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# A Differential game control problem with state constraints

NIDHAL GAMMOUDI\*and HASNAA ZIDANI †

## Abstract

We study the Hamilton-Jacobi (HJ) approach for a two-person zero-sum differential game with state constraints and where controls of the two players are coupled within the dynamics, the state constraints and the cost functions. It is known for such problems that the value function may be discontinuous and its characterization by means of an HJ equation requires some controllability assumptions involving the dynamics and the set of state constraints. In this work, we characterize this value function through an auxiliary differential game free of state constraints. Furthermore, we establish a link between the optimal strategies of the constrained problem and those of the auxiliary problem and we present a general approach allowing to construct approximated optimal feedbacks to the constrained differential game for both players. Finally, an aircraft landing problem in the presence of wind disturbances is given as an illustrative numerical example.

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## 1 Introduction

Two-person zero-sum differential games provide convenient framework for analyzing real conflict situations between two players where the gain of one player corresponds certainly to the loss of the other player. The most classical example is the *Target Problem* where the dynamic is controlled by both players, one player wants the dynamical system to reach, in finite time, a given set called the target while his opponent tries to avoid this target forever (see [13, 12, 14]). Another classical example is the *Pursuit-Evasion* game for which each player controls only half of the system's coordinates and the cost is the *capture time* which is the first time when the first player's coordinates become close enough to those of his opponent (see [15, 5]).

Differential games can be studied in different contexts depending on the information advantage accorded to the two players which defines two values of the game. The most popular class of information advantage is the one of nonanticipative strategies where one of the two players knows, at each instant, the past and present choices of his opponent without having any idea about his future actions, see [17, 16]. Another class of game strategies is called positional strategies where at each instant and before taking any decision, both players know the position of the system, see [4, Chapter VIII].

In the present work, we consider a two-person zero-sum differential game subject to state constraints where the first player is allowed to use nonanticipative strategies which are mappings from the set of controls of the second player,  $\mathcal{B}$ , to the actions set of the first player,  $\mathcal{A}$  (the precise definitions of  $\mathcal{A}$ ,  $\mathcal{B}$  and the set of nonanticipative strategies,  $\Gamma$ , will be given in section 2).

For a given finite time horizon  $T > 0$ , consider the dynamical system

$$\begin{cases} \dot{y}(s) = f(s, y(s), \alpha[b](s), b(s)), a.e. s \in [t, T], \\ y(t) = x, \end{cases} \quad (1)$$

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where  $\alpha[\cdot] \in \Gamma$  is the nonanticipative strategy of the first player,  $b(\cdot) \in \mathcal{B}$  is the action of the second player, and  $f: [t, T] \times \mathbb{R}^d \times A \times B$  is a continuous function (more precise definitions and assumptions are given in section 2).

The absolutely continuous solution of (1) is denoted by  $y_{t,x}^{\alpha[b],b}$  and will be referred as the trajectory corresponding to  $(\alpha[b](\cdot), b(\cdot)) \in \mathcal{A} \times \mathcal{B}$ .

For a given set of state constraints,  $\mathcal{K}$ , a trajectory  $y_{t,x}^{a,b}(\cdot)$ , corresponding to a couple of actions  $(a(\cdot), b(\cdot)) \in \mathcal{A} \times \mathcal{B}$ , is admissible if  $y_{t,x}^{a,b}(s) \in \mathcal{K}$ ,  $\forall s \in [t, T]$ .

We are interested in the following differential game with maximum running cost:

$$v(t, x) := \inf_{\alpha[\cdot] \in \Gamma} \pi(t, x; \alpha) \quad (2)$$

where  $\pi$  is defined by:

$$\pi(t, x; \alpha) := \begin{cases} \sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \phi(s, y_{t,x}^{\alpha[b],b}(s)) \right) \vee \psi(y_{t,x}^{\alpha[b],b}(T)) \right\}, & \text{if } y_{t,x}^{\alpha[b],b}(\cdot) \text{ is admissible } \forall b(\cdot) \in \mathcal{B}, \\ +\infty, & \text{else,} \end{cases}$$

where the cost functions  $\phi: [t, T] \times \mathbb{R}^d \times A \times B \rightarrow \mathbb{R}$  and  $\psi: \mathbb{R}^d \rightarrow \mathbb{R}$  are continuous (see section 2 for precise assumptions). This problem formulation can model the situation where the first player, representing the controller, uses his advantage of information to counteract to unknown disturbances, representing the second player of the game, which can affect the system and the cost functions, see section 5. For such situations, the traditional approach is to represent disturbances via a statistical model and to optimize the expected value of the cost [24] and the references therein. However, this approach may not be effective against some catastrophic cases and it is not always possible to have a good statistical model. Therefore this situation can be modeled by a two-person zero-sum differential game where the first player (the controller) minimizes the cost in the case of the worst behavior of his opponent (unknown disturbances). We can also imagine another game example where the second player objective is to maximize the cost  $\pi$  or to violate state constraints.

Without considering state constraints,  $\mathcal{K} \equiv \mathbb{R}^d$ , this differential game was studied in [27] with a maximum bounded cost function and in [25] with a Lipschitz continuous infimum cost. It was also studied in the case of a single controller in [23] by characterizing the value function epigraph via a viability kernel.

In the presence of state constraints,  $\mathcal{K} \neq \mathbb{R}^d$ , some difficulties appear. The value function  $v$  becomes lower semi continuous and its characterization as unique solution of an HJ equation requires some additional assumptions involving the dynamics  $f$  and  $\mathcal{K}$ . In the case of one controller problems, the most popular assumption, which is the inward pointing condition, was introduced in [28]. This condition imposes, at each point of the boundary of  $\mathcal{K}$ , the existence of a control value that lets the dynamics points in the interior of  $\mathcal{K}$ . Equivalent assumptions in the case of a two-players problem can be found in [15, 6, 7]. Unfortunately, such assumptions cannot be satisfied for several control problems, which complicates the characterization of the value function as solution of an HJ equation.

In this work, we do not assume any controllability assumptions on the dynamics or on the set of state constraints. In addition to that, controls of the two players are allowed to be coupled within the dynamics, the state constraints and the cost functions. Moreover, we consider weak assumptions on  $f$ ,  $\phi$  and  $\psi$  (locally Lipschitz continuous and not essentially bounded). We extend the result of [1] to the case of a two-person games in order to characterize  $v$  through a locally Lipschitz continuous value function of an auxiliary differential game, unique solution of an HJ equation with obstacle term. In particular, in [1], the link between the constrained problem and the auxiliary was established under a convexity assumption on the dynamics  $f$ . Here, we study this link in a more general setting even when  $f$  is not convex. **Without the convexity of the dynamics and without controllability assumption, the set of trajectories is not closed and the value function is not even lower semi-continuous.**

Another contribution of this paper concerns trajectory reconstruction for the constrained differential game based on the knowledge of an approximation to its associated auxiliary value function. In particular, we prove that an optimal strategy of the first player for the auxiliary differential game, starting from a specific

initial position, is also optimal for the constrained differential game. In addition to that, we exploit some ideas from [2, 26] about trajectory reconstruction in the case of dynamical systems with a single controller in order to present here a general approach allowing to construct approximated optimal feedbacks to the constrained differential game for both players by use of the auxiliary differential game value function. More precisely, we present a general reconstruction procedure corresponding to an arbitrary case for which choices of the second player are not essentially optimal and a *specific* reconstruction procedure where the second player takes optimal decisions representing the worst case for the first player. Convergence results of the trajectories sequences reconstructed by both procedures are also provided and discussed in this paper.

As an illustrative example, we study an aircraft landing problem in the presence of windshear. Indeed, the best strategy to avoid a failed landing, that can occurs because of quick changes of the wind velocity, is to steer the aircraft to the maximal altitude that can be reached, during an interval of time, in order to prevent a crash on the ground. In [20, 21], a Chebyshev-type optimal control problem was proposed and an approximate solution is provided. The Hamilton-Jacobi-Bellman approach was applied in [2] to solve this problem after supposing the knowledge of the wind velocity fields. In [9], the aircraft landing problem was formulated as a nonlinear differential game with state constraints and a semi-Lagrangian scheme was applied to compute an approximation of the value function.

In this paper, we propose a 5D differential game model with maximum running cost, where wind disturbances are considered as a second player and the first player tries, by use of nonanticipative strategies, to counteract to dangerous scenarios that can happen because of wind disturbances.

The paper is organized as follows. Section 2 introduces the constrained differential game with maximum running cost and formulates its associated auxiliary problem. Section 3 shows how the auxiliary problem introduced can be used to overcome the difficulties coming from the states constraints. Section 4 presents some results concerning reconstruction of optimal trajectories for the auxiliary problem and for the original constrained problem. The aircraft landing problem in presence of windshear is presented in section 5 as an illustrative numerical example.

**Notation** In this paper we adopt the convention that  $\inf \emptyset = +\infty$  and  $\sup \emptyset = -\infty$ .

For any  $a, b \in \mathbb{R}$ , the notation  $a \vee b$  (resp.  $a \wedge b$ ) stands for  $\max(a, b)$  (resp.  $\min(a, b)$ ). The Euclidean norm (inner product), on  $\mathbb{R}^d$  ( $d \geq 1$ ), is denoted by  $\|\cdot\|$  ( $\langle \cdot, \cdot \rangle$ ) while  $\mathbb{B}$  denotes the unit ball of  $\mathbb{R}^d$ .

For any subset  $Y$  of  $\mathbb{R}^d$ ,  $\overset{\circ}{Y}$  and  $\partial Y$  denote, respectively, the interior and boundary of  $Y$ . Besides,  $d(\cdot, Y)$  and  $d_Y(\cdot)$  denote the distance and the signed distance function to  $Y$  respectively. Recall that the signed distance  $d_Y(\cdot)$  is defined as follows:

$$d_Y(x) := \begin{cases} -d(x, \partial Y) & \text{if } x \in Y, \\ d(x, \partial Y) & \text{else.} \end{cases}$$

To shorten some notations, s.t., l.s.c. and w.r.t. stand for "such that", "lower semi continuous" and "with respect to".

## 2 Problem formulation

### 2.1 Settings of the constrained differential game

Consider a two-person zero-sum differential game with finite time horizon  $T > 0$ . Actions of the first and the second players are measurable functions from  $[0, T]$  and taking values respectively in  $A$ , a compact set of  $\mathbb{R}^p$  ( $p \geq 1$ ), and  $B$ , a compact set of  $\mathbb{R}^q$  ( $q \geq 1$ ). The set of admissible controls of the second player  $\mathcal{B}$  is defined as follows:

$$\mathcal{B} := \{b(\cdot) : [t, T] \rightarrow B, \text{ measurable}\}.$$

The set of admissible controls of the first player  $\mathcal{A}$  can be defined in a similar way to  $\mathcal{B}$ .

In this paper, the first player is assumed to use nonanticipative strategies. Following the formulation of Elliott and Kalton [16], the set of this type of strategies for the first player, denoted by  $\Gamma$ , can be defined by:

$$\Gamma := \left\{ \alpha[\cdot] : \mathcal{B} \rightarrow \mathcal{A}, \text{ s.t. } \forall b(\cdot), b'(\cdot) \in \mathcal{B}, \text{ and } \forall \tau \in [0, T], \right. \\ \left. \left( b(s) = b'(s) \text{ a.e. } s \in [0, \tau] \right) \Rightarrow \left( \alpha[b](s) = \alpha[b'](s) \text{ a.e. } s \in [0, \tau] \right) \right\}.$$

For a choice  $(\alpha[\cdot], b(\cdot)) \in \Gamma \times \mathcal{B}$  of the two players, consider the following dynamical system

$$\begin{cases} \dot{y}(s) = f(s, y(s), \alpha[b](s), b(s)), \text{ a.e. } s \in [t, T], \\ y(t) = x, \end{cases} \quad (3)$$

where  $x \in \mathbb{R}^d$ . The corresponding absolutely continuous solution is denoted by  $y_{t,x}^{\alpha[b],b}(\cdot)$ .

Furthermore, the following hypothesis will be considered throughout this paper:

**(H<sub>1</sub>)** The dynamics  $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^p \times \mathbb{R}^q \mapsto \mathbb{R}^d$  is continuous and for any  $R > 0$ , there exists  $L_f(R) > 0$ , such that for every  $(a, b) \in A \times B$ ,  $s \in [0, T]$  and  $y_1, y_2 \in \mathbb{R}^d$  verifying  $|y_1|, |y_2| \leq R$  :

$$|f(s, y_1, a, b) - f(s, y_2, a, b)| \leq L_f(R)|y_1 - y_2|.$$

Moreover, there exists  $c_f > 0$ , s.t.  $\forall y \in \mathbb{R}^d$ ,

$$\max\{|f(s, y, a, b)|, s \in [0, T], (a, b) \in A \times B\} \leq c_f(1 + |y|).$$

**(H<sub>2</sub>)** The running cost function  $\phi : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}$  is locally Lipschitz continuous, i.e. for any  $R > 0$  there exists  $L_\phi(R) > 0$  s.t. for any  $s \in [0, T]$  and  $y_1, y_2 \in \mathbb{R}^d$ ,  $|y_1|, |y_2| \leq R$  :

$$|\phi(s, y_1) - \phi(s, y_2)| \leq L_\phi(R)|y_1 - y_2|.$$

Moreover, there exists  $c_\phi$  such that  $\forall y \in \mathbb{R}^d$ ,  $\max\{|\phi(s, y)|, s \in [0, T]\} \leq c_\phi(1 + |y|)$ .

**(H<sub>3</sub>)** The final cost function  $\psi : \mathbb{R}^d \mapsto \mathbb{R}$  is locally Lipschitz continuous, i.e. for any  $R > 0$  there exists  $L_\psi(R) > 0$  s.t. for any  $y_1, y_2 \in \mathbb{R}^d$ ,  $|y_1|, |y_2| \leq R$  :

$$|\psi(y_1) - \psi(y_2)| \leq L_\psi(R)|y_1 - y_2|.$$

There exists also  $c_\psi > 0$  such that  $\forall y \in \mathbb{R}^d$ ,  $|\psi(y)| \leq c_\psi(1 + |y|)$ .

