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# Optimal Control of Quasivariational Inequalities with Applications to Contact Mechanics

Mircea Sofonea

**Abstract** This chapter deals with the optimal control of a class of elliptic quasivariational inequalities. We start with an existence and uniqueness result for such inequalities. Then we state an optimal control problem, list the assumptions on the data and prove the existence of optimal pairs. We proceed with a perturbed control problem for which we state and prove a convergence result, under general conditions. Further, we present a relevant particular case for which these conditions are satisfied and, therefore, our convergence result works. Finally, we illustrate the use of these abstract results in the study of a mathematical model which describes the equilibrium of an elastic body in frictional contact with an obstacle, the so-called foundation. The process is static and the contact is modeled with normal compliance and unilateral constraint, associated with the Coulomb's law of dry friction. We prove the existence, uniqueness, and convergence results together with the corresponding mechanical interpretation. We illustrate these results in the study of a one-dimensional example. Finally, we end this chapter with some concluding remarks.

**Keywords** Quasivariational inequality · Optimal pair · Optimal control · Convergence results · Frictional contact · Unilateral constraint · Weak solution

## 13.1 Introduction

Variational inequalities represent a powerful mathematical tool used in the study of various nonlinear boundary value problems with partial differential equations. They are usually formulated by using a set of constraints, a nonlinear operator, and a convex function which could be nondifferentiable. Quasivariational inequalities

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represent a special class of variational inequalities in which the convex function depends on the solution. The theory of variational inequalities was developed based on arguments of monotonicity and convexity, including properties of the subdifferential of a convex function. Basic references in the field are [4, 10, 17, 22, 25], for instance. Application of variational inequalities in mechanics could be found in the books [14, 19–21, 36], for instance.

The optimal control theory deals with the existence and, when possible, the uniqueness of optimal pairs and optimal control. It also deals with the derivation of necessary conditions of optimality or, better, necessary and sufficient conditions of optimality. This means to find an equation or an inequality which characterizes the optimal control. Basic references for the optimal control of systems governed by partial differential equations are the books [24, 35]. Application of the optimal control theory in mechanics could be found in [1, 2, 13, 31], for instance. Optimal control problems for variational inequalities have been discussed in several works, including [7, 9, 16, 29, 30, 34, 43]. Due to the nonsmooth and nonconvex feature of the functional involved, the treatment of optimal control problems for variational inequalities requires the use of their approximation by smooth optimization problems. And, on this matter, establishing convergence results for the optimal pairs represents a topic of major interest.

Processes of contact between deformable bodies abound in industry and everyday life. A few simple examples are brake pads in contact with wheels, tires on roads, and pistons with skirts. Due to the complex phenomena involved, they lead to strongly nonlinear mathematical models, formulated in terms of various classes of inequalities, including variational and quasivariational inequalities. Because of the importance of contact processes in structural and mechanical systems, considerable effort has been put into their modeling, analysis, and numerical simulations and the literature in the field is extensive. It includes the books [14, 15, 19, 23, 32, 36, 38, 41, 42], for instance. The literature concerning optimal control problems in the study of mathematical models of contact is quite limited. The reason is the strong nonlinearities which arise in the boundary conditions included in such models. The results on optimal control for various contact problems with elastic materials can be found in [3, 6, 8, 11, 12, 26–28, 44] and the references therein.

In the current chapter we consider an optimal control problem for a general class of elliptic quasivariational inequalities. Our motivation is given by the fact that such kind of inequalities arises in the study of frictional contact models and, therefore, their optimal control is important in a large number of engineering applications. The functional framework is the following: let  $X$  and  $Y$  be real Hilbert spaces endowed with the inner products  $(\cdot, \cdot)_X$  and  $(\cdot, \cdot)_Y$ , respectively,  $K \subset X$ ,  $A : X \rightarrow X$ ,  $j : X \times X \rightarrow \mathbb{R}$ , and  $\pi : X \rightarrow Y$ . Then, the inequality problem we consider is the following.

**Problem  $\mathcal{P}$ .** Given  $f \in Y$ , find  $u$  such that

$$u \in K, \quad (Au, v - u)_X + j(u, v) - j(u, u) \geq (f, \pi v - \pi u)_Y \quad \forall v \in K. \quad (13.1)$$

Note that the function  $j$  depends on the solution  $u$  and, for this reason, we refer to (13.1) as a quasivariational inequality. We assume in what follows that for each  $f \in Y$  the quasivariational inequality (13.1) has a unique solution  $u = u(f)$ . Sufficient conditions on the data which guarantee this assumption will be provided in Theorem 13.2.12. The set of admissible pairs for inequality (13.1) is given by

$$\mathcal{V}_{ad} = \{ (u, f) \in K \times Y \text{ such that (13.1) holds} \}. \quad (13.2)$$

Consider now a cost functional  $\mathcal{L} : X \times Y \rightarrow \mathbb{R}$ , where, here and below,  $X \times Y$  represents the product of the Hilbert spaces  $X$  and  $Y$ , equipped with the canonical inner product. Then, the optimal control problem we study in this chapter is the following.

**Problem Q.** Find  $(u^*, f^*) \in \mathcal{V}_{ad}$  such that

$$\mathcal{L}(u^*, f^*) = \min_{(u, f) \in \mathcal{V}_{ad}} \mathcal{L}(u, f). \quad (13.3)$$

Our aim in this chapter is threefold. The first one is to formulate sufficient assumptions on the data which guarantee the existence of optimal pairs, i.e., elements  $(u^*, f^*) \in \mathcal{V}_{ad}$  which solve Problem Q. The answer to this question is provided by Theorem 13.3.1. The second aim is to study the dependence of the optimal pairs with respect to perturbations of the set  $K$ , the operator  $A$ , and the functional  $j$ . The answer to this question is provided by Theorem 13.3.5 which provides a convergence result, under general conditions. This result is completed by Theorem 13.3.8, which holds under specific conditions on the data. Finally, our third aim is to illustrate how these abstract results could be useful in the study of mathematical models of contact. The answer to this question is provided by Theorems 13.4.4–13.4.6 and the corresponding mechanical interpretation.

The rest of this chapter is structured in four sections, as follows: In Sect. 13.2 we provide some preliminary results in the study of Problem P. They concern the existence, uniqueness, and convergence of the solution. Then, in Sect. 13.3 we state and prove the existence of optimal pairs to the control problem Q as well as a general convergence result. Next, we present a relevant particular case for which our convergence result holds. In Sect. 13.4 we consider a mathematical model of frictional contact with elastic materials. The process is static and the contact is described with normal compliance and unilateral constraint, associated with a version of Coulomb's law of dry friction. We apply our results in Sects. 13.2 and 13.3 in the study of this problem. Moreover, we illustrate them in the study of a one-dimensional example. Finally, we end this chapter with some concluding remarks, presented in Sect. 13.5.

## 13.2 Quasivariational Inequalities

In this section we provide some results in the study of Problem  $\mathcal{P}$  that we need in the rest of this chapter. We first introduce preliminary material from functional analysis, and then we state and prove an existence and uniqueness result, Theorem 13.2.12. Finally, we study the dependence of the solution with respect to the element  $f$  and we prove a convergence result, Theorem 13.2.13.

### 13.2.1 Notation and Preliminaries

All the linear spaces considered in this chapter including abstract normed spaces, Banach spaces, Hilbert spaces, and various function spaces are assumed to be real linear spaces. For a normed space  $X$  we denote by  $\|\cdot\|_X$  its norm and by  $0_X$  its zero element. In addition, we denote by  $\rightarrow$  and  $\rightharpoonup$  the strong and weak convergence in various normed spaces. For an inner product space  $X$  we denote by  $(\cdot, \cdot)_X$  its inner product and by  $\|\cdot\|_X$  the associated norm. Unless stated otherwise, all the limits, upper and lower limits, below are considered as  $n \rightarrow \infty$ , even if we do not mention it explicitly. The results presented below in this subsection are well known and can be found in many books and survey and, for this reason, we skip their proofs.

