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ON SOME NEW RESULTS ON ANISOTROPIC SINGULAR PERTURBATIONS OF SECOND ORDER ELLIPTIC OPERATOR

DAVID MALTESE¹ AND CHOKRI OGABI²

Abstract. In this article, we deal with some problems involving a class of singularly perturbed elliptic operator. We prove the asymptotic preserving of a general Galerkin method associated to a semilinear problem. We use a particular Galerkin approximation to estimate the convergence rate on the whole domain, for the linear problem. Finally, we study the asymptotic behavior of the semigroup generated.

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1. INTRODUCTION

Anisotropic singular perturbations problems was introduced by Chipot in [1], these problems can model diffusion phenomena when the diffusion parameters become small in certain directions. We refer the reader to [2], [3], [4], [5], [6], [7], [8], [9], [10], [11] for several works on this topic. In this article, we will study some new theoretical aspects which have not been studied before for these problems.

Let us consider the following perturbed elliptic problem

$$\beta(u_\epsilon) - \operatorname{div}(A_\epsilon \nabla u_\epsilon) = f \text{ in } \Omega, \quad (1)$$

supplemented with the boundary condition

$$u_\epsilon = 0 \text{ on } \partial\Omega. \quad (2)$$

Here, $\Omega = \omega_1 \times \omega_2$ where ω_1 and ω_2 are two bounded open sets of \mathbb{R}^q and \mathbb{R}^{N-q} , with $N > q \geq 1$, and $f \in L^2(\Omega)$. We denote by $x = (x_1, \dots, x_N) = (X_1, X_2) \in \mathbb{R}^q \times \mathbb{R}^{N-q}$ i.e. we split the coordinates into two parts. With this notation we set

$$\nabla = (\partial_{x_1}, \dots, \partial_{x_N})^T = \begin{pmatrix} \nabla_{X_1} \\ \nabla_{X_2} \end{pmatrix},$$

where

$$\nabla_{X_1} = (\partial_{x_1}, \dots, \partial_{x_q})^T \text{ and } \nabla_{X_2} = (\partial_{x_{q+1}}, \dots, \partial_{x_N})^T.$$

The function $A = (a_{ij})_{1 \leq i, j \leq N} : \Omega \rightarrow \mathcal{M}_N(\mathbb{R})$ satisfies the ellipticity assumptions

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- There exists $\lambda > 0$ such that for a.e. $x \in \Omega$

$$A\xi \cdot \xi \geq \lambda |\xi|^2 \text{ for any } \xi \in \mathbb{R}^N. \quad (3)$$

- The coefficients of A are bounded that is

$$a_{ij} \in L^\infty(\Omega) \text{ for any } (i, j) \in \{1, 2, \dots, N\}^2. \quad (4)$$

We have decomposed A into four blocks

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11}, A_{22} are $q \times q$ and $(N - q) \times (N - q)$ matrices respectively. For $\epsilon \in (0, 1]$ we have set

$$A_\epsilon = \begin{pmatrix} \epsilon^2 A_{11} & \epsilon A_{12} \\ \epsilon A_{21} & A_{22} \end{pmatrix}.$$

The function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following conditions:

$$\beta \text{ is continuous and nondecreasing with } \beta(0) = 0. \quad (5)$$

$$\exists M \geq 0 : \forall s \in \mathbb{R}, |\beta(s)| \leq M(1 + |s|). \quad (6)$$

The weak formulation of the problem (1)-(2) is

$$\begin{cases} \int_\Omega \beta(u_\epsilon) \varphi dx + \int_\Omega A_\epsilon \nabla u_\epsilon \cdot \nabla \varphi dx = \int_\Omega f \varphi dx, \forall \varphi \in H_0^1(\Omega) \\ u_\epsilon \in H_0^1(\Omega), \end{cases}, \quad (7)$$

where the existence and uniqueness is a consequence of the assumptions (3) – (6).

Taking $\epsilon = 0$ in (1) gives

$$\beta(u) - \operatorname{div}_{X_2}(A_{22} \nabla u) = f \text{ on } \Omega, \quad (8)$$

supplemented with the boundary condition

$$u(X_1, \cdot) = 0 \text{ in } \partial\omega_2, \text{ for } X_1 \in \omega_1. \quad (9)$$

We introduce the functional space

$$H_0^1(\Omega; \omega_2) = \{v \in L^2(\Omega) \text{ such that } \nabla_{X_2} v \in L^2(\Omega)^{N-q} \text{ and for a.e. } X_1 \in \omega_1, v(X_1, \cdot) \in H_0^1(\omega_2)\},$$

equipped with the norm $\|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)^{N-q}}$. Notice that this norm is equivalent to

$$\left(\|(\cdot)\|_{L^2(\Omega)}^2 + \|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)^{N-q}} \right)^{1/2},$$

thanks to Poincaré's inequality

$$\|v\|_{L^2(\Omega)} \leq C_{\omega_2} \|\nabla_{X_2} v\|_{L^2(\Omega)^{N-q}}, \text{ for any } v \in H_0^1(\Omega; \omega_2). \quad (10)$$

One can prove that $H_0^1(\Omega; \omega_2)$ is a Hilbert space. The space $H_0^1(\Omega)$ will be normed by $\|\nabla(\cdot)\|_{L^2(\Omega)^N}$. One can check immediately that the imbedding $H_0^1(\Omega) \hookrightarrow H_0^1(\Omega, \omega_2)$ is continuous.

The weak formulation of the limit problem (8) – (9) is given by

$$\begin{cases} \int_{\omega_2} \beta(u)(X_1, \cdot) \psi dX_2 + \int_{\omega_2} A_{22}(X_1, \cdot) \nabla_{X_2} u(X_1, \cdot) \cdot \nabla_{X_2} \psi dX_2 \\ \quad = \int_{\omega_2} f(X_1, \cdot) \psi dX_2, \forall \psi \in H_0^1(\omega_2) \\ u(X_1, \cdot) \in H_0^1(\omega_2), \text{ for a.e. } X_1 \in \omega_1 \end{cases} \quad (11)$$

This problem has been studied in [9], and the author proved the following (see Proposition 4 in the above reference)

Theorem 1.1. *Under assumptions (3), (4), (5) and (6) we have*

$$u_\epsilon \rightarrow u \text{ in } L^2(\Omega), \quad \epsilon \nabla_{X_1} u_\epsilon \rightarrow 0 \text{ in } L^2(\Omega)^q \text{ and } \nabla_{X_2} u_\epsilon \rightarrow \nabla_{X_2} u \text{ in } L^2(\Omega)^{N-q},$$

where u_ϵ is the unique solution of (7) in $H_0^1(\Omega)$ and u is the unique solution to (11) in $H_0^1(\Omega; \omega_2)$.

Notice that for $\varphi \in H_0^1(\Omega; \omega_2)$, and for a.e X_1 in ω_1 we have $\varphi(X_1, \cdot) \in H_0^1(\omega_2)$, testing with it in (11) and integrating over ω_1 yields

$$\int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in H_0^1(\Omega; \omega_2). \quad (12)$$

This paper is organized as follows:

- As a first main result, we will prove the asymptotic preserving of the general Galerkin method for the elliptic problem (1-2). This concept has been introduced by S. Jin in [12] and it could be illustrated by the following commutative diagram

$$\begin{array}{ccc} P_{\epsilon, n} & \xrightarrow{n \rightarrow \infty} & P_\epsilon \\ \epsilon \rightarrow 0 \downarrow & & \downarrow \epsilon \rightarrow 0 \\ P_n & \xrightarrow{n \rightarrow \infty} & P_0 \end{array} \quad ,$$

here, $P_{\epsilon, n}$ is the Galerkin approximation of the infinite dimensional perturbed problem P_ϵ , and P_n is the Galerkin approximation of the infinite dimensional limit problem P_0 . We will derive an estimation of the error for a general Galerkin method, and by using a C ea's type lemmas we prove the asymptotic-preserving of the method.

- As a second main result, we will prove, in the linear case, a new result on the estimation of the global convergence rate, such a result is of the form $\|\nabla_{X_2}(u_\epsilon - u)\|_{L^2(\Omega)^{N-q}} \leq C\epsilon$. This estimation is an improvement of the local one proved by Chipot and Guesmia in [3]. Our arguments are founded on the use of a particular Galerkin approximation constructed by a tensor product.
- In section 4 we will give our third main result on the asymptotic behavior of the semigroup generated by the perturbed elliptic operator $\text{div}(A_\epsilon \nabla \cdot)$, and we will give a simple application to linear parabolic problems.

Finally, to make the paper readable, we put some intermediate technical lemmas in the appendix.

2. MAIN THEOREMS FOR THE ELLIPTIC PROBLEM

Definition 2.1. *Let (V_n) be a sequence of finite dimensional subspaces of a Hilbert space H . We say that (V_n) approximates H , if for every $w \in H$.*

$$\inf_{v \in V_n} \|w - v\|_H \longrightarrow 0 \text{ as } n \rightarrow \infty$$

For a sequence (V_n) of a finite dimensional spaces of $H_0^1(\Omega)$, and for every $\epsilon \in (0, 1]$ and $n \in \mathbb{N}$, we denote $u_{\epsilon, n}$ the unique solution of

$$\begin{cases} \int_{\Omega} \beta(u_{\epsilon, n}) \varphi dx + \int_{\Omega} A_{\epsilon} \nabla u_{\epsilon, n} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V_n. \\ u_{\epsilon, n} \in V_n. \end{cases} \quad (13)$$

We suppose that

$$\partial_{x_i} a_{ij} \in L^{\infty}(\Omega), \partial_{x_j} a_{ij} \in L^{\infty}(\Omega) \text{ for } i = 1, \dots, q \text{ and } j = q + 1, \dots, N. \quad (14)$$

We have the following

Theorem 2.2. *Let $\Omega = \omega_1 \times \omega_2$ where ω_1 and ω_2 are two bounded open sets of \mathbb{R}^q and \mathbb{R}^{N-q} , with $N > q \geq 1$. Suppose that $f \in L^2(\Omega)$ and assume (3), (4), (5), (6), and (14). Let (V_n) be a sequence of finite dimensional spaces of $H_0^1(\Omega)$ which approximates it in the sense of Definition 2.1. Let $(u_{\epsilon, n})$ be the sequence of solutions of (13) then we have:*

$$\lim_{\epsilon} (\lim_n u_{\epsilon, n}) = \lim_n (\lim_{\epsilon} u_{\epsilon, n}) = u, \text{ in } H_0^1(\Omega; \omega_2),$$

where u is the unique solution of (11) in $H_0^1(\Omega; \omega_2)$.

Our second result concerns the estimation of the rate of convergence for the continuous problem (7) in the linear case, this result could be seen as a refinement of a result proved in [3]. In the above reference, the authors proved the following interior estimation for the linear problem

$$\text{For every } \omega'_1 \subset\subset \omega_1 \text{ open : } \|\nabla_{X_2}(u_{\epsilon} - u)\|_{L^2(\omega'_1 \times \omega_2)} = O(\epsilon), \text{ and } \|\nabla_{X_1}(u_{\epsilon} - u)\|_{L^2(\omega'_1 \times \omega_2)} = O(1). \quad (15)$$

where they have supposed that

$$\nabla_{X_1} f \in L^2(\Omega)^q, \quad (16)$$

assumption (14) and

$$\nabla_{X_1} A_{22} \in L^{\infty}(\Omega).$$

Our contribution consists in extending (15) to the whole domain Ω , to obtain such a result we take an additional hypothesis on A and f , namely:

$$\text{For a.e. } X_2 \in \omega_2 : f(\cdot, X_2) \in H_0^1(\omega_1), \quad (17)$$

and

$$\text{The block } A_{22} \text{ depends only on } X_2. \quad (18)$$

Theorem 2.3. *Let $\Omega = \omega_1 \times \omega_2$ where ω_1 and ω_2 are two bounded open sets of \mathbb{R}^q and \mathbb{R}^{N-q} , with $N > q \geq 1$. Suppose that $\beta = 0$, and let us assume that A satisfies (3), (4), (14) and (18). Let $f \in L^2(\Omega)$ such that (16) and (17), then there exists C depending on f , λ , C_{ω_2} and A such that*

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_2}(u_{\epsilon} - u)\|_{L^2(\Omega)^{N-q}} \leq C\epsilon,$$

where u_{ϵ} is the unique solution of (7) in $H_0^1(\Omega)$ and u is the unique solution to (11) in $H_0^1(\Omega; \omega_2)$. Moreover we have

$$u \in H_0^1(\Omega) \text{ and } \nabla_{X_1}(u_{\epsilon} - u) \rightharpoonup 0 \text{ weakly in } L^2(\Omega)^q.$$

The constant C is of the form $C_1 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)}$ where C_1, C_2 are dependent on A, λ, C_{ω_2} .

The proof of this theorem will be done in two steps. First, we give the proof in the case $f \in H_0^1(\omega_1) \otimes H_0^1(\omega_2)$, and next that we conclude by a density argument. Let us recall this basic density chain rule, which will be used throughout this article: If (E, τ) and (F, τ') are two topological spaces such that $E \subset F$, and E is dense in F and the canonical injection $E \rightarrow F$ is continuous then, every dense subset in (E, τ) is also dense in (F, τ') .

Remark 2.4. The hypothesis (17) is necessary to obtain the global boundedness of $\nabla_{X_1}(u_\epsilon - u)$. We can observe that through this 2d example, we take

$$A = id_2, f : (x_1, x_2) \mapsto \cos(x_1) \sin(x_2), \text{ and } \Omega = (0, \pi) \times (0, \pi).$$

In this case, we have $u(x_1, x_2) = \cos(x_1) \sin(x_2)$. The quantity $\|\nabla_{X_1}(u_\epsilon - u)\|_{L^2(\Omega)^q}$ could not be bounded. Indeed, if we suppose the converse then according to Theorem 1.1 there exists a subsequence still labeled u_ϵ such that $\nabla_{X_1}(u_\epsilon - u) \rightharpoonup 0$ weakly in $L^2(\Omega)^q$, and $\|\nabla_{X_2}(u_\epsilon - u)\|_{L^2(\Omega)^{N-q}} \rightarrow 0$. Whence $u \in H_0^1(\Omega)$ which is a contradiction.

Let us finish by giving this remark which will be used later in section 4.

Remark 2.5. Suppose that $\beta : s \mapsto \mu s$, for some $\mu > 0$, and suppose that assumptions of Theorem 2.3 hold, then we have the same results of Theorem 2.3 with the same constants. Assume, in addition, that the block A_{12} satisfies the following

$$\partial_{x_i x_j}^2 a_{ij} \in L^2(\Omega), \text{ for } i = 1, \dots, q, j = q + 1, \dots, N, \quad (19)$$

then we have the following

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_2}(u_\epsilon - u)\|_{L^2(\Omega)} \leq \frac{\epsilon}{\mu} \left(C'_1 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C'_2 \|f\|_{L^2(\Omega)} \right),$$

where C'_1, C'_2 are only dependent on A, λ, C_{ω_2} .