Let  $\mathcal{K}$  be a non-empty closed subset of  $\mathbb{R}^d$  representing the set of state constraints.

A trajectory  $y_{t,x}^{a,b}(\cdot)$ , associated to a couple of actions of the two players  $(a(\cdot), b(\cdot)) \in \mathcal{A} \times \mathcal{B}$ , is said to be admissible if it remains in  $\mathcal{K}$  at any time instant.

We are interested in the following state-constrained differential game with maximum running cost:

$$v(t, x) := \inf_{\alpha[\cdot] \in \Gamma} \pi(t, x; \alpha) \quad (4)$$

where

$$\pi(t, x; \alpha) := \begin{cases} \sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \phi(s, y_{t,x}^{\alpha[b],b}(s)) \right) \vee \psi(y_{t,x}^{\alpha[b],b}(T)) \right\}, & \text{if } y_{t,x}^{\alpha[b],b}(\cdot) \text{ is admissible } \forall b(\cdot) \in \mathcal{B}, \\ +\infty, & \text{else.} \end{cases}$$

Problem (4) describes the case when the first player is exploiting his information advantage and trying to find nonanticipative strategies that guarantee admissibility of trajectories against any choice of the second player and minimize the cost  $\pi$ .

One can imagine another situation where the players' actions are separated w.r.t. the dynamics and every player chooses his control in the aim of satisfying his own state constraints, see [6].

In general, for such state-constrained optimal control problems ( $\mathcal{K} \neq \mathbb{R}^d$ ), the value function  $v$  is not essentially continuous and may require further controllability assumptions to characterize it as the unique viscosity solution of an HJ equation. An idea about such assumptions in the case of a two-person differential game can be found in [15, 6, 7].

As mentioned earlier, we do not impose any controllability assumptions in this work. Following [1], we introduce an auxiliary control problem free of state constraints with a more regular value function allowing us to characterize  $v$ .

## 2.2 An auxiliary control problem

First consider the augmented dynamics  $\hat{f}$  defined for  $s \in [0, T]$ ,  $\hat{x} := (x, z) \in \mathbb{R}^d \times \mathbb{R}$  and  $(a, b) \in A \times B$  by:

$$\hat{f}(s, \hat{x}, a, b) = \begin{pmatrix} f(s, x, a, b) \\ 0 \end{pmatrix} \quad (5)$$

Denote by  $\hat{y}_{t,x,z}^{\alpha[b],b}(\cdot)$ , for  $(\alpha[\cdot], b(\cdot)) \in \Gamma \times \mathcal{B}$ , the solution of the following augmented differential system:

$$\begin{cases} \dot{\hat{y}}(s) = \hat{f}(s, \hat{y}(s), \alpha[b](s), b(s)), \text{ a.e. } s \in [t, T], \\ \hat{y}(t) = (x, z) \in \mathbb{R}^d \times \mathbb{R}. \end{cases} \quad (6)$$

Since the last component of the augmented dynamics  $\hat{f}$  is equal to zero,  $\hat{y}_{t,x,z}^{\alpha[b],b}(\cdot)$  can be expressed also as  $\hat{y}_{t,x,z}^{\alpha[b],b}(\cdot) := \left( y_{t,x}^{\alpha[b],b}(\cdot), z \right)$ , where  $y_{t,x}^{\alpha[b],b}(\cdot)$  is the solution of (3).

Moreover, the set of constraints  $\mathcal{K}$  is closed. Henceforth, it can be characterized as follows:

$$\forall y \in \mathbb{R}^d, \quad d_{\mathcal{K}}(y) \leq 0 \Leftrightarrow y \in \mathcal{K}, \quad (7)$$

where  $d_{\mathcal{K}}(\cdot)$  is the signed distance to  $\mathcal{K}$  which is Lipschitz continuous. Therefore an admissible trajectory  $y_{t,x}^{\alpha[b],b}(\cdot)$ , corresponding to a couple of controls  $(\alpha[b](\cdot), b(\cdot)) \in \mathcal{A} \times \mathcal{B}$  can be characterized by means of the signed distance:

$$\left[ y_{t,x}^{\alpha[b],b}(s) \in \mathcal{K}, \forall s \in [t, T] \right] \Leftrightarrow \max_{s \in [t, T]} d_{\mathcal{K}}(y_{t,x}^{\alpha[b],b}(s)) \leq 0.$$

The value function of the auxiliary problem,  $w$ , can be defined, for  $t \in [0, T]$  and  $(x, z) \in \mathbb{R}^d \times \mathbb{R}$ , by :

$$w(t, x, z) := \inf_{\alpha[\cdot] \in \Gamma} \sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \hat{\phi}(s, y_{t,x}^{\alpha[b],b}(s), z) \right) \vee \hat{\psi}(y_{t,x}^{\alpha[b],b}(T), z) \right\} \quad (8)$$

where for  $(x, z) \in \mathbb{R}^d \times \mathbb{R}$  and  $s \in [0, T]$ , the functions  $\hat{\phi}$  and  $\hat{\psi}$  are given by:

$$\hat{\phi}(s, x, z) := (\phi(s, x) - z) \vee d_{\mathcal{K}}(x) \quad \text{and} \quad \hat{\psi}(x, z) := \psi(x) - z.$$

**Remark 2.1** When the constrained problem (4) is of type Bolza, the auxiliary problem can be formulated by modifying the augmented dynamics and the functions  $\hat{\phi}$  and  $\hat{\psi}$ . In this case, the objective function  $J$  is given by:

$$J(t, x, a, b) := \int_t^T \ell(s, y_{t,x}^{a,b}(s), a(s), b(s)) + \psi(y_{t,x}^{a,b}(T))$$

for  $(t, x) \in [0, T] \times \mathbb{R}^d$  and  $(a(\cdot), b(\cdot)) \in \mathcal{A} \times \mathcal{B}$  and where  $\ell$  and  $\psi$  are respectively the distributed and the final cost functions.

For  $\hat{x} = (x, z) \in \mathbb{R}^d \times \mathbb{R}$ , consider the augmented dynamics  $\hat{f}$ :

$$\hat{f}(s, \hat{x}, a, b) := \begin{pmatrix} f(s, x, a, b) \\ -\ell(s, x, a, b) \end{pmatrix},$$

and the cost functions  $\hat{\phi}$  and  $\hat{\psi}$ :

$$\hat{\phi}(s, \hat{x}) := d_{\mathcal{K}}(x) \quad \text{and} \quad \hat{\psi}(\hat{x}) = \psi(x) - z.$$

Let  $\hat{y}_{t, \hat{x}}^{\alpha[b], b}(\cdot)$  be the unique continuous solution of the following differential system, associated to  $(\alpha[b](\cdot), b(\cdot)) \in \mathcal{A} \times \mathcal{B}$ :

$$\begin{cases} \dot{\hat{y}}(s) = \hat{f}(s, \hat{y}(s), \alpha[b](s), b(s)), \text{ a.e. in } [t, T], \\ \hat{y}(t) = \hat{x} := (x, z) \in \mathbb{R}^d \times \mathbb{R}. \end{cases}$$

Therefore, the corresponding auxiliary problem is defined by:

$$w(t, \hat{x}) := \inf_{\alpha[\cdot] \in \Gamma} \sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \hat{\phi}(s, \hat{y}_{t, \hat{x}}^{\alpha[b], b}(s)) \right) \vee \hat{\psi}(\hat{y}_{t, \hat{x}}^{\alpha[b], b}(T)) \right\}$$

for  $t \in [0, T]$  and  $\hat{x} \in \mathbb{R}^d \times \mathbb{R}$ .

The above formulation still valid even for a problem of type Mayer ( $\ell \equiv 0$ ). On the other hand, all the results that will be seen in the following sections still true also for a state constrained problem of type Bolza or Mayer.

### 3 Properties of the value functions $v$ and $w$

This section is devoted to some properties of the auxiliary value function,  $w$ , and to show how it can be used to characterize  $v$ , the value function of the constrained problem (4).

The following proposition gives some results concerning the regularity of  $w$  and its characterization through Hamilton Jacobi equations.

**Proposition 3.1** *Assume that hypothesis  $(\mathbf{H}_1)$ ,  $(\mathbf{H}_2)$  and  $(\mathbf{H}_3)$  hold, then:*

(i)  *$w$  verifies a dynamic programming principle. For any  $h \in [0, T - t]$ ,*

$$w(t, x, z) = \inf_{\alpha[\cdot] \in \Gamma} \sup_{b(\cdot) \in \mathcal{B}} \left\{ w(t+h, y_{t, x}^{\alpha[b], b}(t+h), z) \vee \left( \max_{s \in [t, t+h]} \hat{\phi}(s, y_{t, x}^{\alpha[b], b}(s), z) \right) \right\}. \quad (9)$$

(ii) *The value function  $w$  is locally Lipschitz continuous on  $[0, T] \times \mathbb{R}^d \times \mathbb{R}$ .*

(iii)  *$w$  is the unique viscosity solution of the following HJ equation:*

$$\begin{cases} \min \left( -\partial_t w(t, x, z) + H(t, x, D_x w(t, x, z)), w(t, x, z) - \hat{\phi}(t, x, z) \right) = 0, & \text{in } [0, T] \times \mathbb{R}^d \times \mathbb{R}, \\ w(T, x, z) = \hat{\phi}(T, x, z) \vee \hat{\psi}(x, z), & \text{in } \mathbb{R}^d \times \mathbb{R}, \end{cases} \quad (10)$$

where  $\partial_t$  stands for the time derivative,  $D_x$  stands for the derivative w.r.t  $x$  and the hamiltonian  $H$  is given by:

$$H(t, x, p) := \min_{b \in \mathcal{B}} \max_{a \in \mathcal{A}} - \langle f(t, x, a, b), p \rangle, \quad (11)$$

for  $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ .

*Proof.* First, the dynamic programming principle (9) is a classical result that can be found for instance in [4, Chapter VIII]. Moreover, since the auxiliary problem is free of state constraints, one can show that  $w$  is locally Lipschitz continuous by the same arguments used in [1]. Finally, it remains to prove that  $w$  is the unique viscosity solution of (10).

Let  $\xi$  be a function of class  $C^1$  and  $(t, x, z)$  be a local minimum for  $w - \xi$  such that  $w(t, x, z) = \xi(t, x, z)$ . Assume that there exists  $\delta > 0$  such that

$$-\partial_t \xi(t, x, z) + H(t, x, D_x \xi(t, x, z)) = -\delta < 0.$$

In this case, there exists  $b_0 \in B$  such that, for any  $a \in A$ ,

$$-\partial_t \xi(t, x, z) - \langle f(t, x, a, b_0), D_x \xi(t, x, z) \rangle \leq -\delta.$$

Now let fix  $h > 0$  small enough and  $\alpha[\cdot] \in \Gamma$ . For any time instant  $s \in [t, T]$ , we have that  $\alpha[b_0](s) \in A$ . Hence, from the last inequality, we get:

$$-\partial_t \xi(t, x, z) - \langle f(t, x, \alpha[b_0](s), b_0), D_x \xi(t, x, z) \rangle \leq -\delta, \text{ for any } s \in [t, t+h].$$

By continuity of  $\xi$ ,  $f$  and  $D_x \xi$ , we obtain, for any  $s \in [t, t+h]$ :

$$-\partial_t \xi(s, y_{t,x}^{\alpha[b_0], b_0}(s), z) - \langle f(s, y_{t,x}^{\alpha[b_0], b_0}(s), \alpha[b_0](s), b_0), D_x \xi(s, y_{t,x}^{\alpha[b_0], b_0}(s), z) \rangle \leq -\frac{\delta}{2},$$

henceforth

$$\int_t^{t+h} \left( -\partial_t \xi(s, y_{t,x}^{\alpha[b_0], b_0}(s), z) - \langle f(s, y_{t,x}^{\alpha[b_0], b_0}(s), \alpha[b_0](s), b_0), D_x \xi(s, y_{t,x}^{\alpha[b_0], b_0}(s), z) \rangle \right) ds \leq -\frac{\delta h}{2},$$

which implies that

$$\xi(t, x, z) - \xi(t+h, y_{t,x}^{\alpha[b_0], b_0}(t+h), z) \leq -\frac{\delta h}{2}.$$

On the other hand, we have

$$(w - \xi)(t+h, y_{t,x}^{\alpha[b_0], b_0}(t+h), z) \geq (w - \xi)(t, x, z).$$

From the two above inequalities, we deduce that

$$\frac{\delta h}{2} + w(t, x, z) \leq w(t+h, y_{t,x}^{\alpha[b_0], b_0}(t+h), z) \leq w(t+h, y_{t,x}^{\alpha[b_0], b_0}(t+h), z) \bigvee \left( \max_{s \in [t, t+h]} \hat{\phi}(s, y_{t,x}^{\alpha[b_0], b_0}(s), z) \right).$$

Therefore we get

$$\frac{\delta h}{2} + w(t, x, z) \leq \sup_{b(\cdot) \in \mathcal{B}} \left\{ w(t+h, y_{t,x}^{\alpha[b], b}(t+h), z) \bigvee \left( \max_{s \in [t, t+h]} \hat{\phi}(s, y_{t,x}^{\alpha[b], b}(s), z) \right) \right\}.$$

Since the last inequality holds for any  $\alpha[\cdot] \in \Gamma$ , we deduce that

$$\frac{\delta h}{2} + w(t, x, z) \leq \inf_{\alpha[\cdot] \in \Gamma} \sup_{b(\cdot) \in \mathcal{B}} \left\{ w(t+h, y_{t,x}^{\alpha[b], b}(t+h), z) \bigvee \left( \max_{s \in [t, t+h]} \hat{\phi}(s, y_{t,x}^{\alpha[b], b}(s), z) \right) \right\} = w(t, x, z),$$

which is impossible. We conclude that

$$-\partial_t \xi(t, x, z) + H(t, x, D_x \xi(t, x, z)) \geq 0.$$

On the other hand,  $\hat{\phi}(t, x, z) \leq \max_{s \in [t, T]} \hat{\phi}(s, y_{t,x}^{\alpha[b], b}(s), z) \leq w(t, x, z) = \xi(t, x, z)$ .