**Definition 13.2.1** Let  $X$  be a normed space. A subset  $K \subset X$  is called:

- (i) *(strongly) closed* if the limit of each convergent sequence of elements of  $K$  belongs to  $K$ , that is,  $\{u_n\} \subset K, u_n \rightarrow u \text{ in } X \implies u \in K$ .
- (ii) *weakly closed* if the limit of each weakly convergent sequence of elements of  $K$  belongs to  $K$ , that is,  $\{u_n\} \subset K, u_n \rightharpoonup u \text{ in } X \implies u \in K$ .
- (iii) *convex*, if  $u, v \in K \implies (1-t)u + tv \in K \quad \forall t \in [0, 1]$ .

Evidently, every weakly closed subset of  $X$  is (strongly) closed, but the converse is not true, in general. An exception is provided by the class of convex subsets of a Banach space, as shown in the following result.

**Theorem 13.2.2 (The Mazur Theorem)** *A convex subset of a Banach space is (strongly) closed if and only if it is weakly closed.*

We now recall the following important property which represents a particular case of the well-known Eberlein–Smulyan theorem.

**Theorem 13.2.3** *If  $X$  is a Hilbert space, then any bounded sequence in  $X$  has a weakly convergent subsequence.*

It follows that if  $X$  is a Hilbert space and the sequence  $\{u_n\} \subset X$  is bounded, that is,  $\sup_n \|u_n\|_X < \infty$ , then there exists a subsequence  $\{u_{n_k}\} \subset \{u_n\}$  and an element  $u \in X$  such that  $u_{n_k} \rightharpoonup u$  in  $X$ . Furthermore, if the limit  $u$  is independent of the subsequence, then the whole sequence  $\{u_n\}$  converges weakly to  $u$ , as stated in the following result.

**Theorem 13.2.4** *Let  $X$  be a Hilbert space and let  $\{u_n\}$  be a bounded sequence of elements in  $X$  such that each weakly convergent subsequence of  $\{u_n\}$  converges weakly to the same limit  $u \in X$ . Then  $u_n \rightharpoonup u$  in  $X$ .*

We now proceed with the definition of some classes of operators.

**Definition 13.2.5** Let  $X$  be an inner product space and let  $A : X \rightarrow X$  be an operator. The operator  $A$  is said to be:

- (i) *monotone*, if  $(Au - Av, u - v)_X \geq 0 \quad \forall u, v \in X$ .
- (ii) *strongly monotone*, if there exists a constant  $m > 0$  such that

$$(Au - Av, u - v)_X \geq m \|u - v\|_X^2 \quad \forall u, v \in X.$$

- (iii) *bounded*, if  $A$  maps bounded sets into bounded sets.
- (iv) *pseudomonotone*, if it is bounded and  $u_n \rightharpoonup u$  in  $X$  with

$$\limsup_{n \rightarrow \infty} (Au_n, u_n - u)_X \leq 0 \tag{13.4}$$

implies

$$\liminf_{n \rightarrow \infty} (Au_n, u_n - v)_X \geq (Au, u - v)_X \quad \forall v \in X. \tag{13.5}$$

- (v) *Lipschitz continuous* if there exists  $M > 0$  such that

$$\|Au - Av\|_X \leq M \|u - v\|_X \quad \forall u, v \in X.$$

- (vi) *hemicontinuous* if the real valued function

$$\theta \mapsto (A(u + \theta v), w)_X \quad \text{is continuous on } \mathbb{R}, \quad \forall u, v, w \in X.$$

It is easy to see that a strongly monotone operator  $A : X \rightarrow X$  is monotone and a Lipschitz continuous operator  $A : X \rightarrow X$  is bounded and hemicontinuous. Moreover, the following result holds.

**Proposition 13.2.6** *Let  $X$  be an inner product space and  $A : X \rightarrow X$  a monotone hemicontinuous operator. Assume that  $\{u_n\}$  is a sequence of elements in  $X$  which converges weakly to the element  $u \in X$  such that (13.4) holds. Then (13.5) holds, too.*

A proof of Proposition 13.2.6 can be found in [41, p. 21]. As a consequence we obtain the following result which will be used later in this chapter.

**Corollary 13.2.7** *Let  $X$  be an inner product space and  $A : X \rightarrow X$  a monotone Lipschitz continuous operator. Then  $A$  is pseudomonotone.*

Convex lower semicontinuous functions represent a crucial ingredient in the study of variational inequalities. To introduce them, we start with the following definitions.

**Definition 13.2.8** Let  $X$  be a linear space and let  $K$  be a nonempty convex subset of  $X$ . A function  $\varphi : K \rightarrow \mathbb{R}$  is said to be *convex* if

$$\varphi((1-t)u + tv) \leq (1-t)\varphi(u) + t\varphi(v) \quad (13.6)$$

for all  $u, v \in K$  and  $t \in [0, 1]$ . The function  $\varphi$  is *strictly convex* if the inequality in (13.6) is strict for  $u \neq v$  and  $t \in (0, 1)$ .

**Definition 13.2.9** Let  $X$  be a normed space and let  $K$  be a nonempty closed convex subset of  $X$ . A function  $\varphi : K \rightarrow \mathbb{R}$  is said to be *lower semicontinuous (l.s.c.)* at  $u \in K$  if

$$\liminf_{n \rightarrow \infty} \varphi(u_n) \geq \varphi(u) \quad (13.7)$$

for each sequence  $\{u_n\} \subset K$  converging to  $u$  in  $X$ . The function  $\varphi$  is *l.s.c.* if it is l.s.c. at every point  $u \in K$ . When inequality (13.7) holds for each sequence  $\{u_n\} \subset K$  that converges weakly to  $u$ , the function  $\varphi$  is said to be *weakly lower semicontinuous* at  $u$ . The function  $\varphi$  is *weakly l.s.c.* if it is weakly l.s.c. at every point  $u \in K$ .

Since the strong convergence implies the weak convergence, it follows that a weakly lower semicontinuous function is lower semicontinuous. Moreover, the following results hold.

**Proposition 13.2.10** *Let  $X$  be a Banach space,  $K$  a nonempty closed convex subset of  $X$ , and  $\varphi : K \rightarrow \mathbb{R}$  a convex function. Then  $\varphi$  is lower semicontinuous if and only if it is weakly lower semicontinuous.*

The proof of this result is based on Theorem 13.2.2.

## 13.2.2 Existence and Uniqueness

Everywhere in the rest of this chapter we assume that  $X$  is a Hilbert space. Given a subset  $K \subset X$ , an operator  $A : X \rightarrow X$ , a function  $j : X \times X \rightarrow \mathbb{R}$ , and an element  $\tilde{f} \in X$ , we consider the following quasivariational inequality problem: find an element  $u$  such that

$$u \in K, \quad (Au, v - u)_X + j(u, v) - j(u, u) \geq (\tilde{f}, v - u)_X \quad \forall v \in K. \quad (13.8)$$

Quasivariational inequalities of the form (13.8) have been studied by many authors, by using different functional methods, including fixed point and topological degree arguments. The existence and uniqueness results for such inequalities could be

found in [12, 33, 40, 41], for instance, under various assumptions on the function  $j$ . Here, in this chapter, we consider the following assumptions:

$$K \text{ is a nonempty, closed, convex subset of } X. \quad (13.9)$$

$$\left\{ \begin{array}{l} A \text{ is a strongly monotone Lipschitz continuous operator, i.e.,} \\ \text{there exist } m > 0 \text{ and } M > 0 \text{ such that} \\ \text{(a) } (Au - Av, u - v)_X \geq m \|u - v\|_X^2 \quad \forall u, v \in X, \\ \text{(b) } \|Au - Av\|_X \leq M \|u - v\|_X \quad \forall u, v \in X. \end{array} \right. \quad (13.10)$$

$$\left\{ \begin{array}{l} \text{(a) For all } \eta \in X, j(\eta, \cdot) : X \rightarrow \mathbb{R} \text{ is convex and l.s.c.} \\ \text{(b) There exists } \alpha \geq 0 \text{ such that} \\ \quad j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_1) - j(\eta_2, v_2) \\ \quad \leq \alpha \|\eta_1 - \eta_2\|_X \|v_1 - v_2\|_X \quad \forall \eta_1, \eta_2, v_1, v_2 \in X. \end{array} \right. \quad (13.11)$$

$$m > \alpha. \quad (13.12)$$

We recall the following existence and uniqueness result, which guarantees the unique solvability of Problem  $\mathcal{P}$ .

