac{1}{2T\xi_1}\frac{1}{r}\frac{1}{\eta_k - \frac{N\eta_k'^2}{N^2-\beta^2}\cos^2(\Delta\phi)}, k = 1, 2, \tag{19}$$

$$\mathrm{DCRB}_{\mathrm{Rec}}(\theta_1, \theta_2) = 0, \tag{20}$$

  where $\eta_k = \mathbf{d}_k^H \mathbf{d}_k$ and $\eta_k' = \mathbf{d}_k^H \mathbf{a}_{3-k}$, $k = 1, 2$.

- Case $\rho' = \pm 1$ (coherent sources):

$$\mathrm{DCRB}_{\mathrm{Rec}}(\theta_k, \theta_k) = \frac{1}{2T\xi_1}\frac{1}{r}\frac{\beta^2 - N^2}{(\beta^2 - N^2)\eta_k + N\eta_k'^2 - \frac{(\beta\eta_1'\eta_2' - (\beta^2-N^2)\eta_2'')^2\cos^2\Delta\phi}{(\beta^2-N^2)\eta_{3-k} + N\eta_{3-k}'^2}}, k = 1, 2, \tag{21}$$

$$\mathrm{DCRB}_{\mathrm{Rec}}(\theta_1, \theta_2) = \frac{1}{2T\xi_1}\frac{1}{r}\frac{(\beta^2-N^2)(\beta\eta_1'\eta_2' - (\beta^2-N^2)\eta_2'')\cos\Delta\phi}{((\beta^2-N^2)\eta_1 + N\eta_1'^2)((\beta^2-N^2)\eta_2 + N\eta_2'^2) - (\beta\eta_1'\eta_2' - (\beta^2-N^2)\eta_2'')^2\cos^2\Delta\phi} \tag{22}$$

  where $\eta_2'' = \mathbf{d}_1^H \mathbf{d}_2$.

- Case $\beta = 0$ (orthogonal steering vectors):

$$\mathrm{DCRB}_{\mathrm{Rec}}(\theta_k, \theta_k) = \frac{1}{2T\xi_1}\frac{1}{r}\frac{1}{\eta_k - \frac{\rho'^2\eta_k'^2\sin^2\Delta\phi}{N} - \left(\frac{\eta_k'^2}{N} - \frac{N\rho'^2\eta_2''^2}{N\eta_{3-k} - \eta_{3-k}'^2(\rho'^2 + (1-\rho'^2)\cos^2\Delta\phi)}\right)\cos^2\Delta\phi} \tag{23}$$

$$\mathrm{DCRB}_{\mathrm{Rec}}(\theta_1, \theta_2) = -\frac{N\rho'\eta_2''\cos\Delta\phi}{N\eta_2 - \eta_2'^2(\rho'^2 + (1-\rho'^2)\cos^2\Delta\phi)}\mathrm{DCRB}_{\mathrm{Rec}}(\theta_k, \theta_k). \tag{24}$$

  Note that (19), (21) and (23) are functions of $\Delta\phi$, symmetric w.r.t. $\pi/2$ and decreasing from its maximum for $\Delta\phi = 0 \bmod \pi$ to its minimum for $\Delta\phi = \pi/2$.

From (71) and (72) derived from (14), interpretable closed-form expressions of $\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k)$ and $\mathrm{SCRB}_{\mathrm{Cir}}(\theta_1, \theta_2)$ are only possible in the following scenarios:

- Case $\rho = 0$:

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k) = \frac{1}{2T\xi_2}\frac{\alpha_{3-k}(N + r(N^2-\beta^2))((1+Nr)^2 - r^2\beta^2)}{r^2(\alpha_1\alpha_2(N + r(N^2-\beta^2))^2) - \alpha_{1,2}^2\beta^2}, k = 1, 2, \tag{25}$$

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_1, \theta_2) = -\frac{1}{2T\xi_2}\frac{\alpha_{1,2}\beta((1+Nr)^2 - r^2\beta^2)}{r^2(\alpha_1\alpha_2(N + r(N^2-\beta^2))^2 - \alpha_{1,2}^2\beta^2)}. \tag{26}$$

  We note that (25) is an increasing function of β^2, which is minimum [resp., maximum] for $\beta = 0$ (orthogonal steering vectors) [resp., $\beta = N$ (collinear steering vectors)].

- Case $|\rho| = 1$:

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k) = \frac{1}{4T\xi_2} \frac{\alpha_{3-k}(1 + 2r(N + \beta\cos(\phi)))}{r^2(N + \beta\cos(\phi))(\alpha_1\alpha_2 - \alpha_{1,2}^2\cos(\phi)^2)}, k = 1, 2, \tag{27}$$

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_1, \theta_2) = -\frac{1}{4T\xi_2} \frac{\alpha_{1,2}\cos(\phi)(1 + 2r(N + \beta\cos(\phi)))}{r^2(N + \beta\cos(\phi))(\alpha_1\alpha_2 - \alpha_{1,2}^2\cos(\phi)^2))}. \tag{28}$$

  We note that for not too far DOA (such that $\beta > 0$), $\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k)$ is maximum in ϕ for $\phi = \pi$. But its minimum is approached for $\phi = \pi/2$ mod π only for high SNR r.

- Case $\beta = 0$:

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k) = \frac{1}{2T\xi_2} \frac{\alpha_{3-k}(1 + |\rho|^2 + Nr(1 - |\rho|^2))(1 + Nr(2 + Nr(1 - |\rho|^2)))}{Nr^2(\alpha_1\alpha_2(1+|\rho|^2+Nr(1-|\rho|^2))^2 - \alpha_{1,2}^2|\rho|^2(-2+Nr(|\rho|^2-1))^2\cos^2\phi)}, k=1,2, \tag{29}$$

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_1, \theta_2) = -\frac{1}{2T\xi_2} \frac{\alpha_{1,2}|\rho|\cos(\phi)(2 + Nr(|\rho|^2 - 1))(1 + Nr(2 + Nr(|\rho|^2 - 1)))}{Nr^2(\alpha_1\alpha_2(1+|\rho|^2+Nr(1-|\rho|^2))^2 - \alpha_{1,2}^2|\rho|^2(-2+Nr(|\rho|^2-1))^2\cos^2\phi)}. \tag{30}$$

  It is clearly that $\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k)$ are functions of ϕ, symmetric w.r.t. $\pi/2$ and decreasing from its maximum for $\phi = 0$ mod π to its minimum for $\phi = \pi/2$, but it is not easy to assess the dependence in $|\rho|$, although we observe that $\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k)$ are numerically increasing with $|\rho|$ for the ULA and UCA.

It is easy to deduce from these particular expressions the following interpretable closed-form expressions in more specific cases:

- Case $\rho = 0$ and $\beta = 0$:

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k) = \frac{1}{2T\xi_2} \frac{1}{\alpha_k} \frac{1}{r} \left(1 + \frac{1}{Nr}\right), k = 1, 2, \tag{31}$$

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_1, \theta_2) = 0. \tag{32}$$

  So the DOAs are decoupled in the SCRB and this SCRB is similar to those a single source k, which has also the expression (31) where α_k is replaced by $\mathbf{d}_k^H \mathbf{\Pi}_{\mathbf{a}_k}^{\perp} \mathbf{d}_k$. So the SCRB for two sources is larger than for a single source and these SCRB are equal i.i.f. $\mathbf{d}_k^H \mathbf{a}_{3-k} = 0$.

- Case $|\rho| = 1$ and $\beta = 0$:

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k) = \frac{1}{2T\xi_2} \frac{\alpha_{3-k}}{(\alpha_1\alpha_2 - \alpha_{1,2}^2\cos^2\phi)} \frac{1}{r} \left(1 + \frac{1}{2Nr}\right), k = 1, 2, \tag{33}$$

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_1, \theta_2) = -\frac{1}{2T\xi_2} \frac{\alpha_{1,2}\cos^2\phi}{(\alpha_1\alpha_2 - \alpha_{1,2}^2\cos^2\phi)} \frac{1}{r} \left(1 + \frac{1}{2Nr}\right). \tag{34}$$

  So the DOAs are coupled in the SCRB and this SCRB is generally larger than for correlated sources, except for very low SNR r.

Unlike the circular case, interpretable closed-form expressions of $\mathrm{SCRB}_{\mathrm{Rec}}(\theta_k, \theta_k)$ and $\mathrm{SCRB}_{\mathrm{Rec}}(\theta_1, \theta_2)$ can be only found in more specific scenarios for which (15) gives:

- Case $\rho' = 0$ and $\beta = 0$:

$$\mathrm{SCRB}_{\mathrm{Rec}}(\theta_k, \theta_k) = \frac{1}{2T\xi_2} \frac{1}{\eta_k - \frac{\eta_k'^2}{N}\cos^2\Delta\phi} \frac{1}{r}\left(1 + \frac{1}{2Nr}\right), k = 1, 2, \quad (35)$$

$$\mathrm{SCRB}_{\mathrm{Rec}}(\theta_1, \theta_2) = 0. \quad (36)$$

The DOAs are therefore decoupled in the SCRB and the SCRB of θ_k is similar to those of a single source k derived in [51, (19)], which also has the expression (35) by replacing $\eta_k - \frac{\eta_k'^2}{N}\cos^2(\Delta\phi)$ by η_k (because $\mathbf{d}_k^H \mathbf{\Pi}_{\mathbf{a}_k}^{\perp} \mathbf{d}_k = \eta_k$ using $\mathbf{d}_k^H \mathbf{a}_k = 0$ for centro-symmetric arrays). Thus, the SCRB for two sources is larger than for a single source and these SCRB are equal iif $\eta_k' = 0$ or $\Delta\phi = \frac{\pi}{2}$.