As a conclusion,  $\min \left( -\partial_t \xi(t, x, z) + H(t, x, D_x \xi(t, x, z)), (\xi - \hat{\phi})(t, x, z) \right) \geq 0$ , which means that  $w$  is a super-solution of (10).

Now, we will show that  $w$  is a sub-solution of (10). Let  $\xi$  be a function of class  $C^1$  such that  $w - \xi$  has a maximum at  $(t, x, z)$  and  $w(t, x, z) = \xi(t, x, z)$ .

If  $\xi(t, x, z) = w(t, x, z) \leq \hat{\phi}(t, x, z)$ , then  $\xi$  satisfies:

$$\min \left( -\partial_t \xi(t, x, z) + H(t, x, D_x \xi(t, x, z)), (\xi - \hat{\phi})(t, x, z) \right) \leq 0.$$

which means that  $w$  is a sub-solution of (10).

If not, we get  $w(t, x, z) > \hat{\phi}(t, x, z)$ . Henceforth, there exists  $\tau > 0$ , such that for any admissible trajectory  $y_{t,x}^{\alpha[b],b}(\cdot)$ , we have  $w(s, y_{t,x}^{\alpha[b],b}(s), z) > \hat{\phi}(s, y_{t,x}^{\alpha[b],b}(s), z)$ , for any  $s \in [t, t + \tau]$ . In this case and by using the dynamic programming principle (9) verified by  $w$  between  $t$  and  $t + h$ , for any  $0 < h \leq \tau$ , we get

$$w(t, x, z) = \inf_{\alpha[\cdot] \in \Gamma} \sup_{b(\cdot) \in \mathcal{B}} w(t + h, y_{t,x}^{\alpha[b],b}(s), z)$$

From [4, Chapter VIII], we get the following inequality:

$$-\partial_t \xi(t, x, z) + H(t, x, D_x \xi(t, x, z)) \leq 0,$$

which ends the proof.

Uniqueness of  $w$  as a viscosity solution of (10) comes from the comparison result given in Appendix of [1] and which remains true for this class of Hamilton-Jacobi equations.  $\square$

After its characterization,  $w$  can be exploited to get information on the value function of the constrained problem  $v$  and this is the aim of the following theorem.

**Theorem 3.2** *Assume that hypothesis  $(\mathbf{H}_1)$ ,  $(\mathbf{H}_2)$  and  $(\mathbf{H}_3)$  hold. The value function  $v$  can be estimated by means of  $w$  through the following relation :*

$$\inf\{z \in \mathbb{R} \mid w(t, x, z) \leq 0\} \leq v(t, x) \leq \inf\{z \in \mathbb{R} \mid w(t, x, z) < 0\}, \quad (12)$$

for any  $(t, x) \in [0, T] \times \mathcal{K}$ .

*Proof.* Let  $(t, x) \in [0, T] \times \mathcal{K}$ . First, we will assume that  $v(t, x) < +\infty$  and prove the following inequality:  $\inf\{z \in \mathbb{R} \mid w(t, x, z) \leq 0\} \leq v(t, x)$  (the inequality is obvious if  $v(t, x) = +\infty$ ).

Let  $z \in \mathbb{R}$  be such that  $v(t, x) < z$ . By definition of  $v$ , there exists a nonanticipative strategy  $\alpha_0[\cdot] \in \Gamma$  s.t. for any  $b(\cdot) \in \mathcal{B}$ , the trajectory  $y_{t,x}^{\alpha_0[b],b}(\cdot)$  remains in  $\mathcal{K}$  and  $\pi(t, x; \alpha_0) \leq z$ . notice that if  $v(t, x) = +\infty$  then it is obvious that the following inequality holds:

Therefore for any  $b(\cdot) \in \mathcal{B}$  we get

$$\max_{s \in [t, T]} d_{\mathcal{K}}(y_{t,x}^{\alpha_0[b],b}(s)) \leq 0 \text{ and } \sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \phi(s, y_{t,x}^{\alpha_0[b],b}(s)) \right) \vee \psi(y_{t,x}^{\alpha_0[b],b}(T)) \right\} \leq z.$$

From the two above inequalities, we deduce that

$$\sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \hat{\phi}(s, y_{t,x}^{\alpha_0[b],b}(s), z) \right) \vee \hat{\psi}(y_{t,x}^{\alpha_0[b],b}(T), z) \right\} \leq 0.$$

We conclude that  $w(t, x, z) \leq 0$  whenever  $z > v(t, x)$ . By continuity of  $w$  w.r.t.  $z$ , we deduce that

$$\forall z \in \mathbb{R}, \quad z \geq v(t, x) \implies w(t, x, z) \leq 0. \quad (13)$$

In particular, we have  $w(t, x, v(t, x)) \leq 0$ . Henceforth

$$\inf\{z \in \mathbb{R} \mid w(t, x, z) \leq 0\} \leq v(t, x).$$

Now let  $z \in \mathbb{R}$  such that  $w(t, x, z) < 0$ . Let  $\bar{\delta} := -w(t, x, z) > 0$ . From the definition of  $w$  and the infimum property, there exists  $\bar{\alpha}[\cdot] \in \Gamma$  verifying:

$$\sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \hat{\phi}(s, y_{t,x}^{\bar{\alpha}[b], b}(s), z) \right) \bigvee \hat{\psi}(y_{t,x}^{\bar{\alpha}[b], b}(T), z) \right\} \leq w(t, x, z) + \bar{\delta} = 0.$$

Therefore for any  $b(\cdot) \in \mathcal{B}$  the trajectory  $y_{t,x}^{\bar{\alpha}[b], b}(\cdot)$  is admissible and

$$\sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \phi(s, y_{t,x}^{\bar{\alpha}[b], b}(s)) \right) \bigvee \psi(y_{t,x}^{\bar{\alpha}[b], b}(T)) \right\} \leq z,$$

which means that  $\pi(t, x; \bar{\alpha}) \leq z$ . By definition of  $v$ , we deduce that  $v(t, x) \leq z$  for every  $z \in \mathbb{R}$  such that  $w(t, x, z) < 0$ . This proves that

$$v(t, x) \leq \inf\{z \in \mathbb{R} | w(t, x, z) < 0\}.$$

□

**Remark 3.3** For any  $(t, x) \in [0, T] \times \mathcal{K}$ , if  $\inf\{z \in \mathbb{R} | w(t, x, z) \leq 0\} = +\infty$ , thereby  $v(t, x) = +\infty$  which means that there is no strategies of the first player that guarantee admissibility of trajectories for any action of the second player. However if  $\inf\{z \in \mathbb{R} | w(t, x, z) \leq 0\} \in \mathbb{R}$ , the infimum is reached by some  $z \in \mathbb{R}$ . Furthermore, if  $\inf\{z \in \mathbb{R} | w(t, x, z) < 0\} \in \mathbb{R}$  one can prove that:

$$\inf\{z \in \mathbb{R} | w(t, x, z) \leq 0\} = v(t, x) = \inf\{z \in \mathbb{R} | w(t, x, z) < 0\}.$$

On the other hand it may occur that  $\inf\{z \in \mathbb{R} | w(t, x, z) \leq 0\} < \infty$  and  $\inf\{z \in \mathbb{R} | w(t, x, z) < 0\} = +\infty$  and in this case we have no information on  $v(t, x)$ .

**Remark 3.4** In [1], it was shown that when some convexity assumption is verified by  $f$ , a precise connection is established between  $v$  and  $w$ . Theorem 3.2 gives a more general result on the link between those two value functions without assuming any additional assumption.

When some convexity assumption is verified by  $f$ , we can prove a more precise connection. For this, assume that:

(H<sub>4</sub>) The dynamics  $f$  is affine in the first control input  $a^1$  and  $A$  is a convex set of  $\mathbb{R}^p$ .

The aim of the following Theorem is to characterize the value of the constrained problem  $v$  through the auxiliary value function  $w$ . In addition to that, it establishes a link between optimal strategies of the auxiliary and the constrained problems.

**Theorem 3.5** Assume that assumptions (H<sub>1</sub>), (H<sub>2</sub>), (H<sub>3</sub>) and (H<sub>4</sub>) hold and let  $(t, x) \in [0, T] \times \mathcal{K}$ .

(i) Suppose that  $w(t, x, z) \leq 0$  for some  $z \in \mathbb{R}$ , then there exists  $\alpha^*[\cdot] \in \Gamma$  s.t. for any  $b(\cdot) \in \mathcal{B}$ , the trajectory  $y_{t,x}^{\alpha^*[b], b}(\cdot)$  is admissible and

$$\left( \max_{s \in [t, T]} \phi(s, y_{t,x}^{\alpha^*[b], b}(s)) \right) \bigvee \psi(y_{t,x}^{\alpha^*[b], b}(T)) \leq z.$$

(ii) The exact value of  $v$  can be obtained by the following relation:

$$v(t, x) = \inf\{z \in \mathbb{R} | w(t, x, z) \leq 0\}. \quad (14)$$

(iii) If  $v(t, x) < \infty$ , then any optimal strategy for the auxiliary problem (8) on  $[t, T]$  associated to  $(x, v(t, x))$  is optimal for the constrained problem (4) on  $[t, T]$  associated with the initial position  $x$ .

<sup>1</sup> $f$  is of the form  $f(t, x, a, b) := f_0(t, x, b) + f_1(t, x, b)a$ .

*Proof.*

Assertion (i) states that, for every  $(t, x, z) \in [0, T] \times \mathcal{K} \times \mathbb{R}$ , the auxiliary control problem admits an optimal strategy  $\alpha^*[\cdot]$ . This claim holds when assumption **(H<sub>4</sub>)** is satisfied, and it can be proved by adapting arguments from [12, 13, 14, 18]. For the sake of completeness, a sketch of the proof of (i) is given in Appendix A.

Now we will prove (ii). From Theorem 3.2, we already know that

$$\inf\{z \in \mathbb{R} | w(t, x, z) \leq 0\} \leq v(t, x).$$

On the other hand, let  $z \in \mathbb{R}$  s.t.  $w(t, x, z) \leq 0$ . From assertion (i), there exists  $\alpha^*[\cdot] \in \Gamma$  s.t. for any  $b(\cdot) \in \mathcal{B}$ , the trajectory  $y_{t,x}^{\alpha^*[b],b}(\cdot)$  is admissible and

$$\left( \max_{s \in [t, T]} \phi(s, y_{t,x}^{\alpha^*[b],b}(s)) \right) \bigvee \psi(y_{t,x}^{\alpha^*[b],b}(T)) \leq z.$$

which means that  $\pi(t, x; \alpha^*) \leq z$ . By definition of  $v$ , we conclude that  $v(t, x) \leq z$ , for any  $z \in \mathbb{R}$  verifying  $w(t, x, z) \leq 0$ . Henceforth,

$$v(t, x) \leq \inf\{z \in \mathbb{R} | w(t, x, z) \leq 0\}.$$

It remains to prove assertion (iii). For this reason, let  $\alpha^*[\cdot] \in \Gamma$  be an optimal strategy to the auxiliary problem (8) on  $[t, T]$  associated to the initial point  $(x, z)$ , where  $z := v(t, x)$ , which means that

$$w(t, x, z) = \sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \hat{\phi}(s, y_{t,x}^{\alpha^*[b],b}(s), z) \right) \bigvee \hat{\psi}(y_{t,x}^{\alpha^*[b],b}(T), z) \right\}.$$

Since  $z = v(t, x)$ , statement (13) concludes that  $w(t, x, z) \leq 0$  and thus

$$\sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \hat{\phi}(s, y_{t,x}^{\alpha^*[b],b}(s), z) \right) \bigvee \hat{\psi}(y_{t,x}^{\alpha^*[b],b}(T), z) \right\} \leq 0.$$

Therefore, for any  $b(\cdot) \in \mathcal{B}$ , the trajectory  $y_{t,x}^{\alpha^*[b],b}(\cdot)$  is admissible and

$$\sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \phi(s, y_{t,x}^{\alpha^*[b],b}(s)) \right) \bigvee \psi(y_{t,x}^{\alpha^*[b],b}(T)) \right\} \leq v(t, x).$$

In other words,  $\pi(t, x; \alpha^*) \leq v(t, x) = \inf_{\alpha[\cdot] \in \Gamma} \pi(t, x; \alpha)$ . We conclude that  $\alpha^*[\cdot]$  is an optimal strategy for the constrained problem (4) associated to the initial state  $x$ .  $\square$

### Comments: Reduction of the computational domain

Since problem (8) is without state constraints, the auxiliary value function  $w$  is defined on  $[0, T] \times \mathbb{R}^d \times \mathbb{R}$ . Nevertheless, for computational issues, we should restrict the domain of interest of  $w$  to a neighbourhood of  $\mathcal{K} \times \mathbb{R}$ . Therefore, we will follow a technique developed in [2, Section 3.2] for which the auxiliary value function  $w$  will keep a constant value outside a neighbourhood of  $\mathcal{K} \times \mathbb{R}$ . Let  $\mu > 0$  be a fixed parameter and  $\mathcal{K}_\mu$  be a neighbourhood of  $\mathcal{K}$  defined by  $\mathcal{K}_\mu := \mathcal{K} + \mu \mathbb{B}_{\mathbb{R}^d}$ .

The idea consists in introducing a truncation of  $d_{\mathcal{K}}$ ,  $\hat{\phi}$  and  $\hat{\psi}$  to obtain a new control problem free of state constraints with value function  $w_\mu$  taking constant value outside  $\mathcal{K}_\mu$ .