**Theorem 13.2.11** *Assume that (13.9)–(13.12) hold. Then, for each  $\tilde{f} \in X$  the quasivariational inequality (13.8) has a unique solution.*

A proof of Theorem 13.2.11 can be found in [41, p. 49], based on the Banach fixed point argument. We now turn to the study of Problem  $\mathcal{P}$  and, to this end, we consider the following additional assumptions:

$$\left\{ \begin{array}{l} \pi \text{ is a linear continuous operator, i.e.,} \\ \text{there exists } c_0 > 0 \text{ such that} \\ \quad \|\pi v\|_Y \leq c_0 \|v\|_X \quad \forall v \in X. \end{array} \right. \quad (13.13)$$

$$\left\{ \begin{array}{l} \text{There exist } \beta, \gamma \geq 0 \text{ such that} \\ \quad j(\eta, v_1) - j(\eta, v_2) \leq (\beta + \gamma \|\eta\|_X) \|v_1 - v_2\|_X \\ \quad \forall \eta, v_1, v_2 \in X. \end{array} \right. \quad (13.14)$$

$$m > \gamma. \quad (13.15)$$

We have the following result.

**Theorem 13.2.12** *Assume that (13.9)–(13.13) hold. Then, for each  $f \in Y$ , the quasivariational inequality (13.1) has a unique solution. Moreover, if (13.14)*



and (13.15) hold, then the solution satisfies the inequality

$$\|u\|_X \leq \frac{1}{m - \gamma} (\|Au_0\|_X + c_0\|f\|_Y + \gamma\|u_0\|_X + \beta) + \|u_0\|_X, \quad (13.16)$$

for any element  $u_0 \in K$ .

*Proof* Let  $f \in Y$ . We use assumption (13.13) to see that the functional  $v \mapsto (f, \pi v)_Y$  is linear and continuous on  $X$ . Therefore, using the Riesz representation theorem, there exists a unique element  $\tilde{f} \in X$  such that

$$(\tilde{f}, v)_X = (f, \pi v)_Y \quad \forall v \in X. \quad (13.17)$$

Using now Theorem 13.2.11 we deduce that there exists a unique element  $u$  such that

$$u \in K, \quad (Au, v - u)_X + j(u, v) - j(u, u) \geq (\tilde{f}, v - u)_X \quad \forall v \in K. \quad (13.18)$$

The existence and uniqueness part of Theorem 13.2.12 is now a direct consequence of (13.17) and (13.18).

Assume now that (13.14) and (13.15) hold and consider an arbitrary element  $u_0 \in K$ . Then, taking  $v = u_0$  in (13.1) we find that

$$(Au, u - u_0)_X \leq (f, \pi u - \pi u_0)_Y + j(u, u_0) - j(u, u)$$

which implies that

$$(Au - Au_0, u - u_0)_X \leq (Au_0, u_0 - u)_X + (f, \pi u - \pi u_0)_Y + j(u, u_0) - j(u, u).$$

We now use assumptions (13.10)(a), (13.13), and (13.14) to deduce that

$$\begin{aligned} m \|u - u_0\|_X^2 &\leq \|Au_0\|_X \|u - u_0\|_X \\ &\quad + c_0 \|f\|_Y \|u - u_0\|_X + (\beta + \gamma \|u\|_X) \|u - u_0\|_X. \end{aligned}$$

Next, we use the triangle inequality  $\|u\|_X \leq \|u - u_0\|_X + \|u_0\|_X$  to deduce that

$$(m - \gamma) \|u - u_0\|_X \leq \|Au_0\|_X + c_0 \|f\|_Y + \gamma \|u_0\|_X + \beta.$$

This inequality combined with the smallness assumption (13.15) implies the bound (13.16) and concludes the proof.  $\square$

### 13.2.3 A Convergence Result

Theorem 13.2.12 allows us to define the operator  $f \mapsto u(f)$  which associates to each element  $f \in Y$  the solution  $u = u(f) \in K$  of the quasivariational inequality (13.1). An important property of this operator is its weak–strong continuity, which represents a crucial ingredient in the study of the optimal control problem  $\mathcal{Q}$ . It holds under the following additional assumptions:

$$\left\{ \begin{array}{l} \text{For any sequences } \{\eta_k\} \subset X, \{u_k\} \subset X \text{ such that} \\ \eta_k \rightharpoonup \eta \in X, u_k \rightharpoonup u \in X \text{ one has} \\ \limsup_{k \rightarrow \infty} [j(\eta_k, v) - j(\eta_k, u_k)] \leq j(\eta, v) - j(\eta, u) \quad \forall v \in X. \end{array} \right. \quad (13.19)$$

$$\left\{ \begin{array}{l} \text{For any sequence } \{v_k\} \subset X \text{ such that} \\ v_k \rightharpoonup v \text{ in } X \text{ one has } \pi v_k \rightarrow \pi v \text{ in } Y. \end{array} \right. \quad (13.20)$$

Note that assumption (13.19) implies that for all  $\eta \in X$ ,  $j(\eta, \cdot) : X \rightarrow \mathbb{R}$  is lower semicontinuous. Indeed, this property can be easily deduced by taking  $\eta_k = \eta$  in (13.19). Moreover, assumption (13.20) shows that the operator  $\pi : X \rightarrow Y$  is completely continuous.

Our main result in this subsection is the following.

**Theorem 13.2.13** *Assume that (13.9)–(13.15), (13.19), and (13.20) hold. Then,*

$$f_n \rightharpoonup f \text{ in } Y \implies u(f_n) \rightarrow u(f) \text{ in } X, \quad \text{as } n \rightarrow \infty. \quad (13.21)$$

*Proof* The proof of Theorem 13.2.13 will be carried out in several steps that we present in what follows.

- (i) *Weak convergence of a subsequence.* Assume that  $\{f_n\}$  is a sequence of elements in  $Y$  such that

$$f_n \rightharpoonup f \text{ in } Y \text{ as } n \rightarrow \infty \quad (13.22)$$

and, for simplicity, denote  $u(f_n) = u_n$  and  $u(f) = u$ . Then, it follows from (13.22) that  $\{f_n\}$  is a bounded sequence in  $Y$  and, therefore, inequality (13.16) implies that  $\{u_n\}$  is a bounded sequence in  $X$ . Using now Theorem 13.2.3 we deduce that there exists an element  $\tilde{u} \in X$  and a subsequence of  $\{u_n\}$ , again denoted  $\{u_n\}$ , such that

$$u_n \rightharpoonup \tilde{u} \text{ in } X \text{ as } n \rightarrow \infty. \quad (13.23)$$

