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On the emergence of adhesion in asymptotic analysis of piecewise linear anisotropic elastic bonded joints

Frédéric Lebon ^a, Raffaella Rizzoni ^{b,*}

^a Aix-Marseille Univ., CNRS, Centrale Marseille, LMA, Marseille, France

^b Department of Engineering, University of Ferrara, Italy

This work is concerned with the equilibrium problem of a composite body comprising two linear elastic adherents joined by an adhesive of small thickness ε and made of a conewise linear elastic material (Curnier et al., 1995). The elasticity coefficients of the adhesive material are chosen to simulate soft behavior in traction and hard behavior in compression, or vice versa. Shear moduli are always assumed to be soft. The method of matched asymptotic expansions is applied to calculate the transmission conditions approximating the behavior of the adhesive in the limit of vanishing ε . In the plane of the adhesive a classical spring-type law is obtained, linking the tangential components of the stress vector to the jumps of the tangential components of the displacement vector. In the direction perpendicular to the plane of the adhesive, Signorini's contact conditions of non interpenetration are recovered. However, adhesion can emerge for adhesives showing octantwise linear elastic behavior or particular symmetries of half-spacewise behavior.

1. Introduction

In the last decades, adhesive bonded technology has been successfully applied on a large scale for reduction of structural weight, time and manufacturing costs. Due to its versatility, adhesion methodology is used in a variety of products in aerospace, marine, automotive and biomedical industries for joining of composite–composite and composite–metal structural parts or biological tissues (Banea and da Silva, 2009; Brennan, 1991; Budhe et al., 2017; Heshmati et al., 2015; Higgins, 2000; Misra et al., 2004). Main critical difficulties in numerical modeling the behavior of adhesive structural joints are small mesh size, mesh-dependency, large number of degrees of freedom and high computational time. A convenient alternative strategy to volumetric modeling of the adhesive layer is its replacement by an interface model, i.e. transmission conditions linking the adherent surfaces and simulating the behavior of an adhesive layer as its thickness vanishes.

There is a wide literature on interface models for adhesive joints, stemming from the pioneering works on imperfect linear interfaces by Goland and Reissner (1944), and Jones and Whittier (1967). Other studies have proposed several mathematical approaches to obtain interface models: asymptotic expansions (Abdelmoula et al., 1998; Benveniste and Miloh, 2001; Hashin, 2001, 2002; Klarbring, 1991; Klarbring and Movchan, 1998; Lebon and Zaittouni, 2010; Serpilli and Lenci, 2016), Γ -convergence techniques (Caillerie, 1980; Lebon and Rizzoni, 2010; Licht and Michaille, 1997; Geymonat et al., 1999; Mielke

et al., 2012), and energy methods coupled with asymptotic expansions (Lebon and Rizzoni, 2011; Rizzoni et al., 2014), just to cite a few. The method of asymptotic expansions is a rigorous and mathematically elegant way to recover the governing equations of interfaces modeling thin adhesives. The method, proposed by Sanchez-Palencia to derive the homogenized response of composites, is based on the choice of a geometrically small parameter (e.g. the size of the microstructure or the thickness of the adhesive layer) and on the expansion of the relevant fields (displacement, stress and strain) in a power series with respect to the small parameter (Sanchez-Palencia, 1980). For a classical linear elastic material behavior of the adhesive, standard results of asymptotic analysis are the perfect interface model for hard adhesives and the spring-type interface model for soft adhesives (Abdelmoula et al., 1998; Klarbring, 1991; Lebon and Rizzoni, 2010, 2011; Rizzoni et al., 2014). The perfect interface model is characterized by the continuity of the stress and displacement vectors fields across the interface, which is the geometrical limit of the adhesive as its thickness goes to zero. The spring-type interface model prescribes the discontinuity of the displacement vector field across the limit interface, but the continuity of the traction vector field and its proportionality to the jump of the displacement vector.

Aside from linear elasticity, more general adhesive material behaviors have been considered in several studies, for example elasto-plasticity (Edlund, 1994; Edlund et al., 2009; Sonato et al., 2015),

* Corresponding author.

E-mail address: raffaella.rizzoni@unife.it (R. Rizzoni).

nonlinear incompressible elasticity (Ganghoffer et al., 1997), viscoelasticity (Bonetti et al., 2020; Licht et al., 2015), cohesive and damage models (Campilho et al., 2013; Jaillon et al., 2020; Shishesaz and Hosseini, 2020; Bonetti et al., 2017), and multiphysics behavior (Gu and He, 2011; Licht et al., 2015; Serpilli, 2017, 2018, 2019; Serpilli et al., 2019, 2021). Less attention has been devoted to adhesive materials with different Young’s modulus in traction and compression, or bimodular materials. Many engineering materials containing cracks and microvoids or characterized by an inherent microstructure exhibit different properties in traction and compression. In structural adhesives, bimodularity may originate from the mismatch of thermo-elastic material properties of adhesive and adherent composite laminae during interface bonding curing process (Jones, 1977; Shah and Panda, 2017; Vijayakumar and Rao, 1987). Biological adhesives can also show asymmetric behavior in tension and compression, such as fibrin glue (Litvinov and Weisel, 2017; Rosakis et al., 2015), for which bimodularity is caused by microbuckling of individual fibers in compression (Lakes et al., 1993).

In the literature, constitutive relations for bimodular materials have been first proposed by Ambartsumyan and his collaborators (Ambartsumyan, 1965, 1986; Ambartsumyan and Khachatryan, 1966) and further explored by a number of other authors, among which (Bert, 1977; Curnier et al., 1995; Sacco and Reddy, 1992; Shapiro, 1966) and many others. In the present paper, we focus on the approach proposed by Curnier et al. in Curnier et al. (1995), which introduces the definition of conewise linear elastic materials as the proper generalization to two and three dimensions of one-dimensional bimodular models characterized by different Young’s moduli in tension and compression. We thus consider a composite body made of two linear elastic adherents in perfect contact with an adhesive layer of very small thickness ϵ , whose material is characterized by a piecewise linear stress–strain law with elasticity coefficients depending on ϵ . Compared to previous studies, e.g. Curnier et al. (1995), Serpilli and Lenci (2016), the novelty of the presented work is not methodological. We apply the asymptotic expansion method, a consolidated asymptotic technique since Klarbring’s pioneering work on adhesives (Klarbring, 1991), to a bonded joint in which the intermediate thin layer is made of piecewise elastic material in the sense proposed by Curnier et al. in Curnier et al. (1995). In the literature, we could not find any asymptotic study of a bonded joint in which the thin layer follows the constitutive model proposed in Curnier et al. (1995).

For half-spacewise linear elastic materials, the (bulk) elasticity coefficients of the adhesive are assumed to linearly rescale with ϵ in traction, modeling a soft material behavior. In compression, the (bulk) elasticity coefficients are taken to be independent of ϵ , modeling a hard material behavior. The opposite situation is also treated in the paper, i.e. the case of an adhesive material hard in traction (bulk elasticity coefficients independent of ϵ) and soft in compression (bulk elasticity coefficients linearly rescaling with ϵ). In all cases, the shear moduli are assumed to be soft. For octantwise linear elastic materials, the conditions of continuity of the stress–strain law highly constrain the elasticity coefficients, and the following choice of rescaling is thus proposed: shear moduli and off-diagonal bulk coefficients are taken to be soft, diagonal bulk coefficients are taken to be hard.

The method of matched asymptotic expansions is then applied to the equilibrium problem of the composite body. In the plane of the adhesive, the results of the asymptotic analysis are spring-type transmission conditions of imperfect interface for all material models. In the direction perpendicular to the plane of the adhesive, different conditions are obtained, depending on the material model and on the loading mode. For half-spacewise materials, a spring-type interface law is found in the loading direction (of the strain space) characterized by a soft material behavior, for example in the opening mode for materials soft in traction and in the closing mode for materials soft in compression. In the loading direction characterized by a hard material behavior, two different interface models may arise, depending on the

orientation of the hyperplane separating the two different regimes of traction and compression in the strain space: Signorini’s conditions of unilateral contact or adhesion. In both cases, the jump of the relative displacement normal to the plane of the adhesive is found to vanish.

The case of an octantwise linear elastic adhesive material is more complex, because it is characterized by a subdivision of the strain space into eight different domains. Due to the particular choice of the rescaling discussed above, a spring-type interface law is obtained in the plane of the adhesive. Moreover, in the case of orthotropic or cubic symmetry, the contact is always perfect (the jump of the displacement is zero) in the direction normal to the adhesive, but the reaction stress can be non positive, corresponding to Signorini’s contact conditions, or non negative, corresponding to adhesion.

The results obtained in the paper are twofold: first, the particular scaling of the elasticity coefficients allows to obtain Signorini’s contact conditions in the loading direction of the strain space corresponding to a hard behavior. This result, somehow expected, is original and it allows to overcome the drawback of interpenetration presented by the classical spring-type interface law in closing mode. Second, adhesion can emerge for some anisotropies of the adhesive material, both in the case of half-spacewise and octantwise materials. To the authors’ knowledge, adhesion has never been related to piecewise material behavior before.

Classically, adhesion can be viewed arising at the atomic scale, where Van der Waals forces are responsible for surface adhesion, cf. Kendall et al. (2011) and the references therein. At the macroscopic scale, adhesion is typically modeled using energetic arguments like Griffith theory or contact laws not invoking any specific theory about the nature of the adhesive material, as for example in Johnson et al. (1971), Fremond (1988), Talon and Curnier (2003). At an intermediate level, more refined adhesion models have been proposed, exploiting biological evidence of hierarchy and contact splitting as a strategy to enhance the adhesive properties of natural structures (Berardo and Pugno, 2020; Fraldi et al., 2021). In this paper, another viewpoint arises, attributing the origin of macroscopic adhesion to the mechanical presence of an interposed adhesive layer with piecewise linear elastic anisotropic behavior.

The paper is organized as follows. Section 2 is devoted to notation and to formulation of the equilibrium problem for the composite body. Section 3 presents the details of asymptotic analysis for the two types of adhesive materials considered, half-spacewise linear elastic material and octantwise linear elastic material. Section 4 contains a summary of results and some closing remarks.

2. Statement of the equilibrium problem

In the following a composite body made of three deformable solids, two adherents and an adhesive in perfect contact and occupying a smooth bounded domain $\Omega^\epsilon \subset \mathbb{R}^3$, is considered. The dependence of the domain Ω^ϵ on the parameter ϵ will be made precise in the following. An orthonormal Cartesian frame $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is introduced and the triplet (x_1, x_2, x_3) is taken to denote the three coordinates of a particle. The origin lies at the center of the adhesive midplane and the x_3 -axis runs perpendicular to the bounded set $S = \{(x_1, x_2, x_3) \in \Omega^\epsilon : x_3 = 0\}$, which will be identified as the interface in the limit problem. The adhesive or interphase occupies the domain B^ϵ , defined by $B^\epsilon = \{(x_1, x_2, x_3) \in \Omega^\epsilon : |x_3| < \frac{\epsilon}{2}\}$, being $\epsilon > 0$ its thickness. The adherents occupy the domains $\Omega_\pm^\epsilon = \{(x_1, x_2, x_3) \in \Omega^\epsilon : \pm x_3 > \frac{\epsilon}{2}\}$. The two-dimensional domains S_\pm^ϵ are taken to denote the interfaces between the adhesive and the adherents, $S_\pm^\epsilon = \{(x_1, x_2, x_3) \in \Omega^\epsilon : x_3 = \pm \frac{\epsilon}{2}\}$. On a part S_g of the boundary $\partial\Omega^\epsilon$, an external load \mathbf{g} is applied, and on a part S_u of $\partial\Omega^\epsilon$, of strictly positive measure and such that $S_g \cap S_u = \emptyset$, the displacement is imposed to vanish. Moreover, it is assumed that the boundary sets S_g and S_u are located far from the interphase. Finally, a body force \mathbf{f} is applied to Ω_\pm^ϵ , while body forces are neglected in B^ϵ .