First consider the Lipschitz continuous function  $d_{\mathcal{K}}^\mu := d_{\mathcal{K}} \wedge \mu$  which verifies for any  $y \in \mathbb{R}^d$ :

$$d_{\mathcal{K}}^\mu(y) \leq 0 \Leftrightarrow y \in \mathcal{K}, \quad d_{\mathcal{K}}^\mu(y) \leq \mu, \quad \text{and} \quad d_{\mathcal{K}}^\mu(y) = \mu \Leftrightarrow y \notin \overset{\circ}{\mathcal{K}}_\mu.$$

Furthermore, we consider a truncation of  $\hat{\phi}$  and  $\hat{\psi}$  as follows:

$$\hat{\phi}_\mu := \hat{\phi} \wedge \mu \quad \text{and} \quad \hat{\psi}_\mu := \hat{\psi} \wedge \mu. \tag{15}$$

Finally, we define the specific auxiliary value function  $w_\mu$ , for  $(t, x, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$  as:

$$w_\mu(t, x, z) := \inf_{\alpha[\cdot] \in \Gamma} \sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \hat{\phi}_\mu(s, y_{t,x}^{\alpha[b], b}(s), z) \right) \bigvee \hat{\psi}_\mu(y_{t,x}^{\alpha[b], b}(T), z) \right\}, \quad (16)$$

which verifies the following relation

$$w_\mu(t, x, z) = w(t, x, z) \bigwedge \mu.$$

Since, we are interested in the region  $\{z | w(t, x, z) \leq 0\}$ , for  $(t, x) \in [0, T] \times \mathcal{K}$ , which coincides with  $\{z | w_\mu(t, x, z) \leq 0\}$  for any  $\mu > 0$ , it does not matter which auxiliary value function we use ( $w$  or  $w_\mu$ ). Therefore, for the sequel we will confound  $w$  and  $w_\mu$ , for  $\mu > 0$ , denoted simply by  $w$ .

The question now is how to characterize  $w_\mu$  and this is the aim of the following proposition.

**Proposition 3.6**  *$w_\mu$  is the unique viscosity solution of the following HJ equation:*

$$\begin{cases} \min \left( -\partial_t w_\mu + H(t, x, \nabla_x w_\mu), w_\mu(t, x, z) - \hat{\phi}_\mu(t, x, z) \right) = 0, & \text{in } [0, T] \times \overset{\circ}{\mathcal{K}}_\mu \times \mathbb{R}, \\ w_\mu(T, x, z) = \hat{\phi}_\mu(T, x, z) \bigvee \hat{\psi}_\mu(x, z), & \text{in } \overset{\circ}{\mathcal{K}}_\mu \times \mathbb{R}, \\ w_\mu(t, x, z) = \mu, & \text{for all } t \in [0, T], x \notin \overset{\circ}{\mathcal{K}}_\mu \text{ and } z \in \mathbb{R}. \end{cases} \quad (17)$$

The proof of Proposition 3.6 can be done as in [2, Theorem 3.5].

On the other hand, suppose that cost functions  $\phi$  and  $\psi$  are bounded (they take values on some interval  $[m, M]$ ). Thus, to establish estimations for the value function  $v$  or to find its exact value, it is enough to consider the auxiliary variable  $z$  in  $[m, M]$ .

## 4 Reconstruction of trajectories based on the value function

### 4.1 Introduction

Consider a differential game free of state constraints with maximum running and final cost functions. We present in this section a reconstruction procedure based on the knowledge of an approximation of this differential game value function. We propose to discretize in time the continuous problem and to reconstruct approximated optimal feedbacks.

For a given finite time horizon  $T > 0$ , consider the differential system:

$$\begin{cases} \dot{\zeta}(s) = F(s, \zeta(s), a(s), b(s)), & \text{a.e. in } [t, T], \\ \zeta(t) = \chi, \end{cases} \quad (18)$$

where  $\chi \in \mathbb{R}^m$ ,  $m \geq 1$ , and  $(a(\cdot), b(\cdot)) \in \mathcal{A} \times \mathcal{B}$  are the actions of the first and the second player respectively.

The value function of the differential game studied in this section is defined by:

$$u(t, \chi) := \inf_{\alpha[\cdot] \in \Gamma} \sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \Phi(s, \zeta_{t,\chi}^{\alpha[b], b}(s)) \right) \bigvee \Psi(\zeta_{t,\chi}^{\alpha[b], b}(T)) \right\}, \quad (19)$$

where  $\Phi$  and  $\Psi$  are the maximum running and final cost functions respectively. We impose the following hypothesis:

**(H<sub>5</sub>)** Suppose that  $F$ ,  $\Phi$  and  $\Psi$  satisfy respectively hypothesis **(H<sub>1</sub>)**, **(H<sub>2</sub>)** and **(H<sub>3</sub>)**.

Under hypothesis **(H<sub>5</sub>)** and from section 2,  $u$  is the unique solution of the following HJ equation with obstacle term:

$$\begin{cases} \min \left( -\partial_t u(t, \chi) + H(t, \chi, D_\chi u(t, \chi)), u(t, \chi) - \Phi(t, \chi) \right) = 0, & \text{in } [0, T] \times \mathbb{R}^m, \\ u(T, \chi) = \Phi(T, \chi) \bigvee \Psi(\chi), & \text{in } \mathbb{R}^m, \end{cases} \quad (20)$$

where the hamiltonian  $H$  is given by:

$$H(t, \chi, p) := \min_{b \in B} \max_{a \in A} - \langle F(t, \chi, a, b), p \rangle,$$

for  $(t, \chi, p) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}^m$ .

We denote by  $J$  the global cost in (19):

$$J(t, \chi, a, b) := \left( \max_{s \in [t, T]} \Phi(s, \zeta_{t, \chi}^{\alpha[b], b}(s)) \right) \bigvee \Psi(\zeta_{t, \chi}^{\alpha[b], b}(T)), \quad (21)$$

for  $(t, \chi) \in [0, T] \times \mathbb{R}^m$  and  $(a(\cdot), b(\cdot)) \in \mathcal{A} \times \mathcal{B}$ .

Recall that the aim of this section is to approximate optimal feedbacks of the differential game (19). To this end, we will discretize in time and synthesize from the discrete differential game an approximation of the optimal strategy of the first player and the optimal control of the second player.

For simplicity, consider a uniform time partition of  $[0, T]$  with a step time  $h = \frac{T}{N}$ ,  $N \in \mathbb{N}^*$  :  $s_0 = 0$ ,  $s_1 = h, \dots$ ,  $s_k = kh, \dots$ ,  $s_N = Nh = T$ .

The dynamical system (18) can be approximated through the following Euler forward scheme:

$$\begin{cases} \zeta_k = \chi \\ \zeta_{k+1} = \chi + (s_{k+1} - t)F(t, \zeta_k, a_k, b_k) \\ \zeta_{i+1} = \zeta_i + hF(s_i, \zeta_i, a_i, b_i), \quad i = k + 1, \dots, N - 1. \end{cases} \quad (22)$$

for  $t \in [s_k, s_{k+1}]$ , s.t.  $0 \leq k \leq N - 1$ ,  $\chi \in \mathbb{R}^m$ ,  $(a_i) \in A^{N-k}$  and  $(b_k) \in B^{N-k}$ .

More precise approximations of (18) can be considered by using higher order Runge-Kutta schemes.

Finally, even for the discrete game, we attribute to the first player an advantage of information. This advantage of information can be modeled by discrete nonanticipative strategies. Indeed, at each step time  $s_i$  and before choosing his action  $a_i \in A$ , the first player knows the choice of his opponent  $b_i \in B$ . Following the formulation presented in [4, Chapter VIII], the set of discrete nonanticipative strategies of the first player  $\Gamma^h$  can be define as:

$$\Gamma^h := \left\{ \alpha_h[\cdot] : B^N \rightarrow A^N, \text{ s.t. } \forall (b_i), (b'_i) \in B^N, \text{ and } \forall k \in \mathbb{N}, k \leq N - 1, \right. \\ \left. (b_j = b'_j \text{ for any } j \leq k) \Rightarrow (\alpha_h[b]_j = \alpha_h[b']_j \text{ for any } j \leq k) \right\}.$$

Subsection 4.2 deals with the case of trajectory reconstruction by use of a general class of approximation functions  $u_h$  while in subsection 4.3, we show a convergence result when the approximation  $u_h$  verifies a specific criterion.

## 4.2 A general reconstruction procedure

In this section, we consider an approximation  $u_h$  of  $u$ . This approximation could be a numerical solution obtained by solving a discretized scheme of the Hamilton Jacobi equation (20) verified by  $u$ .

Let  $E_h$  denote the uniform error estimate between  $u$  and  $u_h$  given by  $E_h := \|u_h - u\|$ .

(**H<sub>6</sub>**) Suppose that the error estimate  $E_h$  satisfies:  $E_h = o(h)$ .

Notice that several numerical schemes for solving HJ equation (20) have been studied in the literature. Under monotonicity, stability and consistency assumptions, the error estimates  $E_h$  is of order  $O(\sqrt{\Delta T})$  where  $\Delta t$  is the time step size of the scheme, see [8]. Therefore, to satisfy assumption (**H<sub>6</sub>**), it suffices to consider, for example, a convergent numerical scheme of (20) with  $\Delta t = h^3$ .

The reconstruction procedure presented in Algorithm 1 corresponds to the case where choices of the second player are not optimal and take arbitrary values in  $B$ . The first player will observe his opponent

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**Algorithm 1: Arbitrary case**


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**Require:**  $\chi \in \mathbb{R}^m$ .

- 1: Initialise  $\zeta_0 = \chi$ .
- 2: **for**  $i = 0 \cdots N - 1$  **do**
- 3:   An arbitrary choice of the second player  $b_i \in B$ .
- 4:   The optimal reaction of the first player,  $a_i^* \in A$ , s.t.

$$\alpha_h^*[b]_i := a_i^* \in \operatorname{argmin}_{a \in A} \left\{ u_h(s_{i+1}, \zeta_i + hF(s_i, \zeta_i, a, b_i)) \bigvee \Phi(s_i, \zeta_i) \right\}.$$

- 5:   The new state position:  $\zeta_{i+1} = \zeta_i + hF(s_i, \zeta_i, a_i^*, b_i)$ .
  - 6: **end for**
- 

choice and will choose his optimal reaction. This algorithm is given for a particular reconstruction from an initial time  $t = 0$  and an initial position  $\chi \in \mathbb{R}^m$ .

For  $\chi \in \mathbb{R}^m$ , let  $(a_k^*)$ ,  $(b_k)$ ,  $(\zeta_k)$  be the sequences generated by Algorithm 1 and we define the following piecewise constant controls  $(\alpha_h^*[b_h](\cdot), b_h(\cdot)) \in \mathcal{A} \times \mathcal{B}$ , such that  $\alpha_h^*[b_h](s) := a_k^*$  and  $b_h(s) := b_k$  for  $s \in [s_k, s_{k+1}[$  and an approximate trajectory  $\zeta_h(\cdot)$  solution of:

$$\begin{cases} \dot{\zeta}_h(s) = F(s, \zeta_h(s), \alpha_h^*[b_h](s), b_h(s)), a.e. \text{ in } [0, T], \\ \zeta_h(0) = \chi, \end{cases} \quad (23)$$

**Theorem 4.1** *Assume that hypothesis **(H<sub>5</sub>)** and **(H<sub>6</sub>)** hold and that assumption **(H<sub>4</sub>)** is verified for  $F$  and  $A$ . For  $\chi \in \mathbb{R}^m$ , the trajectory  $\zeta_h(\cdot)$ , defined in (23), verifies:*

$$\limsup_{h \rightarrow 0^+} \left\{ \left( \max_{s \in [t, T]} \Phi(s, \zeta_h(s)) \right) \bigvee \Psi(\zeta_h(T)) \right\} \leq u(0, \chi). \quad (24)$$

*Proof.* Let  $\chi \in \mathbb{R}^m$ , and let  $(\zeta_k)$ ,  $(a_k^*)$  and  $(b_k)$  be the sequences of trajectory and players' actions generated by Algorithm 1.

Since  $F$  is locally Lipschitz continuous, there exists  $R > 0$  such that for any  $h > 0$  and any  $0 \leq k \leq N$ , we have  $\|\zeta_k\| \leq R$ . We can choose the constant  $R$  large enough such that any trajectory starting from any initial position  $\zeta_k$  will remain in  $\mathbb{B}_R$ , the ball of  $\mathbb{R}^d$  centred at 0 and with radius  $R$ . Let  $M_R > 0$  be a constant verifying:

$$\|F(s, \zeta, a, b)\| \leq M_R, \forall s \in [0, T], \forall \zeta \in \mathbb{B}_R, \forall (a, b) \in A \times B.$$

**Step 1.** We establish that there exists  $\epsilon_h > 0$ , s.t.  $\lim_{h \rightarrow 0} \epsilon_h = 0$ , and

$$u_h(s_0, \chi) \geq u_h(s_1, \zeta_1) \bigvee \Phi(s_0, \chi) - h\epsilon_h - 2E_h. \quad (25)$$

By the dynamic programming principle verified by  $u$ , between  $s_0 = 0$  and  $s_1 = h$ , we get

$$u(s_0, \chi) = \inf_{\alpha[\cdot] \in \Gamma} \sup_{b(\cdot) \in \mathcal{B}} \left\{ u(s_1, \zeta_{s_0, \chi}^{\alpha[b], b}(s_1)) \bigvee \left( \max_{s \in [s_0, s_1]} \Phi(s, \zeta_{s_0, \chi}^{\alpha[b], b}(s)) \right) \right\},$$

which implies that

$$u(s_0, \chi) \geq \inf_{\alpha[\cdot] \in \Gamma} \sup_{b(\cdot) \in \mathcal{B}} \left\{ u(s_1, \zeta_{s_0, \chi}^{\alpha[b], b}(s_1)) \bigvee \Phi(s_0, \chi) \right\}.$$

For  $\epsilon > 0$ , we pick an  $\epsilon$ -optimal strategy,  $\alpha^\epsilon[\cdot] \in \Gamma$ , and the above inequality becomes:

$$u(s_0, \chi) \geq -\epsilon + \sup_{b(\cdot) \in \mathcal{B}} \left\{ u(s_1, \zeta_{s_0, \chi}^{\alpha^\epsilon[b], b}(s_1)) \bigvee \Phi(s_0, \chi) \right\}.$$