On the other hand, we recall that  $K$  is a closed convex subset of the space  $X$  and  $\{u_n\} \subset K$ . Then, Theorem 13.2.2 and (13.23) imply that

$$\tilde{u} \in K. \quad (13.24)$$

(ii) *Weak convergence of the whole subsequence.* Let  $n \in \mathbb{N}$ . We write (13.1) for  $f = f_n$  to obtain

$$(Au_n, u_n - v)_X \leq j(u_n, v) - j(u_n, u_n) + (f_n, \pi u_n - \pi v)_Y \quad \forall v \in K, \quad (13.25)$$

then we take  $v = \tilde{u} \in K$  to find that

$$(Au_n, u_n - \tilde{u})_X \leq j(u_n, \tilde{u}) - j(u_n, u_n) + (f_n, \pi u_n - \pi \tilde{u})_Y.$$

We now pass to the upper limit and use the convergences (13.22), (13.23) and assumptions (13.19), (13.20). As a result we deduce that

$$\limsup_{n \rightarrow \infty} (Au_n, u_n - \tilde{u})_X \leq 0.$$

Therefore, using assumption (13.10), Corollary 13.2.7, and Definition 13.2.5(iv) we deduce that

$$\liminf_{n \rightarrow \infty} (Au_n, u_n - v)_X \geq (A\tilde{u}, \tilde{u} - v)_X \quad \forall v \in X. \quad (13.26)$$

On the other hand, passing to the upper limit in inequality (13.25) and using the convergences (13.22), (13.23) and assumptions (13.19), (13.20) yields

$$\limsup_{n \rightarrow \infty} (Au_n, u_n - \tilde{u})_X \leq j(\tilde{u}, v) - j(\tilde{u}, \tilde{u}) + (f, \pi \tilde{u} - \pi v)_Y \quad \forall v \in K. \quad (13.27)$$

We now combine the inequalities (13.26) and (13.27) to see that

$$(A\tilde{u}, v - \tilde{u})_X + j(\tilde{u}, v) - j(\tilde{u}, \tilde{u}) \geq (f, \pi v - \pi \tilde{u})_Y \quad \forall v \in K. \quad (13.28)$$

Next, it follows from (13.24) and (13.28) that  $\tilde{u}$  is a solution of inequality (13.1) and, by the uniqueness of the solution of this inequality, guaranteed by Theorem 13.2.12, we obtain that

$$\tilde{u} = u. \quad (13.29)$$

A careful analysis, based on the arguments above, reveals that  $u$  is the weak limit of any weakly convergent subsequence of the sequence  $\{u_n\}$ . Therefore, using Theorem 13.2.4 we deduce that the whole sequence  $\{u_n\}$  converges weakly in  $X$  to  $u$  as  $n \rightarrow \infty$ , i.e.,

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{as } n \rightarrow \infty. \quad (13.30)$$

(iii) *Strong convergence.* Let  $n \in \mathbb{N}$  be given. We take  $v = u$  in inequality (13.25) to see that

$$(Au_n, u_n - u)_X \leq j(u_n, u) - j(u_n, u_n) + (f_n, \pi u_n - \pi u)_Y. \quad (13.31)$$

Next, we use (13.31) and assumption (13.10)(a) to find that

$$\begin{aligned} m \|u_n - u\|_X^2 &\leq (Au_n - Au, u_n - u)_X \\ &= (Au_n, u_n - u)_X - (Au, u_n - u)_X \\ &\leq j(u_n, u) - j(u_n, u_n) + (f_n, \pi u_n - \pi u)_Y - (Au, u_n - u)_X. \end{aligned}$$

We now pass to the upper limit in this inequality and use the convergences (13.22), (13.30) and assumptions (13.19), (13.20) to deduce that

$$\|u_n - u\|_X \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This convergence concludes the proof since, recall,  $u_n = u(f_n)$  and  $u = u(f)$ .  $\square$

### 13.3 Optimal Control of Quasivariational Inequalities

We now move to the study of the optimal control problem  $\mathcal{Q}$ . We start with an existence result for the optimal pairs, Theorem 13.3.1. We proceed with a convergence result, Theorem 13.3.5. Finally, we consider a relevant particular case for which this convergence result holds, Theorem 13.3.8.

#### 13.3.1 Existence of Optimal Pairs

In the study of Problem  $\mathcal{Q}$  we assume that

$$\mathcal{L}(u, f) = g(u) + h(f) \quad \forall u \in X, f \in Y, \quad (13.32)$$

where  $g$  and  $h$  are functions which satisfy the following conditions:

$$\left\{ \begin{array}{l} g : X \rightarrow \mathbb{R} \text{ is continuous, positive, and bounded, i.e.,} \\ \text{(a) } v_n \rightarrow v \text{ in } X \implies g(v_n) \rightarrow g(v). \\ \text{(b) } g(v) \geq 0 \quad \forall v \in X. \\ \text{(c) } g \text{ maps bounded sets in } X \text{ into bounded sets in } \mathbb{R}. \end{array} \right. \quad (13.33)$$

$$\left\{ \begin{array}{l} h : Y \rightarrow \mathbb{R} \text{ is weakly lower semicontinuous and coercive, i.e.,} \\ \text{(a) } f_n \rightharpoonup f \text{ in } Y \implies \liminf_{n \rightarrow \infty} h(f_n) \geq h(f). \\ \text{(b) } \|f_n\|_Y \rightarrow \infty \implies h(f_n) \rightarrow \infty. \end{array} \right. \quad (13.34)$$

Our first result in this section is the following.

**Theorem 13.3.1** *Assume that (13.9)–(13.15), (13.19), (13.20), and (13.32)–(13.34) hold. Then, there exists at least one solution  $(u^*, f^*) \in \mathcal{V}_{ad}$  of Problem  $\mathcal{Q}$ .*

*Proof* Let

$$\theta = \inf_{(u, f) \in \mathcal{V}_{ad}} \mathcal{L}(u, f) \in \mathbb{R} \quad (13.35)$$

and let  $\{(u_n, f_n)\} \subset \mathcal{V}_{ad}$  be a minimizing sequence for the functional  $\mathcal{L}$ , i.e.,

$$\lim_{n \rightarrow \infty} \mathcal{L}(u_n, f_n) = \theta. \quad (13.36)$$

We claim that the sequence  $\{f_n\}$  is bounded in  $Y$ . Arguing by contradiction, assume that  $\{f_n\}$  is not bounded in  $Y$ . Then, passing to a subsequence still denoted  $\{f_n\}$ , we have

$$\|f_n\|_Y \rightarrow +\infty \text{ as } n \rightarrow +\infty. \quad (13.37)$$

We now use equality (13.32) and assumption (13.33)(b) to see that

$$\mathcal{L}(u_n, f_n) \geq h(f_n).$$

Therefore, passing to the limit as  $n \rightarrow +\infty$  and using (13.37) combined with assumption (13.34)(b) we deduce that

$$\lim_{n \rightarrow +\infty} \mathcal{L}(u_n, f_n) = +\infty. \quad (13.38)$$

Equalities (13.36) and (13.38) imply that  $\theta = +\infty$  which is in contradiction with (13.35).