- Case $\rho' = \pm 1$ and $\beta = 0$:

$$\mathrm{SCRB}_{\mathrm{Rec}}(\theta_k, \theta_k) = \frac{1}{2T\xi_2} \frac{1}{\eta_k - \frac{\eta_k'^2}{N} + \frac{N\eta_2''^2\cos^2\Delta\phi}{\eta_{3-k}'^2 - N\eta_{3-k}}} \frac{1}{r}\left(1 + \frac{1}{4Nr}\right) k = 1, 2, \quad (37)$$

$$\mathrm{SCRB}_{\mathrm{Rec}}(\theta_1, \theta_2) = -\frac{1}{2T\xi_2} \frac{N^2\eta_2''\cos(\Delta\phi)}{(\eta_1'^2 - N\eta_1)(\eta_2'^2 - N\eta_2) - N^2\eta_2''^2\cos^2\Delta\phi} \frac{1}{r}\left(1 + \frac{1}{4Nr}\right). \quad (38)$$

The DOAs are therefore coupled in the SCRB and this SCRB is greater than for uncorrelated sources (see (35).

It is clear that $\mathrm{SCRB}_{\mathrm{Rec}}(\theta_k, \theta_k)$ in (35) and (37) are functions of $\Delta\phi$, symmetric w.r.t. $\pi/2$ and decreasing from its maximum for $\Delta\phi = 0$ mod π to its minimum for $\Delta\phi = \pi/2$.

Finally, note that some properties of SCRB and DCRB proved for particular values of the parameters β, $|\rho|$, ρ', $\cos\phi$ and $\cos\Delta\phi$ are also confirmed for arbitrary values of these parameters. For example, extensive numerical experiments with ULA and UCA confirm that for rectilinear signal sources, the different DCRB are functions of $\Delta\phi$, symmetric w.r.t. $\pi/2$ and decreasing from its maximum for $\Delta\phi = 0$ mod π to its minimum for $\Delta\phi = \pi/2$.

B. Approximate closed-form expressions of the DCRB and SCRB for small DOA separation

We examine here the DCRB (12) and (13) when the signal sources are known or unknown and the SCRB (14) and (15) when the DOA separation $\delta\theta \stackrel{\mathrm{def}}{=} \theta_1 - \theta_2$ is small, by expressing the DOA-separation-dependent-coefficients in terms of Taylor series about $\delta\theta = 0$ and identifying the dominant term of the different CRB as $\delta\theta \to 0$. For simplicity[3], we restrict our analysis to the ULA with the following steering vectors

$$\mathbf{a}_k = (e^{-i(N-1)\theta_k/2}, e^{-i(N-3)\theta_k/2}, \ldots, e^{i(N-3)\theta_k/2}, e^{i(N-1)\theta_k/2}), \quad (39)$$

[3]For example, the analysis for the UCA is much more complicated because the different CRB are not functions only of the DOA separation, but also on the mid DOA. In other words, the UCA which is isotropic for a single source is no longer isotropic for two sources.

where $\theta_k = \pi \sin \alpha_k$, with α_k are the DOAs relative to the normal of array broadside and where the coordinate system has its origin at the centroid of the array. With the aid of symbolic algebra and calculus tools, the following asymptotic expressions of the DCRB are respectively obtained[4]:

$$\mathrm{DCRB}^{\mathrm{Kn}}(\theta_k, \theta_k) = \frac{1}{T\xi_1}\frac{1}{r}\frac{6}{N(N^2-1)(1-|\rho|^2\cos^2\phi)} + O((\delta\theta)^2), \tag{40}$$

$$\mathrm{DCRB}^{\mathrm{Kn}}(\theta_1, \theta_2) = -\frac{1}{T\xi_1}\frac{1}{r}\frac{6|\rho|\cos(\phi)}{N(N^2-1)(1-|\rho|^2\cos^2\phi)} + O((\delta\theta)^2), \tag{41}$$

$$\mathrm{DCRB}^{\mathrm{Kn}}_{\mathrm{Rec}}(\theta_k, \theta_k) = \frac{1}{T\xi_1}\frac{1}{r}\frac{6}{N(N^2-1)(1-\rho'^2\cos^2\Delta\phi)} + O((\delta\theta)^2), \tag{42}$$

$$\mathrm{DCRB}^{\mathrm{Kn}}_{\mathrm{Rec}}(\theta_1, \theta_2) = -\frac{1}{T\xi_1}\frac{1}{r}\frac{6\rho'\cos(\Delta\phi)}{N(N^2-1)(1-\rho'^2\cos^2\Delta\phi)} + O((\delta\theta)^2), \tag{43}$$

when the signal sources are known,

$$\mathrm{DCRB}^{\mathrm{Unk}}(\theta_k, \theta_k) = \frac{1}{T\xi_1}\frac{1}{r}\frac{360}{N(N^2-4)(N^2-1)(1-|\rho|^2\cos^2(\phi))(\delta\theta)^2} + O(1), \tag{44}$$

$$\mathrm{DCRB}^{\mathrm{Unk}}(\theta_1, \theta_2) = \frac{1}{T\xi_1}\frac{1}{r}\frac{360|\rho|\cos(\phi)}{N(N^2-4)(N^2-1)(1-|\rho|^2\cos^2(\phi))(\delta\theta)^2} + O(1), \tag{45}$$

$$\mathrm{DCRB}^{\mathrm{Unk}}_{\mathrm{Rec}}(\theta_k, \theta_k) = \frac{1}{T\xi_1}\frac{1}{r}\frac{6}{N(N^2-1)(1-\rho'^2)\sin^2\Delta\phi} + O((\delta\theta)^2), \text{ for } \Delta\phi \neq 0 \tag{46}$$

$$\mathrm{DCRB}^{\mathrm{Unk}}_{\mathrm{Rec}}(\theta_1, \theta_2) = \frac{1}{T\xi_1}\frac{1}{r}\frac{6\rho'\cos(\Delta\phi)}{N(N^2-1)(1-\rho'^2)\sin^2\Delta\phi} + O((\delta\theta)^2), \text{ for } \Delta\phi \neq 0 \tag{47}$$

$$\mathrm{DCRB}^{\mathrm{Unk}}_{\mathrm{Rec}}(\theta_k, \theta_k) = \frac{1}{T\xi_1}\frac{1}{r}\frac{360}{N(N^2-1)(N^2-4)(1-\rho'^2)(\delta\theta)^2} + O(1), \text{ for } \Delta\phi = 0 \tag{48}$$

$$\mathrm{DCRB}^{\mathrm{Unk}}_{\mathrm{Rec}}(\theta_1, \theta_2) = \frac{1}{T\xi_1}\frac{1}{r}\frac{360\rho'}{N(N^2-4)(N^2-1)(1-\rho'^2)(\delta\theta)^2} + O(1), \text{ for } \Delta\phi = 0 \tag{49}$$

when the source signals are unknown.

Using the same tools, the following asymptotic SCRB for circular sources are obtained:

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k) = \frac{1}{T\xi_2}\frac{90(1+2Nr(1+|\rho|\cos\phi))(1+|\rho|^2+2|\rho|\cos(\phi))}{r^2|\rho|^2N^2(N^2-1)(N^2-4)(1+|\rho|\cos(\phi))^2\sin^2\phi}\frac{1}{(\delta\theta)^2} + O(1), \tag{50}$$

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_1, \theta_2) = \frac{1}{T\xi_2}\frac{90(1+2Nr(1+|\rho|\cos\phi))(1-|\rho|^2+2|\rho|\cos\phi(1+|\rho|\cos\phi))}{r^2|\rho|^2N^2(N^2-1)(N^2-4)(1+|\rho|\cos\phi)^2\sin^2(\phi)}\frac{1}{(\delta\theta)^2} + O(1), \tag{51}$$

for $\phi \neq 0 \bmod \pi$ and

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k) = \frac{b_1}{(\delta\theta)^4} + \frac{b_2}{(\delta\theta)^2} + O(1) \tag{52}$$

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_1, \theta_2) = \frac{b_1}{(\delta\theta)^4} + \frac{b_3}{(\delta\theta)^2} + O(1) \tag{53}$$

[4]The exponents Kn and Unk denote respectively known and unknown

for $\phi = 0 \bmod \pi$. As for the SCRB for rectilinear sources, we get:

$$\mathrm{SCRB}_{\mathrm{Rec}}(\theta_k, \theta_k) = \frac{1}{T\xi_2} \frac{3((\rho' + \cos(\Delta\phi))^2 + 2Nr(\rho'^2 - 1)\sin^2(\Delta\phi))}{r^2 N^2 (N^2 - 1)(\rho'^2 - 1)^2 \sin^4(\Delta\phi)} + O((\delta\theta)^2), \quad (54)$$

$$\mathrm{SCRB}_{\mathrm{Rec}}(\theta_1, \theta_2) = \frac{1}{T\xi_2} \frac{3\cos\Delta\phi((1+\rho'^2)\cos\Delta\phi + 2\rho'(1 - Nr(\rho'^2 - 1)\sin^2\Delta\phi))}{r^2 N^2 (N^2 - 1)(\rho'^2 - 1)^2 \sin^4 \Delta\phi} + O((\delta\theta)^2), \quad (55)$$

for $\Delta\phi \neq 0$ and

$$\mathrm{SCRB}_{\mathrm{Rec}}(\theta_k, \theta_k) = \frac{a_1}{(\delta\theta)^4} + \frac{a_2}{(\delta\theta)^2} + O(1) \quad (56)$$

$$\mathrm{SCRB}_{\mathrm{Rec}}(\theta_1, \theta_2) = \frac{a_1}{(\delta\theta)^4} + \frac{a_3}{(\delta\theta)^2} + O(1) \quad (57)$$

for $\Delta\phi = 0$.

Although the dominant terms in (52)-(57) are $b_1/(\delta\theta)^4$ and $a_1/(\delta\theta)^4$, the terms $b_k/(\delta\theta)^2$ and $a_k/(\delta\theta)^2$, $k = 2, 3$, are needed to derive the SRL in Section V because the dominant terms are eliminated in (61). In fact, only the following differences are useful.