The fields of the external forces are endowed with sufficient regularity to ensure the existence of equilibrium configuration.

In the following, \mathbf{u}^ε is taken to denote the displacement field, $\boldsymbol{\sigma}^\varepsilon$ the Cauchy stress tensor and $\mathbf{e}(\mathbf{u}^\varepsilon)$ the strain tensor, belonging to the “vector” space of symmetric second-order tensors, herein denoted \mathcal{E} . Under the small strain hypothesis, we have $e_{ij}(\mathbf{u}^\varepsilon) = \frac{1}{2}(u_{i,j}^\varepsilon + u_{j,i}^\varepsilon)$, where the comma stands for the partial derivative.

The two adherents are supposed to be linear elastic with elasticity tensors \mathbf{a}^\pm independent of ε :

$$\boldsymbol{\sigma}^\varepsilon = \mathbf{a}^\pm(\mathbf{e}(\mathbf{u}^\varepsilon)). \quad (1)$$

The material of the adhesive is assumed to be linear elastic with different behavior in different subregions of the strain space \mathcal{E} . In general, complicated subdivisions could be considered, as in the case of the octantwise linear elastic material considered in Section 3.11. In the present Section, for simplicity of presentation, we restrict to a half-spacewise linear elastic material, modeling a material with different behavior in tension and compression. For this material, the strain space is divided into two half-spaces, \mathcal{E}^\pm , by a hyperplane $\mathcal{S} \subset \mathcal{E}$ characterized by a linear scalar valued function $g^\varepsilon(\mathbf{e}) = 0$. Correspondingly, the piecewise linear stress–strain relationship involves two different fourth order elasticity tensors, \mathbf{b}_+^ε in \mathcal{E}^+ and \mathbf{b}_-^ε in \mathcal{E}^- , such that

$$\boldsymbol{\sigma}^\varepsilon = \begin{cases} \mathbf{b}_+^\varepsilon(\mathbf{e}(\mathbf{u}^\varepsilon)) & \text{if } g^\varepsilon(\mathbf{e}) \geq 0, \\ \mathbf{b}_-^\varepsilon(\mathbf{e}(\mathbf{u}^\varepsilon)) & \text{if } g^\varepsilon(\mathbf{e}) \leq 0. \end{cases} \quad (2)$$

The tensors \mathbf{a}^\pm and $\mathbf{b}_\pm^\varepsilon$ are assumed to be symmetric, with the minor and major symmetries, and positive definite. The positive definiteness of the elasticity tensors ensures the convexity of the energy potential (Curnier, 1993).

Taken $\mathbf{N}^\varepsilon = \nabla g^\varepsilon(\mathbf{e})$ to denote the unit normal to the hyperplane \mathcal{S} , the condition

$$\mathbf{b}_+^\varepsilon - \mathbf{b}_-^\varepsilon = b^\varepsilon(\mathbf{N}^\varepsilon \otimes \mathbf{N}^\varepsilon) \quad (3)$$

ensures the continuity of the stress–strain law across the interface \mathcal{S} , cf. Curnier et al. (1995, Eqn. (4.5)). In (3), b^ε is a constant, and the symbol \otimes is taken to denote the dyadic product of second order tensors.

In the asymptotic analysis proposed in the present paper, the two elasticity tensors, \mathbf{b}_+^ε and \mathbf{b}_-^ε , are assumed to depend on the thickness ε , and different regimes are considered. For example, an adhesive material exhibiting a soft behavior in traction (more precisely in \mathcal{E}^+) can be modeled by taking the elasticity tensor \mathbf{b}_+^ε linearly rescaling with the adhesive thickness:

$$\mathbf{b}_+^\varepsilon = O(\varepsilon) = \varepsilon \mathbf{b}_+^1, \quad (4)$$

with \mathbf{b}_+^1 a fourth order elasticity tensor independent of ε . To model the possible hard behavior of the adhesive material in compression (in \mathcal{E}^-) the following rescaling of the elastic tensor \mathbf{b}_-^ε can be assumed

$$\mathbf{b}_-^\varepsilon = O(1) = \mathbf{b}_-^0 + \varepsilon \mathbf{b}_-^1, \quad (5)$$

with the tensors \mathbf{b}_-^0 and \mathbf{b}_-^1 independent of ε and to be made precise later (cf. (46)).

In the asymptotic analysis, the rescalings (4) and (5) can be interchanged, so as to consider an adhesive material hard in traction and soft in compression.

In the case of octantwise materials, continuity of the stress–strain law across the surfaces separating the octants imposes many restrictions on the elasticity coefficients, cf. (94). Therefore, in this situation, a particular rescaling of the elasticity coefficients is considered, cf. (96).

The governing equations of equilibrium problem are written as follows:

$$\begin{cases} \sigma_{ij,j}^\varepsilon + f_i = 0 & \text{in } \Omega_\pm^\varepsilon \\ \sigma_{ij}^\varepsilon n_j = g_i & \text{on } S_g \\ \sigma_{ij,j}^\varepsilon = 0 & \text{in } B^\varepsilon \\ \sigma_{i3}^\varepsilon = \sigma_{i3}^\varepsilon & \text{on } S_\pm^\varepsilon \\ u_i^\varepsilon = u_i^\varepsilon & \text{on } S_\pm^\varepsilon \\ u_i^\varepsilon = 0 & \text{on } S_u \\ \sigma_{ij}^\varepsilon = a_{ijhk}^\pm e_{hk}(\mathbf{u}^\varepsilon) & \text{in } \Omega_\pm^\varepsilon \\ \sigma_{ij}^\varepsilon = b_{+ijhk}^\varepsilon e_{hk}(\mathbf{u}^\varepsilon), \text{ if } g^\varepsilon(\mathbf{u}^\varepsilon) \geq 0 & \text{in } B^\varepsilon \\ \sigma_{ij}^\varepsilon = b_{-ijhk}^\varepsilon e_{hk}(\mathbf{u}^\varepsilon), \text{ if } g^\varepsilon(\mathbf{u}^\varepsilon) \leq 0 & \text{in } B^\varepsilon \end{cases} \quad (6)$$

Einstein’s convention summation is used in (6) and throughout the paper. Eq. (6)₄ results from localization of balance of linear momentum on the interfaces between the adhesive layer and the adherents. No surface forces are present on these interfaces, so the stress vector is continuous.

3. Asymptotic analysis

Since the thickness of the interphase is very small, it is natural to seek the solution of the equilibrium problem using asymptotic expansions with respect to the small parameter ε . In particular, the following asymptotic series are assumed:

$$\begin{cases} \mathbf{u}^\varepsilon = \mathbf{u}^0 + \varepsilon \mathbf{u}^1 + o(\varepsilon) \\ \boldsymbol{\sigma}^\varepsilon = \boldsymbol{\sigma}^0 + \varepsilon \boldsymbol{\sigma}^1 + o(\varepsilon). \end{cases} \quad (7)$$

The domain is then rescaled using a classical procedure (Ciarlet, 1997).

- In the adhesive, the following change of variable is introduced $(x_1, x_2, x_3) \in B^\varepsilon \rightarrow (z_1, z_2, z_3) \in B$, with $(z_1, z_2, z_3) = (x_1, x_2, \frac{x_3}{\varepsilon})$ and it is set $\hat{\mathbf{u}}^\varepsilon(z_1, z_2, z_3) = \mathbf{u}^\varepsilon(x_1, x_2, x_3)$ and $\hat{\boldsymbol{\sigma}}^\varepsilon(z_1, z_2, z_3) = \boldsymbol{\sigma}^\varepsilon(x_1, x_2, x_3)$, with $B = \{(x_1, x_2, x_3) \in \Omega : |x_3| < \frac{1}{2}\}$.
- In the adherents, the following change of variable is introduced $(x_1, x_2, x_3) \in \Omega_\pm^\varepsilon \rightarrow (z_1, z_2, z_3) \in \Omega_\pm$, with $(z_1, z_2, z_3) = (x_1, x_2, x_3 \pm 1/2 \mp \varepsilon/2)$ and it is set $\bar{\mathbf{u}}^\varepsilon(z_1, z_2, z_3) = \mathbf{u}^\varepsilon(x_1, x_2, x_3)$ and $\bar{\boldsymbol{\sigma}}^\varepsilon(z_1, z_2, z_3) = \boldsymbol{\sigma}^\varepsilon(x_1, x_2, x_3)$, with $\Omega_\pm = \{(x_1, x_2, x_3) \in \Omega : \pm x_3 > \frac{1}{2}\}$.

The external forces are assumed to be independent of ε . As a consequence, it is set $\bar{\mathbf{f}}(z_1, z_2, z_3) = \mathbf{f}(x_1, x_2, x_3)$ and $\bar{\mathbf{g}}(z_1, z_2, z_3) = \mathbf{g}(x_1, x_2, x_3)$.

In view of the change of variables in the adhesive, one has

$$\frac{\partial}{\partial z_1} = \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial z_2} = \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial z_3} = \varepsilon \frac{\partial}{\partial x_3}. \quad (8)$$

The governing equations of the rescaled problem are as follows:

$$\begin{cases} \bar{\sigma}_{ij,j}^\varepsilon + \bar{f}_i = 0 & \text{in } \Omega_\pm \\ \bar{\sigma}_{ij}^\varepsilon n_j = \bar{g}_i & \text{on } \bar{S}_g \\ \hat{\sigma}_{ij,j}^\varepsilon = 0 & \text{in } B \\ \bar{\sigma}_{i3}^\varepsilon = \hat{\sigma}_{i3}^\varepsilon & \text{on } S_\pm \\ \bar{u}_i^\varepsilon = \hat{u}_i^\varepsilon & \text{on } S_\pm \\ \bar{u}_i^\varepsilon = 0 & \text{on } \bar{S}_u \\ \bar{\sigma}_{ij}^\varepsilon = a_{ijhk}^\pm e_{hk}(\bar{u}^\varepsilon) & \text{in } \Omega_\pm \\ \hat{\sigma}_{ij}^\varepsilon = b_{+ijhk}^\varepsilon e_{hk}(\hat{u}^\varepsilon), \text{ if } \hat{g}^\varepsilon(\hat{u}^\varepsilon) \geq 0 & \text{in } B \\ \hat{\sigma}_{ij}^\varepsilon = b_{-ijhk}^\varepsilon e_{hk}(\hat{u}^\varepsilon), \text{ if } \hat{g}^\varepsilon(\hat{u}^\varepsilon) \leq 0 & \text{in } B \end{cases} \quad (9)$$

where $S_\pm = \{(x_1, x_2, x_3) \in \Omega : x_3 = \pm \frac{1}{2}\}$, the superscripts $\hat{\cdot}$, $\bar{\cdot}$ are taken to denote the rescaled operators in the adherents and in the adhesive, respectively.