We conclude from above that the sequence  $\{f_n\}$  is bounded in  $Y$ . Therefore, using Theorem 13.2.3 we deduce that there exists  $f^* \in Y$  such that, passing to a subsequence still denoted  $\{f_n\}$ , we have

$$f_n \rightharpoonup f^* \text{ in } Y \text{ as } n \rightarrow +\infty. \quad (13.39)$$

Let  $u^*$  be the solution of the quasivariational inequality (13.1) for  $f = f^*$ , i.e.,  $u^* = u(f^*)$ . Recall that the existence and uniqueness of this solution is guaranteed

by Theorem 13.2.12. Then, by the definition (13.2) of the set  $\mathcal{V}_{ad}$  we have

$$(u^*, f^*) \in \mathcal{V}_{ad}. \quad (13.40)$$

Moreover, using (13.39) and (13.21) it follows that

$$u_n \rightarrow u^* \quad \text{in } X \quad \text{as } n \rightarrow +\infty. \quad (13.41)$$

We now use the convergences (13.39), (13.41) and the weakly lower semicontinuity of the functional  $\mathcal{L}$ , guaranteed by assumptions (13.33)(a) and (13.34)(a), to deduce that

$$\liminf_{n \rightarrow +\infty} \mathcal{L}(u_n, f_n) \geq \mathcal{L}(u^*, f^*). \quad (13.42)$$

It follows from (13.36) and (13.42) that

$$\theta \geq \mathcal{L}(u^*, f^*). \quad (13.43)$$

In addition, (13.40) and (13.35) yield

$$\theta \leq \mathcal{L}(u^*, f^*). \quad (13.44)$$

We now combine inequalities (13.43) and (13.44) to see that (13.3) holds, which concludes the proof.  $\square$

*Remark 13.3.2* Assume now that  $U \subset Y$  is a nonempty weakly closed subset, i.e., it satisfies the following property:

$$\text{for any sequence } \{f_n\} \subset U \text{ such that } f_n \rightharpoonup f \in Y \quad \text{one has } f \in U. \quad (13.45)$$

Then, careful analysis of the previous proof reveals the fact that the statement of Theorem 13.3.1 still remains valid if we replace the definition (13.2) of admissible pairs for inequality (13.1) with the following one:

$$\mathcal{V}_{ad} = \{ (u, f) \in K \times U \text{ such that (13.1) holds} \}. \quad (13.46)$$

Considering the set (13.46) instead of (13.2) leads to a version of Theorem 13.3.1 which could be useful in various applications, when the control  $f$  is assumed to satisfy some constraints.

### 13.3.2 Convergence of Optimal Pairs

In this subsection we focus on the dependence of the solution of the optimal control  $\mathcal{Q}$  with respect to the set  $K$ , the operator  $A$ , and the function  $j$ . To this end, we assume in what follows that the hypothesis of Theorem 13.3.1 holds. Moreover, for each  $n \in \mathbb{N}$  we consider a perturbation  $K_n$ ,  $A_n$ , and  $j_n$  of  $K$ ,  $A$ , and  $j$ , respectively, which satisfy the following conditions:

$$K_n \text{ is a nonempty, closed, convex subset of } X. \quad (13.47)$$

$$\left\{ \begin{array}{l} A_n \text{ is a strongly monotone Lipschitz continuous operator,} \\ \text{i.e., it satisfies condition (13.10) with } m_n > 0 \text{ and } M_n > 0. \end{array} \right. \quad (13.48)$$

$$\left\{ \begin{array}{l} \text{(a) For all } \eta \in X, j_n(\eta, \cdot) : X \rightarrow \mathbb{R} \text{ is convex.} \\ \text{(b) There exists } \alpha_n \geq 0 \text{ such that} \\ \quad j_n(\eta_1, v_2) - j_n(\eta_1, v_1) + j_n(\eta_2, v_1) - j_n(\eta_2, v_2) \\ \quad \leq \alpha_n \|\eta_1 - \eta_2\|_X \|v_1 - v_2\|_X \quad \forall \eta_1, \eta_2, v_1, v_2 \in X. \\ \text{(c) There exist } \beta_n, \gamma_n \geq 0 \text{ such that} \\ \quad j_n(\eta, v_1) - j_n(\eta, v_2) \leq (\beta_n + \gamma_n \|\eta\|_X) \|v_1 - v_2\|_X \\ \quad \forall \eta, v_1, v_2 \in X. \\ \text{(d) For any sequences } \{\eta_k\} \subset X, \{u_k\} \subset X \text{ such that} \\ \quad \eta_k \rightharpoonup \eta \in X, u_k \rightharpoonup u \in X \quad \text{one has} \\ \quad \limsup_k [j_n(\eta_k, v) - j_n(\eta_k, u_k)] \leq j_n(\eta, v) - j_n(\eta, u) \quad \forall v \in X. \end{array} \right. \quad (13.49)$$

$$m_n > \alpha_n. \quad (13.50)$$

$$m_n > \gamma_n. \quad (13.51)$$

We consider the following perturbation of Problem  $\mathcal{P}$ .

**Problem  $\mathcal{P}_n$ .** Given  $f_n \in Y$ , find  $u_n$  such that

$$\begin{aligned} u_n \in K_n, \quad (A_n u_n, v - u_n)_X + j_n(u_n, v) - j_n(u_n, u_n) \\ \geq (f_n, \pi v - \pi u_n)_Y \quad \forall v \in K_n. \end{aligned} \quad (13.52)$$

It follows from Theorem 13.2.12 that for each  $f_n \in Y$  there exists a unique solution  $u_n = u_n(f_n)$  to the quasivariational inequality (13.52). Moreover, the

solution satisfies

$$\|u_n\|_X \leq \frac{1}{m_n - \gamma_n} (\|A_n u_{0n}\|_X + c_0 \|f_n\|_Y + \gamma_n \|u_{0n}\|_X + \beta_n) + \|u_{0n}\|_X, \quad (13.53)$$

where  $u_{0n}$  denotes an arbitrary element of  $K_n$ . We define the set of admissible pairs for inequality (13.52) by

$$\mathcal{V}_{ad}^n = \{ (u_n, f_n) \in K_n \times Y \text{ such that (13.52) holds} \}. \quad (13.54)$$

Then, the optimal control problem associated with Problem  $\mathcal{P}_n$  is the following.

**Problem  $\mathcal{Q}_n$ .** Find  $(u_n^*, f_n^*) \in \mathcal{V}_{ad}^n$  such that

$$\mathcal{L}(u_n^*, f_n^*) = \min_{(u_n, f_n) \in \mathcal{V}_{ad}^n} \mathcal{L}(u_n, f_n). \quad (13.55)$$

Using Theorem 13.3.1 it follows that for each  $n \in \mathbb{N}$  there exists at least one solution  $(u_n^*, f_n^*) \in \mathcal{V}_{ad}^n$  of Problem  $\mathcal{Q}_n$ . We now consider the following assumptions:

$$f_n \rightharpoonup f \text{ in } Y \implies u_n(f_n) \rightarrow u(f) \text{ in } X, \text{ as } n \rightarrow \infty. \quad (13.56)$$

$$\left\{ \begin{array}{l} \text{There exists } f^0 \in Y \text{ such that} \\ \text{the sequence } \{u_n(f^0)\} \text{ is bounded in } X. \end{array} \right. \quad (13.57)$$

Concerning assumptions (13.56) and (13.57) we have the following remarks.

*Remark 13.3.3* Assumptions (13.56) and (13.57) are not formulated in terms of the data  $K_n$ ,  $A_n$ , and  $j_n$ . They are formulated in terms of the solutions  $u_n$  and  $u$  which are unknown and, therefore, they represent implicit assumptions. We consider these assumptions for their generality. In the next section we shall provide explicit assumptions on  $K_n$ ,  $A_n$ , and  $j_n$  which guarantee that conditions (13.56) and (13.57) hold. Considering such explicit assumptions will lead us to introduce a relevant particular case in which Theorem 13.3.5 holds.

*Remark 13.3.4* Condition (13.56) represents a continuous dependence condition of the solution of (13.1) with respect to the set  $K$ , the operator  $A$ , the function  $j$ , and the element  $f \in Y$ .

The second result in this section is a convergence result for the set of solution of Problem  $\mathcal{Q}$ . Its statement is as follows.