3. THE ANALYSIS OF A GENERAL GALERKIN METHOD

3.1. Preliminaries

Let $V \subset H_0^1(\Omega)$ be a closed subspace of $H_0^1(\Omega, \omega_2)$. Notice that V is closed in $H_0^1(\Omega)$, thanks to the continuous imbedding $H_0^1(\Omega) \hookrightarrow H_0^1(\Omega, \omega_2)$. Let $f \in L^2(\Omega)$, we denote by $u_{\epsilon, V, f}$ the unique solution of

$$\begin{cases} \int_{\Omega} \beta(u_{\epsilon, V, f}) \varphi dx + \int_{\Omega} A_\epsilon \nabla u_{\epsilon, V, f} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V \\ u_{\epsilon, V, f} \in V. \end{cases} \quad (20)$$

We denote by $u_{V, f}$ the unique solution of

$$\begin{cases} \int_{\Omega} \beta(u_{V, f}) \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u_{V, f} \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V \\ u_{V, f} \in V. \end{cases} \quad (21)$$

The existence of $u_{\epsilon, V, f}$ follows from the Schauder fixed point theorem. For the existence of $u_{V, f}$ see Appendix C. The uniqueness, for the two problems, follows immediately from (3) and (5). Now, let us begin by some preliminary lemmas

Lemma 3.1. *Under assumptions of Theorem 1.1 and for any $\epsilon \in (0, 1]$ we have the following bounds*

$$\|\nabla_{X_2} u_{\epsilon, V, f}\|_{L^2(\Omega)^{N-q}} \leq \frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda}, \quad (22)$$

$$\|\nabla_{X_1} u_{\epsilon, V, f}\|_{L^2(\Omega)^q} \leq \frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\epsilon \lambda}, \quad (23)$$

$$\|\nabla_{X_2} u_{V, f}\|_{L^2(\Omega)^{N-q}} \leq \frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda}, \quad (24)$$

$$\|\nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda}, \quad (25)$$

$$\|\beta(u_{\epsilon,V},f)\|_{L^2(\Omega)} \leq M \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\omega_2}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right), \quad (26)$$

$$\|\beta(u_{V,f})\|_{L^2(\Omega)} \leq M \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\omega_2}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right), \quad (27)$$

$$\|\beta(u_f)\|_{L^2(\Omega)} \leq M \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\omega_2}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right), \quad (28)$$

where u_f denotes the unique solution of (11).

Proof. These bounds follow easily from a suitable choice of the test functions, monotonicity and ellipticity assumptions. Let us prove, for example, (25) and (28).

According to Theorem 1.1 one can take $\varphi = u_f$ in (12), using ellipticity assumption and the fact that $\int_{\Omega} \beta(u_f)u_f dx \geq 0$ (thanks to (5)) we obtain

$$\lambda \int_{\Omega} |\nabla_{X_2} u_f|^2 dx \leq \int_{\Omega} f u_f dx.$$

By the Cauchy-Schwartz inequality and Poincaré's inequality (10) we obtain (25). Now, using assumption 6 we obtain

$$|\beta(u_f)|^2 \leq M^2 (1 + |u_f|)^2,$$

integrating over Ω and applying Minkowski inequality, (10) and (25) we obtain (28). \square

Using the above lemma, one can prove the following C ea's type lemma

Lemma 3.2. *Under assumptions of Theorem 1.1 we have*

$$\|\nabla_{X_2}(u_{V,f} - u_f)\|_{L^2(\Omega)^{N-q}} \leq C_{c ea} \left(\inf_{v \in V} \|\nabla_{X_2}(v - u_f)\|_{L^2(\Omega)^{N-q}} \right)^{\frac{1}{2}}, \quad (29)$$

and for any $\epsilon \in (0, 1]$

$$\|\nabla(u_{\epsilon,V},f - u_{\epsilon,f})\|_{L^2(\Omega)^N} \leq \frac{C'_{c ea}}{\epsilon^2} \left(\inf_{v \in V} \|\nabla v - \nabla u_f\|_{L^2(\Omega)^N} \right)^{\frac{1}{2}}. \quad (30)$$

where

$$C_{c ea} = \left[2MC_{\omega_2} \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\omega_2}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right) + \|A_{22}\|_{L^\infty(\Omega)} \frac{2C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda} \right]^{\frac{1}{2}}$$

and

$$C'_{c ea} = \left[2MC_{\Omega} \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\Omega}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right) + \|A\|_{L^\infty(\Omega)} \frac{2C_{\Omega} \|f\|_{L^2(\Omega)}}{\lambda} \right]^{\frac{1}{2}},$$

here C_{Ω} is the Poincar e's constant of Ω , and $u_{\epsilon,f}$ is the unique solution of (7).

Proof. The proofs of these two inequalities are similar, so let us prove the first one. Using the Galerkin orthogonality one has, for every $v \in V$:

$$\begin{aligned} & \int_{\Omega} (\beta(u_{V,f}) - \beta(u_f))(u_{V,f} - u_f) dx + \|\nabla_{X_2}(u_{V,f} - u_f)\|_{L^2(\Omega)^{N-q}}^2 \\ &= \int_{\Omega} (\beta(u_{V,f}) - \beta(u_f))(v - u_f) dx + \int_{\Omega} A_{22} \nabla_{X_2}(u_{V,f} - u_f) \cdot \nabla_{X_2}(v - u_f) dx \end{aligned}$$

Using the fact that $\int_{\Omega} (\beta(u_{V,f}) - \beta(u_f))(u_{V,f} - u_f) dx \geq 0$, then by Cauchy-Schwarz and Poincaré's inequalities we derive

$$\|\nabla_{X_2}(u_{V,f} - u_f)\|_{L^2(\Omega)^{N-q}}^2 \leq \left[C_{\omega_2} \|\beta(u_{V,f}) - \beta(u_f)\|_{L^2(\Omega)} + \|A_{22}\|_{L^\infty(\Omega)} \|\nabla_{X_2}(u_{V,f} - u_f)\|_{L^2(\Omega)^{N-q}} \right] \times \|\nabla_{X_2}(v - u_f)\|_{L^2(\Omega)^{N-q}}.$$

Now, by using (27), (28) and the triangle inequality we obtain

$$\|\nabla_{X_2}(u_{V,f} - u_f)\|_{L^2(\Omega)^{N-q}}^2 \leq 2 \left[MC_{\omega_2} \left(|\Omega|^{\frac{1}{2}} + \frac{C_{\omega_2}^2 \|f\|_{L^2(\Omega)}}{\lambda} \right) + \|A_{22}\|_{L^\infty(\Omega)} \frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda} \right] \times \|\nabla_{X_2}(v - u_f)\|_{L^2(\Omega)^{N-q}}.$$

□

Remark 3.3. 1) If $\beta = 0$ (the linear case) then we have for any $\epsilon \in (0, 1]$

$$\|\nabla u_{\epsilon,V,f} - \nabla u_{\epsilon,f}\|_{L^2(\Omega)^N} \leq \frac{\|A\|_{L^\infty(\Omega)}}{\lambda \epsilon^2} \inf_{v \in V} \|\nabla v - \nabla u_{\epsilon,f}\|_{L^2(\Omega)^N}.$$

$$\|\nabla_{X_2} u_{V,f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \frac{\|A_{22}\|_{L^\infty(\Omega)}}{\lambda} \inf_{v \in V} \|\nabla_{X_2} v - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}}.$$

2) If β is Lipschitz, then we can obtain estimations similar to those of the linear case.

3.2. Error estimates in the Galerkin method

Lemma 3.4. Under assumptions of Theorem 1.1, suppose in addition that (14) holds. Then we have

$$\|\nabla_{X_2}(u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 \|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right)$$

and

$$\|\nabla_{X_1}(u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left(C_1 \|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right)$$

where

$$C_1 = \left(\frac{4(C + C')}{\lambda} \right)^{\frac{1}{2}} \quad \text{and} \quad C_2 = \frac{2\sqrt{C''} C_{\omega_2}}{\lambda^{3/2}}$$

and where C, C', C'' are given by (32), (34) and (35). Notice that these constants are independent of ϵ, V and f .

Proof. By subtracting (21) from (20) we get, for every $v \in V$:

$$\begin{aligned} \int_{\Omega} (\beta(u_{\epsilon,V,f}) - \beta(u_{V,f})) v dx + \epsilon^2 \int_{\Omega} A_{11} \nabla_{X_1} u_{\epsilon,V,f} \cdot \nabla_{X_1} v dx \\ + \epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{\epsilon,V,f} \cdot \nabla_{X_1} v dx + \epsilon \int_{\Omega} A_{21} \nabla_{X_1} u_{\epsilon,V,f} \cdot \nabla_{X_2} v dx \\ + \int_{\Omega} A_{22} \nabla_{X_2} (u_{\epsilon,V,f} - u_{V,f}) \cdot \nabla_{X_2} v dx = 0, \end{aligned}$$

Testing with $v = u_{\epsilon, V, f} - u_{V, f}$ we obtain

$$\begin{aligned} & \int_{\Omega} (\beta(u_{\epsilon, V, f}) - \beta(u_{V, f})) (u_{\epsilon, V, f} - u_{V, f}) dx + \int_{\Omega} A_{\epsilon} \nabla(u_{\epsilon, V, f} - u_{V, f}) \cdot \nabla(u_{\epsilon, V, f} - u_{V, f}) \\ &= -\epsilon^2 \int_{\Omega} A_{11} \nabla_{X_1} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx - \epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx \\ & \quad - \epsilon \int_{\Omega} A_{21} \nabla_{X_1} u_{V, f} \cdot \nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f}) dx. \end{aligned}$$

whence, by using (5) and the ellipticity assumption we get

$$\begin{aligned} & \epsilon^2 \lambda \int_{\Omega} |\nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f})|^2 dx + \lambda \int_{\Omega} |\nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f})|^2 dx \leq \\ & \quad -\epsilon^2 \int_{\Omega} A_{11} \nabla_{X_1} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx - \epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx \\ & \quad - \epsilon \int_{\Omega} A_{21} \nabla_{X_1} u_{V, f} \cdot \nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f}) dx. \end{aligned}$$

Estimating the first and the last term of the second member of the above inequality. By using Young's inequality we obtain,

$$\begin{aligned} & -\epsilon^2 \int_{\Omega} A_{11} \nabla_{X_1} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx \\ & \leq \frac{\epsilon^2 \lambda}{2} \int_{\Omega} |\nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f})|^2 dx + \epsilon^2 \frac{\|A_{11}\|_{L^{\infty}(\Omega)}^2}{2\lambda} \int_{\Omega} |\nabla_{X_1} u_{V, f}|^2 dx, \end{aligned}$$

and

$$\begin{aligned} & -\epsilon \int_{\Omega} A_{21} \nabla_{X_1} u_{V, f} \cdot \nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f}) dx \\ & \leq \epsilon^2 \frac{\|A_{21}\|_{L^{\infty}(\Omega)}^2}{2\lambda} \int_{\Omega} |\nabla_{X_1} u_{V, f}|^2 dx + \frac{\lambda}{2} \int_{\Omega} |\nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f})|^2 dx, \end{aligned}$$

thus

$$\begin{aligned} & \frac{\epsilon^2 \lambda}{2} \|\nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f})\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \|\nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f})\|_{L^2(\Omega)^{N-q}}^2 \\ & \leq C \epsilon^2 \int_{\Omega} |\nabla_{X_1} u_{V, f}|^2 dx - \epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx, \end{aligned} \tag{31}$$

where

$$C = \frac{\|A_{21}\|_{L^{\infty}(\Omega)}^2 + \|A_{11}\|_{L^{\infty}(\Omega)}^2}{2\lambda}. \tag{32}$$

Now, we estimate $-\epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx$. Since $u_{\epsilon, V, f} - u_{V, f} \in H_0^1(\Omega)$ and $\partial_{x_i} a_{ij} \in L^{\infty}(\Omega)$, $\partial_{x_j} a_{ij} \in L^{\infty}(\Omega)$ for $i = 1, \dots, q$ and $j = q + 1, \dots, N$, then we can show by a simple density argument that for

$i = 1, \dots, q$ and $j = q + 1, \dots, N$ $\partial_{x_k}(a_{ij}(u_{\epsilon, V, f} - u_{V, f})) \in L^2(\Omega)$ and

$$\partial_{x_k}(a_{ij}(u_{\epsilon, V, f} - u_{V, f})) = (u_{\epsilon, V, f} - u_{V, f})\partial_{x_i}a_{ij} + a_{ij}\partial_{x_i}(u_{\epsilon, V, f} - u_{V, f}), \text{ for } k = i, j).$$

Whence

$$\begin{aligned} -\epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx &= -\epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} a_{ij} \partial_{x_j} u_{V, f} \partial_{x_i} (u_{\epsilon, V, f} - u_{V, f}) dx \\ &= -\epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} \partial_{x_i} (a_{ij}(u_{\epsilon, V, f} - u_{V, f})) \partial_{x_j} u_{V, f} dx \\ &\quad + \epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} (u_{\epsilon, V, f} - u_{V, f}) \partial_{x_i} a_{ij} \partial_{x_j} u_{V, f} dx \\ &= -\epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} \partial_{x_j} (a_{ij}(u_{\epsilon, V, f} - u_{V, f})) \partial_{x_i} u_{V, f} dx \\ &\quad + \epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} (u_{\epsilon, V, f} - u_{V, f}) \partial_{x_i} a_{ij} \partial_{x_j} u_{V, f} dx, \end{aligned}$$

where we have used $\int_{\Omega} \partial_{x_i} (a_{ij}(u_{\epsilon, V, f} - u_{V, f})) \partial_{x_j} u_{V, f} dx = \int_{\Omega} \partial_{x_j} (a_{ij}(u_{\epsilon, V, f} - u_{V, f})) \partial_{x_i} u_{V, f} dx$ which holds by a simple density argument (recall that $u_{V, f} \in H_0^1(\Omega)$). therefore

$$\begin{aligned} -\epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx &= -\epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} (u_{\epsilon, V, f} - u_{V, f}) \partial_{x_j} a_{ij} \partial_{x_i} u_{V, f} dx \quad (33) \\ &\quad - \epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} a_{ij} \partial_{x_j} (u_{\epsilon, V, f} - u_{V, f}) \partial_{x_i} u_{V, f} dx \\ &\quad + \epsilon \sum_{i=1}^q \sum_{j=q+1}^N \int_{\Omega} (u_{\epsilon, V, f} - u_{V, f}) \partial_{x_i} a_{ij} \partial_{x_j} u_{V, f} dx. \end{aligned}$$