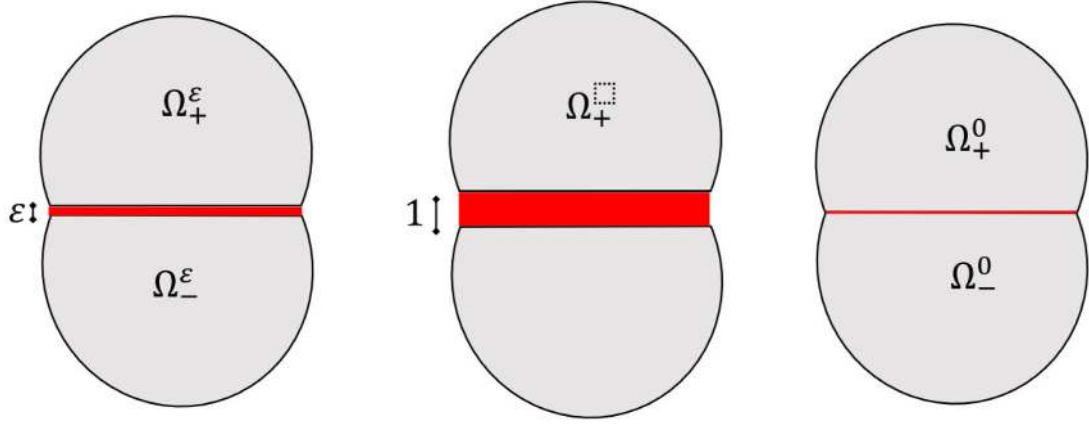


Fig. 1. Left: reference configuration of the bonded joint showing the adherents, occupying the domains $\Omega_{\pm}^{\varepsilon}$, and a thin interphase of small thickness ε . Center: rescaled reference configuration obtained applying the change of variables after which the interphase has fixed thickness (equal to 1). Right: final reference configuration of the bonded joint, composed of the two adherents joined by an interface, the limit configuration of the interphase as its thickness goes to zero. Across the interface, transmission conditions have to be prescribed to simulate the limit behavior of the thin interphase. The goal of the present work is to calculate the form of the transmission conditions.

In view of the asymptotic expansions (7), the rescaled displacement and stress fields can also be written as asymptotic expansions

$$\begin{cases} \hat{\sigma}^{\varepsilon} = \hat{\sigma}^0 + \varepsilon \hat{\sigma}^1 + o(\varepsilon) \\ \hat{\mathbf{u}}^{\varepsilon} = \hat{\mathbf{u}}^0 + \varepsilon \hat{\mathbf{u}}^1 + o(\varepsilon) \\ \bar{\sigma}^{\varepsilon} = \bar{\sigma}^0 + \varepsilon \bar{\sigma}^1 + o(\varepsilon) \\ \bar{\mathbf{u}}^{\varepsilon} = \bar{\mathbf{u}}^0 + \varepsilon \bar{\mathbf{u}}^1 + o(\varepsilon), \end{cases} \quad (10)$$

in the rescaled adhesive and adherents, respectively.

3.1. Kinematics in the adhesive

The gradient of the displacement vector field $\hat{\mathbf{u}}^{\varepsilon}$ in the rescaled interphase is computed as:

$$\nabla \hat{\mathbf{u}}^{\varepsilon} = \varepsilon^{-1} \begin{bmatrix} 0 & \hat{u}_{\alpha,3}^0 \\ 0 & \hat{u}_{3,3}^0 \end{bmatrix} + \begin{bmatrix} \hat{u}_{\alpha,\beta}^1 & \hat{u}_{\alpha,3}^1 \\ \hat{u}_{3,\beta}^1 & \hat{u}_{3,3}^1 \end{bmatrix} + \varepsilon \begin{bmatrix} \hat{u}_{\alpha,\beta}^2 & \hat{u}_{\alpha,3}^2 \\ \hat{u}_{3,\beta}^2 & \hat{u}_{3,3}^2 \end{bmatrix} + O(\varepsilon^2), \quad (11)$$

with $\alpha, \beta = 1, 2$. The strain tensor can be obtained as:

$$\mathbf{e}(\hat{\mathbf{u}}^{\varepsilon}) = \varepsilon^{-1} \hat{\mathbf{e}}^{-1} + \hat{\mathbf{e}}^0 + \varepsilon \hat{\mathbf{e}}^1 + O(\varepsilon^2), \quad (12)$$

with:

$$\hat{\mathbf{e}}^{-1} = \hat{u}_{3,3}^0 \odot \mathbf{i}_3, \quad (13)$$

$$\hat{\mathbf{e}}^k = \hat{u}_{1,1}^k \odot \mathbf{i}_1 + \hat{u}_{2,2}^k \odot \mathbf{i}_2 + \hat{u}_{3,3}^{k+1} \odot \mathbf{i}_3, \quad k = 0, 1, \quad (14)$$

where \odot is the symmetric dyadic product of vectors, and \mathbf{i}_i is the versor of the i axis, $i = 1, 2, 3$.

3.2. Kinematics in the adherents

The gradient tensor of the displacement vector field $\bar{\mathbf{u}}^{\varepsilon}$ in the adherents is computed as:

$$\nabla \bar{\mathbf{u}}^{\varepsilon} = \begin{bmatrix} \bar{u}_{\alpha,\beta}^0 & \bar{u}_{\alpha,3}^0 \\ \bar{u}_{3,\beta}^0 & \bar{u}_{3,3}^0 \end{bmatrix} + \varepsilon \begin{bmatrix} \bar{u}_{\alpha,\beta}^1 & \bar{u}_{\alpha,3}^1 \\ \bar{u}_{3,\beta}^1 & \bar{u}_{3,3}^1 \end{bmatrix} + O(\varepsilon^2), \quad (15)$$

with $\alpha, \beta = 1, 2$. The strain tensor can be obtained as:

$$\mathbf{e}(\bar{\mathbf{u}}^{\varepsilon}) = \varepsilon^{-1} \bar{\mathbf{e}}^{-1} + \bar{\mathbf{e}}^0 + \varepsilon \bar{\mathbf{e}}^1 + O(\varepsilon^2), \quad (16)$$

with:

$$\bar{\mathbf{e}}^{-1} = \mathbf{0}, \quad (17)$$

$$\bar{\mathbf{e}}^k = \bar{u}_{1,1}^k \odot \mathbf{i}_1 + \bar{u}_{2,2}^k \odot \mathbf{i}_2 + \bar{u}_{3,3}^k \odot \mathbf{i}_3, \quad k = 0, 1. \quad (18)$$

3.3. Equilibrium equations in the adhesive

After substituting the stress representation form in the adhesive (first equation in (10)) into the equilibrium equation of the interphase (third equation in (9)) and using (8), one obtains:

$$\begin{aligned} 0 &= \hat{\sigma}_{i\alpha,\alpha}^{\varepsilon} + \varepsilon^{-1} \hat{\sigma}_{i3,3}^{\varepsilon} \\ &= \varepsilon^{-1} \hat{\sigma}_{i3,3}^0 + \hat{\sigma}_{i\alpha,\alpha}^0 + \hat{\sigma}_{i3,3}^1 + \varepsilon \hat{\sigma}_{i\alpha,\alpha}^1 + O(\varepsilon), \end{aligned} \quad (19)$$

for $i = 1, 2, 3$. Eq. (19) has to be satisfied for any value of ε , leading to:

$$\hat{\sigma}_{i3,3}^0 = 0, \quad (20)$$

$$\hat{\sigma}_{i1,1}^0 + \hat{\sigma}_{i2,2}^0 + \hat{\sigma}_{i3,3}^1 = 0. \quad (21)$$

Eq. (20) indicates that $\hat{\sigma}_{i3}^0$ is independent of z_3 in the adhesive, leading to the condition

$$[[\hat{\sigma}_{i3}^0]] = 0, \quad (22)$$

where $[[\cdot]]$ denotes the jump between $z_3 = \frac{1}{2}$ and $z_3 = -\frac{1}{2}$. Eq. (22) means that the stress vector is constant through the thickness of the rescaled adhesive, cf. Klarbring (1991, Equation 4.8).

3.4. Equilibrium equations in the adherents

The stress representation form in the adherents (third equation in (10)) substituted into the equilibrium equation of the adherents (first equation in (9)) gives

$$\bar{\sigma}_{ij,j}^0 + \bar{f}_i = 0, \quad i = 1, 2, 3. \quad (23)$$

3.5. Matching external and internal expansions

Due to the assumption of perfect contact between the adhesive and the adherents, the displacement and stress vector fields and the corresponding rescaled fields are continuous across the surfaces S_{\pm}^{ε} and S_{\pm}^0 , respectively. In particular, the continuity of the displacements gives:

$$\mathbf{u}^{\varepsilon}(\bar{\mathbf{x}}, \pm \frac{\varepsilon}{2}) = \hat{\mathbf{u}}^{\varepsilon}(\bar{\mathbf{z}}, \pm \frac{1}{2}) = \bar{\mathbf{u}}^{\varepsilon}(\bar{\mathbf{z}}, \pm \frac{1}{2}), \quad (24)$$

where $\bar{\mathbf{x}} := (x_1, x_2)$, $\bar{\mathbf{z}} := (z_1, z_2) \in S$. Expanding the displacement in the adherent, \mathbf{u}^{ε} , in Taylor series along the x_3 -direction and taking into account the asymptotic expansion for \mathbf{u}^{ε} in (10), it results:

$$\begin{aligned} \mathbf{u}^{\varepsilon}(\bar{\mathbf{x}}, \pm \frac{\varepsilon}{2}) &= \mathbf{u}^{\varepsilon}(\bar{\mathbf{x}}, 0^{\pm}) \pm \frac{\varepsilon}{2} \mathbf{u}_{,3}^{\varepsilon}(\bar{\mathbf{x}}, 0^{\pm}) + \dots \\ &= \mathbf{u}^0(\bar{\mathbf{x}}, 0^{\pm}) + \varepsilon \mathbf{u}^1(\bar{\mathbf{x}}, 0^{\pm}) \pm \frac{\varepsilon}{2} \mathbf{u}_{,3}^0(\bar{\mathbf{x}}, 0^{\pm}) + \dots \end{aligned} \quad (25)$$

Substituting the expansions of the rescaled displacement fields given in (10) together with formula (25) into the continuity condition (24), we find:

$$\begin{aligned} \mathbf{u}^0(\bar{\mathbf{x}}, 0^\pm) + \varepsilon \mathbf{u}^1(\bar{\mathbf{x}}, 0^\pm) \pm \frac{\varepsilon}{2} \mathbf{u}_{,3}^0(\bar{\mathbf{x}}, 0^\pm) + \dots &= \hat{\mathbf{u}}^0(\bar{\mathbf{z}}, \pm \frac{1}{2}) + \varepsilon \hat{\mathbf{u}}^1(\bar{\mathbf{z}}, \pm \frac{1}{2}) + \dots \\ &= \bar{\mathbf{u}}^0(\bar{\mathbf{z}}, \pm \frac{1}{2}) + \varepsilon \bar{\mathbf{u}}^1(\bar{\mathbf{z}}, \pm \frac{1}{2}) + \dots \end{aligned} \quad (26)$$

After identifying the terms in the same powers of ε , Eq. (26) gives:

$$\mathbf{u}^0(\bar{\mathbf{x}}, 0^\pm) = \hat{\mathbf{u}}^0(\bar{\mathbf{z}}, \pm \frac{1}{2}) = \bar{\mathbf{u}}^0(\bar{\mathbf{z}}, \pm \frac{1}{2}), \quad (27)$$

$$\mathbf{u}^1(\bar{\mathbf{x}}, 0^\pm) \pm \frac{1}{2} \mathbf{u}_{,3}^0(\bar{\mathbf{x}}, 0^\pm) = \hat{\mathbf{u}}^1(\bar{\mathbf{z}}, \pm \frac{1}{2}) = \bar{\mathbf{u}}^1(\bar{\mathbf{z}}, \pm \frac{1}{2}). \quad (28)$$

For the stress vector, similar results can be obtained. We start from the continuity condition of the stress vector in the rescaled configuration, Eq. (9)₄, and we expand the stress vector in Taylor series along the x_3 -direction to obtain the following conditions:

$$\sigma_{i3}^0(\bar{\mathbf{x}}, 0^\pm) = \hat{\sigma}_{i3}^0(\bar{\mathbf{z}}, \pm \frac{1}{2}) = \bar{\sigma}_{i3}^0(\bar{\mathbf{z}}, \pm \frac{1}{2}), \quad (29)$$

$$\sigma_{i3}^1(\bar{\mathbf{x}}, 0^\pm) \pm \frac{1}{2} \sigma_{i3,3}^0(\bar{\mathbf{x}}, 0^\pm) = \hat{\sigma}_{i3}^1(\bar{\mathbf{z}}, \pm \frac{1}{2}) = \bar{\sigma}_{i3}^1(\bar{\mathbf{z}}, \pm \frac{1}{2}), \quad (30)$$

for $i = 1, 2, 3$.