**Theorem 13.3.5** *Assume that (13.9)–(13.15), (13.19), (13.20), and (13.32)–(13.34) hold and, for any  $n \in \mathbb{N}$ , assume that (13.47)–(13.51) hold, too. Moreover, assume that conditions (13.56)–(13.57) are satisfied and let  $\{(u_n^*, f_n^*)\}$  be a sequence of solutions of Problem  $\mathcal{Q}_n$ . Then, there exists a subsequence of the sequence*



$\{(u_n^*, f_n^*)\}$ , again denoted  $\{(u_n^*, f_n^*)\}$ , and an element  $(u^*, f^*) \in X \times Y$  such that

$$f_n^* \rightarrow f^* \quad \text{in } Y \quad \text{as } n \rightarrow \infty, \quad (13.58)$$

$$u_n^* \rightarrow u^* \quad \text{in } X \quad \text{as } n \rightarrow \infty, \quad (13.59)$$

$$(u^*, f^*) \quad \text{is a solution of Problem } \mathcal{Q}. \quad (13.60)$$

*Proof* We claim that the sequence  $\{f_n^*\}$  is bounded in  $Y$ . Arguing by contradiction, assume that  $\{f_n^*\}$  is not bounded in  $Y$ . Then, passing to a subsequence still denoted  $\{f_n^*\}$ , we have

$$\|f_n^*\|_Y \rightarrow +\infty \quad \text{as } n \rightarrow +\infty. \quad (13.61)$$

We use equality (13.32) and assumption (13.33)(b) to see that

$$\mathcal{L}(u_n^*, f_n^*) \geq h(f_n^*).$$

Therefore, passing to the limit as  $n \rightarrow \infty$  in this inequality and using (13.61) combined with assumption (13.34)(b) we deduce that

$$\lim_{n \rightarrow \infty} \mathcal{L}(u_n^*, f_n^*) = +\infty. \quad (13.62)$$

On the other hand, since  $(u_n^*, f_n^*)$  represents a solution to Problem  $\mathcal{Q}_n$ , for each  $n \in \mathbb{N}$  we have

$$\mathcal{L}(u_n^*, f_n^*) \leq \mathcal{L}(u_n, f_n) \quad \forall (u_n, f_n) \in \mathcal{V}_{ad}^n. \quad (13.63)$$

We now use assumption (13.57) and denote by  $u_n^0$  the solution of Problem  $\mathcal{P}_n$  for  $f_n = f^0$ , i.e.,  $u_n^0 = u_n(f^0)$ . Then  $(u_n^0, f^0) \in \mathcal{V}_{ad}^n$  and, therefore, (13.63) and (13.32) imply that

$$\mathcal{L}(u_n^*, f_n^*) \leq g(u_n^0) + h(f^0). \quad (13.64)$$

Then, since (13.57) guarantees that  $\{u_n^0\}$  is a bounded sequence in  $X$ , assumption (13.33)(c) on the function  $g$  implies that there exists  $D > 0$  which does not depend on  $n$  such that

$$g(u_n^0) + h(f^0) \leq D \quad \forall n \in \mathbb{N}. \quad (13.65)$$

Relations (13.62), (13.64), and (13.65) lead to a contradiction, which concludes the claim.

Next, since the sequence  $\{f_n^*\}$  is bounded in  $Y$  we can find a subsequence again denoted  $\{f_n^*\}$  and an element  $f^* \in Y$  such that (13.58) holds. Denote by  $u^*$  the solution of Problem  $\mathcal{P}$  for  $f = f^*$ , i.e.,  $u^* = u(f^*)$ . Then, we have

$$(u^*, f^*) \in \mathcal{V}_{ad} \quad (13.66)$$

and, moreover, assumption (13.56) implies that (13.59) holds, too.

We now prove that  $(u^*, f^*)$  is a solution to the optimal control problem  $\mathcal{Q}$ . To this end we use the convergences (13.58), (13.59) and the weakly lower semicontinuity of the functional  $\mathcal{L}$ , guaranteed by (13.32)–(13.34), to see that

$$\mathcal{L}(u^*, f^*) \leq \liminf_{n \rightarrow \infty} \mathcal{L}(u_n^*, f_n^*). \quad (13.67)$$

Next, we fix a solution  $(u_0^*, f_0^*)$  of Problem  $\mathcal{Q}$  and, in addition, for each  $n \in \mathbb{N}$  we denote by  $\tilde{u}_n^0$  the solution of Problem  $\mathcal{P}_n$  for  $f_n = f_0^*$ . It follows from here that  $(\tilde{u}_n^0, f_0^*) \in \mathcal{V}_{ad}^n$  and, by the optimality of the pair  $(u_n^*, f_n^*)$ , we have that

$$\mathcal{L}(u_n^*, f_n^*) \leq \mathcal{L}(\tilde{u}_n^0, f_0^*) \quad \forall n \in \mathbb{N}.$$

We pass to the upper limit in this inequality to see that

$$\limsup_{n \rightarrow \infty} \mathcal{L}(u_n^*, f_n^*) \leq \limsup_{n \rightarrow \infty} \mathcal{L}(\tilde{u}_n^0, f_0^*). \quad (13.68)$$

Now, remember that  $u_0^*$  is the solution of the inequality (13.1) for  $f = f_0^*$  and  $\tilde{u}_n^0$  is the solution of the inequality (13.52) for  $f_n = f_0^*$ , i.e.,  $\tilde{u}_n^0 = u_n(f_0^*)$  and  $\tilde{u}_n = u_n(f_0^*)$ . Therefore, assumption (13.56) implies that

$$\tilde{u}_n^0 \rightarrow u_0^* \quad \text{in } X \quad \text{as } n \rightarrow \infty$$

and, using the continuity of the functional  $u \mapsto \mathcal{L}(u, f_0^*) : X \rightarrow \mathbb{R}$  yields

$$\lim_{n \rightarrow \infty} \mathcal{L}(\tilde{u}_n^0, f_0^*) = \mathcal{L}(u_0^*, f_0^*). \quad (13.69)$$

We now use (13.67)–(13.69) to see that

$$\mathcal{L}(u^*, f^*) \leq \mathcal{L}(u_0^*, f_0^*). \quad (13.70)$$

On the other hand, since  $(u_0^*, f_0^*)$  is a solution of Problem  $\mathcal{Q}$ , we have

$$\mathcal{L}(u_0^*, f_0^*) = \min_{(u, f) \in \mathcal{V}_{ad}} \mathcal{L}(u, f), \quad (13.71)$$

and, therefore, inclusion (13.66) implies that

$$\mathcal{L}(u_0^*, f_0^*) \leq \mathcal{L}(u^*, f^*). \quad (13.72)$$

We now combine the inequalities (13.70) and (13.72) to see that

$$\mathcal{L}(u^*, f^*) = \mathcal{L}(u_0^*, f_0^*). \quad (13.73)$$

Finally, relations (13.66), (13.73), and (13.71) imply that (13.60) holds, which concludes the proof.  $\square$

We end this subsection with the following remarks.

*Remark 13.3.6* If Problem  $\mathcal{Q}$  has a unique solution  $(u^*, f^*)$  then, under the assumption of Theorem 13.3.5 the convergences (13.58) and (13.59) are valid for the whole sequence  $\{(u_n^*, f_n^*)\}$ . Indeed, a careful analysis of the proof of Theorem 13.2 reveals that the sequence  $\{f_n^*\}$  is bounded in  $Y$  and, moreover, each weakly convergent subsequence of  $\{f_n^*\}$  converges weakly to  $f^*$ . We now use Theorem 13.2.4 to deduce that the whole sequence satisfies (13.58). Finally, using Theorem 13.2.13 it follows that (13.59) holds, too.

*Remark 13.3.7* The statement of Theorem 13.3.5 still remains valid if we replace the definition (13.2) with (13.46) and the definition (13.54) with

$$\mathcal{V}_{ad}^n = \{(u_n, f_n) \in K_n \times U \text{ such that (13.52) holds}\}, \quad (13.74)$$

$U$  being a nonempty weakly closed subset of  $Y$ . The proof of this statement is based on the property (13.45) of the set  $U$ .

### 13.3.3 A Relevant Particular Case

Our aim in this subsection is to present explicit conditions on the family of sets  $K_n$ , operators  $A_n$ , and functionals  $j_n$  which guarantee that assumptions (13.56) and (13.57) hold. We conclude from here that, under these conditions, the abstract result in Theorem 13.3.5 holds.