By Young and Poincaré's inequalities we obtain

$$\begin{aligned} -\epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V, f} \cdot \nabla_{X_1} (u_{\epsilon, V, f} - u_{V, f}) dx &\leq \frac{\lambda}{4} \int_{\Omega} |\nabla_{X_2} (u_{\epsilon, V, f} - u_{V, f})|^2 dx \\ &\quad + C' \epsilon^2 \int_{\Omega} |\nabla_{X_1} u_{V, f}|^2 dx + C'' \epsilon^2 \int_{\Omega} |\nabla_{X_2} u_{V, f}|^2 dx, \end{aligned}$$

where

$$C' = \frac{3 \left[C_{\omega_2} \max_{1 \leq i \leq q, q+1 \leq j \leq N} \|\partial_{x_j} a_{ij}\|_{L^\infty(\Omega)} (N - q) \right]^2 + 3 \left(\max_{1 \leq i \leq q, q+1 \leq j \leq N} \|a_{ij}\|_{L^\infty(\Omega)} (N - q) \right)^2}{\lambda}. \quad (34)$$

and

$$C'' = \frac{3 \left[q C_{\omega_2} \max_{1 \leq i \leq q, q+1 \leq j \leq N} \|\partial_{x_i} a_{ij}\|_{L^\infty(\Omega)} \right]^2}{\lambda}. \quad (35)$$

According to (24) yields

$$\begin{aligned}
& -\epsilon \int_{\Omega} A_{12} \nabla_{X_2} u_{V,f} \cdot \nabla_{X_1} (u_{\epsilon,V,f} - u_{V,f}) dx \leq \\
& \quad \frac{\lambda}{4} \int_{\Omega} |\nabla_{X_2} (u_{\epsilon,V,f} - u_{V,f})|^2 dx + C' \epsilon^2 \int_{\Omega} |\nabla_{X_1} u_{V,f}|^2 dx + \epsilon^2 C'' \left(\frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda} \right)^2.
\end{aligned} \tag{36}$$

Combining (31) and (36) we get

$$\begin{aligned}
& \frac{\epsilon^2 \lambda}{2} \|\nabla_{X_1} (u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)^q}^2 + \frac{\lambda}{4} \|\nabla_{X_2} (u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)^{N-q}}^2 \\
& \leq \epsilon^2 \left((C + C') \int_{\Omega} |\nabla_{X_1} u_{V,f}|^2 dx + C'' \left(\frac{C_{\omega_2} \|f\|_{L^2(\Omega)}}{\lambda} \right)^2 \right),
\end{aligned}$$

and the proof is finished. \square

Using the triangle inequality, Lemma 3.9 and (29) we obtain the following estimation of the global error between $u_{\epsilon,V,f}$ and u_f

Corollary 3.5. *Under assumption of Lemma 3.4 we have for any $\epsilon \in (0, 1]$:*

$$\|\nabla_{X_2} (u_{\epsilon,V,f} - u_f)\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 \|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) + C_{c\acute{e}a} \left(\inf_{v \in V} \|\nabla_{X_2} (v - u_f)\|_{L^2(\Omega)^{N-q}} \right)^{\frac{1}{2}}$$

We give an important remark which will be used to prove the inequality given in Remark 2.5 .

Remark 3.6. *When $\beta(s) = \mu s$ for some $\mu > 0$ and when the bloc A_{12} satisfies assumption (19) then, by performing some integration by part in the last term of (33) and using the fact that*

$$\|u_{V,f}\|_{L^2(\Omega)} \leq \frac{1}{\mu} \|f\|_{L^2(\Omega)}$$

we can obtain the following bound

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_2} (u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)} \leq \epsilon \left(C'_1 \|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^q} + \frac{C'_2}{\mu} \|f\|_{L^2(\Omega)} \right),$$

where $C'_1, C'_2 > 0$ are independent of f, V, μ and ϵ

3.3. Proof of Theorem 2.2

Let (V_n) be a sequence of finite dimensional spaces which approximates $H_0^1(\Omega)$ in the sense of Definition 2.1. Using the density of $H_0^1(\Omega)$ in $H_0^1(\Omega, \omega_2)$ (Lemma A.1, Appendix A), one can check easily that (V_n) approximates $H_0^1(\Omega, \omega_2)$ in the same sense. Therefore, one has:

$$\text{For every } \epsilon \in (0, 1] : \inf_{v \in V_n} \|\nabla(v - u_{\epsilon,f})\|_{L^2(\Omega)^N} \rightarrow 0 \text{ as } n \rightarrow \infty, \tag{37}$$

and

$$\inf_{v \in V_n} \|\nabla_{X_2}(v - u_f)\|_{L^2(\Omega)^{N-q}} \rightarrow 0 \text{ as } n \rightarrow \infty \quad (38)$$

According to Lemma 3.4, (29) and (30) we have, for every $n \in \mathbb{N}$ and $\epsilon \in (0, 1]$:

$$\|\nabla_{X_2}(u_{\epsilon, V_n, f} - u_{V_n, f})\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 \|\nabla_{X_1} u_{V_n, f}\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right), \quad (39)$$

$$\|\nabla_{X_2}(u_{V_n, f} - u_f)\|_{L^2(\Omega)^{N-q}} \leq C_{c\acute{e}a} \left(\inf_{v \in V_n} \|\nabla_{X_2}(v - u_f)\|_{L^2(\Omega)^{N-q}} \right)^{\frac{1}{2}}, \quad (40)$$

and

$$\|\nabla(u_{\epsilon, V_n, f} - u_{\epsilon, f})\|_{L^2(\Omega)^N} \leq \frac{C'_{c\acute{e}a}}{\epsilon^2} \left(\inf_{v \in V_n} \|\nabla(v - u_{\epsilon, f})\|_{L^2(\Omega)^N} \right)^{\frac{1}{2}}, \quad (41)$$

- Fix ϵ and pass to the limit in (41) using (37), one has

$$u_{\epsilon, V_n, f} \rightarrow u_{\epsilon, f} \text{ as } n \rightarrow \infty \text{ in } H_0^1(\Omega),$$

whence, the continuous imbedding $H_0^1(\Omega) \hookrightarrow H_0^1(\Omega, \omega_2)$ gives

$$u_{\epsilon, V_n, f} \rightarrow u_{\epsilon, f} \text{ as } n \rightarrow \infty \text{ in } H_0^1(\Omega, \omega_2).$$

Now, passing to the limit as $\epsilon \rightarrow 0$, using Theorem 1.1, we get

$$\lim_{\epsilon} (\lim_n u_{\epsilon, V_n, f}) = u_f \text{ in } H_0^1(\Omega, \omega_2). \quad (42)$$

- Fix n and passe to the limit as $\epsilon \rightarrow 0$ using (39), we get

$$u_{\epsilon, V_n, f} \rightarrow u_{V_n, f} \text{ as } \epsilon \rightarrow 0 \text{ in } H_0^1(\Omega, \omega_2).$$

Now, passing to the limit as $n \rightarrow \infty$ in (40) by using (38) we get

$$\lim_n (\lim_{\epsilon} u_{\epsilon, V_n, f}) = u_f \text{ in } H_0^1(\Omega, \omega_2). \quad (43)$$

Finally, Theorem 2.2 follows from (42) and (43).

3.4. Proof of Theorem 2.3

Throughout this subsection we will suppose that $\beta = 0$. The key of the proof of Theorem 2.3 is based on the control the quantity $\|\nabla_{X_1} u_{V, f}\|_{L^2(\Omega)^q}$ independently of V . In fact, we need the following

Lemma 3.7. *Let us assume that A satisfies (3), (4), and that A_{22} satisfies (18). Let V_1 and V_2 be two finite dimensional subspaces of $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively. Let $f \in V_1 \otimes V_2$, and let $u_{V, f}$ be the unique solution in $V = V_1 \otimes V_2$ to:*

$$\int_{\Omega} A_{22}(X_2) \nabla_{X_2} u_{V, f} \cdot \nabla_{X_2} v dx = \int_{\Omega} f v dx, \quad \forall v \in V_1 \otimes V_2. \quad (44)$$

Then we have

$$\|\nabla_{X_1} u_{V, f}\|_{L^2(\Omega)^q} \leq C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q}.$$

where C_3 is given by $C_3 = \frac{\sqrt{q} C_{\omega_2}}{\lambda}$.

Proof. The prove is based on the difference quotient method (see for instance [13] page 168). For every $v = \varphi \otimes \psi \in V_1 \otimes V_2$, the function $X_1 \mapsto \int_{\omega_2} A_{22}(X_2) \nabla_{X_2} u_{V,f}(X_1, X_2) \cdot \nabla_{X_2} \psi dX_2$ belongs to V_1 . In fact $u_{V,f} = \sum_{finite} \varphi_i \otimes \psi_i$ and whence $\int_{\omega_2} A_{22}(X_2) \nabla_{X_2} u_{V,f} \cdot \nabla_{X_2} \psi dX_2$ is a linear combination of φ_i , thanks to the linearity of the integral. Similarly, the function $X_1 \mapsto \int_{\omega_2} f(X_1, X_2) \psi dX_2$ belongs to V_1 . Now, from (44) we derive:

$$\int_{\omega_1} \left(\int_{\omega_2} \{A_{22}(X_2) \nabla_{X_2} u_{V,f} \cdot \nabla_{X_2} \psi - f \cdot \psi\} dX_2 \right) \varphi dX_1 = 0,$$

thus, when φ run through a set of an orthogonal basis of the euclidean space V_1 equipped with the $L^2(\omega_1)$ -scalar product, one can deduce that for a.e. $X_1 \in \omega_1$:

$$\int_{\omega_2} A_{22}(X_2) \nabla_{X_2} u_{V,f}(X_1, X_2) \cdot \nabla_{X_2} \psi dX_2 = \int_{\omega_2} f \cdot (X_1, X_2) \psi dX_2, \quad \forall \psi \in V_2$$

Now, fix $i \in \{1, \dots, q\}$. Let $\omega'_1 \subset \subset \omega_1$ open, for any $0 < h < d(\omega'_1, \partial\omega_1)$ and for any $(X_1, X_2) \in \omega'_1 \times \omega_2$ we denote $\tau_h u_{V,f}(x) = u_{V,f}(x_1, \dots, x_i + h, \dots, x_q, X_2)$. According to the above equality we get for a.e. $X_1 \in \omega'_1$ and for every $\psi \in V_2$:

$$\int_{\omega_2} A_{22}(X_2) \nabla_{X_2} \{\tau_h u_{V,f}(X_1, X_2) - u_{V,f}(X_1, X_2)\} \nabla_{X_2} \psi dX_2 = \int_{\omega_2} \{\tau_h f(X_1, X_2) - f(X_1, X_2)\} \psi dX_2$$

For every $w \in V_1 \otimes V_2$, and for every X_1 fixed the function $w(X_1, \cdot)$ belongs to V_2 , so one can take $\psi = \tau_h u_{V,f}(X_1, \cdot) - u_{V,f}(X_1, \cdot)$ as a test function in the above equality. Therefore, by using the Cauchy-Schwarz inequality, the ellipticity assumption and Poincaré's inequality (10) we obtain:

$$\int_{\omega_2} |\tau_h u_{V,f}(X_1, \cdot) - u_{V,f}(X_1, \cdot)|^2 dX_2 \leq \frac{C_{\omega_2}^2}{\lambda^2} \int_{\omega_2} |\tau_h f(X_1, \cdot) - f(X_1, \cdot)|^2 dX_2$$

Now, integrating the above inequality over ω'_1 yields

$$\int_{\omega'_1 \times \omega_2} |\tau_h u_{V,f} - u_{V,f}|^2 dx \leq \frac{C_{\omega_2}^2}{\lambda^2} \int_{\omega'_1 \times \omega_2} |\tau_h f - f|^2 dx$$

Since $\nabla_{X_1} f \in L^2(\Omega)^q$ then

$$\int_{\omega'_1 \times \omega_2} |\tau_h f - f|^2 dx \leq \|\nabla_{X_1} f\|_{L^2(\Omega)^q}^2 h^2$$

Finally we obtain

$$\int_{\omega'_1 \times \omega_2} \left| \frac{\tau_h u_{V,f} - u_{V,f}}{h} \right|^2 dx \leq \frac{C_{\omega_2}^2 \|\nabla_{X_1} f\|_{L^2(\Omega)^q}^2}{\lambda^2}$$

Therefore,

$$\|D_{x_i} u_{V,f}\|_{L^2(\Omega)} \leq \frac{C_{\omega_2}}{\lambda} \|\nabla_{X_1} f\|_{L^2(\Omega)^q},$$

and hence

$$\|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^q} \leq C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q}$$

with $C_3 = \frac{\sqrt{q} C_{\omega_2}}{\lambda}$. □

Remark 3.8. We have a similar result when (44) is replaced by

$$\mu \int_{\Omega} u_{V,f} v dx + \int_{\Omega} A_{22}(X_2) \nabla_{X_2} u_{V,f} \cdot \nabla_{X_2} v dx = \int_{\Omega} f v dx, \quad \forall v \in V_1 \otimes V_2,$$

where $\mu > 0$. In this case we obtain the following

$$\|\nabla_{X_1} u_{V,f}\|_{L^2(\Omega)^q} \leq \frac{\sqrt{q}}{\mu} \|\nabla_{X_1} f\|_{L^2(\Omega)^q}.$$

Now, we can refine the estimations of Lemma 3.4 as follows

Lemma 3.9. *Under assumptions of Lemmas 3.4 and 3.7 we have:*

$$\begin{aligned} \|\nabla_{X_2} u_{\epsilon,V,f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} &\leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) \\ &\quad + \frac{\|A_{22}\|_{L^\infty(\Omega)}}{\lambda} \inf_{v \in V_1 \otimes V_2} \|\nabla_{X_2} v - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}}, \end{aligned}$$

and

$$\|\nabla_{X_1} u_{\epsilon,V,f}\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) + C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q}.$$

Proof. We have

$$\|\nabla_{X_2} u_{\epsilon,V,f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \|\nabla_{X_2} u_{\epsilon,V,f} - \nabla_{X_2} u_{V,f}\|_{L^2(\Omega)^{N-q}} + \|\nabla_{X_2} u_{V,f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}}.$$

Using Lemma 3.4 and Lemma 3.7 we obtain that

$$\|\nabla_{X_2} u_{\epsilon,V,f} - \nabla_{X_2} u_{V,f}\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right),$$

and using Remark 3.3, we deduce

$$\|\nabla_{X_2} u_{V,f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \frac{\|A_{22}\|_{L^\infty(\Omega)}}{\lambda} \inf_{v \in V_1 \otimes V_2} \|\nabla_{X_2} v - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}}.$$

Using the previous inequalities gives the expected result. The second inequality follows by using the triangle inequality and applying Lemma 3.4 and Lemma 3.7. \square

Remark 3.10. When $\beta(s) = \mu s$, for a certain $\mu > 0$, we obtain, by combining Remarks 3.6 and 3.8, the bound

$$\|\nabla_{X_2}(u_{\epsilon,V,f} - u_{V,f})\|_{L^2(\Omega)} \leq \frac{\epsilon}{\mu} \left(\sqrt{q} C'_1 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C'_2 \|f\|_{L^2(\Omega)} \right).$$