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k, \theta_k) - \mathrm{SCRB}_{\mathrm{Cir}}(\theta_1, \theta_2) = \frac{1}{T\xi_2} \frac{180(1 + 2Nr(1 + |\rho|))}{r^2 N^2((N^2 - 1)(N^2 - 4))(1 + |\rho|)^2 (\delta\theta)^2} + O(1) \quad (58)$$

$$\mathrm{SCRB}_{\mathrm{Rec}}(\theta_k, \theta_k) - \mathrm{SCRB}_{\mathrm{Rec}}(\theta_1, \theta_2) = \frac{1}{T\xi_2} \frac{90(1 + 4Nr(1 + \rho'))}{r^2 N^2((N^2 - 1)(N^2 - 4))(1 + \rho')^2 (\delta\theta)^2} + O(1). \quad (59)$$

We see that the DCRB for known arbitrary (40), (41) or rectilinear (42), (43) sources tend to non-zero finite values when the DOA separation tends to zero. This is also the case for the DCRB for unknown rectilinear sources (46), (47) and the SCRB for rectilinear sources (54), (55) when $\Delta\phi \neq 0$. This property of the SCRB for rectilinear sources has been observed numerically in [40] and was later confirmed by the behavior of the non-circular MUSIC algorithm in [53].

V. CRB-Based Statistical resolution limits

A. Derivation of different SRL

In [15], the SRL proposed by Smith was defined as the source separation that equals its own CRB, providing an algorithm-independent resolution bound and was illustrated by the DCRB in the context of two damped exponentials of identical unknown amplitudes. In the context of DOA estimation, the SRL is the DOA separation $\delta\theta = \theta_1 - \theta_2$ solution of the implicit equation:

$$\delta\theta = \sqrt{\mathrm{CRB}(\delta\theta)}, \quad (60)$$

where CRB$(\delta\theta)$ denotes the CRB on the DOA separation $\theta_1-\theta_2$. Because $\mathrm{Var}(\widehat{\theta}_1-\widehat{\theta}_2)=\mathrm{Var}(\widehat{\theta}_1)+\mathrm{Var}(\widehat{\theta}_2)-2\mathrm{Cov}(\widehat{\theta}_1,\widehat{\theta}_2)$, CRB$(\delta\theta)$ can be deduced form the matrix CRB$(\boldsymbol{\theta})$ by:

$$\mathrm{CRB}(\delta\theta)=\mathrm{CRB}(\theta_1,\theta_1)+\mathrm{CRB}(\theta_2,\theta_2)-2\mathrm{CRB}(\theta_1,\theta_2), \tag{61}$$

for which $\mathrm{CRB}(\theta_1,\theta_1)=\mathrm{CRB}(\theta_2,\theta_2)$ for two equal-powered source signals impinging on a ULA, and where the CRB is either the DCRB or the SCRB. Although definition (60) essentially makes sense because the CRB indicates the DOA estimation accuracy, it is not supported by rigorous statistical arguments. Some authors have introduced a scalar factor λ between $\delta\theta$ and $\sqrt{\mathrm{CRB}(\delta\theta)}$ (e.g., $\lambda=0.25$ in [54], $\lambda=4$ in [55]). But using a generalized likelihood ratio test approach, [7] gave a statistical basis to (60) by defining the SRL instead by the solution of the following implicit equation:

$$\delta\theta=\lambda\sqrt{\mathrm{CRB}(\delta\theta)}, \tag{62}$$

where λ is analytically determined by the preassigned constraints on the probability of false alarm and detection. As λ is of the order of unity for the usual values of these probabilities, we will retain here definition (60).

We derive here explicit closed-form expressions for SRLs by solving (60), whose solutions are only possible in the form of approximate solutions for two closely-spaced sources resulting from the different asymptotic expressions (40)-(59). Using $\Delta\theta\stackrel{\mathrm{def}}{=}N\delta\theta/2\sqrt{3}$[5], these approximate solutions are given from the derived DCRB for known (40) (41) and unknown (44) (45) arbitrary sources, respectively, by

$$(\Delta\theta)_{\mathrm{D}}^{\mathrm{Kn}} \approx \left(\frac{1}{T\xi_1}\frac{N}{r(N^2-1)(1-|\rho|\cos\phi)}\right)^{1/2}, \tag{63}$$

$$(\Delta\theta)_{\mathrm{D}}^{\mathrm{Unk}} \approx \left(\frac{1}{T\xi_1}\frac{5N^3}{r(N^2-1)(N^2-4)(1+|\rho|\cos\phi)}\right)^{1/4}, \tag{64}$$

and for known (42) (43) and unknown (46)-(49) rectilinear sources, respectively, by:

$$(\Delta\theta)_{\mathrm{D,Rec}}^{\mathrm{Kn}} \approx \left(\frac{1}{T\xi_1}\frac{N}{r(N^2-1)(1-\rho'\cos\Delta\phi)}\right)^{1/2}, \tag{65}$$

$$(\Delta\theta)_{\mathrm{D,Rec}}^{\mathrm{Unk}} \approx \left(\frac{1}{T\xi_1}\frac{N(1-\rho'\cos\Delta\phi)}{r(N^2-1)(1-\rho'^2)\sin^2\Delta\phi}\right)^{1/2}, \text{ for } \Delta\phi\neq 0 \tag{66}$$

$$(\Delta\theta)_{\mathrm{D,Rec}}^{\mathrm{Unk}} \approx \left(\frac{1}{T\xi_1}\frac{5N^3}{r(N^2-1)(N^2-4)(1+\rho')}\right)^{1/4}, \text{ for } \Delta\phi=0. \tag{67}$$

As a comparison, $(\Delta\theta)_{\mathrm{D}}^{\mathrm{Kn}}$ given in [20, rel. (26)] has complicated and uninterpretable expressions while (63) is an interpretable closed-form expression obtained thanks to the choice of the origin of the phases in the middle

[5]This normalization has been introduced in [4] and then taken up by [56], so we also use it in order to simplify comparisons with the literature.

of the ULA.

The approximate SRL solution of (60) are given from the derived SCRB for circular sources (50)-(53) by

$$(\Delta\theta)_{\mathrm{S,Cir}} \approx \left(\frac{1}{2T\xi_2}\frac{5N^2(1+2Nr(1+|\rho|\cos\phi))}{r^2(N^2-1)(N^2-4)(1+|\rho|\cos\phi)^2}\right)^{1/4}, \tag{68}$$

for all ϕ, and for rectilinear sources (54)-(57), by:

$$(\Delta\theta)_{\mathrm{S,Rec}} \approx \left(\frac{1}{2T\xi_2}\frac{(1+\rho'^2+2Nr(1-\rho'^2)(1-\rho'\cos\Delta\phi))}{r^2(N^2-1)(1-\rho'^2)^2\sin^2\Delta\phi}\right)^{1/2}, \text{ for } \Delta\phi \neq 0, \tag{69}$$

$$(\Delta\theta)_{\mathrm{S,Rec}} \approx \left(\frac{1}{4T\xi_2}\frac{5N^2(1+4Nr(\rho'+1))}{r^2(N^2-1)(N^2-4)(\rho'+1)^2}\right)^{1/4}, \text{ for } \Delta\phi = 0, \tag{70}$$

B. General comments

This section sheds light on the influence of various parameters involved in the SRLs expressions (63)-(70) such as the number T of snapshots, the number N of sensors and signal and noise parameters (r, ξ_1, ξ_2, ρ, ϕ, ρ', $\Delta\phi$). It also compares SRLs deduced from SCRBs to those deduced from DCRBs.

1) Impact of parameters r, N and T on SRLs: The SNR r impacts the DCRB-derived SRL in a manner similar as the number of snapshots T. As for the SCRB-derived SRL, its impact is more complex to analyze. But it is similar for a large SNR and on the other hand, for a weak SNR and N, r^2 impacts the SRL in a manner similar to T.

Moreover for large values of N, the number of sensors N impacts all the SRLs in a similar way to T because for the DCRB-derived SLRs $\frac{N}{N^2-1} \approx \frac{1}{N}$ and $\frac{N^3}{(N^2-1)(N^2-4)} \approx \frac{1}{N}$ in (63)-(66) and (67), respectively, and for the SCRB-derived SLRs $\frac{N^2(1+2Nr(1+|\rho|\cos\phi))}{2r(N^2-1)(N^2-4)} \approx \frac{1+|\rho|\cos\phi}{N}$, $\frac{(1+\rho'^2+2Nr(1-\rho'^2)(1-\rho'\cos\Delta\phi))}{r(N^2-1)} \approx \frac{2(1-\rho'^2)(1-\rho'\cos\Delta\phi)}{N}$ and $\frac{N^2(1+4Nr(\rho'+1))}{r(N^2-1)(N^2-4)} \approx \frac{4(\rho'+1)}{N}$ in (68), (69), and (70), respectively.

Now, the dependence of the SRL on T and therefore on r and N depends largely on the data assumptions of the model, for which we have two types of dependencies. For the deterministic model with arbitrary unknown sources and the stochastic model with circular sources, the SRLs are proportional to the inverse of the fourth root of T. This behavior is similar for deterministic unknown and stochastic rectilinear sources with $\Delta\phi = 0$. In contrast, for deterministic unknown rectilinear and stochastic rectilinear sources with $\Delta\phi \neq 0$, the SRLs are proportional to the inverse of the square root of T, which can give a much lower SRL. This behavior is similar when the sources are known (arbitrary or rectilinear).

2) Impact of parameters ξ_1 and ξ_2 on SRLs: The non-Gaussianity of noise in the deterministic data model and of observations in the stochastic data model impacts the SRLs through the coefficients ξ_1 and ξ_2, respectively. These coefficients influence the SRL in an equivalent way to the numbers of samples $T\xi_1$ and $T\xi_2$, respectively.