Everywhere in this subsection we assume that (13.9)–(13.15), (13.19), and (13.20) hold and, for each  $f \in Y$ , we denote by  $u = u(f)$  the solution of inequality (13.1), guaranteed by Theorem 13.2.12. Moreover, for each  $n \in \mathbb{N}$  we consider the set  $K_n \subset X$ , the operator  $A_n : X \rightarrow X$ , and the functional  $j_n : X \times X \rightarrow \mathbb{R}$  such that the followings hold:

$$K_n = c_n K \quad \text{with} \quad c_n > 0. \quad (13.75)$$

$$\left\{ \begin{array}{l} \text{(a) } A_n = A + T_n. \\ \text{(b) } A : X \rightarrow X \text{ satisfies condition (13.10) with} \\ \quad m > 0 \text{ and } M > 0. \\ \text{(c) } T_n : X \rightarrow X \text{ is a monotone Lipschitz continuous operator.} \end{array} \right. \quad (13.76)$$

$$\left\{ \begin{array}{l} j_n \text{ satisfies condition (13.49) with } \alpha_n \geq 0, \beta_n \geq 0, \gamma_n \geq 0 \\ \text{such that } m > \alpha_n, m > \gamma_n. \end{array} \right. \quad (13.77)$$

With this choice we consider Problem  $\mathcal{P}_n$ . It is easy to see that for each  $n \in \mathbb{N}$  the set  $K_n \subset X$  satisfies condition (13.47). Moreover, the operator  $A_n$  satisfies condition (13.48) with  $m_n = m$  and  $M_n = M + L_{T_n}$ ,  $L_{T_n}$  being the Lipschitz constant of the operator  $T_n$ . We now use assumption (13.77) to see that conditions (13.49)–(13.51) are also satisfied. Therefore, using Theorem 13.2.12 we deduce that for each  $f_n \in Y$  there exists a unique solution  $u_n = u_n(f_n)$  to the quasivariational inequality (13.52).

On the other hand, if (13.32)–(13.34) hold, then Theorem 13.3.1 guarantees the existence of at least one solution  $(u^*, f^*)$  of Problem  $\mathcal{Q}$  and, for each  $n \in \mathbb{N}$ , the existence of at least one solution  $(u_n^*, f_n^*)$  to Problem  $\mathcal{Q}_n$ .

We now consider the following additional assumptions:

$$\lim_{n \rightarrow \infty} c_n = 1. \quad (13.78)$$

$$j(u, \lambda v) = \lambda j(u, v) \quad \forall \lambda \geq 0, u, v \in X. \quad (13.79)$$

$$\left\{ \begin{array}{l} \text{For any } n \in \mathbb{N} \text{ there exists } F_n \geq 0 \text{ and } \delta_n \geq 0 \text{ such that} \\ \text{(a) } \|T_n v\|_X \leq F_n (\|v\|_X + \delta_n) \quad \forall v \in X. \\ \text{(b) } \lim_{n \rightarrow \infty} F_n = 0. \\ \text{(c) The sequence } \{\delta_n\} \subset \mathbb{R} \text{ is bounded.} \end{array} \right. \quad (13.80)$$

$$\left\{ \begin{array}{l} \text{For any } n \in \mathbb{N} \text{ there exists } G_n \geq 0 \text{ and } H_n \geq 0 \text{ such that} \\ \text{(a) } j_n(v_1, v_2) - j_n(v_1, v_1) + j(v_2, v_1) - j(v_2, v_2) \\ \quad \leq G_n + H_n \|v_1 - v_2\|_X + \alpha \|v_1 - v_2\|_X^2 \\ \quad \quad \forall v_1, v_2 \in X. \\ \text{(b) } \lim_{n \rightarrow \infty} G_n = \lim_{n \rightarrow \infty} H_n = 0. \end{array} \right. \quad (13.81)$$

Moreover, we reinforce assumption (13.77) by assuming that there exist two constants  $\beta_0$  and  $\gamma_0$  such that

$$\beta_n \leq \beta_0, \quad \gamma_n \leq \gamma_0 < m, \quad \forall n \in \mathbb{N}. \quad (13.82)$$

We have the following result.

**Theorem 13.3.8** *Assume that (13.9)–(13.15), (13.19), (13.20), and (13.32)–(13.34) hold and, for any  $n \in \mathbb{N}$ , assume that (13.75)–(13.77) hold, too. Assume moreover that conditions (13.78)–(13.82) are satisfied and let  $\{(u_n^*, f_n^*)\}$  be a sequence of solutions of Problem  $\mathcal{Q}_n$ . Then, there exists a subsequence of the sequence  $\{(u_n^*, f_n^*)\}$ , again denoted  $\{(u_n^*, f_n^*)\}$ , and an element  $(u^*, f^*) \in X \times Y$  such that (13.58)–(13.60) hold.*

The proof is carried out in several steps, based on the abstract result provided by Theorem 13.3.5. The first step of the proof is the following.

**Lemma 13.3.9** *Under the assumption of Theorem 13.3.8, if the sequence  $\{f_n\}$  is bounded in  $Y$ , then the sequence  $\{u_n(f_n)\}$  is bounded in  $X$ .*

*Proof* Let  $u_0$  be a given element of  $K$  and let  $n \in \mathbb{N}$ . Condition (13.75) guarantees that  $c_n u_0 \in K_n$  and, therefore, using inequality (13.53) with  $u_{0n} = c_n u_0$  yields

$$\|u_n\|_X \leq \frac{1}{m_n - \gamma_n} (\|A_n(c_n u_0)\|_X + c_0 \|f_n\|_Y + \gamma_n \|c_n u_0\|_X + \beta_n) + \|c_n u_0\|_X.$$

We now use assumption (13.82) and equality  $m_n = m$  which, recall, follows from assumption (13.76). In this way we deduce that

$$\|u_n\|_X \leq \frac{1}{m - \gamma_0} (\|A_n(c_n u_0)\|_X + c_0 \|f_n\|_Y + \gamma_0 c_n \|u_0\|_X + \beta_0) + c_n \|u_0\|_X. \quad (13.83)$$

Recall that  $A_n(c_n u_0) = A(c_n u_0) + T_n(c_n u_0)$  and, therefore,

$$\|A_n(c_n u_0)\|_X \leq \|A(c_n u_0)\|_X + \|T_n(c_n u_0)\|_X. \quad (13.84)$$

We now write

$$\|A(c_n u_0)\|_X \leq \|A(c_n u_0) - Au_0\|_X + \|Au_0\|_X,$$

then we use assumption (13.10)(b) to deduce that

$$\|A(c_n u_0)\|_X \leq (M|c_n - 1| \|u_0\|_X + \|Au_0\|_X). \quad (13.85)$$

Moreover, using (13.80) we have that

$$\|T_n(c_n u_0)\|_X \leq F_n(c_n \|u_0\|_X + \delta_n). \quad (13.86)$$

Next, we combine inequalities (13.83)–(13.86) to find that

$$\begin{aligned} & \|u_n\|_X \tag{13.87} \\ & \leq \frac{1}{m - \gamma_0} (M|c_n - 1| \|u_0\|_X + \|Au_0\|_X + F_n(c_n \|u_0\|_X + \delta_n) \\ & \quad + \frac{1}{m - \gamma_0} (c_0 \|f_n\|_Y + \gamma_0 c_n \|u_0\|_X + \beta_0) + c_n \|u_0\|_X. \end{aligned}$$

Lemma 13.3.9 is now a direct consequence of inequality (13.87) and assumptions (13.78), (13.80)(b).  $\square$

We proceed with the following result.