Let  $b_0(\cdot) \in \mathcal{B}$  be a constant control such that  $b_0(\cdot) \equiv b_0 \in B$  ( $b_0$  generated by Algorithm 1). The above inequality becomes:

$$u(s_0, \chi) \geq -\epsilon + u(s_1, \zeta_{s_0, \chi}^{a^\epsilon, b_0}(s_1)) \bigvee \Phi(s_0, \chi). \quad (26)$$

We set  $a^\epsilon(\cdot) := \alpha^\epsilon[b_0](\cdot) \in \mathcal{A}$ . On the other hand, since the set  $F(s_0, \chi, A, b_0)$  is convex (assumption  $(\mathbf{H}_4)$ ), there exists  $a_0 \in A$  such that:

$$\zeta_0 + \int_{s_0}^{s_1} F(s_0, \chi, a^\epsilon(s), b_0) ds = \chi + hF(s_0, \chi, a_0, b_0)$$

Consider now the trajectory  $\zeta_{s_0, \chi}^{a^\epsilon, b_0}(\cdot)$ . We have  $\|\zeta_{s_0, \chi}^{a^\epsilon, b_0}(s) - \chi\| \leq M_R h$ , for  $s \in [s_0, s_1]$ , and

$$\|\zeta_{s_0, \chi}^{a^\epsilon, b_0}(s_1) - (\chi + hF(s_0, \chi, a_0, b_0))\| \leq \int_{s_0}^{s_1} \|F(s, \zeta_{s_0, \chi}^{a^\epsilon, b_0}(s), a^\epsilon(s), b_0) - F(s_0, \chi, a^\epsilon(s), b_0)\| ds.$$

From the Lipschitz continuity of  $F$  there exists  $\delta(h) \geq 0$ , the modulus of continuity of  $F$ , defined as:

$$\delta(h) := \max \left\{ \|F(s, \zeta, a, b) - F(s', \zeta, a, b)\|, \text{ for } \zeta \in \mathbb{B}_R, (a, b) \in A \times B \text{ and } s, s' \in [0, T] \text{ with } |s - s'| \leq h \right\}.$$

Combining the last inequality with the definition of  $\delta(h)$  above gives:

$$\|\zeta_{s_0, \chi}^{a^\epsilon, b_0}(s_1) - (\chi + hF(s_0, \chi, a_0, b_0))\| \leq h\delta(h) + \int_{s_0}^{s_1} L_F(R) \|\zeta_{s_0, \chi}^{a^\epsilon, b_0}(s) - \chi\| ds \leq h\delta(h) + L_F(R)M_R h^2.$$

From the Lipschitz continuity of  $u$  and the above inequality, we get:

$$u(s_1, \zeta') - hL_u(R)(\delta(h) + L_F(R)M_R h) \leq u(s_1, \zeta_{s_0, \chi}^{a^\epsilon, b_0}(s_1)), \quad (27)$$

where  $\zeta' := \chi + hF(s_0, \chi, a_0, b_0)$  and  $L_u(R)$  is the Lipschitz constant of  $u$ . We set  $\epsilon_h := L_u(R)(\delta(h) + L_F(R)M_R h)$ .

Notice that for any  $x, y, z \in \mathbb{R}$  with  $z \geq 0$ , we have

$$(x - z) \bigvee y \geq x \bigvee y - z.$$

Therefore from (26) and (27), we deduce:

$$u(s_0, \chi) \geq -\epsilon - h\epsilon_h + u(s_1, \zeta') \bigvee \Phi(s_0, \chi).$$

Now, by using the fact that  $\|u_h - u\| = E_h$  we obtain:

$$u_h(s_0, \chi) \geq -\epsilon - h\epsilon_h - 2E_h + u_h(s_1, \zeta') \bigvee \Phi(s_0, \chi).$$

From Algorithm 1 and since  $a_0^*$  minimizes  $a \mapsto u_h(s_1, \chi + hF(s_0, \chi, a, b_0)) \bigvee \Phi(s_0, \chi)$ , we obtain:

$$u_h(s_1, \zeta_1) \bigvee \Phi(s_0, \chi) \leq u_h(s_1, \zeta') \bigvee \Phi(s_0, \chi),$$

where  $\zeta_1$  is defined in Algorithm 1. Therefore

$$u_h(s_0, \chi) \geq -\epsilon - h\epsilon_h - 2E_h + u_h(s_1, \zeta_1) \bigvee \Phi(s_0, \chi).$$

which implies (25) because  $\epsilon$  is chosen arbitrary.

This result can be generalized with the same arguments for all  $k = 0, \dots, N - 1$ :

$$u_h(s_k, \zeta_k) \geq u_h(s_{k+1}, \zeta_{k+1}) \bigvee \Phi(s_k, \zeta_k) - h\epsilon_h - 2E_h, \quad (28)$$

where  $\zeta_k$  is defined in Algorithm 1, for any  $k = 0, \dots, N$ .

**Step 2.** From (28), and by using the fact that  $(x - z) \vee y \geq x \vee y - z$  for any  $x, y, z \in \mathbb{R}$  s.t.  $z \geq 0$ , we get by induction:

$$u_h(s_0, \chi) \geq \left( u_h(s_N, \zeta_N) \vee \Phi(s_0, \chi) \vee \dots \vee \Phi(s_{N-1}, \zeta_{N-1}) \right) - Nh\epsilon_h - 2NE_h. \quad (29)$$

Recall that  $E_h$  is the uniform error between  $u$  and  $u_h$ . Therefore, we deduce from (29):

$$u_h(s_0, \chi) \geq (u(s_N, \zeta_N) - E_h) \vee \left( \max_{0 \leq k \leq N-1} \Phi(s_k, \zeta_k) \right) - Nh\epsilon_h - 2NE_h.$$

Since  $s_N = T = Nh$  and  $u(T, \zeta_N) = \Phi(T, \zeta_N) \vee \Psi(\zeta_N)$ , the last inequality becomes:

$$u_h(s_0, \chi) \geq \left( \max_{0 \leq k \leq N} \Phi(s_k, \zeta_k) \right) \vee \Psi(\zeta_N) - T\epsilon_h - \left( \frac{2T}{h} + 1 \right) E_h,$$

By hypothesis (**H<sub>6</sub>**), we conclude that:

$$u(0, \chi) \geq \limsup_{h \rightarrow 0^+} \left( \max_{0 \leq k \leq N} \Phi(s_k, \zeta_k) \right) \vee \Psi(\zeta_N). \quad (30)$$

**Step 3.** In this step, we will establish an estimation between  $(\zeta_k)$ , the trajectory generated by Algorithm 1 and  $\zeta_h(\cdot)$ , solution of (23). We claim that for any  $k = 0, \dots, N - 1$ , we have:

$$\max_{s \in [s_k, s_{k+1}]} \|\zeta_h(s) - \zeta_k\| \leq \mathcal{O}(h) \quad \text{and} \quad \|\zeta_h(T) - \zeta_N\| \leq \mathcal{O}(h). \quad (31)$$

We start by proving the claim by induction. For  $s \in [s_0, s_1]$ , we have:

$$\|\zeta_h(s) - \chi\| \leq \int_{s_0}^s \|F(\theta, \zeta_h(\theta), \mathbf{a}_0^*, \mathbf{b}_0)\| d\theta \leq M_R(s - s_0) \leq M_R h = \mathcal{O}(h),$$

which gives the result for  $k = 0$ , i.e.  $\max_{s \in [s_0, s_1]} \|\zeta_h(s) - \chi\| \leq \mathcal{O}(h)$ .

Suppose that (31) is verified for  $0 \leq k \leq N - 2$ , and let's prove it for  $k + 1$ . For  $s \in [s_{k+1}, s_{k+2}]$ :

$$\begin{aligned} \|\zeta_h(s) - \zeta_{k+1}\| &= \|\zeta_h(s_{k+1}) + \int_{s_{k+1}}^s F(\theta, \zeta_h(\theta), \mathbf{a}_{k+1}^*, \mathbf{b}_{k+1}) d\theta - hF(s_k, \zeta_k, \mathbf{a}_k^*, \mathbf{b}_k) - \zeta_k\| \\ &\leq 2M_R h + \|\zeta_h(s_{k+1}) - \zeta_k\| \end{aligned}$$

Using the induction's hypothesis for  $\|\zeta_h(s_{k+1}) - \zeta_k\|$  concludes the proof. On the other hand, we have:

$$\|\zeta_h(T) - \zeta_N\| \leq \|\zeta_h(T) - \zeta_{N-1}\| + h\|F(s_{N-1}, \zeta_{N-1}, \mathbf{a}_{N-1}^*, \mathbf{b}_{N-1})\| \leq M_R h + \mathcal{O}(h) = \mathcal{O}(h).$$

**Step 4.** For any  $k = 0 \dots N - 1$  and  $s \in [s_k, s_{k+1}]$ , we have:

$$\|\zeta_h(s) - \zeta_h(s_k)\| \leq \int_{s_k}^s \|F(\theta, \zeta_h(\theta), \mathbf{a}_k^*, \mathbf{b}_k)\| d\theta \leq M_R h,$$

which gives:

$$\max_{s \in [s_k, s_{k+1}]} \|\zeta_h(s) - \zeta_h(s_k)\| \leq M_R h. \quad (32)$$

On the other hand, we have:

$$\left| \max_{0 \leq k \leq N-1} \Phi(s_k, \zeta_k) - \max_{s \in [0, T]} \Phi(s, \zeta_h(s)) \right| = \left| \max_{0 \leq k \leq N-1} \Phi(s_k, \zeta_k) - \max_{0 \leq k \leq N-1} \left( \max_{s \in [s_k, s_{k+1}]} \Phi(s, \zeta_h(s)) \right) \right|$$

$$\begin{aligned}
&\leq \left| \max_{0 \leq k \leq N-1} \Phi(s_k, \zeta_k) - \max_{0 \leq k \leq N-1} \Phi(s_k, \zeta_h(s_k)) \right| + \left| \max_{0 \leq k \leq N-1} \Phi(s_k, \zeta_h(s_k)) - \max_{0 \leq k \leq N-1} \left( \max_{s \in [s_k, s_{k+1}]} \Phi(s, \zeta_h(s)) \right) \right| \\
&\leq \max_{0 \leq k \leq N-1} |\Phi(s_k, \zeta_k) - \Phi(s_k, \zeta_h(s_k))| + \max_{0 \leq k \leq N-1} |\Phi(s_k, \zeta_h(s_k)) - \max_{s \in [s_k, s_{k+1}]} \Phi(s, \zeta_h(s))| \\
&\leq \max_{0 \leq k \leq N-1} L_\Phi(R) \|\zeta_k - \zeta_h(s_k)\| + \max_{0 \leq k \leq N-1} L_\Phi(R) \max_{s \in [s_k, s_{k+1}]} \|\zeta_h(s_k) - \zeta_h(s)\|,
\end{aligned}$$

where  $L_\Phi(R)$  is the Lipschitz constant of  $\Phi$ . From (31) and (32), we conclude that:

$$\left| \max_{0 \leq k \leq N-1} \Phi(s_k, \zeta_k) - \max_{s \in [0, T]} \Phi(s, \zeta_h(s)) \right| \leq \mathcal{O}(h).$$

Finally, using the fact that  $\|\zeta_h(T) - \zeta_N\| \leq \mathcal{O}(h)$ , we get:

$$\left| \left( \max_{0 \leq k \leq N-1} \Phi(s_k, \zeta_k) \right) \bigvee \Psi(\zeta_N) - \left( \max_{s \in [0, T]} \Phi(s, \zeta_h(s)) \right) \bigvee \Psi(\zeta_h(T)) \right| \leq \mathcal{O}(h) \quad (33)$$

By combining the estimates (30) and (33), we obtain:

$$u(0, \chi) \geq \limsup_{h \rightarrow 0^+} \left\{ \left( \max_{s \in [0, T]} \Phi(s, \zeta_h(s)) \right) \bigvee \Psi(\zeta_h(T)) \right\}.$$

□

### 4.3 Reconstruction of robust optimal trajectories (worst case)

In this section, we consider a reconstruction algorithm for the worst case. We adapt some arguments introduced in [4, Chapter VIII] to our case with finite time horizon and maximum running cost. We consider a specific approximation  $u_h$  of the value function  $u$  which verifies a discrete dynamic programming principle. Then, we prove the existence of an optimal strategy for the first player and an optimal control for the second player obtained from  $u_h$ . Henceforth, we deduce an approximation of optimal feedbacks for problem (19).

First, we start by defining the specific approximation  $u_h$  of the value function  $u$  on  $[0, T] \times \mathbb{R}^m$ . We keep the uniform time partition of  $[0, T]$ . Let  $J_h$  be an approximation of the global cost  $J$  defined by:

$$J_h(t, \chi, (a_i), (b_i)) = \Phi(t, \chi) \bigvee \left( \max_{i=k+1 \dots N} \Phi(s_i, \zeta_i) \right) \bigvee \Psi(\zeta_N).$$

for  $t \in [s_k, s_{k+1}[$ ,  $\chi \in \mathbb{R}^m$  and  $((a_i), (b_i)) \in A^{N-k} \times B^{N-k}$ .

Therefore, we consider the following approximation of the value function  $u$ :

$$u_h(t, \chi) = \inf_{\alpha_h[\cdot] \in \Gamma^h} \sup_{(b_i) \in B^{N-k}} J_h(t, \chi, \alpha_h[b], b) \quad (34)$$

for  $t \in [s_k, s_{k+1}[$  and  $\chi \in \mathbb{R}^m$ .

Now, we present the following algorithm corresponding to a reconstruction procedure based on the value function  $u_h$  in the worst case where the second player takes optimal decisions which corresponds to the worst situation for the first player. This algorithm is presented in a general form for some initial time  $t \in [0, T[$  from an initial position  $\chi \in \mathbb{R}^m$ . Let  $k \leq N - 1$  such that  $t \in [s_k, s_{k+1}[$ .