While ξ_1 is always greater than or equal to one, ξ_2 can be greater or less than one [43]. This proves that the Gaussian distributed observations (i.e., $\xi_2 = 1$) does not lead to the largest SRL based on the SCRB. As examples, we prove in the Appendix that the expressions of ξ_1 and ξ_2 for the normalized complex Student's $t-$distribution of degree of freedom $\nu > 2$, are given by $\xi_1 = \frac{\nu/2}{(\nu/2)-1}\frac{(\nu/2)+N}{(\nu/2)+N+1} > 1$ and $\xi_2 = \frac{(\nu/2)+N}{(\nu/2)+N+1} < 1$, and for the normalized generalized Gaussian distribution with exponent $s > 0$, are given by $\xi_1 = \frac{\Gamma(2+\frac{N-1}{s})\Gamma(\frac{N+1}{s})}{(\Gamma(1+\frac{N}{s}))^2} > 1$ and $\xi_2 = \frac{N+s}{N+1}$ where $\xi_2 < 1$ [resp., $\xi_2 > 1$] if $0 < s < 1$ [resp., $s > 1$].

Finally, note that the complex angular elliptical (CAE) distribution which is obtained by normalizing any centered C-CES distribution [33] is an interesting case because the ML estimate of its scatter matrix is the Tyler's M-estimator [57]. Based on the Fisher information analysis [42, eq. (20)], we deduce that the SCRB for the CAE distribution, which is independent of the C-CES generating function $g(.)$, is given by (14) with $\xi_2 = \frac{N}{N+1}$, despite this distribution is not a C-CES distribution. This result is consistent with the invariance and the efficiency of the ML estimator, which allows us to directly deduce $\xi_2 = \xi_{2,\text{Tyler}} = \frac{N}{N+1}$ from the asymptotic distribution of the Tyler's M-estimator. Note further that for the CAE distribution the SCRB for DOA is equal to the asymptotic minimum variance bound (AMVB) [58] based on the Tyler's M statistics. It follows from the C-CES distribution-free property of the asymptotic distribution of the Tyler's M estimator, that for arbitrary second-order C-CES distributed observations $\mathbf{y}_t$, the AMVB for DOA based on the Tyler's M statistics is equal to the SCRB given by (14) where $\xi_2 = \frac{N}{N+1}$. As a result, all the analysis of the SCRB-derived SRL apply to AMVB-derived SRL based on the Tyler's M statistics.

3) Impact of parameters ρ, ϕ, ρ' and $\Delta\phi$ on SRLs: The different SRLs are functions of the magnitude of the correlation of sources, but also of the phase and this dependence depends on the considered SRL. Thus, SRLs for known arbitrary sources (63) and known rectilinear sources (65) are respectively increasing functions of $|\rho|\cos\phi$ and $\rho'\cos\Delta\phi$, which is unbounded for ρ and $\rho' e^{i\Delta\phi}$ approaching one, respectively.

In contrast, the SRL (64) deduced from the DCRB for arbitrary unknown sources and the SRL (68) deduced from the SCRB for circular sources with large SNR are both decreasing functions of $|\rho|\cos\phi$. Note that, for $|\rho| \neq 1$, the DCRB-derived SRL (64) and the SCRB-derived SRL (68) are both minimal for $\phi = 0$, and are maximum for $\phi = \pi$. On the other hand, for $|\rho| = 1$, these SRLs are minimal for $\phi = 0$ and go to infinity for $\phi = \pi$.

It can also be seen that the DCRB-derived SRL (66) (67) and SCRB-derived SRL (69) (70) for rectilinear sources depend strongly on the correlation phase. A non-zero correlation phase greatly improves the SRL due to the proportionality of SRL in $(.)^{1/2}$ instead of $(.)^{1/4}$ for zero-phase. Note that the DCRB-derived SRL (66) and the SCRB-derived SRL (69) are respectively minimum for $\Delta\phi = \tan^{-1}(\frac{\sqrt{2(\rho'^2+\sqrt{1-\rho'^2}-1)}}{1-\sqrt{1-\rho'^2}})$ and $\Delta\phi = \tan^{-1}[(\frac{2Nr\rho'\left(1-\rho'^2\right)}{2Nr\left(1-\rho'^2\right)+\rho'^2+1-\sqrt{4N^2r^2\left(1-\rho'^2\right)^3+4Nr\left(1-\rho'^4\right)+\left(\rho'^2+1\right)^2}})^2 - 1]^{1/2}$. These values become equal for

a high SNR and are equal to $\pi/2$ for uncorrelated sources ($\rho' = 0$). These SRLs are both maximum for $\Delta\phi = 0$.

4) Comparisons between SRL deduced from DCRB and SCRB: Comparing the SRL derived from the DCRB when the sources are unknown arbitrary (64), unknown rectilinear with $\Delta\phi \neq 0$ (66) and $\Delta\phi = 0$ (67) to that deduced from the SCRB when the sources are respectively circular (68), rectilinear with $\Delta\phi \neq 0$ (69) and $\Delta\phi = 0$ (70) in Gaussian data models (i.e., $\xi_1 = \xi_2 = 1$), we see that these SRLs each tends to the same limit when r increases. This property is consistent with the general result [26, R9] proved in the Gaussian framework that states that the DCRB and SCRB tend to the same limit as all SNRs increase.

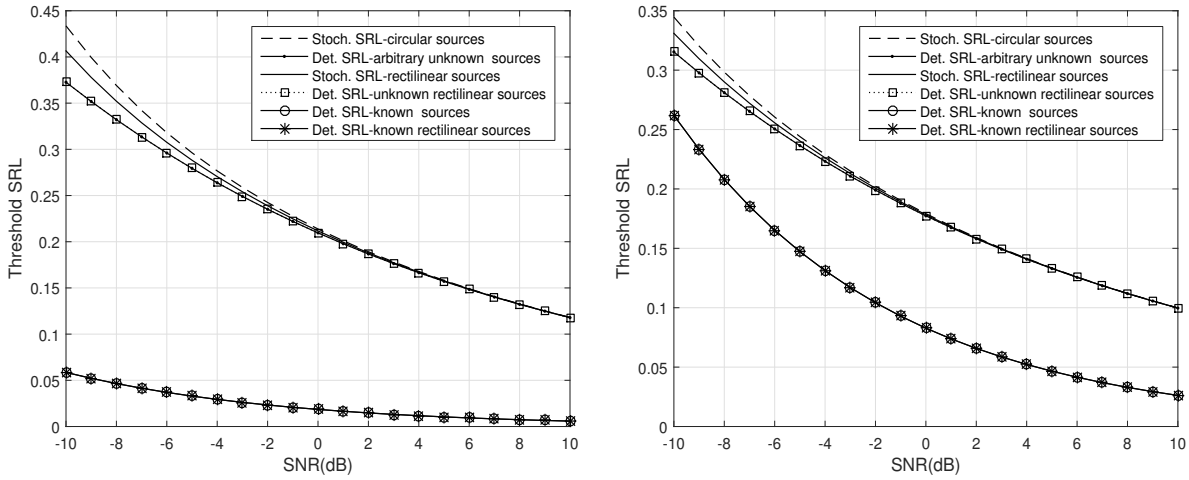
As for the SRL (63) deduced from the DCRB with known source signals which is the only expression of SLR [20] for correlated sources published in the literature, we see that the knowledge of rectilinearity of the source signals which adds an unknown phase parameter does not modify this SRL (65). Naturally, these SRLs are more optimistic than the SRLs resulting from the DCRB with arbitrary unknown sources and SCRB for circular sources, due to the proportionality of SRL in $(.)^{1/2}$ instead of $(.)^{1/4}$.

VI. Numerical illustrations

This section illustrates the dependence of the derived DCRB (and SCRB)-based SRLs expressions (63)-(70) on various parameters such as the number of sensors, the number of snapshots and the signal and noise parameters. Throughout this section (except in Figs. 6 and 10), the number of sensors N is fixed at 6 and that of snapshots T at 500. Note first that our interpretable closed-form expressions (64)-(70) of SRLs only give approximate solutions of (60). Their relative precisions depend on the different parameters, but from our different calculations, we can say that they are increasing functions of the different SRLs and that they are better than 1% as soon as our calculated values of SRL are lower than 0.5rd. This good precision of our approximations can be explained by the expansions in δ that are even, and truncated to order two or four.

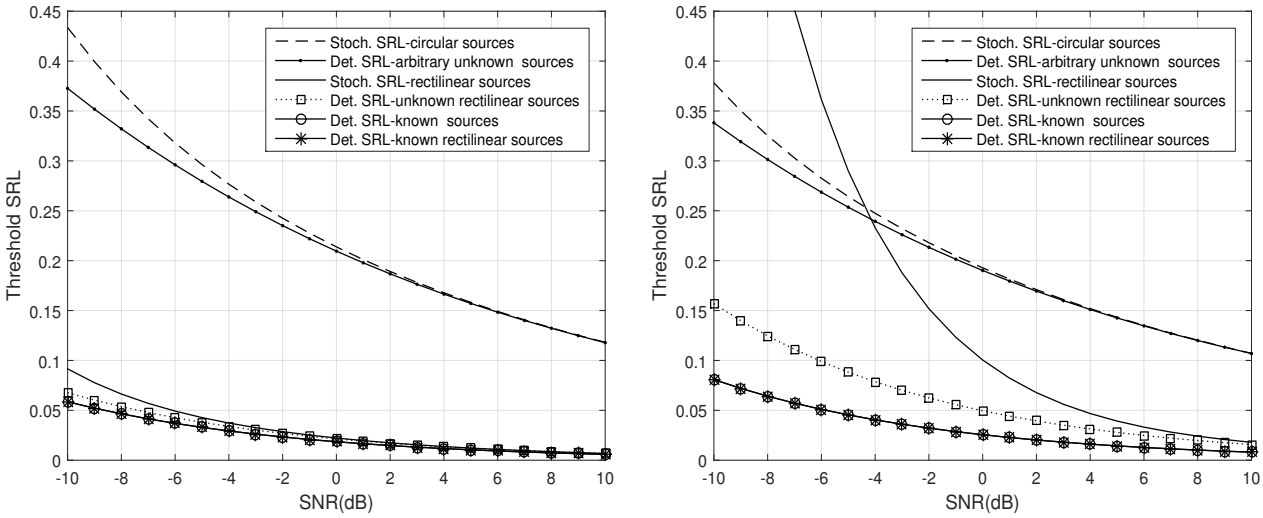
In the first experiment, Figs. 1, 2, 3 and 4 compare the different SRLs (63)-(70) with respect to the SNR under the assumption of Gaussian noise or observations (i.e., $\xi_1 = \xi_2 = 1$). Figs. 1 and 2 clearly show that the SRL derived under the assumption of known sources, which is the only result published in the literature [20] is very optimistic with respect to other SCRB (and DCRB)-derived SRLs, especially for a correlation or non-circularity phase equal to zero. On the other hand, the DCRB-derived SRLs are lower than the associated SCRB-derived SRLs for both circular and rectilinear unknown sources, similarly to the behavior of the associated CRBs. Note however that the DCRB (and SCRB)-derived SRLs are very close, except in the case of strongly correlated rectilinear sources with non-zero phase (see Fig. 2b).