Now, we are able to give the first convergence result

Lemma 3.11. *Suppose that assumptions of Lemmas 3.4 and 3.7 hold. Let $f \in H_0^1(\omega_1) \otimes H_0^1(\omega_2)$. Then we have, for any $\epsilon \in (0, 1]$:*

$$\|\nabla_{X_2} u_{\epsilon,f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right),$$

and

$$\|\nabla_{X_1} u_{\epsilon,f}\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) + C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q}.$$

Proof. Let $(V_n^{(1)})_{n \geq 0}$ and $(V_n^{(2)})_{n \geq 0}$ be two nondecreasing sequences of finite dimensional subspaces of $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively, whose the union of each one is dense in the corresponding space and such that $f \in V_0^{(1)} \otimes V_0^{(2)}$, such a sequence always exists. Indeed, let $\{e_i^{(1)}\}_{i \in \mathbb{N}}$ and $\{e_i^{(2)}\}_{i \in \mathbb{N}}$ be a Hilbert basis of $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively, we know that $\cup_{n \geq 0} \text{span}(e_0^{(1)}, \dots, e_n^{(1)})$ and $\cup_{n \geq 0} \text{span}(e_0^{(2)}, \dots, e_n^{(2)})$ are dense in $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively, in the other hand we have $f = \sum_{i=0}^m f_i^{(1)} \times f_i^{(2)}$ for some $m \in \mathbb{N}$ and $f_i^{(1)} \in H_0^1(\omega_1)$, $f_i^{(2)} \in H_0^1(\omega_2)$ for $i = 0, \dots, m$. Then we set, for every $n \in \mathbb{N}$:

$$\begin{aligned} V_n^{(1)} &:= \text{span}(e_0^{(1)}, \dots, e_n^{(1)}, f_0^{(1)}, \dots, f_m^{(1)}), \\ V_n^{(2)} &:= \text{span}(e_0^{(2)}, \dots, e_n^{(2)}, f_0^{(2)}, \dots, f_m^{(2)}). \end{aligned}$$

Now, since f belongs to each $V_n^{(1)} \otimes V_n^{(2)}$ then according to Lemma 3.9 with $V_n := V_n^{(1)} \otimes V_n^{(2)}$ one has, for every $\epsilon \in (0, 1]$, $n \in \mathbb{N}$:

$$\begin{aligned} \|\nabla_{X_2} u_{\epsilon, V_n, f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} &\leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) \\ &\quad + \inf_{v \in V_n} \|\nabla_{X_2} v - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}}. \end{aligned}$$

According to Corollary A.5 in Appendix A $\cup_{n \geq 0} (V_n^{(1)} \otimes V_n^{(2)})$ is dense in $H_0^1(\Omega)$. Using the fact that the sequence $(V_n)_{n \geq 0}$ is nondecreasing then we obtain that

$$\forall \epsilon \in (0, 1] : \lim_{n \rightarrow \infty} \inf_{v \in V_n} \|\nabla v - \nabla u_{\epsilon, f}\|_{L^2(\Omega)^N} = 0,$$

and therefore, by using (30) we get

$$\forall \epsilon \in (0, 1] : \lim_{n \rightarrow \infty} \|\nabla u_{\epsilon, V_n, f} - \nabla u_{\epsilon, f}\|_{L^2(\Omega)^N} = 0,$$

and thus

$$\forall \epsilon \in (0, 1] : \lim_{n \rightarrow \infty} \|\nabla_{X_2} u_{\epsilon, V_n, f} - \nabla_{X_2} u_{\epsilon, f}\|_{L^2(\Omega)^{N-q}} = 0. \text{ and } \lim_{n \rightarrow \infty} \|\nabla_{X_1} u_{\epsilon, V_n, f} - \nabla_{X_1} u_{\epsilon, f}\|_{L^2(\Omega)^q} = 0.$$

Using the fact that $H_0^1(\Omega)$ is dense in $H_0^1(\Omega, \omega_2)$ (Lemma A.1, Appendix A) and that the imbedding $H_0^1(\Omega) \hookrightarrow H_0^1(\Omega, \omega_2)$ is continuous then $\cup_{n \geq 0} (V_n^{(1)} \otimes V_n^{(2)})$ is dense in $H_0^1(\Omega, \omega_2)$. Using the fact that the sequence $(V_n)_{n \geq 0}$ is nondecreasing then we obtain that

$$\lim_{n \rightarrow \infty} \inf_{v \in V_n} \|\nabla_{X_2} v - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} = 0.$$

Then the passage to the limit as $n \rightarrow \infty$ in the above inequality gives

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_2} u_{\epsilon, f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right).$$

Finally, by using the second inequality of Lemma 3.9 we get

$$\forall \epsilon \in (0, 1] : \|\nabla_{X_1} u_{\epsilon, V_n, f}\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) + C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q},$$

and the passage to limit as $n \rightarrow \infty$ shows the second estimation of the lemma. \square

Now, we are able to give the proof of Theorem 2.3. Let us introduce the space

$$H_0^1(\Omega; \omega_1) = \{v \in L^2(\Omega) \text{ such that } \nabla_{X_1} v \in L^2(\Omega)^q \text{ and for a.e. } X_2 \in \omega_2, v(\cdot, X_2) \in H_0^1(\omega_1)\},$$

normed by the Hilbert norm $\|\nabla_{X_1}(\cdot)\|_{L^2(\Omega)^q}$. We have the Poincaré's inequality

$$\|v\|_{L^2(\Omega)} \leq C_{\omega_1} \|\nabla_{X_1} v\|_{L^2(\Omega)^q} \text{ for any } v \in H_0^1(\Omega; \omega_1) \quad (45)$$

Let $f \in L^2(\Omega)$ such that (16) and (17), thus $f \in H_0^1(\Omega; \omega_1)$. According to Lemma A.3 of Appendix A $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ is dense in $H_0^1(\Omega)$, and according to Remark A.2 of Appendix A $H_0^1(\Omega)$ is dense in $H_0^1(\Omega; \omega_1)$, then it follows that $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ is dense in $H_0^1(\Omega; \omega_1)$, thanks to the continuous imbedding $H_0^1(\Omega) \hookrightarrow H_0^1(\Omega; \omega_1)$. Therefore, for $\delta > 0$ there exists $g_\delta \in H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ such that

$$\|\nabla_{X_1}(f - g_\delta)\|_{L^2(\Omega)^q} \leq \delta. \quad (46)$$

Let u_{ϵ, g_δ} be the unique solution of (7) with f replaced by g_δ . Testing with $u_{\epsilon, f} - u_{\epsilon, g_\delta}$ in the difference of weak formulations

$$\int_{\Omega} A_\epsilon \nabla(u_{\epsilon, f} - u_{\epsilon, g_\delta}) \cdot \nabla \varphi dx = \int_{\Omega} (f - g_\delta) \varphi dx, \quad \forall \varphi \in H_0^1(\Omega),$$

we obtain

$$\|\nabla_{X_2} u_{\epsilon, f} - \nabla_{X_2} u_{\epsilon, g_\delta}\|_{L^2(\Omega)^{N-q}} \leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda} \delta, \text{ and } \|\nabla_{X_1} u_{\epsilon, f} - \nabla_{X_1} u_{\epsilon, g_\delta}\|_{L^2(\Omega)^q} \leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda \epsilon} \delta,$$

where we have used the ellipticity assumption, Poincaré's inequalities (10), (45) and (46). By a passage to the limit as $\epsilon \rightarrow 0$ in the first above inequality, using Theorem 1.1, we get

$$\|\nabla_{X_2} u_f - \nabla_{X_2} u_{g_\delta}\|_{L^2(\Omega)^{N-q}} \leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda} \delta.$$

Applying Lemma 3.11 on u_{ϵ, g_δ} and u_{g_δ} we obtain

$$\|\nabla_{X_2} u_{\epsilon, g_\delta} - \nabla_{X_2} u_{g_\delta}\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} g_\delta\|_{L^2(\Omega)^q} + C_2 \|g_\delta\|_{L^2(\Omega)} \right),$$

and from (46) we derive

$$\|\nabla_{X_2} u_{\epsilon, g_\delta} - \nabla_{X_2} u_{g_\delta}\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 C_3 (\|\nabla_{X_1} f\|_{L^2(\Omega)^q} + \delta) + C_2 \|g_\delta\|_{L^2(\Omega)} \right).$$

Notice that $\|g_\delta\|_{L^2(\Omega)} \rightarrow \|f\|_{L^2(\Omega)}$ as $\delta \rightarrow 0$, thanks to (46) and Poincaré's inequality (45). Finally the triangle inequality gives

$$\begin{aligned} \|\nabla_{X_2} u_{\epsilon, f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} &\leq \|\nabla_{X_2} u_{\epsilon, f} - \nabla_{X_2} u_{\epsilon, g_\delta}\|_{L^2(\Omega)^{N-q}} \\ &\quad + \|\nabla_{X_2} u_{\epsilon, g_\delta} - \nabla_{X_2} u_{g_\delta}\|_{L^2(\Omega)^{N-q}} + \|\nabla_{X_2} u_{g_\delta} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \\ &\leq \epsilon \left(C_1 C_3 (\|\nabla_{X_1} f\|_{L^2(\Omega)^q} + \delta) + C_2 \|g_\delta\|_{L^2(\Omega)} \right) + 2 \frac{C_{\omega_1} C_{\omega_2}}{\lambda} \delta. \end{aligned}$$

Passing to the limit as $\delta \rightarrow 0$ we obtain

$$\|\nabla_{X_2} u_{\epsilon, f} - \nabla_{X_2} u_f\|_{L^2(\Omega)^{N-q}} \leq \epsilon \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right),$$

which is the estimation given in Theorem 2.3.

For the estimation in the first direction, we have

$$\begin{aligned} \|\nabla_{X_1} u_{\epsilon, f}\|_{L^2(\Omega)^q} &\leq \|\nabla_{X_1} u_{\epsilon, f} - \nabla_{X_1} u_{\epsilon, g\delta}\|_{L^2(\Omega)^q} + \|\nabla_{X_1} u_{\epsilon, g\delta}\|_{L^2(\Omega)^q} \\ &\leq \frac{C_{\omega_1} C_{\omega_2}}{\lambda\epsilon} \delta + \frac{1}{\sqrt{2}} \left(C_1 C_3 \|\nabla_{X_1} g\delta\|_{L^2(\Omega)^q} + C_2 \|g\delta\|_{L^2(\Omega)} \right) + C_3 \|\nabla_{X_1} g\delta\|_{L^2(\Omega)^q}, \end{aligned}$$

where we have applied, the triangle inequality and Lemma 3.11. Passing to the limit as $\delta \rightarrow 0$, by using (46) we obtain

$$\|\nabla_{X_1} u_{\epsilon, f}\|_{L^2(\Omega)^q} \leq \frac{1}{\sqrt{2}} \left(C_1 C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q} + C_2 \|f\|_{L^2(\Omega)} \right) + C_3 \|\nabla_{X_1} f\|_{L^2(\Omega)^q}.$$

Hence, by a passage to the limit in $L^2(\Omega) - weak$ as $\epsilon \rightarrow 0$, up to a subsequence, we show that u_f belongs to $H_0^1(\Omega)$, and by a contradiction argument, using the metrisability (for the weak topology) of weakly closed bounded subsets in separable Hilbert spaces, one can show that the global sequence $(\nabla_{X_1} u_{\epsilon, f})_{\epsilon}$ converges weakly to $\nabla_{X_1} u_f$ in $L^2(\Omega)^q$, and this completes the proof of Theorem 2.3.

Remark 3.12. In the case $\beta(s) = \mu s$ with $\mu > 0$ We repeat the same arguments of this subsection by using Remark 3.10 and we obtain the inequality of Remark 2.5.

4. PERTURBATIONS OF SEMIGROUPS OF LINEAR OPERATORS

4.1. Preliminaries

For the standard basic theory of semigroups of bounded linear operators, we refer the reader to [14]. Let us begin by some reminders. Let E be a real Banach space. An undounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset E \rightarrow E$ is said to be closed if for every sequence (x_n) of $D(\mathcal{A})$ such that (x_n) and $(\mathcal{A}(x_n))$ converge in E , we have $\lim x_n \in D(\mathcal{A})$ and $\lim \mathcal{A}(x_n) = \mathcal{A}(\lim x_n)$. An operator is said to be densely defined on E if its domain $D(\mathcal{A})$ is dense in E . Let $\mu \in \mathbb{R}$ we said that μ belongs to the resolvent set of \mathcal{A} if $(\mu I - \mathcal{A}) : D(\mathcal{A}) \rightarrow E$ is one-to-one and onto and such that $R_\mu = (\mu I - \mathcal{A})^{-1} : E \rightarrow D(\mathcal{A}) \subset E$ is a bounded operator on E . Notice that R_μ and \mathcal{A} commute on $D(\mathcal{A})$, that is $\forall x \in D(\mathcal{A}) : R_\mu \mathcal{A}x = \mathcal{A}R_\mu x$. Let \mathcal{A} be a densely defined closed operator. The bounded operator

$$\mathcal{A}_\mu = \mu \mathcal{A}(\mu I - \mathcal{A})^{-1} = \mu \mathcal{A}R_\mu = \mu^2 R_\mu - \mu I,$$

is called the Yosida approximation of \mathcal{A} . We check immediately that \mathcal{A}_μ and \mathcal{A} commute on $D(\mathcal{A})$ that is for every $x \in D(\mathcal{A})$ we have $\mathcal{A}_\mu x \in D(\mathcal{A})$ and $\mathcal{A}\mathcal{A}_\mu x = \mathcal{A}_\mu \mathcal{A}x$. Furthermore, since \mathcal{A} is closed then $e^{t\mathcal{A}_\mu}$ and \mathcal{A} commute on $D(\mathcal{A})$ that is

$$\forall t \in \mathbb{R}, \forall x \in D(\mathcal{A}), e^{t\mathcal{A}_\mu} x \in D(\mathcal{A}), \quad (47)$$

and

$$\mathcal{A}e^{t\mathcal{A}_\mu} x = e^{t\mathcal{A}_\mu} \mathcal{A}x = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\mathcal{A}_\mu)^k \mathcal{A}x,$$

indeed, we can check by induction that if $x \in D(\mathcal{A})$ then $(\mathcal{A}_\mu)^k x \in D(\mathcal{A})$, and that $(\mathcal{A}_\mu)^k$ and \mathcal{A} commute on $D(\mathcal{A})$. Recall also that if $(\mu I - \mathcal{A})^{-1}$ exists for $\mu > 0$ and such that $\|(\mu I - \mathcal{A})^{-1}\| \leq \frac{1}{\mu}$ then

$$\forall t \geq 0 : \|e^{t\mathcal{A}_\mu}\| = \|e^{t\mu^2 R_\mu}\| \times \|e^{-\mu t I}\| \leq e^{t\mu^2 \|R_\mu\|} \times e^{-\mu t} \leq 1,$$

where $\|\cdot\|$ is the operator norm of $\mathcal{L}(E)$. A C_0 semigroup of bounded linear operators on E is a family of bounded operators $(S(t))_{t \geq 0}$ of $\mathcal{L}(E)$ such that: $S(0) = I$, for every $t, s \geq 0 : S(t+s) = S(t)S(s)$, and for every $x \in E : \|S(t)x - x\|_E \rightarrow 0$ as $t \rightarrow 0$. $(S(t))_{t \geq 0}$ is called a semigroup of contractions if for every $t \geq 0 : \|S(t)\|_E \leq 1$. Now, let us recall the well-known Hill-Yosida theorem in its Hilbertian (real) version: An unbounded operator \mathcal{A} is the infinitesimal generator of a C_0 semigroup of contraction $(S(t))_{t \geq 0}$ if and only

if \mathcal{A} is maximal dissipative, that is when $\mu I - \mathcal{A}$ is surjective for every $\mu > 0$ and for every $x \in D(\mathcal{A})$: $\langle \mathcal{A}x, x \rangle \leq 0$. Recall that, in this case $D(\mathcal{A})$ is dense and \mathcal{A} is closed whose the resolvent set contains $]0, +\infty[$. Furthermore, for every $t \geq 0$, $e^{t\mathcal{A}\mu}$ converges, in the strong operator topology, to $S(t)$, as $\mu \rightarrow +\infty$ that is $\forall x \in E : e^{t\mathcal{A}\mu}x \rightarrow S(t)x$ in E as $\mu \rightarrow +\infty$.