The following Proposition presents some results verified by the approximated value function  $u_h$ , the reconstructed trajectory  $(\zeta_k^*)$ , the discrete strategy  $\alpha_h^*[\cdot] \in \Gamma^h$  and control  $(b_i^*) \in B^{N-k}$  generated by Algorithm 2:

---

**Algorithm 2: Worst case**


---

**Require:**  $t \in [s_k, s_{k+1}[$ ,  $k \leq N - 1$  and  $\chi \in \mathbb{R}^m$ .

- 1: Initialise  $\zeta_k^* = \chi$ .
- 2: **for**  $i = k \cdots N - 1$  **do**
- 3:   **if**  $i = k$  **then**
- 4:     The optimal choice of the second player,  $b_k^* \in B$ , s.t.

$$b_k^* \in \operatorname{argmax}_{b \in B} \min_{a \in A} \left\{ u_h(s_{k+1}, \chi + (s_{k+1} - t)F(t, \chi, a, b)) \bigvee \Phi(t, \chi) \right\}.$$

- 5:     The optimal reaction of the first player,  $a_i^* \in A$ , s.t.

$$\alpha_h^*[b^*]_k := a_k^* \in \operatorname{argmin}_{a \in A} \left\{ u_h(s_{k+1}, \chi + (s_{k+1} - t)F(t, \chi, a, b_k^*)) \bigvee \Phi(t, \chi) \right\}.$$

- 6:     The new state position:  $\zeta_{k+1}^* = \chi + (s_{k+1} - t)F(t, \chi, a_k^*, b_k^*)$ .
  - 7:   **else**
  - 8:      $b_i^* \in \operatorname{argmax}_{b \in B} \min_{a \in A} \left\{ u_h(s_{i+1}, \zeta_i^* + hF(s_i, \zeta_i^*, a, b)) \bigvee \Phi(s_i, \zeta_i^*) \right\}$ .
  - 9:      $\alpha_h^*[b^*]_i := a_i^* \in \operatorname{argmin}_{a \in A} \left\{ u_h(s_{i+1}, \zeta_i^* + hF(s_i, \zeta_i^*, a, b_i^*)) \bigvee \Phi(s_i, \zeta_i^*) \right\}$ .
  - 10:     $\zeta_{i+1}^* = \zeta_i^* + hF(s_i, \zeta_i^*, a_i^*, b_i^*)$ .
  - 11:   **end if**
  - 12: **end for**
- 

**Proposition 4.2** *Assume that hypothesis **(H<sub>5</sub>)** holds. For  $t \in [s_k, s_{k+1}[ \subset [0, T[$  and  $\chi \in \mathbb{R}^m$ , we have:*

(i)  $u_h$  verifies the following discrete dynamic programming principle:

$$u_h(t, \chi) = \max_{b \in B} \min_{a \in A} \left\{ u_h(s_{k+1}, \chi + (s_{k+1} - t)F(t, \chi, a, b)) \bigvee \Phi(t, \chi) \right\} \quad (35)$$

(ii) Furthermore,  $J_h(t, \chi, \alpha_h^*[b], b) \leq u_h(t, \chi)$ , for any  $(b_i) \in B^{N-k}$ . The equality holds when  $(b_i) = (b_i^*)$ . In this case  $(\alpha_h^*[\cdot], (b_i^*))$  represents a Nash equilibrium for (34)<sup>2</sup>, in the terminology of the theory of noncooperative games.

(iii) Finally,  $u_h$  converges to  $u$ , when  $h \rightarrow 0$ , over compact subsets of  $[0, T] \times \mathbb{R}^m$ .

*Proof.* [Proof of Proposition 4.2] The results of this Proposition have been established in [4, Chapter VIII] for unconstrained Bolza problem with infinite time horizon. The proof can be adapted to the case of finite time-horizon problem with maximum running cost. We give here a very succinct sketch of the proof.

The dynamic programming principle is a classical result. From (i) and Algorithm 2, we obtain:

$$\begin{aligned} u_h(t, \chi) &= \max_{b \in B} \min_{a \in A} \left\{ u_h(s_{k+1}, \chi + (s_{k+1} - t)F(t, \chi, a, b)) \bigvee \Phi(\chi) \right\} \\ &= \min_{a \in A} \left\{ u_h(s_{k+1}, \chi + (s_{k+1} - t)F(s, \chi, a, b_k^*)) \bigvee \Phi(\chi) \right\} = u_h(s_{k+1}, \zeta_{k+1}^*) \bigvee \Phi(\chi). \end{aligned}$$

By the same argument, we get for any  $i \geq k + 1$ :

$$u_h(s_i, \zeta_i^*) = u_h(s_{i+1}, \zeta_{i+1}^*) \bigvee \Phi(\zeta_i^*).$$

From the above equalities, we conclude:

$$u_h(t, \chi) = \Phi(\chi) \bigvee \left( \max_{i=k+1, \dots, N} \Phi(\zeta_i^*) \right) \bigvee \Psi(\zeta_N^*) = J_h(t, \chi, \alpha_h^*[b^*], b^*).$$

---

<sup>2</sup>This means that every player cannot improve his guaranteed outcome, given by  $u_h(t, \chi)$ , by any unilateral deviation from his optimal choice  $(\alpha_h^*[\cdot])$  and  $(b_i^*)$  for the first and the second player respectively.)

In a similar way, one can prove that for any  $(b_i)_i \in B^{N-k}$ :

$$J_h(t, \chi, \alpha_h^*[b], b) \leq u_h(t, \chi).$$

This concludes claim (ii). To prove statement (iii), we first consider the weak limits

$$\bar{u}(t, \chi) := \limsup_{(s,y) \rightarrow (t,\chi), h \rightarrow 0^+} u_h(s, y) \quad \underline{u}(t, \chi) := \liminf_{(s,y) \rightarrow (t,\chi), h \rightarrow 0^+} u_h(s, y).$$

By viscosity arguments, we can prove that  $\bar{u}$  is a subsolution and  $\underline{u}$  is a supersolution of the HJ equation (20). The conclusion comes from the comparison principle [1, Appendix] which gives  $\bar{u} \leq u \leq \underline{u}$ , and the inequality  $\underline{u} \leq \bar{u}$  is obvious.  $\square$

**Theorem 4.3** *Assume that  $(\mathbf{H}_5)$  is satisfied and assume that  $u_h$  is given by (34). Let  $t \in [0, T]$  and  $\xi \in \mathbb{R}^d$ , there exist  $(\tilde{\alpha}_h^*[\cdot], b_h^*(\cdot)) \in \Gamma \times \mathcal{B}$  verifying*

$$\lim_{h \rightarrow 0^+} J(t, \chi, \tilde{\alpha}_h^*[b_h^*], b_h^*) = u(t, \chi).$$

*Proof.* We define  $\mathcal{B}_h$ , a subset of  $\mathcal{B}$ , by:

$$\mathcal{B}_h := \left\{ b(\cdot) \in \mathcal{B}, \text{ s.t. } b(s) = b(kh), \forall s \in [kh, (k+1)h[, \text{ for } k = 0, \dots, N-1 \right\},$$

and let  $\mathcal{A}_h$  be the subset of  $\mathcal{A}$  defined in a similar way to  $\mathcal{B}_h$ . Now, let's define  $(\tilde{\alpha}_h^*[\cdot], b_h^*(\cdot)) \in \Gamma \times \mathcal{B}$  by:

$$b_h^*(s) := b_{\lfloor s/h \rfloor}^* \quad \text{and} \quad \tilde{\alpha}_h^*[b](s) := \alpha_h^*[\hat{b}]_{\lfloor s/h \rfloor}$$

where  $\alpha_h^*[\cdot] \in \Gamma_h$  and  $(b_i^*)_i \in B^{N-k}$  are generated by Algorithm 2 and  $(\hat{b}_i)_i \in B^{N-k}$  is defined by  $\hat{b}_i := b(ih)$ , for  $i = k, \dots, N-1$ , for any  $b(\cdot) \in \mathcal{B}$ . Notice that  $b_h^*(\cdot) \in \mathcal{B}_h$  and  $\tilde{\alpha}_h^*[b](\cdot) \in \mathcal{A}_h$ , for any  $b(\cdot) \in \mathcal{B}$ . Furthermore, for any  $(a(\cdot), b(\cdot)) \in \mathcal{A}_h \times \mathcal{B}_h$ , we define  $((\hat{a}_i)_i, (\hat{b}_i)_i) \in A^{N-k} \times B^{N-k}$  by:

$$\hat{a}_i = a(ih) \quad \text{and} \quad \hat{b}_i = b(ih) \quad \text{for } i = k, \dots, N-1.$$

By exploiting the Lipschitz continuity of  $F$ ,  $\Phi$  and  $\Psi$ , and by following the same arguments used in **Step 3 & 4** from the proof of Theorem 4.3, we get:

$$|J(t, \chi, a, b) - J_h(t, \chi, \hat{a}, \hat{b})| \leq \mathcal{O}(h). \quad (36)$$

This claim implies that

$$|J(t, \chi, \tilde{\alpha}_h^*[b_h^*], b_h^*) - J_h(t, \chi, \alpha_h^*[b^*], b^*)| \leq \mathcal{O}(h). \quad (37)$$

Then, from (ii) and (iii) of Proposition 4.2 we have:

$$J_h(t, \chi, \alpha_h^*[b^*], b^*) = u_h(t, \chi) \quad \text{and} \quad \lim_{h \rightarrow 0^+} u_h(t, \chi) = u(t, \chi).$$

Finally, combining the two above equalities with inequality (37) gives the desired result. Now, it remains to justify the claim (36).  $\square$

Theorem 4.3 shows the convergence of the trajectory reconstruction procedure of Algorithm 2. However, this result requires the approximation  $u_h$  of  $u$  to satisfy a dynamic programming principle. If this requirement is not satisfied, then the statement of Theorem 4.3 is no longer valid.

## 5 Application to the aircraft landing problem

### 5.1 Introduction

Aircraft accidents can happen because of quick changes of the wind velocity at low altitudes which present a real danger. For this reason, it is important to look for the best flying configurations to avoid a failed landing. It consists in steering the aircraft to the maximum altitude that can be reached, during an interval of time, in order to prevent a crash on the ground.

In papers [21, 20], a Chebyshev-type optimal control problem was proposed and an approximate solution of the problem is computed in order to deduce an approximate feedback control.

In paper [2], it was supposed that the wind velocity fields are known and a precise model for these fields is provided in Appendix A of [2]. The Hamilton-Jacobi-Bellman approach was used to characterize and compute the value function of the control problem.

A more realistic situation can be found in [9] where a nonlinear differential game with integral payoff functional and state constraints was studied. In particular, the dynamic programming approach was applied to the problem of an aircraft control during take-off in a windshear. For this case, the first player (the minimizer, the pilot) uses pure feedback strategies, while the second player (the maximizer, the wind) uses feedback counter strategies (the wind is permitted to measure the current value of the first player's control). To solve the problem, a semi-Lagrangian scheme is applied to compute an approximation of the value function.

Consider the flight of an aircraft in a vertical plane. Different forces are acting on the center of gravity of the aircraft. Among those forces one can cite:

- the thrust force  $\mathbf{F}_T$  with a modulus of the form  $F_T(t, v) := \beta(t)F_T(v)$  where  $v$  is the modulus of the aircraft velocity and  $\beta(t) \in [0, 1]$  is the power setting of the engine.
- the lift and drag forces  $\mathbf{F}_L$  and  $\mathbf{F}_D$  with modulus  $F_L$  and  $F_D$  depending on  $v$  and the angle of attack  $\alpha$ .
- the weight force  $\mathbf{F}_P$  with modulus  $F_P = mg$ , where  $m$  is the aircraft mass and  $g$  the gravitational constant.

Precise expressions of  $F_T$ ,  $F_L$  and  $F_D$  can be found in [2, 10, 11].

From the Newton's law, we deduce the following equations of motion (see [2, 10]):

$$\begin{cases} \dot{x}(s) = v(s) \cos(\gamma(s)) + w_x(s) \\ \dot{h}(s) = v(s) \sin(\gamma(s)) + w_h(s) \\ \dot{v}(s) = \frac{\beta F_T(v(s))}{m} \cos(\alpha(s) + \delta) - \frac{F_D(v(s), \alpha(s))}{m} - g \sin(\gamma(s)) - \dot{w}_x(s) \cos(\gamma(s)) - \dot{w}_h(s) \sin(\gamma(s)) \\ \dot{\gamma}(s) = \frac{1}{v(s)} \frac{\beta F_T(v(s))}{m} \sin(\alpha(s) + \delta) + \frac{F_L(v(s), \alpha(s))}{m} - g \cos(\gamma(s)) + \dot{w}_x(s) \sin(\gamma(s)) - \dot{w}_h(s) \cos(\gamma(s)) \\ \dot{\alpha}(s) = a(s), \end{cases}$$

where  $x$  is the horizontal distance,  $h$  denotes the altitude,  $\gamma$  is the relative path inclination,  $\delta > 0$  is a parameter of the model,  $w_x$  and  $w_h$  are respectively the horizontal and the vertical components of the wind velocity vector,  $\dot{w}_x$  and  $\dot{w}_h$  are their derivatives and  $a$  represents the control variable.

### 5.2 5D differential game model

In this paper, we propose a differential game model with maximum running cost in which wind disturbances are considered as a second player and our aim is to steer the aircraft to the maximum altitude that can be reached during an interval of time, by means of nonanticipative strategies, in order to prevent a crash on the ground.