Figs. 3 and 4 clarify this point by plotting the ratios $(\Delta\theta)_{\mathrm{D}}^{\mathrm{Unk}}/(\Delta\theta)_{\mathrm{S,circ}}$ and $(\Delta\theta)_{\mathrm{D,rect}}^{\mathrm{Unk}}/(\Delta\theta)_{\mathrm{S,rect}}$. One can observe that these ratios deviate all the more than one as the SNR is lower, as it has already been noticed in [26] when analyzing the relation between $\mathrm{DCRB}^{\mathrm{Unk}}(\theta_k)$ and $\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k)$. Note also that the magnitude and



(a) $|\rho| = \rho' = 0$ and $\phi = \Delta\phi = 0$ (b) $|\rho| = \rho' = 0.95$ and $\phi = \Delta\phi = 0$

Fig. 1. Comparisons between stochastic SRLs for circular sources (68), rectilinear sources (69), and between deterministic SRLs for known arbitrary sources (63), unknown arbitrary sources (64), known rectilinear sources (65) and unknown rectilinear sources (66) for either Gaussian noise or observations (i.e., $\xi_1 = \xi_2 = 1$) as function of SNR, with $N = 6$.

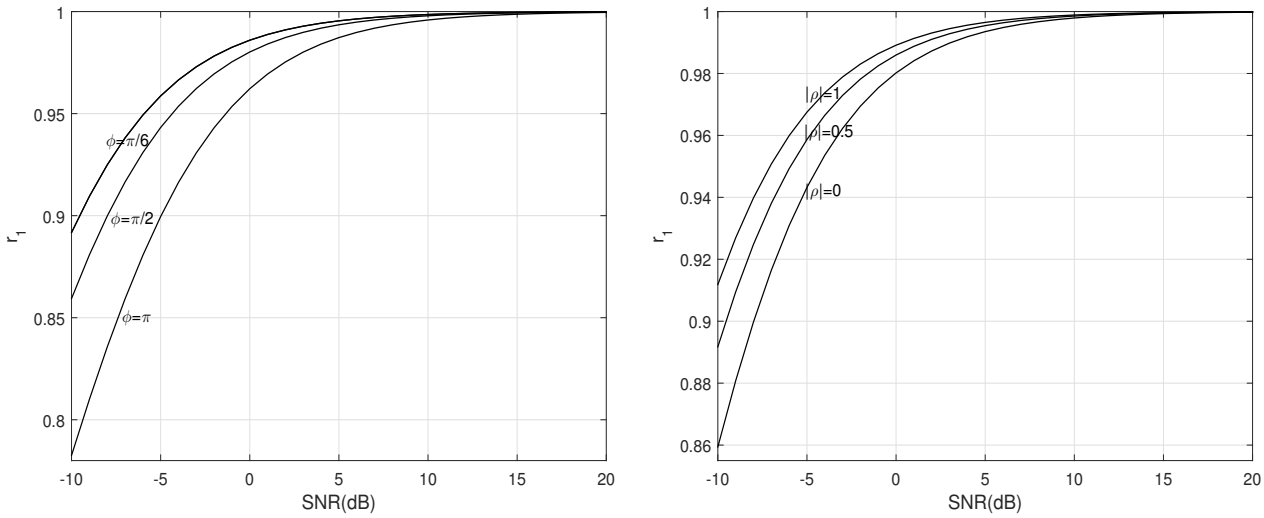


(a) $|\rho| = \rho' = 0$ (b) $|\rho| = \rho' = 0.95$

Fig. 2. Comparisons between stochastic SRLs for circular sources (68), rectilinear sources (69), and between deterministic SRLs for known arbitrary sources (63), unknown arbitrary sources (64), known rectilinear sources (65) and unknown rectilinear sources (66) for either Gaussian noise or observations (i.e., $\xi_1 = \xi_2 = 1$) as function of SNR, with $N = 6$ and $\phi = \Delta\phi = \pi/3$.

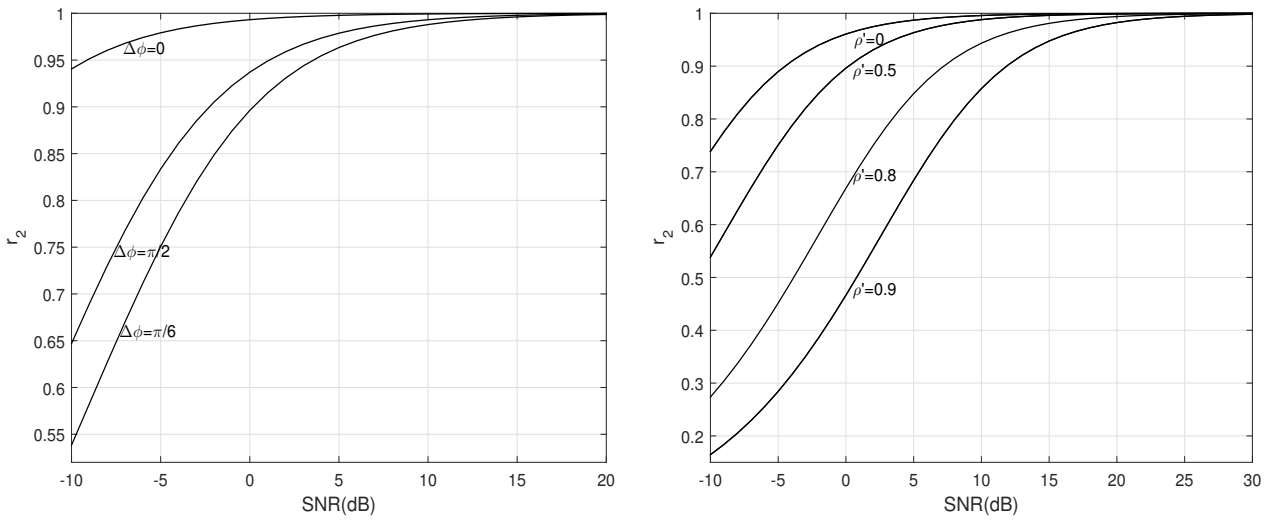
phase of the correlation impact these ratios in different ways for circular or rectilinear source signals. This ratio is lowest for uncorrelated circular sources ($\rho = 0$) and for strongly correlated rectilinear sources of correlation phase not close to zero ($\rho' \approx 1$ and $\Delta\phi \neq 0$).

In the second experiment, Figs. 5 and 6 illustrate the impact of the non-Gaussianity of the noise on the DCRB-derived SRL implied by the influence of the coefficient ξ_1 and of the observation on the SCRB-derived SRL



(a) $|\rho| = 0.5$ (b) $\phi = \pi/6$

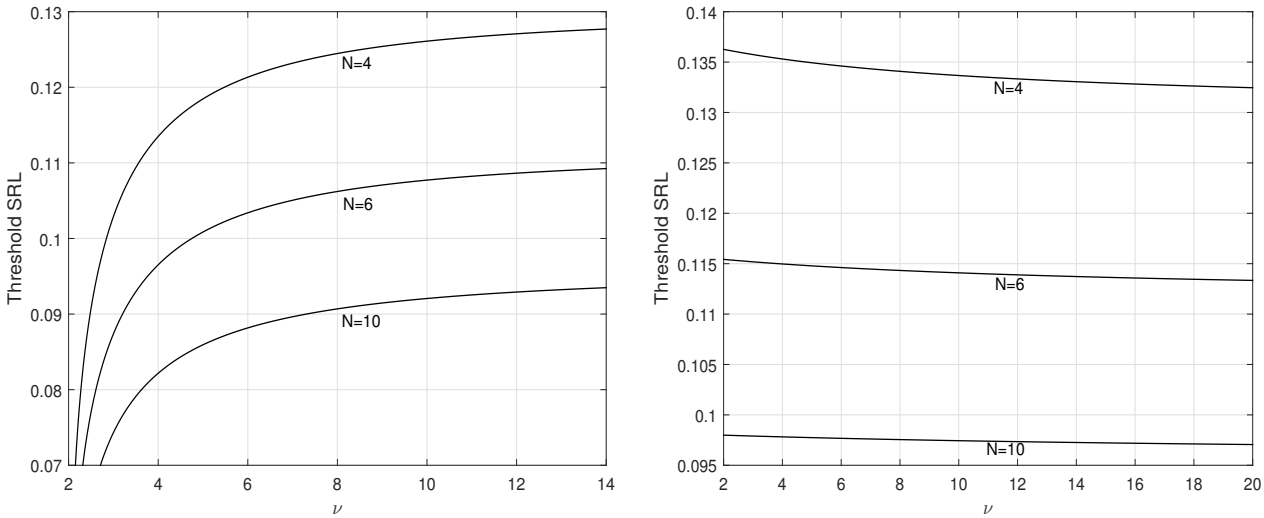
Fig. 3. Ratio $r_1 = (\Delta\theta)_{\mathrm{D}}^{\mathrm{Unk}}/(\Delta\theta)_{\mathrm{S,circ}}$ (i.e., (64)/(68)) for either complex Gaussian noise or observations (i.e., $\xi_1 = \xi_2 = 1$) as a function of SNR with $N = 6$ and $T = 500$.