Using the above results, it is possible to rewrite Eqs. (22) in the following form:

$$[\sigma_{i3}^0] = 0, \quad i = 1, 2, 3, \quad (31)$$

where $[f] := f(\bar{\mathbf{x}}, 0^+) - f(\bar{\mathbf{x}}, 0^-)$ is taken to denote the jump across the surface S of a generic function f defined on the limit configuration obtained as $\varepsilon \rightarrow 0$.

Matching external and internal expansions allows to pass from transmission conditions posed on the rescaled configuration of the bonded joint (shown at the center of Fig. 1) to interface conditions posed on the limit configuration obtained as $\varepsilon \rightarrow 0$ (shown on the right-hand side of Fig. 1). In the limit configuration the adherents occupy the domains Ω_\pm^0 .

The results obtained so far are general, being independent of the constitutive behavior of the composite material.

3.6. Constitutive equation of the adherents

Substituting the expansions of the displacement and stress fields into account the constitutive behavior of the adherents, cf. Eq. (1), we obtain:

$$\bar{\sigma}_{ij}^0 = a_{ijhk}^\pm e_{hk}(\bar{\mathbf{u}}^0). \quad (32)$$

3.7. Asymptotic analysis for an orthotropic half-spacewise linear elastic adhesive

Following (Curnier et al., 1995), the orthotropic symmetry can be characterized by the texture tensors

$$\mathbf{M}_i = \mathbf{m}_i \otimes \mathbf{m}_i, \quad |\mathbf{m}_i| = 1, \quad i = 1, 2, 3, \quad (33)$$

$$\mathbf{m}_i \otimes \mathbf{m}_j = 0, \quad i \neq j, \quad (34)$$

with \mathbf{m}_i , $i = 1, 2, 3$, the directions of orthotropy, here assumed to coincide with the directions of the coordinate axes, i.e. $\mathbf{m}_i = \mathbf{i}_i$, $i = 1, 2, 3$. In (33), (34), the symbol \otimes is taken to denote the dyadic product of vectors. The linear function \hat{g}^ε is a linear combination of the linear invariants $\mathbf{M}_i \cdot \mathbf{e}$, $i = 1, 2, 3$, and the hyperplane normal \mathbf{N}^ε is a linear combination of the texture tensors:

$$\hat{g}^\varepsilon(\mathbf{e}(\hat{\mathbf{u}}^\varepsilon)) = \gamma_i^\varepsilon (\mathbf{M}_i \cdot \mathbf{e}(\hat{\mathbf{u}}^\varepsilon)), \quad (35)$$

$$\mathbf{N}^\varepsilon = \gamma_i^\varepsilon \mathbf{M}_i, \quad \sqrt{\sum_{i=1}^3 (\gamma_i^\varepsilon)^2} = 1. \quad (36)$$

The piecewise linear stress-strain law takes the following form:

$$\hat{\sigma}^\varepsilon = \lambda_{ij}^\varepsilon (\mathbf{N}^\varepsilon \cdot \mathbf{e}(\hat{\mathbf{u}}^\varepsilon)) (\mathbf{M}_i \cdot \mathbf{e}(\hat{\mathbf{u}}^\varepsilon)) \mathbf{M}_j + \mu_i^\varepsilon (\mathbf{M}_i \mathbf{e}(\hat{\mathbf{u}}^\varepsilon) + \mathbf{e}(\hat{\mathbf{u}}^\varepsilon) \mathbf{M}_i), \quad (37)$$

with

$$\lambda_{ij}^\varepsilon (\mathbf{N}^\varepsilon \cdot \mathbf{e}(\hat{\mathbf{u}}^\varepsilon)) = \begin{cases} \lambda_{+ij}^\varepsilon & \text{if } \mathbf{N}^\varepsilon \cdot \mathbf{e}(\hat{\mathbf{u}}^\varepsilon) \geq 0, \\ \lambda_{-ij}^\varepsilon & \text{if } \mathbf{N}^\varepsilon \cdot \mathbf{e}(\hat{\mathbf{u}}^\varepsilon) \leq 0, \end{cases} \quad (38)$$

and $\lambda_{\pm ij}^\varepsilon = \lambda_{\pm ji}^\varepsilon$, $i, j = 1, 2, 3$. The (rescaled) elasticity tensors corresponding to the stress-strain law (37) take the form

$$\mathbf{b}_\pm^\varepsilon = \mu_i^\varepsilon (\mathbf{M}_i \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{M}_i) + \lambda_{ij}^\varepsilon [\mathbf{N}^\varepsilon \cdot \mathbf{e}(\hat{\mathbf{u}}^\varepsilon)] (\mathbf{M}_i \otimes \mathbf{M}_j), \quad (39)$$

where the tensor product $\mathbf{A} \otimes \mathbf{B}$ is defined by $(\mathbf{A} \otimes \mathbf{B})\mathbf{X} = 1/2(\mathbf{A}\mathbf{X}\mathbf{B}^T + \mathbf{B}\mathbf{A}^T\mathbf{X})$, for any second-order tensor symmetric \mathbf{X} , cf. Curnier (1993), Del Piero (1979). Substituting (39) into (3), we obtain following restrictions among the elasticity coefficients:

$$\lambda_{+ij}^\varepsilon - \lambda_{-ij}^\varepsilon = b^\varepsilon \gamma_i^\varepsilon \gamma_j^\varepsilon. \quad (40)$$

Solving for γ_j^ε and using the second in (36) give

$$|\gamma_i^\varepsilon| = \sqrt{(\lambda_{+ii}^\varepsilon - \lambda_{-ii}^\varepsilon)/b^\varepsilon}, \quad i = 1, 2, 3, \quad (41)$$

$$b^\varepsilon = \lambda_{+jj}^\varepsilon - \lambda_{-jj}^\varepsilon, \quad (42)$$

$$\begin{aligned} \lambda_{+ij}^\varepsilon - \lambda_{-ij}^\varepsilon &= -\text{sgn}(\gamma_i^\varepsilon \gamma_j^\varepsilon) \sqrt{(\lambda_{+ii}^\varepsilon - \lambda_{-ii}^\varepsilon)(\lambda_{+jj}^\varepsilon - \lambda_{-jj}^\varepsilon)}, \\ & \quad i \neq j = 1, 2, 3. \end{aligned} \quad (43)$$

As already observed in Curnier (1993, Prop. 4.1), relations (41), (42) indicate that the orientation of the hyperplane interface, separating the two regimes of traction and compression, is determined by the jumps in the diagonal bulk constants $\lambda_{\pm ii}^\varepsilon$, $i = 1, 2, 3$, which must be consistent with the off-diagonal ones, $\lambda_{\pm ij}^\varepsilon$, $i \neq j = 1, 2, 3$, as specified by (43). Note also that, unlike in Curnier (1993, Eqns. (4.18), (4.19)), we consider the possibility that $\gamma_i^\varepsilon < 0$ for some $i \in \{1, 2, 3\}$.

The constants entering (37) are three shear Lamé constants, μ_i^ε , $i = 1, 2, 3$, and twelve bulk Lamé constants $\lambda_{\pm ij}^\varepsilon$, $i, j = 1, 2, 3$. In view of (43), which provide three additional conditions, only nine of the twelve bulk coefficients are linearly independent. The positive definiteness of the elasticity tensors (39) implies that

$$2\mu_i^\varepsilon + \lambda_{\pm ii}^\varepsilon > 0, \quad i = 1, 2, 3, \quad (44)$$

$$\mu_i^\varepsilon + \mu_j^\varepsilon > 0, \quad i \neq j = 1, 2, 3. \quad (45)$$

Consistently with (4) and (5), we choose the following scalings

$$\begin{aligned} \mu_i^\varepsilon &= \mu_i^1 \varepsilon, \quad i = 1, 2, 3, \\ \lambda_{+ij}^\varepsilon &= \lambda_{+ij}^1 \varepsilon, \quad i, j = 1, 2, 3, \\ \lambda_{-ij}^\varepsilon &= \lambda_{-ij}^0 + \lambda_{-ij}^1 \varepsilon, \quad i, j = 1, 2, 3. \end{aligned} \quad (46)$$

which can model an adhesive hard in compression and soft in traction.

In view of (44), we have $\lambda_{-ii}^0 > 0$, $i = 1, 2, 3$. Substituting the last two scalings in (46) into (41) and (43) gives the following conditions at the orders zero, respectively:

$$|\gamma_i^0| = \sqrt{\frac{\lambda_{-ii}^0}{\sum_{j=1}^3 \lambda_{-jj}^0}}, \quad i = 1, 2, 3, \quad (47)$$

$$\lambda_{-ij}^0 = \text{sgn}(\gamma_i^0 \gamma_j^0) \sqrt{\lambda_{-ii}^0 \lambda_{-jj}^0}, \quad i \neq j = 1, 2, 3. \quad (48)$$

Consider now the case with $\mathbf{N}^\varepsilon \cdot \mathbf{e}(\hat{\mathbf{u}}^\varepsilon) \geq 0$. At the lowest order (-1), this condition gives the following inequality:

$$\sum_{i=1}^3 \gamma_i^0 \mathbf{M}_i \cdot \hat{\mathbf{e}}^{-1} \geq 0. \quad (49)$$

Taking into account (13), condition (49) simplifies as

$$\gamma_3^0 \hat{u}_{3,3}^0 \geq 0. \quad (50)$$

Assuming γ_3^0 independent of z_3 , integration along the z_3 -direction gives

$$\gamma_3^0 [[\hat{u}_3^0]] \geq 0. \quad (51)$$

At the lowest order (zero), the constitutive Eq. (37) gives the following relations:

$$\hat{\sigma}_{\alpha 3}^0 = \frac{1}{2}(\mu_\alpha^1 + \mu_3^1)\hat{u}_{\alpha,3}^0, \quad \alpha = 1, 2, \quad (52)$$

$$\hat{\sigma}_{33}^0 = (2\mu_3^1 + \lambda_{+33}^1)\hat{u}_{3,3}^0, \quad (53)$$

which, after integration along the z_3 -direction and use of (22), provide a spring-type linear relation between the traction vector $\hat{\sigma}^0 \mathbf{e}_3$ and the jump of the displacement across the rescaled interface:

$$\hat{\sigma}^0 \mathbf{e}_3 = \begin{pmatrix} \frac{1}{2}(\mu_1^1 + \mu_3^1) & 0 & 0 \\ 0 & \frac{1}{2}(\mu_2^1 + \mu_3^1) & 0 \\ 0 & 0 & (2\mu_3^1 + \lambda_{+33}^1) \end{pmatrix} \begin{pmatrix} [[\hat{u}_1^0]] \\ [[\hat{u}_2^0]] \\ [[\hat{u}_3^0]] \end{pmatrix}. \quad (54)$$