### 5.2.1 Model presentation and differential game

Wind disturbances are represented by  $w_x$  and  $w_h$ , the derivatives the horizontal and the vertical components of the wind velocity vector. To simplify notations, we denote by  $b(\cdot) = (b_1(\cdot), b_2(\cdot)) := (\dot{w}_x(\cdot), \dot{w}_h(\cdot))$  taking values in a compact set  $B$  of the form:

$$B := B_1 \times B_2 = [b_{1,\min}, b_{1,\max}] \times [b_{2,\min}, b_{2,\max}].$$

Now, we consider new state variables represented by a vector of  $\mathbb{R}^5$ :

$$y(\cdot) = (h(\cdot), v(\cdot), \gamma(\cdot), w_h(\cdot), \alpha(\cdot))^\top.$$

The admissible set is of the form:

$$\mathcal{K} = [h_{\min}, h_{\max}] \times [v_{\min}, v_{\max}] \times [\gamma_{\min}, \gamma_{\max}] \times [wh_{\min}, wh_{\max}] \times [\alpha_{\min}, \alpha_{\max}].$$

In our case, the above 5-D differential system is transformed to:

$$\begin{cases} \dot{h}(s) = v(s) \sin \gamma(s) + w_h(s) & (38) \\ \dot{v}(s) = \frac{\beta F_T(v(s))}{m} \cos(\alpha(s) + \delta) - \frac{F_D(v(s), \alpha(s))}{m} - g \sin \gamma(s) - b_1(s) \cos \gamma(s) - b_2(s) \sin \gamma(s) & (39) \\ \dot{\gamma}(s) = \frac{\beta F_T(v(s))}{mv(s)} \sin(\alpha(s) + \delta) + \frac{F_L(v(s), \alpha(s))}{mv(s)} - \frac{g}{v(s)} \cos(\gamma(s)) b_1(s) \sin(\gamma(s)) - \frac{b_2(s)}{v(s)} \cos(\gamma(s)) & (40) \\ \dot{w}_h(s) = b_2(s) & (41) \\ \dot{\alpha}(s) = a(s), & (42) \end{cases}$$

where  $a(\cdot)$ , the control of the first player, takes values in a compact and convex set  $A := [-a_{\min}, a_{\max}]$ .

This differential system can be expressed differently:

$$\dot{y}(t) = f(t, y(t), a(t), b(t)) := g_0(y(t)) + b_1(t)g_1(y(t)) + b_2(t)g_2(y(t)) + a(t)e_5$$

where  $e_5$  is the fifth element of the canonical basis of  $\mathbb{R}^5$  and

$$g_0(y) = \begin{pmatrix} v \sin \gamma \\ \frac{\beta F_T(v)}{m} \cos(\alpha + \delta) - \frac{F_D}{m} - g \sin \gamma \\ \frac{1}{v} \left( \frac{\beta F_T(v)}{m} \sin(\alpha + \delta) + \frac{F_L(v, \alpha)}{m} - g \cos \gamma \right) \\ 0 \\ 0 \end{pmatrix}, g_1(y) = \begin{pmatrix} 0 \\ -\cos \gamma \\ \frac{1}{v} \sin \gamma \\ 0 \\ 0 \end{pmatrix} \text{ and } g_2(y) = \begin{pmatrix} 0 \\ -\sin \gamma \\ \frac{1}{v} \cos \gamma \\ 1 \\ 0 \end{pmatrix}.$$

In order to transform our problem into a minimization problem, the maximum raunning cost function  $\phi$  is defined as  $\phi(y) := H_r - h$ , where  $h$  is the aircraft altitude (the first component of the state vector  $y$ ) and  $H_r > 0$  is a given reference altitude.

### 5.2.2 Numerical resolution

To determinate the intervals in which the state variables, the control of the first player and the wind disturbances take values, we are based on the wind model presented in [2, Appendix A]. Therefore, we obtain the following constraints on state and controls, presented respectively in Tables 1 and 2:

**Remark 5.1** *Since  $\phi$  is bounded, the auxiliary variable  $z$  will take values in an interval of the form  $[z_{\min}, z_{\max}]$ . In this case,  $z \in [H_r - h_{\max}, H_r - h_{\min}]$ .*

State variable	$h(\text{ft})$	$v(\text{ft s}^{-1})$	$\gamma$ (deg)	$w_h(\text{ft s}^{-1})$	$\alpha$ (deg)
min	450	160	-7.0	-100.0	0.0
max	1000	260	15.0	50.0	17.2

Table 1: State constraints: domain  $\mathcal{K}$ .

Control variables	$a$ (deg $s^{-1}$ )	$b_1 := \dot{w}_x(\text{ft s}^{-2})$	$b_2 := \dot{w}_h(\text{ft s}^{-2})$
min	-3.0	0.0	-2.0
max	3.0	7.7	2.0

Table 2: Control constraints: sets  $A$  and  $B$ .

To obtain an approximation of the auxiliary value function, we use a finite differences scheme for solving the HJ equation (10). More precisely, we consider an Essentially Non Oscillatory scheme of second order (ENO2) (see [22]). First, since  $a_{\max} = -a_{\min}$ , we can calculate explicitly the Hamiltonian function  $H(y, p)$ , for  $(y, p) \in \mathbb{R}^5 \times \mathbb{R}^5$ :

$$\begin{aligned} H(y, p) &= \min_{b \in B} \max_{a \in A} - \langle f(t, y, a, b), p \rangle \\ &= -\langle g_0(y), p \rangle - \max_{b_1 \in B_1} b_1 \langle g_1(y), p \rangle - \max_{b_2 \in B_2} b_2 \langle g_2(y), p \rangle + a_{\max} |p_5|. \end{aligned}$$

Therefore, the following numerical Hamiltonian can be used to solve a discretized form of the HJ equation verified by the problem value function:

$$\mathcal{H}(y, p^-, p^+) = \sum_{i=1}^{i=4} \max \left( -f_i(y, b_{opt}), 0 \right) p_i^- + \min \left( -f_i(y, b_{opt}), 0 \right) p_i^+ + a_{\max} \max \left( p_5^-, -p_5^+, 0 \right),$$

where  $b_{opt} \in \operatorname{argmax}_{b \in B} \langle (b_1 g_1(y) + b_2 g_2(y)), p \rangle$ , for  $p = \frac{p^- + p^+}{2}$ .

Then, we extend the computational domain in all directions to obtain  $\mathcal{K}_\mu := \mathcal{K} + \mu B_\infty$ , where  $B_\infty := [-1, 1]^d$  and the parameter  $\mu$  is a small fixed positive value (in this case,  $\mu = 0.05$ ).

Consider also the following mesh steps  $\Delta := (\delta t, (\delta y_i)_{1 \leq i \leq d}, \delta z)$ . For a given multi-index  $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$ , let  $y_i := y_{\min, i} + i \delta y_i$ ,  $z_j := z_{\min} + j \delta z$ ,  $j \in \mathbb{Z}$  and  $t_n := n \delta t$  for  $n = 0, \dots, N$ , where  $N$  is the integer part of  $T/\delta t$ . Therefore, we define the grid of  $\mathcal{K}_\mu \times [z_{\min}, z_{\max}]$ :

$$\mathcal{G} := \{(y_i, z_j), i \in \mathbb{Z}^d, j \in \mathbb{Z}, (y_i, z_j) \in \mathcal{K}_\mu \times [z_{\min}, z_{\max}]\}$$

Let denote  $\hat{\phi}_{i,j} := \hat{\phi}(y_i, z_j)$  and  $w_{i,j}^n$  an approximation of the auxiliary value function  $w$ , defined in (8), at  $(t_n, y_i, z_j)$ .

Finally, to solve numerically the HJ equation verified by  $w$ , we follow the explicit scheme (see [2, 8]):

$$\begin{cases} w_{i,j}^N = \hat{\phi}_{i,j} \\ w_{i,j}^n = \max \left( w_{i,j}^{n+1} - \delta t \mathcal{H}(y_i, D^- w_{i,j}^{n+1}, D^+ w_{i,j}^{n+1}), \hat{\phi}_{i,j} \right), n \in \{N-1, \dots, 0\}, (y_i, z_j) \in \mathcal{G}, \end{cases} \quad (43)$$

where the terms  $D^\pm w_{i,j}^n$  are approximated through a second order ENO scheme, see [22], given by the following expression:

$$D_k^\pm w_{i,j}^n := \pm \frac{w_{i \pm e_k, j}^n - w_{i,j}^n}{\delta y_k} \mp \frac{\delta y_k}{2} \sigma(D_{k,0}^2 w_{i,j}^n, D_{k,\pm 1}^2 w_{i,j}^n)$$

with  $D_{k,\epsilon}^2 w_{i,j}^n := (w_{i+(\epsilon-1)e_k, j}^n + 2w_{i+2\epsilon e_k, j}^n - w_{i+(\epsilon+1)e_k, j}^n)/(\delta y_k)^2$  and  $\sigma$  is defined, for any  $a, b \in \mathbb{R}$ , as follows:

$$\sigma(a, b) = \begin{cases} a & \text{if } ab > 0 \text{ and } |a| \leq |b|, \\ b & \text{if } ab > 0 \text{ and } |a| > |b|, \\ 0 & \text{if } ab \leq 0. \end{cases}$$

**Remark 5.2** *The numerical Hamiltonian  $\mathcal{H}$  is consistent with  $H$ , i.e.  $\mathcal{H}(y, p, p) = H(y, p)$ , monotone, i.e. for  $k = 1, \dots, 5$ ,  $\frac{\partial \mathcal{H}}{\partial p_k^-}(y, p^-, p^+) \geq 0$  and  $\frac{\partial \mathcal{H}}{\partial p_k^+}(y, p^-, p^+) \leq 0$  and Lipschitz continuous w.r.t. all its arguments. Furthermore, the time step  $\delta t$  is chosen in order to satisfy the Courant-Friedrich-Levy condition:*

$$\delta t \left( \sum_{i=1}^4 \frac{\|f_i(y)\|}{\delta y_i} + \frac{a_{\max}}{\delta y_5} \right) \leq CFL,$$

where  $CFL \leq 1$  is a constant value (In our model,  $CFL = 0.5$ ). Under those conditions, the approximation  $w_{i,j}^n$  converges to  $w$ , the unique viscosity solution of (10) (see [8]).

To solve the constrained problem (4), we proceed as follows:

1. First, we compute an approximation of the auxiliary value function  $w$  by solving (43). Denote this approximation by  $W^\Delta$ .
2. Then, thanks to assertion (ii) of Theorem 3.5, we get an approximation of the value function  $v$  at an initial state  $y_0$ :

$$z^\Delta := \min\{z \in [z_{\min}, z_{\max}] \mid W^\Delta(0, y_0, z) \leq 0\}$$

3. Finally, we apply our reconstruction procedure (Algorithm 1 or 2) with the approximation  $W^\Delta$ , starting at time  $t = 0$  from  $(y_0, z^\Delta)$ , to get approximated optimal strategies of the first player for the constrained problem (4), starting at time  $t = 0$  from the initial state  $y_0$  (see assertion (iii) of Theorem 3.5).

### 5.2.3 Numerical test: Reconstruction of optimal trajectories and controls

The numerical simulations presented in this section have been performed by using a c++ code with 20 OPEN-MP parallel threads on a Intel Xeon Gold (3.50GHZ) processor.

The final time horizon is set to  $T = 8$ . We consider a grid  $\mathcal{G}$  which contains  $40 \times 20^4 \times 5$  points. An approximation  $W^\Delta$  is obtained by solving (43). Then we perform the trajectory reconstruction for some initial positions by using Algorithms 1 and 2.

In general, the computation of the minimal values  $a_i^*$  and max-min values  $b_i^*$ , in Algorithms 1 & 2, might be numerically very challenging. In our simulation, the dimension of the control space is low, and the admissible sets of control values  $A$  and  $B$  have simple forms. The minimal values  $a_i^*$  in Algorithms 1 & 2, and max-min values  $b_i^*$  in Algorithm 2 are approximated by an exhaustive search method with 100 control inputs in  $A$  and 100 input values in  $B$ . Notice that while the computation of  $W^\Delta$  requires huge numerical resources, as it is obtained by solving a PDE in dimension 6, the reconstruction procedure takes less than 1 minute.

Figures 1, 2 and 3 present the results obtained with the following initial points:

$$y_1 = (600, 239.7, -2.249, -26.0, 7.373), \quad y_2 = (650, 230, -2.249, -28.0, 7.373)$$

and

$$y_3 = (800, 200.0, -2.249, -30.0, 7.373).$$

First, we shall mention that reconstruction of admissible trajectories is not possible for all initial positions. Indeed, there are some initial points from which we cannot guarantee existence of strategies of the first player corresponding to admissible trajectories, for any perturbation of the wind. Those initial points correspond to a positive value of  $w$ , for any value taken by the auxiliary variable  $z$ , and hence to an infinite value of  $v$ , the value function of the constrained problem (4).

One can get an idea about all initial positions for which there exists at least one strategy of the first player that corresponds to trajectories satisfying state constraints for any perturbation of the wind. The set of such initial positions is called the feasible set which can be defined by the negative level of the auxiliary value function  $w$ .

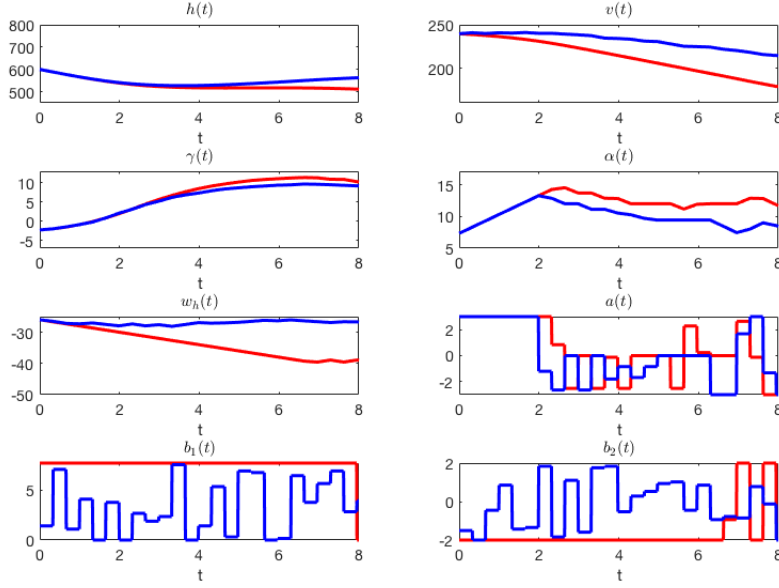


Figure 1: Trajectories and controls reconstruction as a response to arbitrary disturbances (random distribution, Algorithm 1, in blue) and to the worst case (Algorithm 2, in red) from  $y_1$ .