(a) $\rho' = 0.5$ (b) $\Delta\phi = \pi/6$

Fig. 4. Ratio $r_2 = (\Delta\theta)_{\mathrm{D,rect}}^{\mathrm{Unk}}/(\Delta\theta)_{\mathrm{S,rect}}$ (i.e., (66)/(69)) for either complex Gaussian noise or observations (i.e., $\xi_1 = \xi_2 = 1$) as a function of SNR with $N = 6$ and $T = 500$.

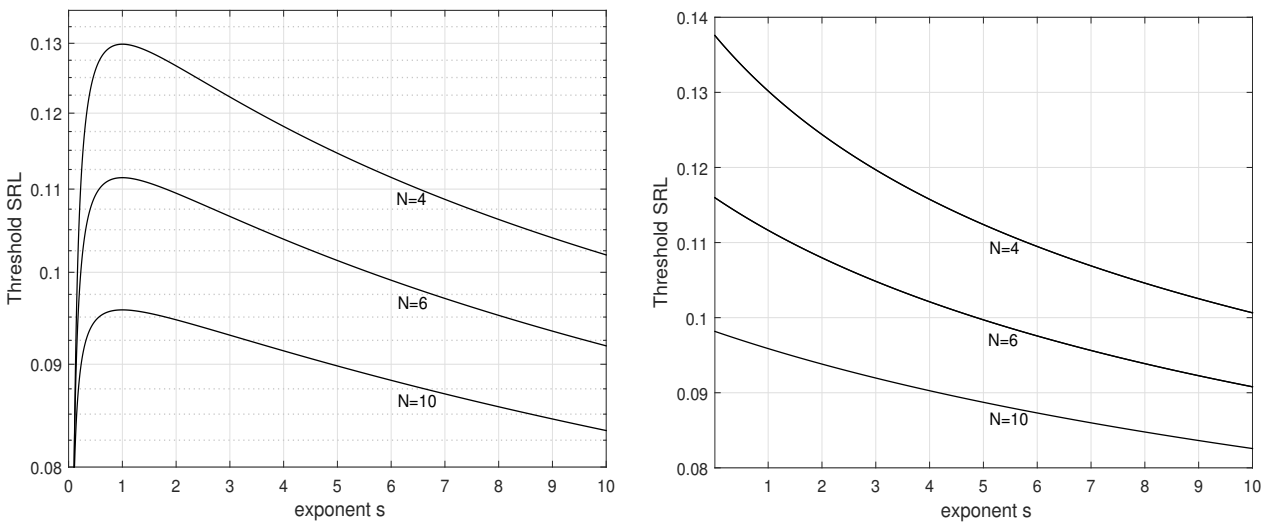
implied by the influence of the coefficient ξ_2. We consider here the normalized complex Student's $t-$distribution of degree of freedom $\nu > 2$ and the normalized complex generalized Gaussian distribution of exponent $s > 0$, which each include the Gaussian distribution for $\nu \to \infty$ and $s = 1$, respectively. For these two distributions, the expressions of ξ_1 and ξ_2 depend not only on the parameter of the distributions, but also on N that are calculated in Appendix. One can observe from Figs. 5 and 6 that plot the SRLs as a function of ν or s for three values of N, that: (i) similar to the well-known result on DCRB in which the Gaussian distribution leads



(a) Deterministic SRL for unknown arbitrary sources

(b) Stochastic SRL for circular source

Fig. 5. (a) Deterministic SRL for unknown arbitrary sources (64) and (b) stochastic SRL for circular sources (68) for either normalized complex Student's t-distributed noise or observations as function of $\nu > 2$ for three values of N with $|\rho| = 0.5$, $\phi = \pi/3$, SNR = 10dB and $T = 500$.

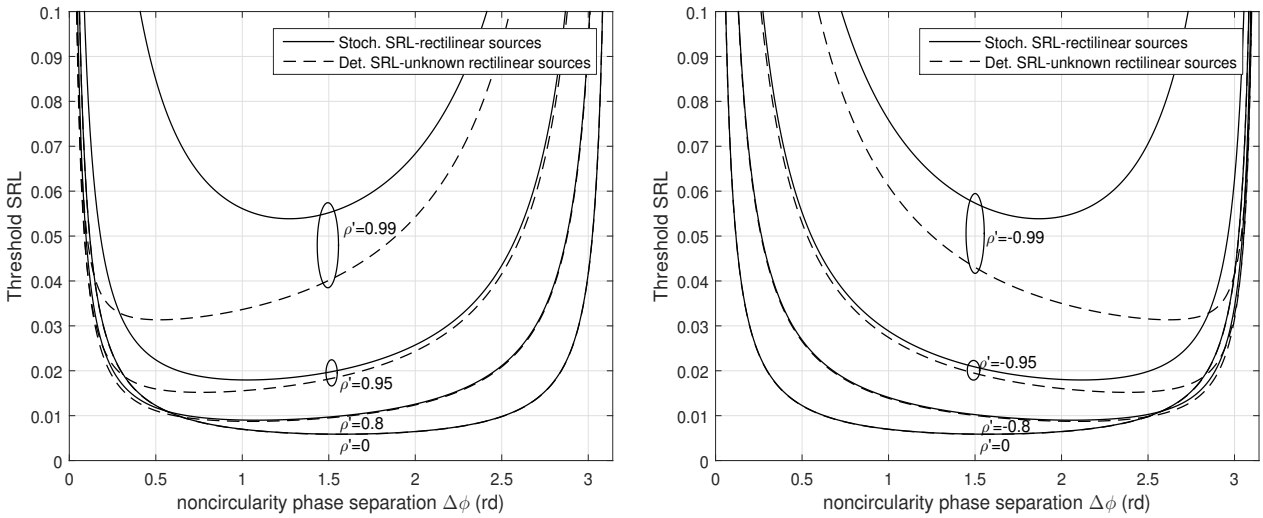


(a) Deterministic for SRL unknown arbitrary sources

(b) Stochastic SRL for circular source

Fig. 6. (a) Deterministic SRL for unknown arbitrary sources (64) and (b) stochastic SRL for circular sources (68) for either complex normalized generalized Gaussian distributed noise or observations as function of exponent $s > 0$ for three values of N with $|\rho| = 0.5$, $\phi = \pi/3$, SNR = 10dB and $T = 500$.

to the largest DCRB ($\xi_1 \geq 1$), the DCRB-derived SRL is maximum for the Gaussian distribution and takes very small values in the case of very heavy-tailed distributions (i.e., ν close to 2 and s close to 0); (ii) similar to the less known result on SCRB in which the Gaussian distribution does not always lead to the largest SCRB (for the normalized complex Student's $t-$distribution $\xi_2 < 1$ and for normalized complex generalized Gaussian distribution $\xi_2 < 1$ for $s < 1$ and $\xi_2 > 1$ for $s > 1$), the SCRB-derived SRL is minimum for $\nu \to \infty$ (Gaussian



(a) (b)

Fig. 7. Deterministic SRL (66) with unknown rectilinear sources and stochastic SRL with rectilinear sources (69) for either complex Gaussian noise or observations (i.e., $\xi_1 = \xi_2 = 1$) as a function of the noncircularity phase separation $\Delta\phi$ with $N = 6$, SNR = 10dB and $T = 500$ for positive (a) and negative (b) values of ρ'.

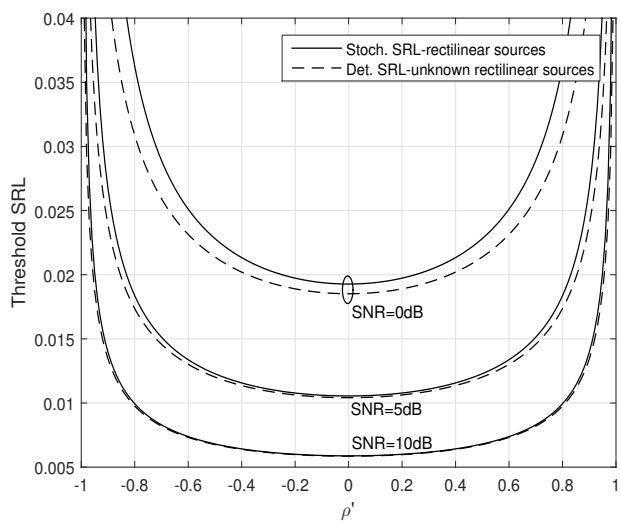
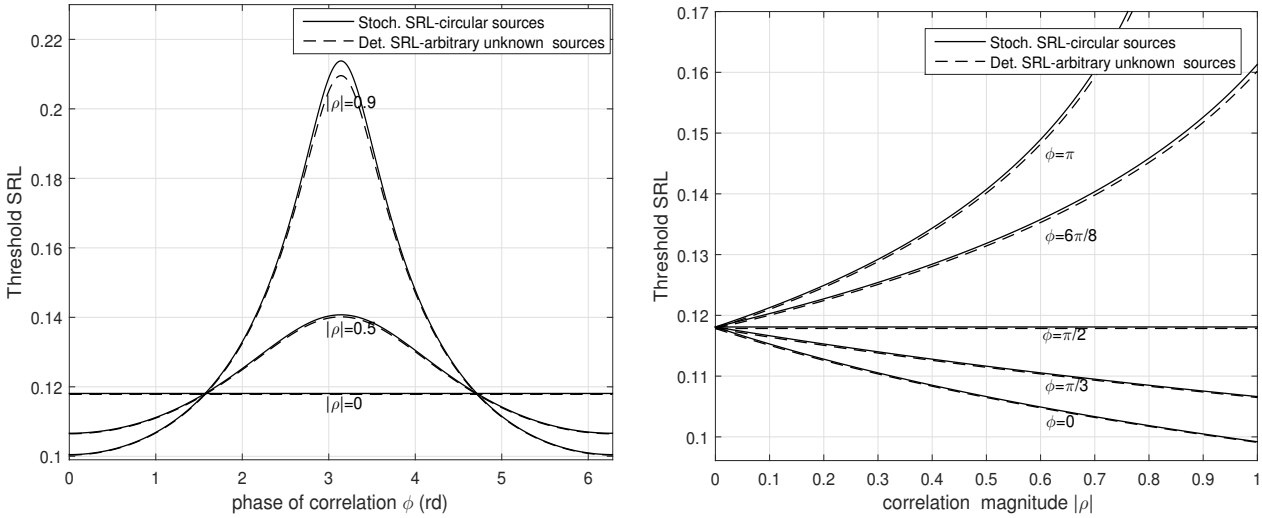


Fig. 8. Deterministic SRL (66) with unknown rectilinear sources and stochastic SRL with rectilinear sources (69) for either complex Gaussian noise or observations (i.e., $\xi_1 = \xi_2 = 1$) as a function of ρ' with $\Delta\phi = \pi/2$, $N = 6$ and $T = 500$.