Now we consider the case, $\mathbf{N}^e \cdot \mathbf{e}(\hat{\mathbf{u}}^e) \leq 0$. If this condition is developed, we obtain the following inequalities at the lowest orders (-1 and 0):

$$\gamma_3^0 \hat{u}_{3,3}^0 \leq 0, \quad (55)$$

$$\gamma_\alpha^0 \hat{u}_{\alpha,\alpha}^0 + \gamma_3^0 \hat{u}_{3,3}^1 + \gamma_3^1 \hat{u}_{3,3}^0 \leq 0. \quad (56)$$

Developing the constitutive Eq. (37), at the lowest orders (again -1 and 0), we obtain the conditions

$$\hat{u}_{3,3}^0 = 0, \quad (57)$$

$$\hat{\sigma}_{33}^0 = \lambda_{-a3}^0 \hat{u}_{\alpha,\alpha}^0 + \lambda_{-33}^0 \hat{u}_{3,3}^1, \quad (58)$$

$$\hat{\sigma}_{13}^0 = \frac{1}{2}(\mu_1^1 + \mu_3^1) \hat{u}_{1,3}^0, \quad (59)$$

$$\hat{\sigma}_{23}^0 = \frac{1}{2}(\mu_2^1 + \mu_3^1) \hat{u}_{2,3}^0. \quad (60)$$

After integrating along the z_3 direction and rewriting in compact form, the latter relations become

$$\hat{\sigma}^0 \mathbf{e}_3 = \begin{pmatrix} \frac{1}{2}(\mu_1^1 + \mu_3^1) & 0 & 0 \\ 0 & \frac{1}{2}(\mu_2^1 + \mu_3^1) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} [[\hat{u}_1^0]] \\ [[\hat{u}_2^0]] \\ 0 \end{pmatrix} + \hat{\tau}_{33}^0 \mathbf{e}_3, \quad (61)$$

with

$$\hat{\tau}_{33}^0 = \lambda_{-a3}^0 \langle \hat{u}_{\alpha,\alpha}^0 \rangle + \lambda_{-33}^0 [[\hat{u}_3^1]], \quad (62)$$

and $\langle f \rangle := \int_{-1/2}^{1/2} f dz_3$. Eqs. (54), (61) and (62) can be condensed into the following form:

$$\hat{\sigma}^0 \mathbf{e}_3 = \begin{pmatrix} \frac{1}{2}(\mu_1^1 + \mu_3^1) & 0 & 0 \\ 0 & \frac{1}{2}(\mu_2^1 + \mu_3^1) & 0 \\ 0 & 0 & (2\mu_3^1 + \lambda_{+33}^1) \end{pmatrix} \begin{pmatrix} [[\hat{u}_1^0]] \\ [[\hat{u}_2^0]] \\ [[\hat{u}_3^0]] \end{pmatrix} + \hat{\tau}_{33}^0 \mathbf{e}_3, \quad (63)$$

$$\hat{\tau}_{33}^0 [[\hat{u}_3^0]] = 0, \quad \text{sgn}(\gamma_3^0) [[\hat{u}_3^0]] \geq 0, \quad \hat{\tau}_{33}^0 = \lambda_{-a3}^0 \langle \hat{u}_{\alpha,\alpha}^0 \rangle + \lambda_{-33}^0 [[\hat{u}_3^1]]. \quad (63)$$

Using the matching conditions obtained in Section 3.5, the conditions (63) can be rewritten in terms of $\sigma_{i3}^0, u_i, i = 1, 2, 3$ (instead of $\hat{\sigma}_{i3}^0, \hat{u}_i, i = 1, 2, 3$):

$$\sigma^0 \mathbf{e}_3 = \begin{pmatrix} \frac{1}{2}(\mu_1^1 + \mu_3^1) & 0 & 0 \\ 0 & \frac{1}{2}(\mu_2^1 + \mu_3^1) & 0 \\ 0 & 0 & (2\mu_3^1 + \lambda_{+33}^1) \end{pmatrix} \begin{pmatrix} [u_1^0] \\ [u_2^0] \\ [u_3^0] \end{pmatrix} + \tau_{33}^0 \mathbf{e}_3, \quad (64)$$

$$\tau_{33}^0 [u_3^0] = 0, \quad \text{sgn}(\gamma_3^0) [u_3^0] \geq 0, \quad \tau_{33}^0 = \lambda_{-a3}^0 S(u_{\alpha,\alpha}^0) + \lambda_{-33}^0 ([u_3^1] + S(u_{3,3}^0)), \quad (64)$$

with $S(f) := 1/2(f(0^+) + f(0^-))$. Notably, the conditions (64) are supplemented by the constraint (56), which, using the matching conditions and Eq. (57), can be rewritten as

$$\gamma_\alpha^0 S(u_{\alpha,\alpha}^0) + \gamma_3^0 ([u_3^1] + S(u_{3,3}^0)) \leq 0. \quad (65)$$

We now distinguish two cases: $\gamma_3^0 > 0$ and $\gamma_3^0 < 0$.

In the first case, $\gamma_3^0 > 0$, relations (47) and (48) can be rewritten as

$$\gamma_\alpha^0 = \text{sgn}(\gamma_\alpha^0) \sqrt{\frac{\lambda_{-a\alpha}^0}{\sum_{j=1}^3 \lambda_{-jj}^0}}, \quad \alpha = 1, 2, \quad (66)$$

$$\gamma_3^0 = \sqrt{\frac{\lambda_{-33}^0}{\sum_{j=1}^3 \lambda_{-jj}^0}}, \quad (67)$$

$$\lambda_{-a3}^0 = \text{sgn}(\gamma_\alpha^0) \sqrt{\lambda_{-a\alpha}^0 \lambda_{-33}^0}, \quad \alpha = 1, 2, \quad (68)$$

which, after substitution into (65) and the last of (64) and simplification of the non negative term $1/\sqrt{\sum_{j=1}^3 \lambda_{-jj}^0}$, give:

$$\tau_{33}^0 = \sqrt{\lambda_{-33}^0} \left(\text{sgn}(\gamma_\alpha^0) \sqrt{\lambda_{-a\alpha}^0} S(u_{\alpha,\alpha}^0) + \sqrt{\lambda_{-33}^0} ([u_3^1] + S(u_{3,3}^0)) \right), \quad (69)$$

$$\text{sgn}(\gamma_\alpha^0) \sqrt{\lambda_{-a\alpha}^0} S(u_{\alpha,\alpha}^0) + \sqrt{\lambda_{-33}^0} ([u_3^1] + S(u_{3,3}^0)) \leq 0, \quad (70)$$

implying that τ_{33}^0 is always non positive. In the second case, $\gamma_3^0 < 0$, a similar calculation gives

$$\tau_{33}^0 = \sqrt{\lambda_{-33}^0} \left(-\text{sgn}(\gamma_\alpha^0) \sqrt{\lambda_{-a\alpha}^0} S(u_{\alpha,\alpha}^0) + \sqrt{\lambda_{-33}^0} ([u_3^1] + S(u_{3,3}^0)) \right), \quad (71)$$

$$\text{sgn}(\gamma_\alpha^0) \sqrt{\lambda_{-a\alpha}^0} S(u_{\alpha,\alpha}^0) - \sqrt{\lambda_{-33}^0} ([u_3^1] + S(u_{3,3}^0)) \leq 0, \quad (72)$$

implying that τ_{33}^0 is always non negative.

To summarize, for a piecewise linear elastic adhesive material soft in traction and hard in compression we obtain contact conditions in the following forms:

• $\gamma_3^0 > 0$:

$$\sigma^0 \mathbf{e}_3 = \begin{pmatrix} \frac{1}{2}(\mu_1^1 + \mu_3^1) & 0 & 0 \\ 0 & \frac{1}{2}(\mu_2^1 + \mu_3^1) & 0 \\ 0 & 0 & (2\mu_3^1 + \lambda_{+33}^1) \end{pmatrix} \begin{pmatrix} [u_1^0] \\ [u_2^0] \\ [u_3^0]_+ \end{pmatrix} + \tau_{33}^0 \mathbf{e}_3, \quad (73)$$

$[u_3^0] \geq 0, \quad \tau_{33}^0 [u_3^0] = 0, \quad \tau_{33}^0 \leq 0;$

• $\gamma_3^0 < 0$:

$$\sigma^0 \mathbf{e}_3 = \begin{pmatrix} \frac{1}{2}(\mu_1^1 + \mu_3^1) & 0 & 0 \\ 0 & \frac{1}{2}(\mu_2^1 + \mu_3^1) & 0 \\ 0 & 0 & (2\mu_3^1 + \lambda_{+33}^1) \end{pmatrix} \begin{pmatrix} [u_1^0] \\ [u_2^0] \\ [u_3^0]_- \end{pmatrix} + \tau_{33}^0 \mathbf{e}_3, \quad (74)$$

$[u_3^0] \leq 0, \quad \tau_{33}^0 [u_3^0] = 0, \quad \tau_{33}^0 \geq 0.$

Both these contact conditions prescribe a spring-type law along the tangential directions of the adhesive, augmented with an inequality involving the jump of the displacement along the 3-axis, i.e. the direction of the thickness of the adhesive.

Eqs. (73) and (74) specify traction–separation laws both in the normal and tangential directions of the adhesive layer. In particular, the first two equations in (73) (or (74)) specify classic spring-type traction–separation laws in the tangential directions of the interface, because they link the tangential stress components with the corresponding jumps of the tangential displacement components. The third equation in (73) (or (74)) specifies a generalized traction–separation law in the direction normal to the interface, because it links the normal stress component with the jump of the displacement component along the direction normal to the interface. In particular, in (73), the contact force τ_{33}^0 vanishes when the jump of the displacement u_3^0 is strictly positive (opening mode), resulting in a spring-like behavior of the thin layer in traction. In compression, the jump of u_3^0 vanishes and σ_{33}^0 equals a negative stress τ_{33}^0 , corresponding to the (unknown) reaction of a rigid obstacle, as in the classical Signorini's conditions of non interpenetration (Signorini, 1959). In (74), the contact force τ_{33}^0 vanishes when the jump of the displacement u_3^0 is strictly negative (closing mode), resulting in a spring-like behavior of the thin layer in compression.

In traction, the jump of u_3^0 vanishes and now σ_{33}^0 equals a positive stress τ_{33}^0 , corresponding to the emergence of adhesion. So, one can conclude that the third equation in (73) (or (74)) is a generalized traction–separation law in the direction normal to the interface because it extends the classic spring-type traction–separation law to incorporate different behaviors in opening and closing modes.