Let Ω as in the introduction. The basic Hilbert space in the sequel is $E = L^2(\Omega)$. For any $\epsilon \in (0, 1]$, we introduce the operator \mathcal{A}_ϵ acting on $L^2(\Omega)$ and given by the formula

$$\mathcal{A}_\epsilon u = \operatorname{div}(A_\epsilon \nabla u),$$

where A_ϵ is given as in the introduction of this paper. The domain of \mathcal{A}_ϵ is given by

$$D(\mathcal{A}_\epsilon) = \{u \in H_0^1(\Omega) \mid \operatorname{div}(A_\epsilon \nabla u) \in L^2(\Omega)\},$$

where $\operatorname{div}(A_\epsilon \nabla u) \in L^2(\Omega)$ is taken in the distributional sense. Now, we introduce the operator \mathcal{A}_0 defined on

$$D(\mathcal{A}_0) = \{u \in H_0^1(\Omega; \omega_2) \mid \operatorname{div}_{X_2}(A_{22} \nabla_{X_2} u) \in L^2(\Omega)\},$$

by the formula

$$\mathcal{A}_0 u = \operatorname{div}_{X_2}(A_{22} \nabla_{X_2} u).$$

We check immediatly, by using assumptions (3–4), that \mathcal{A}_ϵ and \mathcal{A}_0 are maximal dissipative and therefore, they are the infinitesimal generators of a C_0 semigroups of contractions on $L^2(\Omega)$, denoted $(S_\epsilon(t))_{t \geq 0}$ and $(S_0(t))_{t \geq 0}$ respectively. For $\mu > 0$ we denote by $R_{\epsilon, \mu}$ the resolvent of \mathcal{A}_ϵ . Similarly, we denote by $R_{0, \mu}$ the resolvent of \mathcal{A}_0 . For $f \in L^2(\Omega)$, we denote $u_{\epsilon, \mu}$ the unique solution in $H_0^1(\Omega)$ to

$$\mu \int_{\Omega} u_{\epsilon, \mu} \varphi dx + \int_{\Omega} A_\epsilon \nabla u_{\epsilon, \mu} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in H_0^1(\Omega),$$

we have $R_{\epsilon, \mu} f = u_{\epsilon, \mu}$ and $\|R_{\epsilon, \mu}\| \leq \frac{1}{\mu}$, where $\|\cdot\|$ is the operator norm of $\mathcal{L}(L^2(\Omega))$. Similarly, let $u_{0, \mu}$ be the unique solution in $H_0^1(\Omega; \omega_2)$ to

$$\mu \int_{\Omega} u_{0, \mu} \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u_{0, \mu} \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in H_0^1(\Omega; \omega_2), \quad (48)$$

we have $R_{0, \mu} f = u_{0, \mu}$ and $\|R_{0, \mu}\| \leq \frac{1}{\mu}$. According to Remark 2.5, we have the following

Lemma 4.1. *Assume (3), (4), (14), (18) and (19). Let $f \in H_0^1(\Omega; \omega_1)$, there exists $C_{A, \Omega} > 0$ only depending on A and Ω . such that:*

$$\forall \epsilon \in (0, 1], \forall \mu > 0 : \|R_{\epsilon, \mu} f - R_{0, \mu} f\|_{L^2(\Omega)} \leq C_{A, \Omega} \times \frac{\epsilon}{\mu} \times \left(\|\nabla_{X_1} f\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right). \quad (49)$$

4.2. The asymptotic behavior of the perturbed semigroup

In this subsection, we study the relationship between the semigroups $(S_\epsilon(t))_{t \geq 0}$ and $(S_0(t))_{t \geq 0}$. We will assume that

$$A \text{ is lipschitz on } \Omega. \quad (50)$$

Notice that (50) shows that, for any $\epsilon \in (0, 1]$:

$$H_0^1(\Omega) \cap H^2(\Omega) \subset D(\mathcal{A}_0) \cap D(\mathcal{A}_\epsilon).$$

Remark also that (50) implies (14). Now, we can give the main theorem of this section.

Theorem 4.2. *Assume that $\Omega = \omega_1 \times \omega_2$ is a bounded domain of $\mathbb{R}^q \times \mathbb{R}^{N-q}$. Assume (3–4), (18), (19) and (50) then for every $g \in L^2(\Omega)$ and $T \geq 0$ we have:*

$$\sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

In particular, for $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ we have:

$$\sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \leq C_{g, A, \Omega} \times T \times \epsilon.$$

Let us begin by this important lemma

Lemma 4.3. *Suppose that assumptions of Theorem 4.2 hold. Let $f \in H_0^1(\Omega) \cap D(\mathcal{A}_0)$ such that*

$$\operatorname{div}_{X_1}(A_{11}\nabla_{X_1}f), \operatorname{div}_{X_1}(A_{12}\nabla_{X_2}f), \operatorname{div}_{X_2}(A_{21}\nabla_{X_1}f) \in L^2(\Omega),$$

and $\mathcal{A}_0f \in H_0^1(\Omega; \omega_1)$. Then, there exists a constant $C_{f, A, \Omega} > 0$ such that for every $\mu > 0$, $\epsilon \in (0, 1]$ we have:

$$\|\mathcal{A}_{\epsilon, \mu}f - \mathcal{A}_{0, \mu}f\|_{L^2(\Omega)} \leq C_{f, A, \Omega} \times \epsilon,$$

where $\mathcal{A}_{\epsilon, \mu}$ and $\mathcal{A}_{0, \mu}$ are the Yosida approximations of \mathcal{A}_ϵ and \mathcal{A}_0 respectively and

$$\begin{aligned} C_{f, A, \Omega} &= \|\operatorname{div}_{X_1}(A_{11}\nabla_{X_1}f)\|_{L^2(\Omega)} + \|\operatorname{div}_{X_1}(A_{12}\nabla_{X_2}f)\|_{L^2(\Omega)} \\ &\quad + \|\operatorname{div}_{X_2}(A_{21}\nabla_{X_1}f)\|_{L^2(\Omega)} + C_{A, \Omega} \left(\|\nabla_{X_1}\mathcal{A}_0f\|_{L^2(\Omega)} + \|\mathcal{A}_0f\|_{L^2(\Omega)} \right). \end{aligned}$$

Proof. Let $\epsilon \in (0, 1]$ and $\mu > 0$. The bounded operators $\mathcal{A}_{\epsilon, \mu}$, $\mathcal{A}_{0, \mu}$ of $\mathcal{L}(L^2(\Omega))$ are given by:

$$\mathcal{A}_{\epsilon, \mu} = \mu\mathcal{A}_\epsilon R_{\epsilon, \mu} \text{ and } \mathcal{A}_{0, \mu} = \mu\mathcal{A}_0 R_{0, \mu}$$

Now, under the above hypothesis we obtain that $f \in D(\mathcal{A}_\epsilon) \cap D(\mathcal{A}_0)$, and

$$\begin{aligned} \|\mathcal{A}_{\epsilon, \mu}f - \mathcal{A}_{0, \mu}f\|_{L^2(\Omega)} &= \mu \|\mathcal{A}_\epsilon R_{\epsilon, \mu}f - \mathcal{A}_0 R_{0, \mu}f\|_{L^2(\Omega)} = \mu \|R_{\epsilon, \mu}\mathcal{A}_\epsilon f - R_{0, \mu}\mathcal{A}_0f\|_{L^2(\Omega)} \\ &\leq \mu \|R_{\epsilon, \mu}\mathcal{A}_\epsilon f - R_{\epsilon, \mu}\mathcal{A}_0f\|_{L^2(\Omega)} + \mu \|R_{\epsilon, \mu}\mathcal{A}_0f - R_{0, \mu}\mathcal{A}_0f\|_{L^2(\Omega)} \\ &\leq \mu \|R_{\epsilon, \mu}\| \times \|\mathcal{A}_\epsilon f - \mathcal{A}_0f\|_{L^2(\Omega)} + \mu \|R_{\epsilon, \mu}\mathcal{A}_0f - R_{0, \mu}\mathcal{A}_0f\|_{L^2(\Omega)}. \end{aligned}$$

Since $\mathcal{A}_0f \in H_0^1(\Omega; \omega_1)$ by hypothesis, then by using (49) (where we replace f by \mathcal{A}_0f) and the fact that $\|R_{\epsilon, \mu}\| \leq \frac{1}{\mu}$ we obtain

$$\begin{aligned} \|\mathcal{A}_{\epsilon, \mu}f - \mathcal{A}_{0, \mu}f\|_{L^2(\Omega)} &\leq \|\mathcal{A}_\epsilon f - \mathcal{A}_0f\|_{L^2(\Omega)} + \epsilon C_{A, \Omega} \left(\|\nabla_{X_1}\mathcal{A}_0f\|_{L^2(\Omega)} + \|\mathcal{A}_0f\|_{L^2(\Omega)} \right) \\ &= \epsilon \left(\begin{aligned} &\epsilon \|\operatorname{div}_{X_1}(A_{11}\nabla_{X_1}f)\|_{L^2(\Omega)} + \|\operatorname{div}_{X_1}(A_{12}\nabla_{X_2}f)\|_{L^2(\Omega)} \\ &+ \|\operatorname{div}_{X_2}(A_{21}\nabla_{X_1}f)\|_{L^2(\Omega)} + C_{A, \Omega} \left(\|\nabla_{X_1}\mathcal{A}_0f\|_{L^2(\Omega)} + \|\mathcal{A}_0f\|_{L^2(\Omega)} \right) \end{aligned} \right) \\ &\leq C_{f, A, \Omega} \times \epsilon. \end{aligned}$$

where we have used

$$\mathcal{A}_\epsilon f - \mathcal{A}_0f = \epsilon^2 \operatorname{div}_{X_1}(A_{11}\nabla_{X_1}f) + \epsilon \operatorname{div}_{X_1}(A_{12}\nabla_{X_2}f) + \epsilon \operatorname{div}_{X_2}(A_{21}\nabla_{X_1}f),$$

and the proof of the lemma is finished. \square

Lemma 4.4. *Under assumptions of Theorem 4.2, we have for any $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$:*

$$\forall \mu > 0, \forall t \geq 0, \forall \epsilon \in (0, 1] : \|e^{t\mathcal{A}_{\epsilon,\mu}} g - e^{t\mathcal{A}_0,\mu} g\|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times t \times \epsilon,$$

where $C_{g,A,\Omega}$ is independent of μ and ϵ .

Proof. Let $\mu > 0$ and $t \geq 0$ and $\epsilon \in (0, 1]$, we have

$$\begin{aligned} e^{t\mathcal{A}_0,\mu} - e^{t\mathcal{A}_{\epsilon,\mu}} &= \int_0^t \frac{d}{ds} \left(e^{(t-s)\mathcal{A}_{\epsilon,\mu}} e^{s\mathcal{A}_0,\mu} \right) ds \\ &= \int_0^t e^{(t-s)\mathcal{A}_{\epsilon,\mu}} (\mathcal{A}_0,\mu - \mathcal{A}_{\epsilon,\mu}) e^{s\mathcal{A}_0,\mu} ds. \end{aligned}$$

Hence for $g \in L^2(\Omega)$ we have

$$\|e^{t\mathcal{A}_{\epsilon,\mu}} g - e^{t\mathcal{A}_0,\mu} g\|_{L^2(\Omega)} \leq \int_0^t \|\mathcal{A}_0,\mu e^{s\mathcal{A}_0,\mu} g - \mathcal{A}_{\epsilon,\mu} e^{s\mathcal{A}_0,\mu} g\|_{L^2(\Omega)} ds \quad (51)$$

where we have used the fact that $\|e^{(t-s)\mathcal{A}_{\epsilon,\mu}}\| \leq 1$, since $t - s \geq 0$.

Now, we suppose that $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ (remark that $g \in D(\mathcal{A}_0)$) and for $s \geq 0$ we set

$$f_g := e^{s\mathcal{A}_0,\mu} g$$

We can prove that f_g satisfies the same hypothesis satisfied by the function f of Lemma 4.3 and moreover, for every $i, j = 1, \dots, q$ we have :

$$\left\| D_{x_i x_j}^2 f_g \right\|_{L^2(\Omega)} \leq \left\| D_{x_i x_j}^2 g \right\|_{L^2(\Omega)}, \quad \|D_{x_i} f_g\|_{L^2(\Omega)} \leq \|D_{x_i} g\|_{L^2(\Omega)},$$

and

$$\|(\mathcal{A}_0 f_g)\|_{L^2(\Omega)} \leq \|\mathcal{A}_0 g\|_{L^2(\Omega)}, \quad \|D_{x_i}(\mathcal{A}_0 f_g)\|_{L^2(\Omega)} \leq \|D_{x_i}(\mathcal{A}_0 g)\|_{L^2(\Omega)},$$

and for every $i = 1, \dots, q; j = q+1, \dots, N$ we have:

$$\|D_{x_j} f_g\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|\mathcal{A}_0 g\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)} \quad \text{and} \quad \left\| D_{x_j x_j}^2 f_g \right\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|D_{x_i} \mathcal{A}_0 g\|_{L^2(\Omega)} \|D_{x_i} g\|_{L^2(\Omega)}.$$

The proof of these assertions follows from the identity $e^{s\mathcal{A}_0,\mu}(g_1 \otimes g_2) = g_1 \otimes e^{s\mathcal{A}_0,\mu} g_2$ (see Appendix B).