In figures 1 and 2, starting from  $y_1$  and  $y_2$  respectively, as expected, we observe that altitudes reached in the worst case (Algorithm 2) are always lower than those obtained in the random case (Algorithm 1).

In the worst case, perturbations aim to decrease as much as possible the altitude  $h$ . Therefore, its objective is to have  $\dot{h} \leq 0$ . Since  $\dot{h}(s) = v(s) \sin \gamma(s) + w_h(s)$ , wind perturbations in this case will take values that decrease  $w_h$  and  $v \sin(\gamma)$ . For this reason, we observe that  $b_1(\cdot) = b_{1,max} > 0$  *a.e.* to decrease  $v$  along time. However,  $b_2(\cdot) = b_{2,min} < 0$  *a.e.* to decrease  $w_h$  since  $\dot{w}_h(s) = b_2(s)$ .

In figure 3, starting from  $y_3$ , we remark that perturbations values in the worst case do not change compared to the previous initial flight configurations ( $y_1$  and  $y_2$ ). Nevertheless, we observe that altitudes reached in the random case (Algorithm 1) are similar to those obtained in the worst case (at some moments, there are sensibly lower). This observation can be explained by the high initial altitude for  $y_3$  with respect to  $y_1$  and  $y_2$  ( $h_0 = 800ft$ ). In other words, when the initial altitude of the aircraft is higher enough, worst perturbations cannot have the same effect as in previous situations (initial flight configuration with lower altitude such  $y_1$  and  $y_2$ ).

## A Proof of the claim (i) in Theorem 3.5

First, let  $\Lambda : \mathcal{B} \rightarrow \mathcal{A}$  be a set-valued map defined, for any  $b(\cdot) \in \mathcal{B}$ , by:

$$\Lambda(b) := \left\{ a(\cdot) \in \mathcal{A} \mid \left( \max_{s \in [t, T]} \hat{\phi}(y_{t,x}^{a,b}(s), z) \right) \vee \hat{\psi}(y_{t,x}^{a,b}(T), z) \leq 0 \right\}.$$

The map  $\Lambda(\cdot)$  is said to be nonanticipative if for any  $\tau \in [0, T - t]$ , for any  $b_1(\cdot), b_2(\cdot) \in \mathcal{B}$  which coincide almost everywhere on  $[t, t + \tau]$  and for any  $a_1(\cdot) \in \Lambda(b_1)$ , one can find  $a_2(\cdot) \in \Lambda(b_2)$  which coincides with  $a_1(\cdot)$  almost everywhere on  $[t, t + \tau]$  (see [12, 13, 14]).

Now, let  $(t, x, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$  be such that  $w(t, x, z) \leq 0$ . By definition of  $w$ , for any  $n \in \mathbb{N}^*$ , there

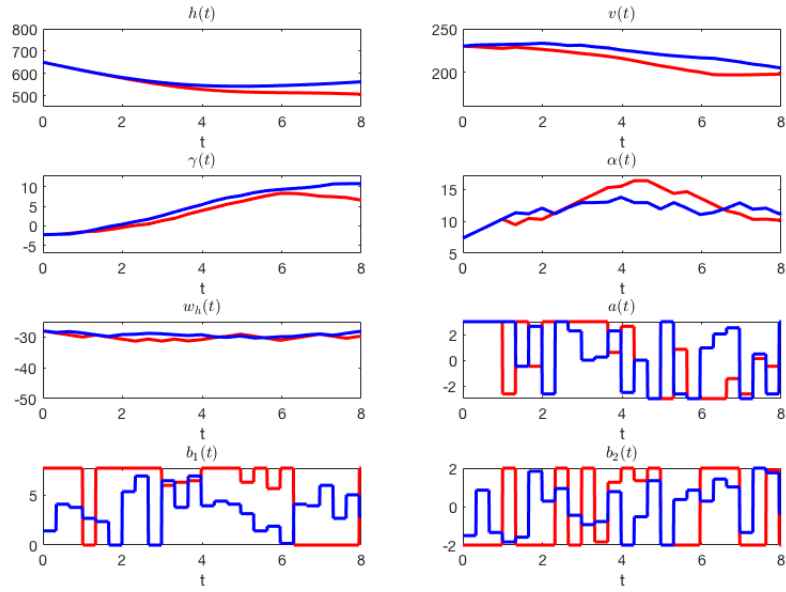


Figure 2: Trajectories and controls reconstruction as a response to arbitrary disturbances (random distribution, Algorithm 1, in blue) and to the worst case (Algorithm 2, in red) from  $y_2$ .

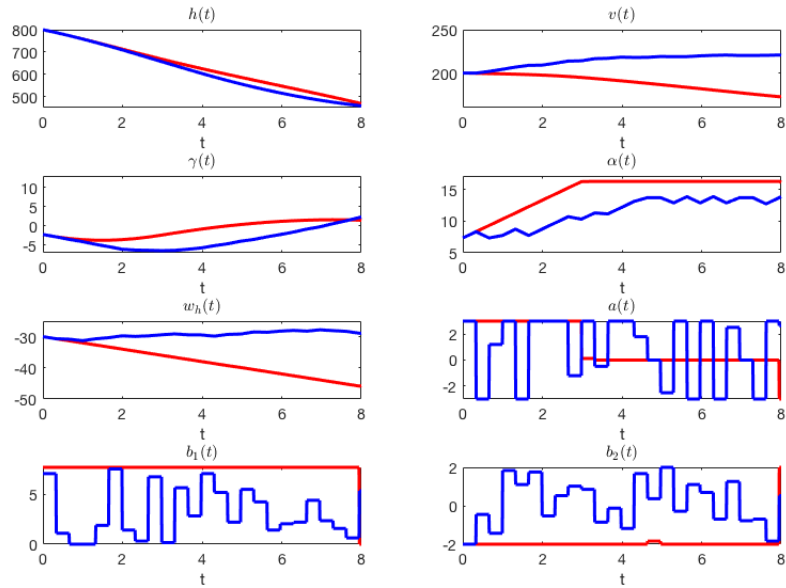


Figure 3: Trajectories and controls reconstruction as a response to arbitrary disturbances (random distribution, Algorithm 1, in blue) and to the worst case (Algorithm 2, in red) from  $y_3$ .

exists  $\alpha_n[\cdot] \in \Gamma$  such that:

$$\sup_{b(\cdot) \in \mathcal{B}} \left\{ \left( \max_{s \in [t, T]} \hat{\phi}(y_{t,x}^{\alpha_n[b], b}(s), z) \right) \vee \hat{\psi}(y_{t,x}^{\alpha_n[b], b}(T), z) \right\} \leq \frac{1}{n}.$$

We start by showing that the set-valued map  $\Lambda(\cdot)$  has nonempty values. Indeed, let fix  $b(\cdot) \in \mathcal{B}$  and let  $y_n(\cdot)$  be the solution of:

$$\begin{cases} \dot{y}_n(s) = f(s, y_n(s), \alpha_n[b](s), b(s)), & a.e. \ s \in [t, T], \\ y_n(t) = x. \end{cases}$$

Therefore, for any  $n \in \mathbb{N}^*$ ,  $y_n(\cdot)$  verifies:

$$\left( \max_{s \in [t, T]} \hat{\phi}(y_n(s), z) \right) \vee \hat{\psi}(y_n(T), z) \leq \frac{1}{n}.$$

Now, denote by  $S_{[t, T]}(x)$  the set of absolutely continuous solutions of:

$$\begin{cases} \dot{y}(s) \in f(s, y(s), A, b(s)), & a.e. \ s \in [t, T], \\ y(t) = x. \end{cases}$$

Under hypothesis **(H<sub>1</sub>)** and **(H<sub>4</sub>)**,  $S_{[t, T]}(x)$  is a compact subset of  $W^{1,1}([0, T])$  for the topology of  $C([0, T]; \mathbb{R}^d)$  (this is a consequence of [3, Theorem 0.3.4] and the Convergence theorem [3, Theorem 1.4.1], see also [19, Proposition 2.1]). Therefore, there exists a subsequence, still denoted  $(y_n(\cdot))_n$ , that converges uniformly to  $y^*(\cdot) \in S_{[t, T]}(x)$  solution of:

$$\begin{cases} \dot{y}^*(s) \in f(s, y^*(s), A, b(s)), & a.e. \ s \in [t, T], \\ y^*(t) = x. \end{cases}$$

By the measurable selection theorem from [18], there exists a control of the first player,  $a_b(\cdot) \in \mathcal{A}$ , depending on  $b(\cdot) \in \mathcal{B}$  which is already fixed, verifying  $y^*(\cdot) = y_{t,x}^{a_b, b}(\cdot)$  almost everywhere on  $[t, T]$ . By continuity of  $\hat{\phi}$  and  $\hat{\psi}$ , we conclude that:

$$\left( \max_{s \in [t, T]} \hat{\phi}(y_{t,x}^{a_b, b}(s), z) \right) \vee \hat{\psi}(y_{t,x}^{a_b, b}(T), z) \leq 0,$$

which means that  $a_b(\cdot) \in \Lambda(b)$  and therefore  $\Lambda(\cdot)$  has nonempty values.

Then, we claim that the set-valued map  $\Lambda(\cdot)$  is nonanticipative. To prove this claim, let  $\tau \in [0, T - t]$ ,  $b_1(\cdot), b_2(\cdot) \in \mathcal{B}$  coinciding almost everywhere on  $[t, t + \tau]$  and consider  $a_1(\cdot) \in \Lambda(b_1)$ . From the definition of  $\Lambda$ , the trajectory  $y_{t,x}^{a_1, b_2}(\cdot)$  is admissible (will remain in  $\mathcal{K}$  on  $[t, t + \tau]$ ) and verifies:

$$\left( \max_{s \in [t, t + \tau]} \phi(y_{t,x}^{a_1, b_2}(s)) \right) \vee \psi(y_{t,x}^{a_1, b_2}(t + \tau)) \leq z.$$

Now, consider  $a_2(\cdot) \in \mathcal{A}$  such that  $a_2(\cdot) := a_1(\cdot)$  on  $[t, t + \tau]$ . Starting at time  $t + \tau$  from  $x' := y_{t,x}^{a_1, b_2}(t + \tau)$  and by exploiting the same arguments already used to show that  $\Lambda(\cdot)$  has nonempty values, there exists  $a_{b_2}(\cdot) \in \mathcal{A}$  s.t.  $y_{t+\tau, x'}^{a_{b_2}, b_2}(\cdot)$  is admissible *i.e.* will remain in  $\mathcal{K}$  on  $[t + \tau, T]$  and verifies:

$$\left( \max_{s \in [t + \tau, T]} \phi(y_{t+\tau, x'}^{a_{b_2}, b_2}(s)) \right) \vee \psi(y_{t+\tau, x'}^{a_{b_2}, b_2}(T)) \leq z.$$

Define the control  $a_2(\cdot) \in \mathcal{A}$  by:

$$a_2(\cdot) = \begin{cases} a_1(\cdot) & \text{on } [t, t + \tau], \\ a_{b_2}(\cdot) & \text{on } [t + \tau, T] \end{cases}$$

which belongs to  $\Lambda(b_2)$  and coincides almost everywhere with  $a_1(\cdot)$  on  $[t, t + \tau]$ . Henceforth we conclude that  $\Lambda(\cdot)$  is nonanticipative.

Finally,  $\Lambda(\cdot)$  has closed values for the weak topology of  $L^2([0, T], A)$ . Indeed, let  $(a_n(\cdot))_n$  be a sequence of  $\Lambda(b)$ , for a fixed control of the second player  $b(\cdot) \in \mathcal{B}$ , that converges, for the weak topology of  $L^2([0, T], A)$ , to some control  $a(\cdot) \in \mathcal{A}$ .

Since for any  $n \in \mathbb{N}$ ,  $a_n(\cdot) \in \Lambda(b)$ , the trajectory  $y_{t,x}^{a_n, b}(\cdot)$  verifies:

$$\left( \max_{s \in [t, T]} \hat{\phi}(y_{t,x}^{a_n, b}(s), z) \right) \bigvee \hat{\psi}(y_{t,x}^{a_n, b}(T), z) \leq 0.$$

Under hypothesis **(H<sub>1</sub>)** and **(H<sub>4</sub>)**, the sequence  $(y_{t,x}^{a_n, b}(\cdot))_n$  will converge, for the compact convergence, to  $y_{t,x}^{a, b}(\cdot)$ . By continuity of  $\hat{\phi}$  and  $\hat{\psi}$ , we deduce that:

$$\left( \max_{s \in [t, T]} \hat{\phi}(y_{t,x}^{a, b}(s), z) \right) \bigvee \hat{\psi}(y_{t,x}^{a, b}(T), z) \leq 0.$$

Therefore,  $a(\cdot)$  belongs to  $\Lambda(b)$ . We conclude that the set-valued map  $\Lambda(\cdot)$  has closed values for the weak topology of the Hilbert space  $L^2([0, T], A)$ .

To end this proof, it is enough to use [13, Lemma 4.1] that guarantees the existence of a nonanticipative selection  $\alpha^*[\cdot]$  such that for any  $b(\cdot) \in \mathcal{B}$ ,  $\alpha^*[b](\cdot) \in \Lambda(b)$ . We conclude that there exists  $\alpha^*[\cdot] \in \Gamma$  s.t. for any  $b(\cdot) \in \mathcal{B}$ , the trajectory  $y_{t,x}^{\alpha^*[b], b}(\cdot)$  is admissible and

$$\left( \max_{s \in [t, T]} \phi(y_{t,x}^{\alpha^*[b], b}(s)) \right) \bigvee \psi(y_{t,x}^{\alpha^*[b], b}(T)) \leq z.$$

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