The asymptotic analysis for an adhesive hard in tension and soft in compression follows steps similar to the ones seen so far. In particular, it can be shown that the rescalings

$$\begin{aligned}\mu_i^\varepsilon &= \mu_i^1 \varepsilon, \quad i = 1, 2, 3, \\ \lambda_{+ij}^\varepsilon &= \lambda_{+ij}^0 + \lambda_{+ij}^1 \varepsilon, \quad i, j = 1, 2, 3, \\ \lambda_{-ij}^\varepsilon &= \lambda_{-ij}^1 \varepsilon, \quad i, j = 1, 2, 3,\end{aligned}\quad (75)$$

lead to the following results

- $\gamma_3^0 > 0$:

$$\begin{aligned}\sigma^0 \mathbf{e}_3 &= \begin{pmatrix} \frac{1}{2}(\mu_1^1 + \mu_3^1) & 0 & 0 \\ 0 & \frac{1}{2}(\mu_2^1 + \mu_3^1) & 0 \\ 0 & 0 & (2\mu_3^1 + \lambda_{-33}^1) \end{pmatrix} \begin{pmatrix} [u_1^0] \\ [u_2^0] \\ [u_3^0]_- \end{pmatrix} \\ &\quad + \tau_{33}^0 \mathbf{e}_3, \\ [u_3^0]_- &\leq 0, \quad \tau_{33}^0 [u_3^0]_- = 0, \quad \tau_{33}^0 \geq 0;\end{aligned}\quad (76)$$

- $\gamma_3^0 < 0$:

$$\begin{aligned}\sigma^0 \mathbf{e}_3 &= \begin{pmatrix} \frac{1}{2}(\mu_1^1 + \mu_3^1) & 0 & 0 \\ 0 & \frac{1}{2}(\mu_2^1 + \mu_3^1) & 0 \\ 0 & 0 & (2\mu_3^1 + \lambda_{-33}^1) \end{pmatrix} \begin{pmatrix} [u_1^0] \\ [u_2^0] \\ [u_3^0]_+ \end{pmatrix} \\ &\quad + \tau_{33}^0 \mathbf{e}_3, \\ [u_3^0]_+ &\geq 0, \quad \tau_{33}^0 [u_3^0]_+ = 0, \quad \tau_{33}^0 \leq 0.\end{aligned}\quad (77)$$

Again, we obtain interface laws characterized by spring-like behavior in the plane of the adhesive, and by unilateral contact à la Signorini or by adhesion in the direction perpendicular to the plane of the film.

3.8. Asymptotic analysis for a transversely isotropic half-spacewise linear elastic adhesive

Let us now consider the case of a transversely isotropic half-spacewise linear material, and let us assume that the isotropy axis is the direction of \mathbf{i}_3 . Material symmetry gives the following additional conditions between the elasticity coefficients:

$$\begin{aligned}\mu_2^\varepsilon &= \mu_1^\varepsilon, \\ \lambda_{\pm 11}^\varepsilon &= \lambda_{\pm 22}^\varepsilon, \\ \lambda_{\pm 13}^\varepsilon &= \lambda_{\pm 23}^\varepsilon, \\ \mu_1^\varepsilon &= (\lambda_{\pm 11}^\varepsilon - \lambda_{\pm 12}^\varepsilon)/2.\end{aligned}\quad (78)$$

Substituting into (40), we obtain that $\gamma_1^\varepsilon = \gamma_2^\varepsilon$, and the additional condition (43) written for $i = 1$ and $j = 3$:

$$\lambda_{+13}^\varepsilon - \lambda_{-13}^\varepsilon = -\operatorname{sgn}(\gamma_1^0 \gamma_3^0) \sqrt{(\lambda_{+11}^\varepsilon - \lambda_{-11}^\varepsilon)(\lambda_{+33}^\varepsilon - \lambda_{-33}^\varepsilon)}.\quad (79)$$

Therefore, there are seven linearly independent elasticity constants: the two shear moduli μ_1^ε and μ_3^ε and five bulk constants, $\lambda_{\pm 11}^\varepsilon$ (or $\lambda_{\pm 22}^\varepsilon$), $\lambda_{\pm 33}^\varepsilon$ and $\lambda_{+13}^\varepsilon$ (or $\lambda_{-13}^\varepsilon$). Consistently with the assumption of adhesive material soft in traction and hard in compression, we choose the following scalings:

$$\begin{aligned}\mu_i^\varepsilon &= \mu_i^1 \varepsilon, \quad i = 1, 3, \\ \lambda_{+ii}^\varepsilon &= \lambda_{+ii}^1 \varepsilon, \quad i = 1, 3, \\ \lambda_{+13}^\varepsilon &= \lambda_{+13}^1 \varepsilon, \\ \lambda_{-ii}^\varepsilon &= \lambda_{-ii}^0 + \lambda_{-ii}^1 \varepsilon, \quad i = 1, 3.\end{aligned}\quad (80)$$

Note that the condition (79) implies that

$$\lim_{\varepsilon \rightarrow 0} \lambda_{+13}^\varepsilon = \operatorname{sgn}(\gamma_1^0 \gamma_3^0) \sqrt{\lambda_{-11}^0 \lambda_{-33}^0} =: \lambda_{-13}^0.\quad (81)$$

Relations (47) still hold true with $\gamma_1^0 = \gamma_2^0$. The asymptotic analysis for the transversely isotropic case follows steps similar to those followed for the orthotropic case. In the end, the following contact condition is obtained:

$$\begin{aligned}\sigma^0 \mathbf{e}_3 &= \begin{pmatrix} \frac{1}{2}(\mu_1^1 + \mu_3^1) & 0 & 0 \\ 0 & \frac{1}{2}(\mu_1^1 + \mu_3^1) & 0 \\ 0 & 0 & (2\mu_3^1 + \lambda_{+33}^1) \end{pmatrix} \begin{pmatrix} [u_1^0] \\ [u_2^0] \\ [u_3^0]_+ \end{pmatrix} + \tau_{33}^0 \mathbf{e}_3, \\ \tau_{33}^0 [u_3^0]_+ &= 0, \quad \operatorname{sgn}(\gamma_3^0) [u_3^0]_+ \geq 0, \quad \operatorname{sgn}(\gamma_3^0) \tau_{33}^0 \leq 0,\end{aligned}\quad (82)$$

condensing to a single relation the two cases $\gamma_3^0 > 0$ and $\gamma_3^0 < 0$. Here too, the classical Signorini's contact conditions are recovered for $\gamma_3^0 > 0$, while adhesion forces can emerge if $\gamma_3^0 < 0$. The opposite can be obtained for an adhesive hard in traction and soft in compression.

3.9. Asymptotic analysis for a cubic half-spacewise linear elastic adhesive

In the special case of cubic symmetry, the number of material constants reduces to five: a shear modulus $\mu^\varepsilon (= \mu_i^\varepsilon, i = 1, 2, 3)$, and four bulk moduli, $\lambda_{\pm 33}^\varepsilon (= \lambda_{\pm 11}^\varepsilon = \lambda_{\pm 22}^\varepsilon)$ and $\lambda_{\pm 13}^\varepsilon (= \lambda_{\pm 12}^\varepsilon = \lambda_{\pm 23}^\varepsilon)$. Conditions (43) give an additional restriction, so the number of linearly independent elasticity constants is now four. From (40) we obtain that $(\gamma_i^\varepsilon)^2 = 1$, $\gamma_i^\varepsilon = \gamma_j^\varepsilon$ $i \neq j = 1, 2, 3$, and thus we find that $\gamma_i^\varepsilon = \sqrt{3}/3$, $i = 1, 2, 3$, implying that the orientation of the hyperplane separating the two regimes of traction and compression is independent of ε . Consistently with the assumption of adhesive material soft in traction and hard in compression, we now choose the following scalings:

$$\begin{aligned}\mu^\varepsilon &= \mu^1 \varepsilon, \\ \lambda_{+33}^\varepsilon &= \lambda_{+33}^1 \varepsilon, \\ \lambda_{+13}^\varepsilon &= \lambda_{+13}^1 \varepsilon, \\ \lambda_{-33}^\varepsilon &= \lambda_{-33}^0 + \lambda_{-33}^1 \varepsilon,\end{aligned}\quad (83)$$

and from (43) we obtain that $\lambda_{-13}^0 = \lambda_{-33}^0$. Using this condition, the asymptotic analysis gives the following contact condition:

$$\begin{aligned}\sigma^0 \mathbf{e}_3 &= \begin{pmatrix} \mu^1 & 0 & 0 \\ 0 & \mu^1 & 0 \\ 0 & 0 & (2\mu^1 + \lambda_{+33}^1) \end{pmatrix} \begin{pmatrix} [u_1^0] \\ [u_2^0] \\ [u_3^0]_+ \end{pmatrix} + \tau_{33}^0 \mathbf{e}_3, \\ \tau_{33}^0 [u_3^0]_+ &= 0, \quad [u_3^0]_+ \geq 0, \quad \tau_{33}^0 \leq 0.\end{aligned}\quad (84)$$

The classical Signorini's contact conditions are now recovered, and, contrary to the previous cases of orthotropic and transversely isotropic materials, adhesion forces cannot emerge for an adhesive material soft in traction and hard in compression.

The situation of an adhesive material soft in compression and hard in traction is different. Indeed, assuming

$$\begin{aligned}\mu^\varepsilon &= \mu^1 \varepsilon, \\ \lambda_{+33}^\varepsilon &= \lambda_{+33}^0 + \lambda_{+33}^1 \varepsilon, \\ \lambda_{-33}^\varepsilon &= \lambda_{-33}^1 \varepsilon, \\ \lambda_{-13}^\varepsilon &= \lambda_{-13}^1 \varepsilon,\end{aligned}\quad (85)$$

an asymptotic analysis similar to the previous one leads to the following contact conditions

$$\begin{aligned}\sigma^0 \mathbf{e}_3 &= \begin{pmatrix} \mu^1 & 0 & 0 \\ 0 & \mu^1 & 0 \\ 0 & 0 & (2\mu^1 + \lambda_{-33}^1) \end{pmatrix} \begin{pmatrix} [u_1^0] \\ [u_2^0] \\ [u_3^0]_- \end{pmatrix} + \tau_{33}^0 \mathbf{e}_3, \\ \tau_{33}^0 [u_3^0]_- &= 0, \quad [u_3^0]_- \leq 0, \quad \tau_{33}^0 \geq 0,\end{aligned}\quad (86)$$

corresponding to the emergence of adhesion forces.

3.10. Asymptotic analysis for an isotropic half-spacewise linear elastic adhesive

The isotropic piecewise linear material is characterized by only three independent elastic constants: λ_+^ϵ , λ_-^ϵ and μ^ϵ . The scalings modeling an adhesive material soft in traction and hard in compression now are

$$\begin{aligned}\mu^\epsilon &= \mu^1 \epsilon, \\ \lambda_+^\epsilon &= \lambda_+^1 \epsilon, \\ \lambda_-^\epsilon &= \lambda_-^0 + \lambda_-^1 \epsilon,\end{aligned}\quad (87)$$

with $\lambda_-^0 > 0$. Specializing the results of the asymptotic analysis for the cubic case to the isotropic one gives

$$\begin{aligned}\sigma^0 \mathbf{e}_3 &= \begin{pmatrix} \mu^1 & 0 & 0 \\ 0 & \mu^1 & 0 \\ 0 & 0 & 2\mu^1 + \lambda_+^1 \end{pmatrix} \begin{pmatrix} [u_1^0] \\ [u_2^0] \\ [u_3^0]_+ \end{pmatrix} + \tau_{33}^0 \mathbf{e}_3, \\ \tau_{33}^0 [u_3^0] &= 0, \quad [u_3^0] \geq 0 \quad \tau_{33}^0 \leq 0,\end{aligned}\quad (88)$$

As for the cubic case, the stress term τ_{33}^0 is always negative, an occurrence corresponding to the classical Signorini's conditions.