Applying Lemma 4.3, and using the above inequalities with (50) we get

$$\begin{aligned} \|\mathcal{A}_0,\mu e^{s\mathcal{A}_0,\mu} g - \mathcal{A}_{\epsilon,\mu} e^{s\mathcal{A}_0,\mu} g\|_{L^2(\Omega)} &\leq \epsilon \left(\begin{aligned} &\epsilon \|\operatorname{div}_{X_1}(\mathcal{A}_{11} \nabla_{X_1} f_g)\|_{L^2(\Omega)} + \|\operatorname{div}_{X_1}(\mathcal{A}_{12} \nabla_{X_2} f_g)\|_{L^2(\Omega)} \\ &+ \|\operatorname{div}_{X_2}(\mathcal{A}_{21} \nabla_{X_1} f_g)\|_{L^2(\Omega)} + C_{A,\Omega} \left(\|\nabla_{X_1} \mathcal{A}_0 f_g\|_{L^2(\Omega)} + \|\mathcal{A}_0 f_g\|_{L^2(\Omega)} \right) \end{aligned} \right) \\ &\leq C_{g,A,\Omega} \times \epsilon. \end{aligned}$$

Notice that $C_{g,A,\Omega}$ does not depend in s, ϵ and μ . Finally, integrating the above inequality in s over $[0, t]$ and by using (51) we get the desired result. \square

Now, we are able to prove Theorem 4.2. First we prove the case when $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ and we conclude by a density argument. So let g as mentioned above, by Lemma 4.4 we have

$$\forall \mu > 0, \forall t \geq 0, \forall \epsilon \in (0, 1] : \|e^{t\mathcal{A}_{\epsilon,\mu}} g - e^{t\mathcal{A}_0,\mu} g\|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times t \times \epsilon. \quad (52)$$

Therefore, by passing to the limit in (52) as $\mu \rightarrow +\infty$ we get (see the preliminaries, the abstract part)

$$\forall t \geq 0, \forall \epsilon \in (0, 1] : \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times t \times \epsilon,$$

whence for $T \geq 0$ fixed we obtain

$$\forall \epsilon \in (0, 1] : \sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \leq C_{g,A,\Omega} \times T \times \epsilon. \quad (53)$$

Whence

$$\sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (54)$$

Now, let $g \in L^2(\Omega)$ and let $\delta > 0$, by density there exists $g_\delta \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ such that

$$\|g - g_\delta\|_{L^2(\Omega)} \leq \frac{\delta}{4}.$$

According to (54) there exists $\epsilon_\delta > 0$ such that

$$\forall \epsilon \in (0, \epsilon_\delta] : \sup_{t \in [0, T]} \|S_\epsilon(t)g_\delta - S_0(t)g_\delta\|_{L^2(\Omega)} \leq \frac{\delta}{2}.$$

Whence, by the triangle inequality we get

$$\forall \epsilon \in (0, \epsilon_\delta] : \sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \leq \frac{\delta}{2} + \sup_{t \in [0, T]} (\|S_\epsilon(t)\| + \|S_0(t)\|) \times \|g_\delta - g\|_{L^2(\Omega)}.$$

Using the fact that the semigroups $(S_\epsilon(t))_{t \geq 0}$ and $(S_0(t))_{t \geq 0}$ are of contraction, we get

$$\forall \epsilon \in (0, \epsilon_\delta] : \sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \leq \delta.$$

So, $\sup_{t \in [0, T]} \|S_\epsilon(t)g - S_0(t)g\|_{L^2(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0$. The second assertion of the theorem is given by (53) and the proof of the theorem is completed.

4.3. An application to a linear parabolic equation

Theorem 4.2 gives an opening for the study of anisotropic singular perturbations of evolution partial differential equations from the semigroup's point of view. In this subsection we just give a simple and short application to the linear parabolic equation

$$\frac{\partial u_\epsilon}{\partial t} - \operatorname{div}(A_\epsilon \nabla u_\epsilon) = 0, \quad (55)$$

supplemented with the boundary and the initial conditions

$$u_\epsilon(t, \cdot) = 0 \text{ in } \partial\Omega \text{ for } t \in (0, +\infty) \quad (56)$$

$$u_\epsilon(0, \cdot) = u_{\epsilon,0}. \quad (57)$$

The limit problem is

$$\frac{\partial u}{\partial t} - \operatorname{div}_{X_2}(A_{22} \nabla_{X_2} u) = 0, \quad (58)$$

supplemented with the boundary and the initial conditions

$$u(t, \cdot) = 0 \text{ in } \omega_1 \times \partial\omega_2 \text{ for } t \in (0, +\infty) \quad (59)$$

$$u(0, \cdot) = u_0. \quad (60)$$

The operator form of (55) – (57) and (58) – (60) reads

$$\frac{du_\epsilon}{dt} - \mathcal{A}_\epsilon u_\epsilon = 0, \text{ with } u_\epsilon(0) = u_{\epsilon,0}, \quad (61)$$

and

$$\frac{du}{dt} - \mathcal{A}_0 u = 0, \text{ with } u(0) = u_0. \quad (62)$$

Suppose that $u_0 \in D(\mathcal{A}_0)$ and $u_{\epsilon,0} \in D(\mathcal{A}_\epsilon)$. Assume that (3), (4) and then it follows that (61), (62) have a unique classical solutions

$$u_\epsilon \in C^1([0, +\infty); L^2(\Omega)) \cap C([0, +\infty); D(\mathcal{A}_\epsilon)),$$

and

$$u \in C^1([0, +\infty); L^2(\Omega)) \cap C([0, +\infty); D(\mathcal{A}_0)).$$

We have the following convergence result.

Proposition 4.5. *Suppose that $u_0 \in D(\mathcal{A}_0)$ and $u_{\epsilon,0} \in D(\mathcal{A}_\epsilon)$ such that $u_{\epsilon,0} \rightarrow u_0$ in $L^2(\Omega)$, then under assumptions of Theorem 4.2, we have for any $T \geq 0$:*

$$\sup_{t \in [0, T]} \|u_\epsilon(t) - u(t)\|_{L^2(\Omega)} \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (63)$$

Moreover, if $u_{\epsilon,0}$ and u_0 are in $H^2(\Omega)$ and such that $(u_{\epsilon,0})$ is bounded in $H^2(\Omega)$ and $\|\nabla_{X_2}(u_{\epsilon,0} - u_0)\|_{L^2(\Omega)} \rightarrow 0$, $\|\nabla_{X_2}^2(u_{\epsilon,0} - u_0)\|_{L^2(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0$, then we have

$$\sup_{t \in [0, T]} \left\| \frac{d}{dt}(u_\epsilon(t) - u(t)) \right\|_{L^2(\Omega)} \rightarrow 0.$$

Proof. It is well known that the solutions u_ϵ, u are given by

$$u_\epsilon(t) = S_\epsilon(t)u_{\epsilon,0} \text{ and } u_0(t) = S_0(t)u_0, \text{ for every } t \geq 0.$$

Let $T \geq 0$, we have

$$\begin{aligned} \sup_{t \in [0, T]} \|u_\epsilon(t) - u(t)\|_{L^2(\Omega)} &\leq \sup_{t \in [0, T]} \|S_\epsilon(t)u_{\epsilon,0} - S_\epsilon(t)u_0\|_{L^2(\Omega)} + \sup_{t \in [0, T]} \|S_\epsilon(t)u_0 - S_0(t)u_0\|_{L^2(\Omega)} \\ &\leq \|u_{\epsilon,0} - u_0\|_{L^2(\Omega)} + \sup_{t \in [0, T]} \|S_\epsilon(t)u_0 - S_0(t)u_0\|_{L^2(\Omega)}. \end{aligned}$$

Passing to the limit as $\epsilon \rightarrow 0$ by using Theorem 4.2, we get $\sup_{t \in [0, T]} \|u_\epsilon(t) - u(t)\|_{L^2(\Omega)} \rightarrow 0$.

For the second affirmation, we have

$$\begin{aligned} \left\| \frac{d}{dt}(u_\epsilon(t) - u(t)) \right\|_{L^2(\Omega)} &= \|S_\epsilon(t)\mathcal{A}_\epsilon u_{\epsilon,0} - S_0(t)\mathcal{A}_0 u_0\|_{L^2(\Omega)} \\ &\leq \|\mathcal{A}_\epsilon u_{\epsilon,0} - \mathcal{A}_0 u_0\|_{L^2(\Omega)} + \sup_{t \in [0, T]} \|S_\epsilon(t)\mathcal{A}_0 u_0 - S_0(t)\mathcal{A}_0 u_0\|_{L^2(\Omega)}. \end{aligned}$$

As $(u_{\epsilon,0})$ is bounded in $H^2(\Omega)$, $u_0 \in H^2(\Omega)$ and $\|\nabla_{X_2}(u_{\epsilon,0} - u_0)\|_{L^2(\Omega)} \rightarrow 0$, $\|\nabla_{X_2}^2(u_{\epsilon,0} - u_0)\|_{L^2(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0$, then by using (50) we get immediately $\|\mathcal{A}_\epsilon u_{\epsilon,0} - \mathcal{A}_0 u_0\|_{L^2(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0$, and we conclude by applying Theorem 4.2. \square

Remark 4.6. Consider the nonhomogeneous parabolic equations associated to (55) and (58) with second member $f(t, x)$. Suppose that f is regular enough, for example $f \in Lip([0, T]; L^2(\Omega))$, then the associated classical solutions u_ϵ and u exist and they are unique. In this case, we have the same convergence result (63). The proof follows immediately from the use of the following integral representation formulas of the solutions

$$u_\epsilon(t) = S_\epsilon(t)u_{\epsilon,0} + \int_0^t S_\epsilon(t-r)f(r)dr,$$

$$u(t) = S_0(t)u_0 + \int_0^t S_0(t-r)f(r)dr,$$

Theorem 4.2, and Lebesgue's theorem.

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APPENDIX A. DENSITY LEMMAS

Let ω_1 and ω_2 be two open bounded subsets of \mathbb{R}^q and \mathbb{R}^{N-q} respectively. Recall that

$$H_0^1(\Omega; \omega_2) = \{u \in L^2(\Omega) \mid \nabla_{X_2} u \in L^2(\Omega), \text{ a.e. } X_1 \in \omega_1, u(X_1, \cdot) \in H_0^1(\omega_2)\},$$

normed by $\|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)}$. We have the following

Lemma A.1. *The space $H_0^1(\Omega)$ is dense in $H_0^1(\Omega; \omega_2)$.*

Proof. Let $u \in H_0^1(\Omega; \omega_2)$ fixed. Let l be the linear form defined on $H_0^1(\Omega)$ by

$$\forall \varphi \in H_0^1(\Omega) : l(\varphi) = \int_\Omega \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx$$

l is continuous on $H_0^1(\Omega)$, indeed we have

$$\forall \varphi \in H_0^1(\Omega) : |l(\varphi)| \leq \|\nabla_{X_2} u\|_{L^2(\Omega)} \|\nabla_{X_2} \varphi\|_{L^2(\Omega)}$$

and then,

$$\forall \varphi \in H_0^1(\Omega) : |l(\varphi)| \leq \|\nabla_{X_2} u\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)}.$$

For every $n \in \mathbb{N}^*$, we denote u_n the unique solution of

$$\begin{cases} \frac{1}{n^2} \int_\Omega \nabla_{X_1} u_n \cdot \nabla_{X_1} \varphi dx + \int_\Omega \nabla_{X_2} u_n \cdot \nabla_{X_2} \varphi dx = l(\varphi), \forall \varphi \in H_0^1(\Omega) \\ u_n \in H_0^1(\Omega) \end{cases} \quad (64)$$

where the existence and uniqueness follows from the Lax-Milgram theorem. Testing with u_n in (64) we get, for every $n \in \mathbb{N}^*$

$$\frac{1}{n^2} \int_\Omega |\nabla_{X_1} u_n|^2 dx + \int_\Omega |\nabla_{X_2} u_n|^2 dx \leq \|\nabla_{X_2} u\|_{L^2(\Omega)} \|\nabla_{X_2} u_n\|_{L^2(\Omega)},$$

then, we deduce that

$$\forall n \in \mathbb{N}^* : \|\nabla_{X_2} u_n\|_{L^2(\Omega)} \leq \|\nabla_{X_2} u\|_{L^2(\Omega)}, \quad (65)$$

and

$$\forall n \in \mathbb{N}^* : \frac{1}{n} \|\nabla_{X_1} u_n\|_{L^2(\Omega)} \leq \|\nabla_{X_2} u\|_{L^2(\Omega)}. \quad (66)$$

Using (65) and Poincaré's inequality we obtain:

$$\forall n \in \mathbb{N}^* : \|u_n\|_{L^2(\Omega)} \leq C_{\omega_2} \|\nabla_{X_2} u\|_{L^2(\Omega)}. \quad (67)$$

Reflexivity of $L^2(\Omega)$ shows that there exists, $u_\infty, u'_\infty, u''_\infty \in L^2(\Omega)$ and a subsequence still labeled (u_n) such that

$$u_n \rightharpoonup u_\infty, \nabla_{X_2} u_n \rightharpoonup u'_\infty \text{ and } \frac{1}{n} \nabla_{X_1} u_n \rightharpoonup u''_\infty \text{ in } L^2(\Omega) \text{ weakly.}$$

Since the derivation on $\mathcal{D}'(\Omega)$ is continuous we get

$$u_n \rightharpoonup u_\infty, \nabla_{X_2} u_n \rightharpoonup \nabla_{X_2} u_\infty \text{ and } \frac{1}{n} \nabla_{X_1} u_n \rightharpoonup 0 \text{ in } L^2(\Omega) \text{ weakly.} \quad (68)$$

1) we have $u_\infty \in H_0^1(\Omega; \omega_2)$:

By the Mazur Lemma, there exists a sequence (U_n) of convex combinations of $\{u_n\}$ such that

$$\nabla_{X_2} U_n \rightarrow \nabla_{X_2} u_\infty \text{ in } L^2(\Omega) \text{ strongly,} \quad (69)$$

then by the Lebesgue theorem there exists a subsequence (U_{n_k}) such that:

$$\text{For a.e. } X_1 \in \omega_1 : \nabla_{X_2} U_{n_k}(X_1, \cdot) \rightarrow \nabla_{X_2} u_\infty(X_1, \cdot) \text{ in } L^2(\omega_2) \text{ strongly.} \quad (70)$$

Now, since $(U_{n_k}) \in H_0^1(\Omega)^\mathbb{N}$ then

$$\text{For a.e. } X_1 \in \omega_1 : (U_{n_k}(X_1, \cdot)) \in H_0^1(\omega_2)^\mathbb{N}. \quad (71)$$

Combining (70) and (71) we deduce:

$$\text{For a.e. } X_1 \in \omega_1, u_\infty(X_1, \cdot) \in H_0^1(\omega_2),$$

and the proof of $u_\infty \in H_0^1(\Omega; \omega_2)$ is finished.