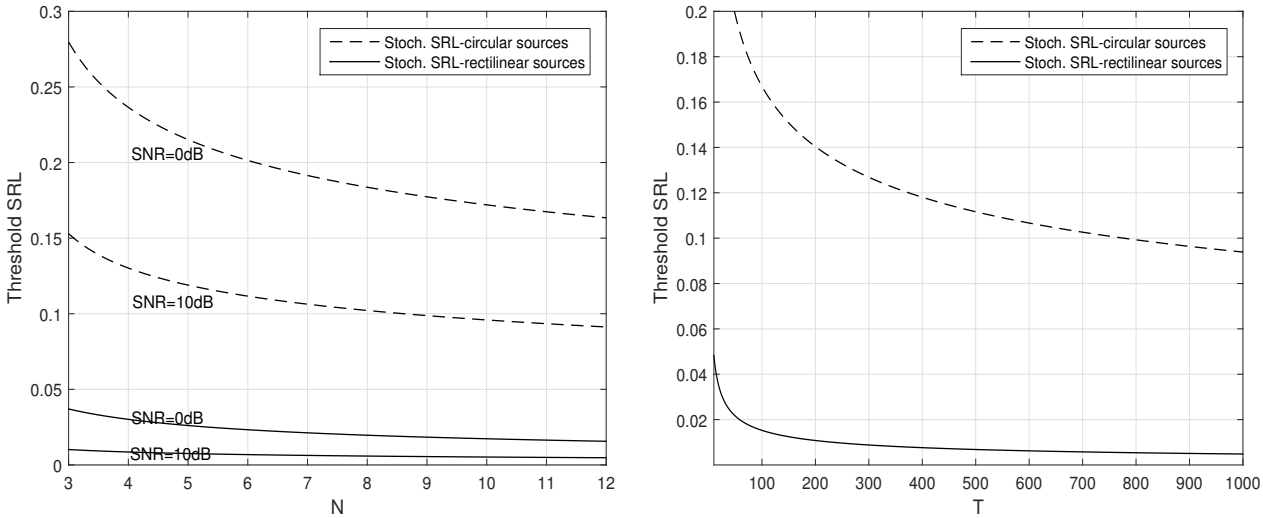
distribution) and for $s \to \infty$ light tail distribution.

In the third experiment, Figs. 7, 8, 9 and 10 illustrate the impact of the correlation (phase and magnitude), the SNR, the number of snapshots and sensors on the DCRB (and SCRB)-derived SRLs. Figs. 7 and 8, dedicated to SRLs depending on rectilinear sources for which $\rho = \rho' e^{i\Delta\phi}$ with $\rho' \in [-1, +1]$ and $\Delta\phi \in [0, \pi]$ and Fig. 9 dedicated to SRLs depending on unknown arbitrary or circular sources for which $\rho = |\rho| e^{i\phi}$ with $\phi \in [0, 2\pi]$, present the important role played by the correlation of the sources. It can be observed from Figs. 7 and 8 that



(a) (b)

Fig. 9. Deterministic SRL with unknown arbitrary sources (64) and stochastic SRL with circular sources (68) for either complex Gaussian noise or observations (i.e., $\xi_1 = \xi_2 = 1$) as a function of the angle (a) and magnitude (b) of the correlation with $N = 6$, SNR = 10dB and $T = 500$.



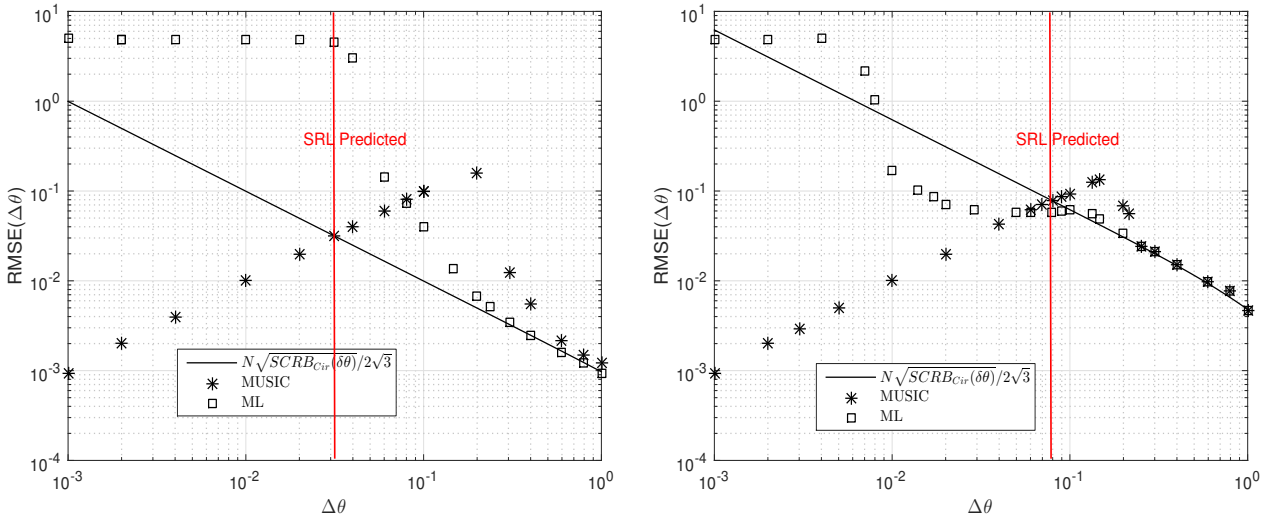
(a) $T = 500$ (b) SNR$= 10dB$

Fig. 10. Stochastic SRL with circular sources (68) and stochastic SRL with rectilinear sources (69) for complex Gaussian observations (i.e., $\xi_2 = 1$) as a function of number of sensors N (a) and number of snapshots T (b) with $N = 6$, $|\rho| = \rho' = 0.5$ and $\phi = \Delta\phi = \pi/3$.

the SRLs increase with ρ' but not symmetric in $\Delta\phi$ and leads to a minimum SRL depending on ρ', N and SNR as indicated in paragraph 3 of subsection V-B, and are minimum for $\Delta\phi = \pi/2$ only for $\rho' = 0$. Fig. 9 shows that the SRLs increase with $|\rho|$ and it is symmetric with respect to $\phi = \pi/2$ and are minimum for $\phi = 0$ and maximum for $\phi = \pi$ as depicted in paragraph 3 of subsection V-B. Fig.10 compares the SCRB-derived SRLs for circular and rectilinear sources for different values of SNRs, number of snapshots and sensors. It

shows that similar to the well-known behavior of SCRB, the SRL is much smaller for rectilinear sources than for circular sources when the phase of correlation is not zero.

Finally, to illustrate that the correlation phase of the sources impacts not only the CRB and thus the SRL, but also the resolution performance of the ML and MUSIC algorithms, a Monte Carlo simulation is presented in Fig.11. This figure shows the predicted SRL $(\Delta\theta)_{\mathrm{S,Cir}}$ given by (68), the $N\sqrt{SCRB_{\mathrm{Cir}}(\delta\theta)}/2\sqrt{3}$ and the RMSE (1000 Monte-Carlo runs are performed for each simulation point) of the difference of the DOA estimates versus $\Delta\theta \stackrel{\mathrm{def}}{=} N\delta\theta/2\sqrt{3}$ for two values of correlation phase $\phi = 0$ and $\phi = \pi$. We can see from the two figures: the dependence of the SRL on the correlation phase which moves from 0.032 when $\phi = 0$ to 0.078 when $\phi = \pi$, and that the RMSE associated with the ML reaches the CRB in the resolvable region and that the reached region is reduced for $\phi = 0$. It can also be observed that the resolution performance of the ML estimator outperforms that of the MUSIC algorithm which is strongly affected by the phase correlation. While in the unresolved region where the two sources are no longer resolved, the ML breaks away from the CRB as well as the MUSIC algorithm. These results ensure that the SRL can not be achieved in general as was discussed in [15].



(a) $\phi = 0$ (b) $\phi = \pi$

Fig. 11. The predicted SRL (68), the $N\sqrt{SCRB_{\mathrm{Cir}}(\delta\theta)}/2\sqrt{3}$ and the RMSE (1000 Monte-Carlo runs are performed for each simulation point) of the difference of the DOA estimates versus $\Delta\theta$ for the ML estimator and the MUSIC algorithm for complex circular Gaussian observations (i.e., $\xi_2 = 1$) considering the two cases (a) $\phi = 0$ and (b) $\phi = \pi$ with $N = 6$, $|\rho| = 0.95$, $T = 500$, and SNR$= 30dB$.