For a half-spacewise isotropic adhesive material hard in traction and soft in compression, adhesion emerges:

$$\begin{aligned}\sigma^0 \mathbf{e}_3 &= \begin{pmatrix} \mu^1 & 0 & 0 \\ 0 & \mu^1 & 0 \\ 0 & 0 & (2\mu^1 + \lambda_-^1) \end{pmatrix} \begin{pmatrix} [u_1^0] \\ [u_2^0] \\ [u_3^0]_- \end{pmatrix} + \tau_{33}^0 \mathbf{e}_3, \\ \tau_{33}^0 [u_3^0] &= 0, \quad [u_3^0] \leq 0, \quad \tau_{33}^0 \geq 0.\end{aligned}\quad (89)$$

As a final remark, consider the engineering elastic constants E_+^ϵ , E_-^ϵ , ν_+^ϵ , ν_-^ϵ , which are related to the elastic constants as follows (cf. Curnier et al. (1995, Eqns. (4.32), (4.33))):

$$\begin{aligned}2\mu^\epsilon &= \frac{E_+^\epsilon}{1 + \nu_+^\epsilon} = \frac{E_-^\epsilon}{1 + \nu_-^\epsilon}, \\ \lambda_+^\epsilon &= 2\mu^\epsilon \frac{\nu_+^\epsilon}{1 - 2\nu_+^\epsilon}, \\ \lambda_-^\epsilon &= 2\mu^\epsilon \frac{\nu_-^\epsilon}{1 - 2\nu_-^\epsilon}.\end{aligned}\quad (90)$$

Introducing the rescaling (87) into the latter relations, we find

$$\begin{aligned}E_+^\epsilon &= \frac{\mu^1(3\lambda_+^1 + 2\mu^1)}{(\lambda_+^1 + \mu^1)} \epsilon + o(\epsilon^3), \\ E_-^\epsilon &= 3\mu^1 \epsilon + o(\epsilon), \\ \nu_+^\epsilon &= \frac{\lambda_+^1}{2(\lambda_+^1 + \mu^1)} \epsilon + o(\epsilon^3), \\ \nu_-^\epsilon &= \frac{1}{2} - \frac{\mu^1}{2\lambda_-^0} \epsilon + o(\epsilon).\end{aligned}\quad (91)$$

These relations imply that Signorini's complementarity conditions in (88) are associated to a quasi-incompressibility of the adhesive material, being ν_-^ϵ close to $1/2$ as $\epsilon \rightarrow 0$.

For an adhesive material soft in compression and hard in traction, an interface condition analogous to (86) found for a cubic material is obtained, leading again to adhesion.

3.11. Asymptotic analysis for an octantwise linear elastic adhesive

For an orthotropic octantwise linear elastic material, Curnier et al. (1995) assume a subdivision of the strain space into eight octants delimited by the three orthogonal hyperplanes normal to three texture tensors. As in the case of orthotropic half-spacewise linear elastic material, we assume that the texture tensors are given by

$$\mathbf{M}_i = \mathbf{i}_i \otimes \mathbf{i}_i, \quad i = 1, 2, 3, \quad (92)$$

with \mathbf{i}_i , $i = 1, 2, 3$, the directions of orthotropy, here assumed to coincide with the directions of the coordinate axes. Herein, we will assume the following numbering of the eight octants:

$$\begin{aligned}\mathcal{E}^1 &= \{\mathbf{e} \in \mathcal{E} \mid \mathbf{M}_1 \cdot \mathbf{e} \geq 0, \quad \mathbf{M}_2 \cdot \mathbf{e} \geq 0, \quad \mathbf{M}_3 \cdot \mathbf{e} \geq 0\}, \\ \mathcal{E}^2 &= \{\mathbf{e} \in \mathcal{E} \mid \mathbf{M}_1 \cdot \mathbf{e} \geq 0, \quad \mathbf{M}_2 \cdot \mathbf{e} \geq 0, \quad \mathbf{M}_3 \cdot \mathbf{e} \leq 0\}, \\ \mathcal{E}^3 &= \{\mathbf{e} \in \mathcal{E} \mid \mathbf{M}_1 \cdot \mathbf{e} \geq 0, \quad \mathbf{M}_2 \cdot \mathbf{e} \leq 0, \quad \mathbf{M}_3 \cdot \mathbf{e} \geq 0\}, \\ \mathcal{E}^4 &= \{\mathbf{e} \in \mathcal{E} \mid \mathbf{M}_1 \cdot \mathbf{e} \geq 0, \quad \mathbf{M}_2 \cdot \mathbf{e} \leq 0, \quad \mathbf{M}_3 \cdot \mathbf{e} \leq 0\}, \\ \mathcal{E}^5 &= \{\mathbf{e} \in \mathcal{E} \mid \mathbf{M}_1 \cdot \mathbf{e} \leq 0, \quad \mathbf{M}_2 \cdot \mathbf{e} \geq 0, \quad \mathbf{M}_3 \cdot \mathbf{e} \geq 0\}, \\ \mathcal{E}^6 &= \{\mathbf{e} \in \mathcal{E} \mid \mathbf{M}_1 \cdot \mathbf{e} \leq 0, \quad \mathbf{M}_2 \cdot \mathbf{e} \geq 0, \quad \mathbf{M}_3 \cdot \mathbf{e} \leq 0\}, \\ \mathcal{E}^7 &= \{\mathbf{e} \in \mathcal{E} \mid \mathbf{M}_1 \cdot \mathbf{e} \leq 0, \quad \mathbf{M}_2 \cdot \mathbf{e} \leq 0, \quad \mathbf{M}_3 \cdot \mathbf{e} \geq 0\}, \\ \mathcal{E}^8 &= \{\mathbf{e} \in \mathcal{E} \mid \mathbf{M}_1 \cdot \mathbf{e} \leq 0, \quad \mathbf{M}_2 \cdot \mathbf{e} \leq 0, \quad \mathbf{M}_3 \cdot \mathbf{e} \leq 0\}.\end{aligned}\quad (93)$$

The stress-strain law is defined in terms of eight elasticity tensors, denoted \mathbf{b}_i^ϵ in the i th octant. As shown in Curnier et al. (1995), the continuity of the stress-strain relation requires that the eight elasticity tensors are related as follows:

$$\begin{aligned}\mathbf{b}_1^\epsilon &= \mathbf{b}_8^\epsilon - b_i^\epsilon \mathbf{M}_i \otimes \mathbf{M}_i, \\ \mathbf{b}_2^\epsilon &= \mathbf{b}_8^\epsilon - b_1^\epsilon \mathbf{M}_1 \otimes \mathbf{M}_1 - b_2^\epsilon \mathbf{M}_2 \otimes \mathbf{M}_2, \\ \mathbf{b}_3^\epsilon &= \mathbf{b}_8^\epsilon - b_1^\epsilon \mathbf{M}_1 \otimes \mathbf{M}_1 - b_3^\epsilon \mathbf{M}_3 \otimes \mathbf{M}_3, \\ \mathbf{b}_4^\epsilon &= \mathbf{b}_8^\epsilon - b_1^\epsilon \mathbf{M}_1 \otimes \mathbf{M}_1, \\ \mathbf{b}_5^\epsilon &= \mathbf{b}_8^\epsilon - b_2^\epsilon \mathbf{M}_2 \otimes \mathbf{M}_2 - b_3^\epsilon \mathbf{M}_3 \otimes \mathbf{M}_3, \\ \mathbf{b}_6^\epsilon &= \mathbf{b}_8^\epsilon - b_2^\epsilon \mathbf{M}_2 \otimes \mathbf{M}_2, \\ \mathbf{b}_7^\epsilon &= \mathbf{b}_8^\epsilon - b_3^\epsilon \mathbf{M}_3 \otimes \mathbf{M}_3.\end{aligned}\quad (94)$$

These relations show that an orthotropic octantwise linear stress-strain law involves twelve independent constants: the nine elasticity constants for \mathbf{b}_8^ϵ , and the three jumps b_i^ϵ , $i = 1, 2, 3$ across the three hyperplanes \mathbf{M}_i , $i = 1, 2, 3$. The nine elasticity constants for \mathbf{b}_8^ϵ are three shear moduli μ_i^ϵ and six bulk moduli λ_{ij}^ϵ :

$$\mathbf{b}_8^\epsilon = \mu_i^\epsilon (\mathbf{M}_i \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{M}_i) + \lambda_{ij}^\epsilon (\mathbf{M}_i \otimes \mathbf{M}_j). \quad (95)$$

Relations (94) imply that the shear moduli are the same in all subdivision of the strain space, as in the case of half-spacewise materials, but now the bulk coefficients λ_{ij}^ϵ with $i \neq j$, $i, j = 1, 2, 3$, are identical in all octants. The diagonal bulk coefficients λ_{ii}^ϵ , however, can be different. This implies that we are somehow limited in choosing the rescaling, and the distinction between materials soft in traction/ hard in compression and hard in traction/soft in compression is not as clear as before. For the sake of simplicity, we choose the following scalings:

$$\begin{aligned}\mu_i^\epsilon &= \mu_i^1 \epsilon, \quad i = 1, 2, 3, \\ \lambda_{ij}^\epsilon &= \lambda_{ij}^1 \epsilon, \quad i \neq j = 1, 2, 3, \\ \lambda_{ii}^\epsilon &= \lambda_{ii}^0 + \lambda_{ii}^1 \epsilon, \quad i, j = 1, 2, 3, \\ b_i^\epsilon &= b_i^1 \epsilon, \quad i = 1, 2, 3,\end{aligned}\quad (96)$$

corresponding to a material soft in shear, as before, with soft bulk components λ_{ij}^ϵ , $i \neq j$, and hard bulk components λ_{ii}^ϵ .

Following (Curnier et al., 1995), the eight elasticity tensors are assumed symmetric and positive definite, implying in particular $\lambda_{ii}^\epsilon > 0$, $i = 1, 2, 3$, and thus $\lambda_{ii}^0 > 0$, $i = 1, 2, 3$.

We now proceed with the asymptotic analysis. In all eight octants, relations (22) are still valid. Moreover, substituting the scalings (96) and the asymptotic expansions (10)₁ and (12) into the constitutive equation in each octant, at the order -1 we obtain that

$$\hat{u}_{3,3}^0 = 0, \quad (97)$$

which, integrated along the z_3 -direction, gives

$$[[\hat{u}_3^0]] = 0 \quad (98)$$

in the rescaled configuration, and

$$[\hat{u}_3^0] = 0 \quad (99)$$

in the limit geometrical configuration as $\varepsilon \rightarrow 0$. At the order 0, substituting the scalings (96) and the asymptotic expansions (10)₁ and (12) into the constitutive equation in each octant gives the relations

$$\begin{aligned} \hat{\sigma}_{\alpha 3}^0 &= \frac{1}{2}(\mu_\alpha^1 + \mu_3^1)\hat{u}_{\alpha,3}^0, \quad \alpha = 1, 2, \\ \hat{\sigma}_{33}^0 &= \lambda_{33}^0 \hat{u}_{3,3}^1. \end{aligned} \quad (100)$$

From the conditions defining the eight octants, we have that

$$\begin{aligned} \hat{u}_{3,3}^1 &\leq 0 \quad \text{in } \mathcal{E}^2, \mathcal{E}^4, \mathcal{E}^6, \mathcal{E}^8, \\ \hat{u}_{3,3}^1 &\geq 0 \quad \text{in } \mathcal{E}^1, \mathcal{E}^3, \mathcal{E}^5, \mathcal{E}^7. \end{aligned} \quad (101)$$

After integrating along the z_3 direction and using the matching conditions obtained in Subsection 2.3, relations (100) and (101) can be condensed into the following form:

$$\begin{aligned} \sigma^0 \mathbf{e}_3 &= \begin{pmatrix} \frac{1}{2}(\mu_1^1 + \mu_3^1) & 0 & 0 \\ 0 & \frac{1}{2}(\mu_2^1 + \mu_3^1) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} [u_1^0] \\ [u_2^0] \\ 0 \end{pmatrix} + \tau_{33}^0 \mathbf{e}_3, \\ [u_3^0] &= 0, \quad \tau_{33}^0 \begin{matrix} \geq \\ < \end{matrix} 0, \end{aligned} \quad (102)$$

with $\tau_{33}^0 := \lambda_{33}^0 ([u_3^0] + S(u_{3,3}^0))$ non negative or non positive depending upon the octant. Indeed, in view of (101) and of the positivity of λ_{33}^0 , the stress τ_{33}^0 is non negative in octants $\mathcal{E}^1, \mathcal{E}^3, \mathcal{E}^5, \mathcal{E}^7$ and non positive in octants $\mathcal{E}^2, \mathcal{E}^4, \mathcal{E}^6, \mathcal{E}^8$. In conclusion, relation (102) is a spring-type contact law in the tangential direction, prescribing perfect contact in the normal direction. However, the sign of the stress is not *a priori* assigned like in the case of the unilateral contact *à la* Signorini, corresponding to $\tau_{33}^0 \leq 0$: indeed, adhesion emerges if $\tau_{33}^0 \geq 0$.