2) we have $u_\infty = u$:

Passing to the limit in (64) by using (68) we obtain

$$\int_{\Omega} \nabla_{X_2} u_\infty \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx, \forall \varphi \in H_0^1(\Omega). \quad (72)$$

For every $\varphi_1 \in H_0^1(\omega_1)$ and $\varphi_2 \in H_0^1(\omega_2)$ taking, $\varphi = \varphi_1 \otimes \varphi_2$ in (72) we obtain, for a.e. $X_1 \in \omega_1$

$$\int_{\omega_2} \nabla_{X_2} u_\infty(X_1, \cdot) \cdot \nabla_{X_2} \varphi_2 dX_2 = \int_{\omega_2} \nabla_{X_2} u(X_1, \cdot) \cdot \nabla_{X_2} \varphi_2 dX_2, \forall \varphi_2 \in H_0^1(\omega_2).$$

For a.e. $X_1 \in \omega_1$, taking $\varphi_2 = u_\infty(X_1, \cdot) - u(X_1, \cdot)$, which belong to $H_0^1(\omega_2)$, in the above equality yields:

$$\int_{\omega_2} |\nabla_{X_2} (u_\infty(X_1, \cdot) - u(X_1, \cdot))|^2 dX_2 = 0.$$

Integrating over ω_1 we deduce

$$\int_{\Omega} |\nabla_{X_2} (u_\infty - u)|^2 dx = 0.$$

and finally since $\|\nabla_{X_2}(\cdot)\|_{L^2(\Omega)}$ is a norm on $H_0^1(\Omega; \omega_2)$ we get,

$$u_\infty = u. \quad (73)$$

combining (69) and (73) we get the desired result. \square

Remark A.2. By symmetry, $H_0^1(\Omega)$ is dense in the space

$$H_0^1(\Omega; \omega_1) = \{u \in L^2(\Omega) \mid \nabla_{X_1} u \in L^2(\Omega), \text{ and for a.e. } X_2 \in \omega_2, u(\cdot, X_2) \in H_0^1(\omega_1)\},$$

normed by $\|\nabla_{X_1}(\cdot)\|_{L^2(\Omega)}$.

Lemma A.3. *The space $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ is dense in $H_0^1(\Omega)$.*

Proof. For a functions $\varphi : \omega_1 \rightarrow \mathbb{R}$, $\psi : \omega_2 \rightarrow \mathbb{R}$ we denote by $\varphi \otimes \psi$ the function defined on Ω by $(\varphi \otimes \psi)(X_1, X_2) = \varphi(X_1) \times \psi(X_2)$, the tensor product $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ is the vector space generated by the elements of the form $\varphi \otimes \psi$ with φ and ψ in $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively.

It is well know that $D(\omega_1) \otimes D(\omega_2)$ is dense in $D(\omega_1 \times \omega_2)$, here $D(\omega_1 \times \omega_2)$ is equipped with its natural topology (the inductive limit topology). It is clear that the injection of $D(\omega_1 \times \omega_2)$ in $H_0^1(\omega_1 \times \omega_2)$ is continuous, thanks to the inequality

$$\forall u \in D(\Omega) : \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}} \leq \sqrt{N \times \text{mes}(\Omega)} \times \left(\max_{1 \leq i \leq N} \sup_{\text{Support}(u)} |\partial_{x_i} u| \right).$$

Hence, by the density chain rule we obtain the density of $D(\omega_1) \otimes D(\omega_2)$ in $H_0^1(\Omega)$, and finally since $D(\omega_1) \otimes D(\omega_2) \subset H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ we get the desired result. \square

Lemma A.4. *Let $(V_n^{(1)})$ and $(V_n^{(2)})$ be two sequences of subspaces (not necessarily of finite dimension) of $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively. If $\cup V_n^{(1)}$ and $\cup V_n^{(2)}$ are dense in $H_0^1(\omega_1)$ and $H_0^1(\omega_2)$ respectively, then $\text{vect} \left(\bigcup_{n,m} (V_n^{(1)} \otimes V_m^{(2)}) \right)$ is dense in $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ for the induced topology of $H_0^1(\Omega)$. In particular, if $(V_n^{(1)})$ and $(V_n^{(2)})$ are nondecreasing then $\bigcup_n (V_n^{(1)} \otimes V_n^{(2)})$ is dense in $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$.*

Proof. Let's start by a useful inequality. For $u \otimes v$ in $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ we have :

$$\begin{aligned} \|u \otimes v\|_{H_0^1(\Omega)}^2 &= \int_{\Omega} |\nabla_{X_1}(u \otimes v)|^2 dx + \int_{\Omega} |\nabla_{X_2}(u \otimes v)|^2 dx \\ &= \left(\int_{\omega_2} v^2 dX_2 \right) \times \left(\int_{\omega_1} |\nabla_{X_1} u|^2 dX_1 \right) \\ &\quad + \left(\int_{\omega_1} u^2 dX_1 \right) \times \left(\int_{\omega_2} |\nabla_{X_2} v|^2 dX_2 \right) \\ &\leq C \|u\|_{H_0^1(\omega_1)}^2 \times \|v\|_{H_0^1(\omega_2)}^2, \end{aligned} \tag{74}$$

where we have used Fubini's theorem and Poincaré's inequality, here $C = C_{\omega_1}^2 + C_{\omega_2}^2 > 0$. Now, fix $\eta > 0$ and let $\varphi \otimes \psi \in H_0^1(\omega_1) \otimes H_0^1(\omega_2)$, by density of $\cup V_n^{(1)}$ in $H_0^1(\omega_1)$ there exists $n \in \mathbb{N}$ and $\varphi_n \in V_n^{(1)}$ such that:

$$\|\psi\|_{H_0^1(\omega_2)} \times \|\varphi_n - \varphi\|_{H_0^1(\omega_1)} \leq \frac{\eta}{2\sqrt{C}}.$$

Similarly by density of $\cup V_n^{(2)}$ in $H_0^1(\omega_2)$, there exists $m \in \mathbb{N}$ (which depends on n and ψ) and $\psi_m \in V_m^{(2)}$ such that

$$\|\varphi_n\|_{H_0^1(\omega_1)} \times \|\psi_m - \psi\|_{H_0^1(\omega_2)} \leq \frac{\eta}{2\sqrt{C}}.$$

Whence, by using the triangle inequality and (74) we obtain

$$\|\varphi \otimes \psi - \varphi_n \otimes \psi_m\|_{H_0^1(\Omega)} \leq \eta. \quad (75)$$

Now, since every element of $H_0^1(\omega_1) \otimes H_0^1(\omega_2)$ could be written as $\sum_{i=1}^l \varphi_i \otimes \psi_i$, then by using the triangle inequality and using (75) with η replaced by $\frac{\eta}{l}$, one gets the desired result. \square

Corollary A.5. *vect* $\left(\bigcup_{n,m} (V_n^{(1)} \otimes V_m^{(2)}) \right)$ *is dense in* $H_0^1(\Omega)$. *in particular, if* $(V_n^{(1)})$ *and* $(V_n^{(2)})$ *are nondecreasing then* $\bigcup_n (V_n^{(1)} \otimes V_n^{(2)})$ *is dense in* $H_0^1(\Omega)$.

APPENDIX B. SEMIGROUP

Lemma B.1. *Assume (3–4), (18) and let* $f_1 \in L^2(\omega_1)$, $f_2 \in L^2(\omega_2)$ *then for every* $\mu > 0$ *we have*

$$R_{0,\mu}(f_1 \otimes f_2) = f_1 \otimes (R_{0,\mu}f_2).$$

Notice that $R_{0,\mu}f_2 \in H_0^1(\omega_2)$. *Moreover, we have*

$$\mathcal{A}_{0,\mu}(f_1 \otimes f_2) = f_1 \otimes (\mathcal{A}_{0,\mu}f_2).$$

Notice also that $\mathcal{A}_{0,\mu}f_2 \in L^2(\omega_2)$. *Here,* $\mathcal{A}_{0,\mu}$ *is the Yosida approximation of* \mathcal{A}_0 *that is* $\mathcal{A}_{0,\mu} = \mu\mathcal{A}_0R_{0,\mu}$.

Proof. Let $v_2 \in H_0^1(\omega_2)$ be the unique solution in $H_0^1(\omega_2)$ to

$$\mu \int_{\omega_2} v_2 \varphi_2 dX_2 + \int_{\omega_2} A_{22}(X_2) \nabla_{X_2} v_2 \cdot \nabla_{X_2} \varphi_2 dX_2 = \int_{\omega_2} f_2 \varphi_2 dX_2, \quad \forall \varphi_2 \in H_0^1(\omega_2), \quad (76)$$

Let $\varphi \in H_0^1(\Omega; \omega_2)$, then $\varphi(X_1, \cdot) \in H_0^1(\omega_2)$ for a.e. $X_1 \in \omega_1$. Let $f_1 \in L^2(\omega_1)$, multiplying (76) by f_1 , testing in (76) with $\varphi(X_1, \cdot)$ and integrating over ω_1 yields

$$\mu \int_{\Omega} f_1 v_2 \varphi dx + \int_{\Omega} A_{22}(X_2) \nabla_{X_2} (f_1 v_2) \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f_1 f_2 \varphi dx.$$

It is clear that $f_1 v_2 \in H_0^1(\Omega; \omega_2)$ whence, $R_{0,\mu}(f_1 \otimes f_2) = f_1 \otimes v_2$, in particular when $f_1 = 1$ we have $R_{0,\mu}(f_2) = v_2$, and therefore $R_{0,\mu}(f_1 \otimes f_2) = f_1 \otimes R_{0,\mu}(f_2)$. The second assertion follows immediately from the first one, in fact

$$\mathcal{A}_{0,\mu}(f_1 \otimes f_2) = \mu\mathcal{A}_0R_{0,\mu}(f_1 \otimes f_2) = \mu\mathcal{A}_0(f_1 \otimes R_{0,\mu}f_2).$$

We have $R_{0,\mu}f_2 \in D(\mathcal{A}_0) \cap H_0^1(\omega_2)$ then by (18) (the operator \mathcal{A}_0 does not depend on the X_1 direction), we get

$$\mathcal{A}_0(f_1 \otimes R_{0,\mu}f_2) = f_1 \otimes \mathcal{A}_0(R_{0,\mu}f_2),$$

notice that $\mathcal{A}_0(R_{0,\mu}f_2) \in L^2(\omega_2)$. Finally we get

$$\mathcal{A}_{0,\mu}(f_1 \otimes f_2) = \mu f_1 \otimes \mathcal{A}_0(R_{0,\mu}f_2) = f_1 \otimes \mathcal{A}_{0,\mu}(f_2).$$

\square

Now, for $s \geq 0$, $\mu > 0$ and any $g \in L^2(\Omega)$, we denote $f_g := e^{s\mathcal{A}_{0,\mu}}g$

Lemma B.2. Assume (3–4), (18). Let $g = g_1 \otimes g_2 \in L^2(\omega_1) \otimes L^2(\omega_2)$ then for $s \geq 0$, $\mu > 0$ we have:

$$f_g = g_1 \otimes e^{s\mathcal{A}_{0,\mu}} g_2.$$

Notice that $e^{s\mathcal{A}_{0,\mu}} g_2 \in L^2(\omega_2)$.

Proof. we have

$$f_g = e^{s\mathcal{A}_{0,\mu}} g = \sum_{k=0}^{\infty} \frac{s^k}{k!} \mathcal{A}_{0,\mu}^k g,$$

where the series converges in $L^2(\Omega)$. By an immediate induction we get by using B.1

$$\forall k \in \mathbb{N} : \mathcal{A}_{0,\mu}^k g = g_1 \otimes \mathcal{A}_{0,\mu}^k g_2,$$

with $\mathcal{A}_{0,\mu}^k g_2 \in L^2(\omega_2)$ for every $k \in \mathbb{N}$, and the Lemma follows. \square

Lemma B.3. Assume (3–4), (18). Let $g \in H^2(\omega_1) \otimes L^2(\omega_2)$ then for $s, \mu > 0$, $i, j = 1, \dots, q$ we have: $D_{x_i x_j}^2 f_g, D_{x_i} f_g \in L^2(\Omega)$, such that

$$D_{x_i x_j}^2 f_g = e^{s\mathcal{A}_{0,\mu}} (D_{x_i x_j}^2 g), \quad D_{x_i} f_g = e^{s\mathcal{A}_{0,\mu}} (D_{x_i} g). \quad (77)$$

and :

$$\left\| D_{x_i x_j}^2 f_g \right\|_{L^2(\Omega)} \leq \left\| D_{x_i x_j}^2 g \right\|_{L^2(\Omega)}, \quad \|D_{x_i} f_g\|_{L^2(\Omega)} \leq \|D_{x_i} g\|_{L^2(\Omega)}. \quad (78)$$

Proof. 1) Suppose the simple case when $g = g_1 \otimes g_2$. So let $g = g_1 \otimes g_2 \in H^2(\omega_1) \otimes L^2(\omega_2)$ and let us prove assertions (77). By Lemma B.2 we get

$$f_g = g_1 \otimes e^{s\mathcal{A}_{0,\mu}} (g_2),$$

with $e^{s\mathcal{A}_{0,\mu}} g_2 \in L^2(\omega_2)$. Hence, for $i, j = 1, \dots, q$ we have $D_{x_i x_j}^2 f_g \in L^2(\Omega)$ and $D_{x_i x_j}^2 f_g = \left(D_{x_i x_j}^2 g_1 \right) \otimes e^{s\mathcal{A}_{0,\mu}} g_2$, and applying B.2 we get

$$D_{x_i x_j}^2 f_g = e^{s\mathcal{A}_{0,\mu}} (D_{x_i x_j}^2 g).$$

Similarly we get $D_{x_i} f_g = e^{s\mathcal{A}_{0,\mu}} (D_{x_i} g)$, and assertion (77) follows.