VII. Conclusion

Simple exact and asymptotic (for small DOA separation) interpretable closed-form expressions are presented for the DCRB and SCRB of the DOA for two equi-powered correlated known, arbitrary, circular or rectilinear

sources in CES data models. The dependence of these bounds on the magnitude and phase of the correlation is examined, and the values of the phase leading to the larger and smaller bounds are obtained. The asymptotic expressions of DCRB and SCRB allow us to give interpretable closed-form expressions of the SRL based on the Smith criterion in different scenarios. Comments to explain how different parameters impact the derived SRLs among them the phase and magnitude of the correlation of the sources, and how also the SRLs derived from the SCRBs are much less optimistic than those that have so far been deduced only from the DCRB under the assumption of known sources are discussed. Finally, numerical illustrations clarify the obtained theoretical results.

APPENDIX

Derivation of the SCRB for two equipowered sources:

After some algebraic manipulations, one arrives at the following expressions of the diagonal $\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k,\theta_k)$, $k=1,2$, and the anti-diagonal $\mathrm{SCRB}_{\mathrm{Cir}}(\theta_1,\theta_2)$ elements of $\mathrm{SCRB}_{\mathrm{Cir}}(\boldsymbol{\theta})$ for two equal-power circular correlated sources deduced from (14).

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_k,\theta_k) = \frac{1}{2T\xi_2}\frac{\gamma\chi_k}{\chi_1\chi_2-\chi_{1,2}^2}, k=1,2, \tag{71}$$

$$\mathrm{SCRB}_{\mathrm{Cir}}(\theta_1,\theta_2) = -\frac{1}{2T\xi_2}\frac{\gamma\chi_{1,2}}{\chi_1\chi_2-\chi_{1,2}^2}. \tag{72}$$

with $\chi_k=\alpha_{3-k}\nu_1$, $k=1,2$, and $\chi_{1,2}=\alpha_{1,2}\nu_2$ where

$$\nu_1 = r(1-|\rho|^2)(N^2-\beta^2)+2\beta|\rho|\cos(\phi)+N(1+|\rho|^2), \tag{73}$$

$$\nu_2 = \beta+r|\rho|\cos(\phi)(1-|\rho|^2)(N^2-\beta^2)+|\rho|(2N\cos(\phi)+\beta|\rho|\cos(2\phi)), \tag{74}$$

$$\gamma = (1-|\rho|^2)(N^2-\beta^2)+2(N+\beta|\rho|\cos(\phi))/r+1/r^2, \tag{75}$$

where α_k and $\alpha_{1,2}$ are the geometric-dependent array coefficients given by $\alpha_k=\mathbf{d}_k^H\mathbf{\Pi}_{\mathbf{A}}^{\perp}\mathbf{d}_k$ and $\alpha_{1,2}=\mathbf{d}_1^H\mathbf{\Pi}_{\mathbf{A}}^{\perp}\mathbf{d}_2$.

Similarly, the diagonal $\mathrm{SCRB}_{\mathrm{Rec}}(\theta_k,\theta_k)$, $k=1,2$, and the anti-diagonal $\mathrm{SCRB}_{\mathrm{Rec}}(\theta_1,\theta_2)$ elements of $\mathrm{SCRB}_{\mathrm{Rec}}(\boldsymbol{\theta})$ for two equal-power rectilinear sources deduced from (15) can be written as:

$$\mathrm{SCRB}_{\mathrm{Rec}}(\theta_k,\theta_k) = \frac{1}{T\xi_2}\frac{\gamma c_1}{c_3}, k=1,2, \tag{76}$$

$$\mathrm{SCRB}_{\mathrm{Rec}}(\theta_1,\theta_2) = \frac{1}{T\xi_2}\frac{\gamma c_2}{c_3}, \tag{77}$$

with

$$c_1 = \zeta_5(\zeta_4^2+\zeta_7^2)-\zeta_2(\zeta_5^2-\zeta_9^2)-2\zeta_4\zeta_8\zeta_9, \tag{78}$$

$$c_2 = \zeta_6\zeta_5^2-\zeta_6\zeta_9^2-\zeta_3\zeta_8\zeta_5+\zeta_3\zeta_4\zeta_9+\zeta_7\zeta_8\zeta_9-\zeta_7\zeta_4\zeta_5, \tag{79}$$

$$\begin{aligned} c_3 &= \zeta_5\left(\zeta_1\left(-\zeta_2\zeta_5+\zeta_8^2+\zeta_4^2\right)+\zeta_6^2\zeta_5\right)+\zeta_9^2\left(\zeta_1\zeta_2-\zeta_6^2\right)-2\zeta_1\zeta_8\zeta_4\zeta_9 \\ &+2\zeta_7(\zeta_6\zeta_8\zeta_9-\zeta_6\zeta_4\zeta_5-\zeta_2\zeta_3\zeta_9+\zeta_3\zeta_8\zeta_4)+2\zeta_6\zeta_3(\zeta_4\zeta_9-\zeta_8\zeta_5)+\zeta_3^2\left(\zeta_2\zeta_5-\zeta_4^2\right)+\zeta_7^2\left(\zeta_2\zeta_5-\zeta_8^2\right) \end{aligned} \tag{80}$$

with $\zeta_k=\tilde{\alpha}_k\nu_1$, $k=1,2,3,4,5$, and $\zeta_k=\tilde{\alpha}_k\nu_2$, $k=6,7,8,9$, where ν_1, ν_2 and γ are given, respectively, by (73), (73) and (75) after replacing N with $2N$, $\rho\in\mathbb{C}$ with $\rho'\in(-1,\ 1)$, and β with $\tilde{\beta}\stackrel{\text{def}}{=}\tilde{\mathbf{a}}_1^H\tilde{\mathbf{a}}_2=2\beta\cos(\Delta\phi)$. The $\tilde{\alpha}_k$ are the geometric and phase-dependent array coefficients given by $\tilde{\alpha}_1=\tilde{\mathbf{d}}_1^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{d}}_1$, $\tilde{\alpha}_2=\tilde{\mathbf{d}}_2^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{d}}_2$, $\tilde{\alpha}_3=\tilde{\mathbf{d}}_1^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{d}}_{\phi_1}$, $\tilde{\alpha}_4=\tilde{\mathbf{d}}_2^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{d}}_{\phi_2}$, $\tilde{\alpha}_5=\tilde{\mathbf{d}}_{\phi_1}^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{d}}_{\phi_1}$, $\tilde{\alpha}_6=\tilde{\mathbf{d}}_1^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{d}}_2$, $\tilde{\alpha}_7=\tilde{\mathbf{d}}_1^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{d}}_{\phi_2}$, $\tilde{\alpha}_8=\tilde{\mathbf{d}}_2^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{d}}_{\phi_1}$ and $\tilde{\alpha}_9=\tilde{\mathbf{d}}_{\phi_1}^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{d}}_{\phi_2}$.

Derivation of $\xi_1\stackrel{\text{def}}{=}\frac{\mathrm{E}[\phi^2(\mathcal{Q}_t)\mathcal{Q}_t]}{N}$ *and* $\xi_2\stackrel{\text{def}}{=}\frac{\mathrm{E}[\phi^2(\mathcal{Q}_t)\mathcal{Q}_t^2]}{N(N+1)}$*:*

Normalized complex Student's $t-$*distribution:*

Note first that the usual zero mean complex Student's $t-$distribution used in [33], [41] of the data $\mathbf{y}_t$ associated with the scatter matrix $\mathbf{\Sigma}$ satisfies $\mathrm{E}(\mathbf{y}_t\mathbf{y}_t^H)=\frac{\nu}{\nu-2}\mathbf{\Sigma}$ for a degree of freedom $\nu>2$. Consequently, we need to normalize this usual distribution so that its covariance is equal to the scatter matrix. Its density generator then becomes $g(t)=\left(1+\frac{2t}{\nu-2}\right)^{-(N+\nu/2)}$ and hence $\phi(t)=\frac{(\nu/2)+N}{(\nu/2)+t-1}$. The 2nd-order modular variate $\mathcal{Q}_t$ then has a scaled $F-$distribution with $2N$ and ν degrees of freedom, $\mathcal{Q}_t=_d\frac{\nu-2}{\nu}NF_{2N,\nu}$ with p.d.f $p(t)=\frac{1}{((\nu/2)-1)^N B(N,\nu/2)}t^{N-1}\left(1+\frac{2t}{\nu-2}\right)^{-(N+\nu/2)}$, $t\geq 0$, where $B(x,y)$ is the Beta function. A straightforward calculation proves that $\xi_1=\frac{\nu/2}{(\nu/2)-1}\frac{(\nu/2)+N}{(\nu/2)+N+1}$ and $\xi_2=\frac{(\nu/2)+N}{(\nu/2)+N+1}$.

Normalized complex generalized Gaussian distribution:

The general density generator for zero mean complex generalized Gaussian distribution with exponent $s>0$ and scale b associated with the scatter matrix $\mathbf{\Sigma}$ is given in [33] by $g(t)=e^{-t^s/b}$ and hence $\phi(t)=\frac{s}{b}t^{s-1}$. The 2nd-order modular variate $\mathcal{Q}_t=_d\mathcal{G}_t^{1/s}$ where $\mathcal{G}_t$ is gamma distributed with shape N/s and scale b. The p.d.f. of $\mathcal{Q}_t$ is given by $p(t)=\frac{s}{\Gamma(\frac{N}{s})b^{N/s}}t^{N-1}g(t)$. Note that b, which controls the scale of the density generator, ensures that the covariance is equal to the scatter matrix for $b=\left(\frac{N\Gamma(\frac{N}{s})}{\Gamma(\frac{N+1}{s})}\right)^s$. With this value of b, a straightforward calculation proves that $\xi_1=\frac{\Gamma(2+\frac{N-1}{s})\Gamma(\frac{N+1}{s})}{(\Gamma(1+\frac{N}{s}))^2}$ and $\xi_2=\frac{N+s}{N+1}$.

Finally, note that ξ_1 depends on the normalization of the CES distributions, unlike ξ_2 for which its normalization does not impact it (see ξ_2 given in [41] and [42]).

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