As a final remark, we note that other symmetries different from the orthotropic one could be studied. For example, in the cubic case the interface function defining the subdivision of the strain space is still defined by the same three texture tensors and constitutive relation of the orthotropic case, but the number of elastic material constants reduces from twelve to four (Soltz, 2000). The asymptotic analysis then would give transmission conditions of the form (102) but with $\mu_1^1 = \mu_2^1 = \mu_3^1$. For symmetry reasons, the case of isotropic material reduces the isotropic half-spacewise material already studied.

4. Summary of results and closing remarks

Putting together all the results obtained so far, the following equilibrium problem is obtained in the configuration made of the two adherents in contact through the interface S , geometrical limit of the adhesive as ε goes to zero (cf. Fig.1):

$$\begin{cases} \sigma_{ij,j}^0 + f_i = 0 & \text{in } \Omega_\pm^0, \\ \sigma_{ij}^0 n_j = g_i & \text{on } S_g^0, \\ u_i^0 = 0 & \text{on } S_u^0, \\ \sigma_{ij}^0 = a_{ijhk}^\pm e_{hk}(u^0) & \text{in } \Omega_\pm^0. \end{cases} \quad (103)$$

This equilibrium problem is augmented by transmission conditions depending on the particular piecewise adhesive material, as follows.

- For a half-spacewise orthotropic adhesive material soft in traction and hard in compression, the transmission conditions are the following (cf. Fig. 2):

$$\begin{cases} \sigma_{\alpha 3}^0 = \frac{1}{2}(\mu_\alpha^1 + \mu_3^1)[u_\alpha^0], \quad \alpha = 1, 2, & \text{on } S \\ \sigma_{33}^0 = (2\mu_3^1 + \lambda_{-33}^1)[u_3^0]_{\text{sgn}(\gamma_3^0)} + \tau_{33}^0, \quad \gamma_3^0 [u_3^0] \geq 0, & \\ \tau_{33}^0 [u_3^0] = 0, \quad \gamma_3^0 \tau_{33}^0 \leq 0 & \text{on } S; \end{cases} \quad (104)$$

- for a half-spacewise orthotropic adhesive material hard in traction and soft in compression, the transmission conditions are the following:

$$\begin{cases} \sigma_{\alpha 3}^0 = \frac{1}{2}(\mu_\alpha^1 + \mu_3^1)[u_\alpha^0], \quad \alpha = 1, 2, & \text{on } S \\ \sigma_{33}^0 = (2\mu_3^1 + \lambda_{-33}^1)[u_3^0]_{\text{sgn}(\gamma_3^0)} + \tau_{33}^0, \quad \gamma_3^0 [u_3^0] \leq 0, & \\ \tau_{33}^0 [u_3^0] = 0, \quad \gamma_3^0 \tau_{33}^0 \geq 0 & \text{on } S; \end{cases} \quad (105)$$

- for a half-spacewise transversely adhesive material (x_3 axis of transverse isotropy), transmission conditions analogous to ones for the orthotropic case are obtained with the further simplification $\mu_1^1 = \mu_2^1$;
- for a half-spacewise cubic or isotropic adhesive material, soft in traction and hard in compression, the transmission conditions are the following:

$$\begin{cases} \sigma_{\alpha 3}^0 = \mu^1 [u_\alpha^0], \quad \alpha = 1, 2, & \text{on } S \\ \sigma_{33}^0 = (2\mu^1 + \lambda_+^1)[u_3^0]_+ + \tau_{33}^0, \quad [u_3^0] \geq 0, & \\ \tau_{33}^0 [u_3^0] = 0, \quad \tau_{33}^0 \leq 0 & \text{on } S; \end{cases} \quad (106)$$

- for a half-spacewise cubic or isotropic adhesive material, hard in traction and soft in compression, the transmission conditions are the following:

$$\begin{cases} \sigma_{\alpha 3}^0 = \mu^1 [u_\alpha^0], \quad \alpha = 1, 2, & \text{on } S \\ \sigma_{33}^0 = (2\mu^1 + \lambda_-^1)[u_3^0]_- + \tau_{33}^0, \quad [u_3^0] \leq 0, & \\ \tau_{33}^0 [u_3^0] = 0, \quad \tau_{33}^0 \geq 0 & \text{on } S; \end{cases} \quad (107)$$

- for an octantwise orthotropic adhesive material, the transmission conditions are the following:

$$\begin{cases} \sigma_{\alpha 3}^0 = \frac{1}{2}(\mu_\alpha^1 + \mu_3^1)[u_\alpha^0], \quad \alpha = 1, 2, & \text{on } S \\ \sigma_{33}^0 = \tau_{33}^0, \quad [u_3^0] = 0, \quad \tau_{33}^0 \begin{matrix} \geq \\ < \end{matrix} 0 & \text{on } S; \end{cases} \quad (108)$$

- for an octantwise transversely isotropic adhesive material, transmission conditions analogous to the previous ones for the orthotropic case are obtained with the further simplification $\mu_1^1 = \mu_2^1$;
- for an octantwise isotropic adhesive material, transmission conditions analogous to the ones for the half-spacewise isotropic case are obtained.

Inspection of the transmission conditions shows a common property between them: spring-type interface conditions are always obtained in the tangential directions of the adhesive plane, here the directions of the x_1 and x_2 axes. This is a direct consequence of the chosen rescaling, which always prescribes the shear moduli to be soft, i.e. to rescale with ε . In the direction perpendicular to the plane of the adhesive, the x_3 axes, different situations can occur. For half-spacewise materials, a spring-type condition is always obtained in the loading direction characterized by a soft material behavior; this implies the occurrence of a jump in the relative displacement perpendicular to the adhesive plane and its proportionality to the normal stress component. In the loading direction characterized by a hard material behavior, Signorini's conditions of unilateral contact or adhesion can emerge; thus, the jump of the relative displacement normal to the plane of the adhesive vanishes, and the normal stress component is negative in the case of Signorini's conditions of unilateral contact, and positive in the case of adhesion. For an octantwise material, the jump of the relative displacement normal to the plane of the adhesive always vanishes, and the sign of the normal stress component can be negative (Signorini) or positive (adhesion). This result depends on the particular rescaling chosen for the octantwise material, and in particular on the choice of hard diagonal elasticity coefficients of the stiffness tensor.

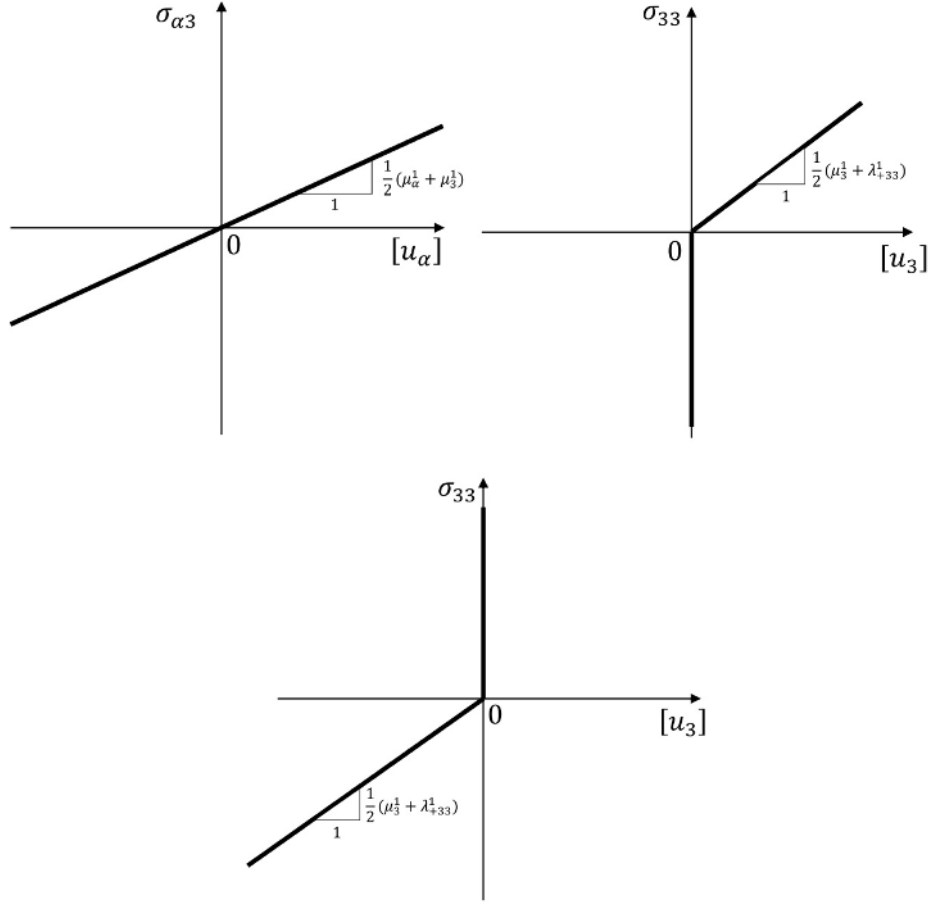


Fig. 2. Sketch of the stress-separation law (104) calculated for a piecewise linear elastic adhesive material soft in traction and hard in compression. Top left: form of the stress-separation law linking the tangential stress components $\sigma_{\alpha 3}$, $\alpha = 1, 2$ to the corresponding jumps of the tangential displacement components u_{α} , $\alpha = 1, 2$. Top right: form of the stress-separation law linking the normal stress component σ_{33} to the jump of the normal displacement component u_3 in the case $\gamma_3^0 > 0$. Bottom: form of the stress-separation law linking the normal stress component σ_{33} to the jump of the normal displacement component u_3 in the case $\gamma_3^0 < 0$.

Notably, the models of piecewise or octantwise material used in this paper and proposed by Curnier et al. in Curnier et al. (1995) are characterized by the convexity of the strain energy. In the limit of vanishing ε , this property is clearly lost, because of the vanishing of the soft elasticity coefficients. However, the equilibrium problems augmented by transmission conditions involving Sinorini's unilateral contact are still characterized by existence and uniqueness of the equilibrium solution. This can be proved using arguments *à la* Duvaut-Lions, cf. Duvaut and Lions (1976).

In this paper, the x_3 axis is assumed to be an axis of orthotropy, but more general situations could be explored. In addition, orthotropy is the maximum asymmetry considered in the asymptotic analysis presented in the paper. However, the study could be easily extended to more general anisotropies, as the monoclinic case. Clearly, the equations involved would be more complex; for example, in soft loading directions the spring-type law would couple tangential and normal components of the displacement jumps.

In the asymptotic analysis proposed in this paper, the adhesive thickness was assumed to be uniform for simplicity, but the results obtained could be easily generalized to a more realistic situation of a non uniform adhesive thin layer of varying thickness by accounting for a smooth rough interface, as in Dumont et al. (2017). Finally, viscoelasticity or viscoplasticity typical of many structural and biological adhesives could be considered. Future research will address these aspects.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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