2) Now, let $g \in H^2(\omega_1) \otimes L^2(\omega_2)$, since g is a finite sum of elements of the form $g_1 \otimes g_2$, then by linearity we get

$$D_{x_i x_j}^2 f_g = e^{s\mathcal{A}_{0,\mu}} (D_{x_i x_j}^2 g), \quad D_{x_i} f_g = e^{s\mathcal{A}_{0,\mu}} (D_{x_i} g),$$

therefore

$$\left\| D_{x_i x_j}^2 f_g \right\|_{L^2(\Omega)} \leq \|e^{s\mathcal{A}_{0,\mu}}\| \left\| D_{x_i x_j}^2 g \right\|_{L^2(\Omega)} \leq \left\| D_{x_i x_j}^2 g \right\|_{L^2(\Omega)},$$

and similarly we obtain the second inequality of (78). \square

Lemma B.4. Assume (3–4), (18) and (50). Let $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ then, for $s \geq 0$, $\mu > 0$:

$$f_g \in D(\mathcal{A}_0), \quad \mathcal{A}_0(f_g) \in H_0^1(\Omega; \omega_1), \quad \text{and} \quad D_{x_i}(\mathcal{A}_0 f_g) = e^{s\mathcal{A}_{0,\mu}} (D_{x_i} \mathcal{A}_0 g), \quad (79)$$

$$\|(\mathcal{A}_0 f_g)\|_{L^2(\Omega)} \leq \|\mathcal{A}_0 g\|_{L^2(\Omega)} \quad \text{and} \quad \|D_{x_i}(\mathcal{A}_0 f_g)\|_{L^2(\Omega)} \leq \|D_{x_i} \mathcal{A}_0 g\|_{L^2(\Omega)}, \quad i = 1, \dots, q, \quad (80)$$

Proof. 1) Suppose $g = g_1 \otimes g_2 \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ and let us prove (79). Since $g \in D(\mathcal{A}_0)$, thanks to (50), then $f_g = e^{s\mathcal{A}_{0,\mu}} g \in D(\mathcal{A}_0)$ and $\mathcal{A}_0 f_g = e^{s\mathcal{A}_{0,\mu}} \mathcal{A}_0 g$ (thanks to (47)). Now, we have

$$\mathcal{A}_0 f_g = \mathcal{A}_0 (e^{s\mathcal{A}_{0,\mu}} g) = \mathcal{A}_0 (g_1 \otimes e^{s\mathcal{A}_{0,\mu}} g_2).$$

Notice that, $g_2 \in D(\mathcal{A}_0)$, thanks to (50) then $e^{s\mathcal{A}_{0,\mu}} g_2 \in D(\mathcal{A}_0)$ (thanks to (47)), hence

$$\mathcal{A}_0 f_g = g_1 \mathcal{A}_0 e^{s\mathcal{A}_{0,\mu}} g_2,$$

where we have used the fact that \mathcal{A}_0 is independent of the X_1 – direction. Using the fact that $e^{s\mathcal{A}_0, \mu}$ and \mathcal{A}_0 commute on $D(\mathcal{A}_0)$, we get

$$\mathcal{A}_0 f_g = g_1 e^{s\mathcal{A}_0, \mu} \mathcal{A}_0 g_2.$$

Now, we have $\mathcal{A}_0 g_2 \in L^2(\omega_2)$ then $e^{s\mathcal{A}_0, \mu} \mathcal{A}_0 g_2 \in L^2(\omega_2)$ (thanks to Lemma B.2), however $g_1 \in H_0^1(\omega_1)$, then

$$\mathcal{A}_0 f_g \in H_0^1(\Omega; \omega_1).$$

Whence, for $i = 1, \dots, q$ we have

$$D_{x_i}(\mathcal{A}_0 f_g) = D_{x_i} g_1 \otimes e^{s\mathcal{A}_0, \mu} \mathcal{A}_0 g_2,$$

and hence by B.2 we get

$$\begin{aligned} D_{x_i}(\mathcal{A}_0 f_g) &= e^{s\mathcal{A}_0, \mu} (D_{x_i} g_1 \otimes \mathcal{A}_0 g_2) \\ &= e^{s\mathcal{A}_0, \mu} (D_{x_i} \mathcal{A}_0 g). \end{aligned}$$

(Remark that $D_{x_i} \mathcal{A}_0 g \in L^2(\Omega)$ since $g_1 \in H_0^1(\omega_1)$ and $\mathcal{A}_0 g_2 \in L^2(\omega_2)$). 2) Now, for a general $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$, assertion (79) follows by linearity. Finally, we show (80). We have

$$\begin{aligned} \|(\mathcal{A}_0 f_g)\|_{L^2(\Omega)} &= \|e^{s\mathcal{A}_0, \mu} (\mathcal{A}_0 g)\|_{L^2(\Omega)} \leq \|e^{s\mathcal{A}_0, \mu}\| \|\mathcal{A}_0 g\|_{L^2(\Omega)} \\ &\leq \|\mathcal{A}_0 g\|_{L^2(\Omega)}. \end{aligned}$$

For $i = 1, \dots, q$ we get

$$\begin{aligned} \|D_{x_i}(\mathcal{A}_0 f_g)\|_{L^2(\Omega)} &= \|e^{s\mathcal{A}_0, \mu} (D_{x_i} \mathcal{A}_0 g)\|_{L^2(\Omega)} \leq \|e^{s\mathcal{A}_0, \mu}\| \|D_{x_i} \mathcal{A}_0 g\|_{L^2(\Omega)} \\ &\leq \|D_{x_i} \mathcal{A}_0 g\|_{L^2(\Omega)}. \end{aligned}$$

□

Lemma B.5. Assume (3–4), (18) and (50). Let $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$ then, for $s \geq 0$, $\mu > 0$, $i = 1, \dots, q$, $j = q+1, \dots, N$ we have $D_{x_j} f_g, D_{x_i x_j}^2 f_g \in L^2(\Omega)$ and

$$\|D_{x_j} f_g\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|\mathcal{A}_0 g\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}, \quad \|D_{x_i x_j}^2 f_g\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|D_{x_i} \mathcal{A}_0 g\|_{L^2(\Omega)} \|D_{x_i} g\|_{L^2(\Omega)}. \quad (81)$$

Proof. 1) Let us show the first inequality of(81). Suppose $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$. Notice that $g \in D(\mathcal{A}_0)$, thanks to (50) then according to (47) we have $f_g \in D(\mathcal{A}_0) \subset H_0^1(\Omega; \omega_2)$, i.e. for $j \in \{q+1, \dots, N\}$ fixed, we have $D_{x_j} f_g \in L^2(\Omega)$, and

$$\begin{aligned} \|D_{x_j} f_g\|_{L^2(\Omega)}^2 &\leq \frac{1}{\lambda} \langle -\mathcal{A}_0 f_g, f_g \rangle_{L^2(\Omega)} \\ &\leq \frac{1}{\lambda} \|\mathcal{A}_0 f_g\|_{L^2(\Omega)} \|f_g\|_{L^2(\Omega)}. \end{aligned}$$

We have, $\|\mathcal{A}_0 f_g\|_{L^2(\Omega)} = \|\mathcal{A}_0 e^{s\mathcal{A}_0, \mu} g\|_{L^2(\Omega)} = \|e^{s\mathcal{A}_0, \mu} \mathcal{A}_0 g\|_{L^2(\Omega)} \leq \|\mathcal{A}_0 g\|_{L^2(\Omega)}$, and $\|f_g\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)}$, therefore

$$\|D_{x_j} f_g\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|\mathcal{A}_0 g\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

2) Now, let $1 \leq i \leq q$ fixed, then according to Lemma B.3 we have $D_{x_i} f_g = e^{s\mathcal{A}_0, \mu} (D_{x_i} g)$, notice that $D_{x_i} g = D_{x_i} g_1 \otimes g_2 \in D(\mathcal{A}_0)$ and hence, $D_{x_i} f_g \in D(\mathcal{A}_0)$, in particular $D_{x_i} f_g \in H_0^1(\Omega; \omega_2)$, and for $q+1 \leq j \leq N$

we have

$$\begin{aligned} \left\| D_{x_j x_i}^2 f_g \right\|_{L^2(\Omega)}^2 &\leq \frac{1}{\lambda} \langle -\mathcal{A}_0 D_{x_i} f_g, D_{x_i} f_g \rangle_{L^2(\Omega)} \\ &\leq \frac{1}{\lambda} \|\mathcal{A}_0 D_{x_i} f_g\|_{L^2(\Omega)} \|D_{x_i} f_g\|_{L^2(\Omega)}. \end{aligned}$$

We have,

$$\begin{aligned} \|\mathcal{A}_0 D_{x_i} f_g\|_{L^2(\Omega)} &= \|\mathcal{A}_0 e^{s\mathcal{A}_0, \mu} (D_{x_i} g)\|_{L^2(\Omega)} = \|e^{s\mathcal{A}_0, \mu} (\mathcal{A}_0 D_{x_i} g)\|_{L^2(\Omega)} \\ &\leq \|(D_{x_i} \mathcal{A}_0 g)\|_{L^2(\Omega)}, \end{aligned}$$

according to (78) we have $\|D_{x_i} f_g\|_{L^2(\Omega)} \leq \|D_{x_i} g\|_{L^2(\Omega)}$, finally we obtain

$$\left\| D_{x_j x_i}^2 f_g \right\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda} \|(D_{x_i} \mathcal{A}_0 g)\|_{L^2(\Omega)} \|D_{x_i} g\|_{L^2(\Omega)}.$$

□

Lemma B.6. *Under assumptions of Lemma B.5, we have for $g \in (H_0^1 \cap H^2(\omega_1)) \otimes (H_0^1 \cap H^2(\omega_2))$:*

$$f_g \in H_0^1(\Omega) \cap D(\mathcal{A}_0), \quad (82)$$

and

$$\operatorname{div}_{X_1}(A_{11} \nabla_{X_1} f), \operatorname{div}_{X_1}(A_{12} \nabla_{X_2} f), \operatorname{div}_{X_2}(A_{21} \nabla_{X_1} f) \in L^2(\Omega). \quad (83)$$

Proof. Let us prove (82). In Lemma B.4 we proved that $f_g \in D(\mathcal{A}_0)$. Let us show that $f_g \in H_0^1(\Omega)$. Suppose the simple case $g = g_1 \otimes g_2$, we have $f_g = g_1 \otimes e^{s\mathcal{A}_0, \mu} g_2$. Since $g_2 \in D(\mathcal{A}_0)$, then $e^{s\mathcal{A}_0, \mu} g_2 \in D(\mathcal{A}_0)$, in particular we have $e^{s\mathcal{A}_0, \mu} g_2 \in H_0^1(\Omega; \omega_2)$ however, according to Lemma B.2 $e^{s\mathcal{A}_0, \mu} g_2 \in L^2(\omega_2)$, hence $e^{s\mathcal{A}_0, \mu} g_2 \in H_0^1(\omega_2)$, finally as $g_1 \in H_0^1(\omega_1)$ we get $f_g \in H_0^1(\Omega)$. For a general g in the tensor product space, the proof follows by linearity.

Now, let us show (83). According to Lemmas B.3, B.5 all these derivatives $D_{x_i} f_g$, $D_{x_i x_j}^2 f_g$ for $1 \leq i, j \leq q$, and $D_{x_j} f_g$, $D_{x_i x_j}^2 f_g$ for $1 \leq i \leq q$, $q+1 \leq j \leq N$ belong to $L^2(\Omega)$. Whence, combining that with (50) we get (83). □

APPENDIX C. EXISTENCE THEOREMS

Let $V \subset H_0^1(\Omega)$ a subspace. We consider also the problem

$$\begin{cases} \int_{\Omega} \beta(u) \varphi dx + \int_{\Omega} A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V \\ u \in V. \end{cases} \quad (84)$$

with A_{22} and β as in the introduction.

Proposition C.1. *If V is closed in $H_0^1(\Omega; \omega_2)$ then there exists a solution to (84).*

Proof. We consider the perturbed problem

$$\begin{cases} \int_{\Omega} \beta(u_\epsilon) \varphi dx + \int_{\Omega} \tilde{A}_\epsilon \nabla u_\epsilon \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx, \forall \varphi \in V \\ u_\epsilon \in V. \end{cases} \quad (85)$$

with

$$\tilde{A}_\epsilon = \begin{pmatrix} \epsilon^2 I_q & 0 \\ 0 & A_{22} \end{pmatrix}$$

The space V is closed in $H_0^1(\Omega)$ and \tilde{A}_ϵ is bounded and coercive, then by the Schauder fixed point theorem there exists a solution u_ϵ to (85). This solution is unique in V thanks to monotonicity and coercivity of \tilde{A}_ϵ . Testing with u_ϵ we obtain

$$\epsilon \|\nabla_{X_1} u_\epsilon\|_{L^2(\Omega)}, \|\nabla_{X_2} u_\epsilon\|_{L^2(\Omega)}, \|u_\epsilon\|_{L^2(\Omega)} \leq C,$$

where C is independent of ϵ , we have used that $\int_\Omega \beta(u_\epsilon) u_\epsilon dx \geq 0$ (thanks to monotonicity of β and $\beta(0) = 0$). And we also have

$$\|\beta(u_\epsilon)\|_{L^2(\Omega)} \leq M(|\Omega|^{\frac{1}{2}} + C),$$

so there exists $v \in L^2(\Omega)$, $u \in L^2(\Omega)$, $\nabla_{X_2} u \in L^2(\Omega)$ and a subsequence $(u_{\epsilon_k})_{k \in \mathbb{N}}$ such that

$$\beta(u_{\epsilon_k}) \rightharpoonup v, \epsilon_k \nabla_{X_1} u_{\epsilon_k} \rightharpoonup 0, \nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u, u_{\epsilon_k} \rightharpoonup u \text{ in } L^2(\Omega)\text{-weak} \quad (86)$$

Passing to the limit in the weak formulation of (85) we get

$$\int_\Omega v \varphi dx + \int_\Omega A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_\Omega f \varphi dx, \forall \varphi \in V \quad (87)$$

Take $\varphi = u_{\epsilon_k}$ in the previous equality and passing to the limit we get

$$\int_\Omega v u dx + \int_\Omega A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} u dx = \int_\Omega f u dx \quad (88)$$

Let us consider the quantity

$$\begin{aligned} 0 \leq I_k &= \int_\Omega \epsilon^2 |\nabla_{X_1} u_{\epsilon_k}|^2 dx + \int_\Omega A_{22} \nabla_{X_2} (u_{\epsilon_k} - u) \cdot \nabla_{X_2} (u_{\epsilon_k} - u) \\ &\quad + \int_\Omega (\beta(u_{\epsilon_k}) - \beta(u))(u_{\epsilon_k} - u) dx \\ &= \int_\Omega f u_{\epsilon_k} dx - \int_\Omega A_{22} \nabla_{X_2} u_{\epsilon_k} \cdot \nabla_{X_2} u dx - \int_\Omega A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} u_{\epsilon_k} dx \\ &\quad + \int_\Omega f u dx - \int_\Omega v u dx - \int_\Omega \beta(u) u_{\epsilon_k} dx \\ &\quad - \int_\Omega \beta(u_{\epsilon_k}) u dx + \int_\Omega \beta(u) u dx \end{aligned}$$

Remark that this quantity is positive thanks to the ellipticity and monotonicity assumptions. Passing to the limit as $k \rightarrow \infty$ using (86), (88) we get

$$\lim I_k = 0$$

And finally the ellipticity assumption shows that

$$\|\epsilon_k \nabla_{X_1} u_{\epsilon_k}\|_{L^2(\Omega)}, \|u_{\epsilon_k} - u\|_{L^2(\Omega)}, \|\nabla_{X_2} (u_{\epsilon_k} - u)\|_{L^2(\Omega)} \rightarrow 0 \quad (89)$$

and therefore,

$$\beta(u_{\epsilon_k}) \rightarrow \beta(u) \text{ strongly in } L^2$$

Whence (87) becomes

$$\int_\Omega \beta(u) \varphi dx + \int_\Omega A_{22} \nabla_{X_2} u \cdot \nabla_{X_2} \varphi dx = \int_\Omega f \varphi dx, \forall \varphi \in V$$

$\|\nabla_{X_2} (u_{\epsilon_k} - u)\|_{L^2(\Omega)} \rightarrow 0$ shows that $u \in H_0^1(\Omega; \omega_2)$, and therefore since V is closed in $H_0^1(\Omega; \omega_2)$ then $u \in V$. \square

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