**Lemma 13.3.10** *Under the assumption of Theorem 13.3.8, condition (13.56) holds.*

*Proof* Let  $\{f_n\} \subset Y$ ,  $f \in Y$  such that

$$f_n \rightharpoonup f \quad \text{in } Y \quad \text{as } n \rightarrow \infty. \tag{13.88}$$

Let  $n \in \mathbb{N}$ . Besides Problems  $\mathcal{P}$  and  $\mathcal{P}_n$  we consider the intermediate problems of finding two elements  $\bar{u}_n$  and  $\tilde{u}_n$  such that

$$\begin{aligned} \bar{u}_n \in K, \quad & (A\bar{u}_n, v - \bar{u}_n)_X + j(\bar{u}_n, v) - j(\bar{u}_n, \bar{u}_n) \tag{13.89} \\ & \geq (f_n, \pi v - \pi \bar{u}_n)_Y \quad \forall v \in K. \end{aligned}$$

$$\begin{aligned} \tilde{u}_n \in K_n, \quad & (A\tilde{u}_n, v_n - \tilde{u}_n)_X + j(\tilde{u}_n, v_n) - j(\tilde{u}_n, \tilde{u}_n) \tag{13.90} \\ & \geq (f_n, \pi v_n - \pi \tilde{u}_n)_Y \quad \forall v_n \in K_n. \end{aligned}$$

Note that Theorem 13.2.12 guarantees the existence of a unique solution  $\bar{u}_n$  and  $\tilde{u}_n$  to the quasivariational inequalities (13.89) and (13.90), respectively. Our aim in what follows is to establish estimates for the norms  $\|u_n - \tilde{u}_n\|_X$  and  $\|\tilde{u}_n - \bar{u}_n\|_X$ .

Let  $n \in \mathbb{N}$ . We take  $v_n = u_n$  in (13.90),  $v_n = \tilde{u}_n$  in (13.52), then we add the resulting inequalities to obtain that

$$(A_n u_n - A\tilde{u}_n, u_n - \tilde{u}_n)_X \leq j_n(u_n, \tilde{u}_n) - j_n(u_n, u_n) + j(\tilde{u}_n, u_n) - j(\tilde{u}_n, \tilde{u}_n).$$

We use now assumption (13.76)(a) to see that  $A_n u_n = Au_n + T_n u_n$  and, therefore, we deduce that

$$\begin{aligned} & (Au_n - A\tilde{u}_n, u_n - \tilde{u}_n)_X \leq (T_n u_n, \tilde{u}_n - u_n)_X \\ & \quad + j_n(u_n, \tilde{u}_n) - j_n(u_n, u_n) + j(\tilde{u}_n, u_n) - j(\tilde{u}_n, \tilde{u}_n). \end{aligned}$$

Next, we use conditions (13.10)(a), (13.80)(a), and (13.81)(a) to find that

$$\begin{aligned} m \|u_n - \tilde{u}_n\|_X^2 &\leq F_n(\|u_n\|_X + \delta_n) \|u_n - \tilde{u}_n\|_X \\ &+ G_n + H_n \|u_n - \tilde{u}_n\|_X + \alpha \|u_n - \tilde{u}_n\|_X^2. \end{aligned} \quad (13.91)$$

On the other hand, assumption (13.88) and Lemma 13.3.9 imply that there exists  $E > 0$  which does not depend on  $n$  such that  $\|u_n\|_X \leq E$ . Therefore, since  $m > \alpha$ , inequality (13.91) yields

$$\|u_n - \tilde{u}_n\|_X^2 \leq \left( \frac{H_n}{m - \alpha} + \frac{(E + \delta_n)F_n}{m - \alpha} \right) \|u_n - \tilde{u}_n\|_X + \frac{G_n}{m - \alpha}.$$

Next, the elementary inequality

$$x^2 \leq ax + b \implies x \leq a + \sqrt{b} \quad \forall x, a, b \geq 0$$

combined with assumptions (13.80)(b),(c) and (13.81)(b) implies that

$$\|u_n - \tilde{u}_n\|_X \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (13.92)$$

On the other hand, condition (13.75) allows us to test in (13.90) with  $v_n = c_n \bar{u}_n \in K_n$ . As a result we deduce that

$$(A\tilde{u}_n, c_n \bar{u}_n - \tilde{u}_n)_X + j(\tilde{u}_n, c_n \bar{u}_n) - j(\tilde{u}_n, \tilde{u}_n) \geq (f_n, c_n \pi \bar{u}_n - \pi \tilde{u}_n)_Y. \quad (13.93)$$

We now use condition (13.75), again, to test in (13.89) with  $v = \frac{1}{c_n} \tilde{u}_n \in K$ . Then, we multiply the resulting inequality with  $c_n > 0$  and use assumption (13.79) on  $j$  to find that

$$(A\bar{u}_n, \tilde{u}_n - c_n \bar{u}_n)_X + j(\bar{u}_n, \tilde{u}_n) - j(\bar{u}_n, c_n \bar{u}_n) \geq (f_n, \pi \tilde{u}_n - c_n \pi \bar{u}_n)_Y. \quad (13.94)$$

We now add inequalities (13.93) and (13.94) to deduce that

$$\begin{aligned} &(A\tilde{u}_n - A\bar{u}_n, \tilde{u}_n - c_n \bar{u}_n)_X \\ &\leq j(\tilde{u}_n, c_n \bar{u}_n) - j(\tilde{u}_n, \tilde{u}_n) + j(\bar{u}_n, \tilde{u}_n) - j(\bar{u}_n, c_n \bar{u}_n), \end{aligned}$$

then we use assumption (13.11)(b) to obtain that

$$(A\tilde{u}_n - A\bar{u}_n, \tilde{u}_n - c_n \bar{u}_n)_X \leq \alpha \|\tilde{u}_n - \bar{u}_n\|_X \|\tilde{u}_n - c_n \bar{u}_n\|_X. \quad (13.95)$$

Next, we write

$$\tilde{u}_n - c_n \bar{u}_n = \tilde{u}_n - \bar{u}_n + (1 - c_n) \bar{u}_n,$$

then we substitute this equality in (13.95) and use condition (13.10)(a) to find that

$$\begin{aligned} m \|\tilde{u}_n - \bar{u}_n\|_X^2 &\leq (A\tilde{u}_n - A\bar{u}_n, (c_n - 1)\bar{u}_n)_X \\ &+ \alpha \|\tilde{u}_n - \bar{u}_n\|_X^2 + \alpha |1 - c_n| \|\tilde{u}_n - \bar{u}_n\|_X \|\bar{u}_n\|_X. \end{aligned}$$

We now use assumption (13.10)(b) and the smallness assumption (13.12) to see that

$$\|\tilde{u}_n - \bar{u}_n\|_X \leq \frac{M + \alpha}{m - \alpha} |1 - c_n| \|\bar{u}_n\|_X. \quad (13.96)$$

Next, consider an element  $u_0 \in K$ . Condition (13.75) guarantees that  $c_n u_0 \in K_n$  and, therefore, using inequality (13.16) for the variational inequality (13.90) yields

$$\|\bar{u}_n\|_X \leq \frac{1}{m - \gamma} (\|A(c_n u_0)\|_X + c_0 \|f_n\|_Y + \gamma \|c_n u_0\|_X + \beta) + \|c_n u_0\|_X.$$

We use assumption (13.10) and convergences (13.78), (13.88) to deduce that the sequence  $\{\bar{u}_n\}$  is bounded in  $X$ , i.e., there exists  $\bar{E} > 0$  such that

$$\|\bar{u}_n\|_X \leq \bar{E} \quad \forall n \in \mathbb{N}. \quad (13.97)$$

We now combine inequalities (13.96) and (13.97), then we use assumption (13.78) to deduce that

$$\|\tilde{u}_n - \bar{u}_n\|_X \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (13.98)$$

Finally, assumption (13.88) and Theorem 13.2.13 yield

$$\|\bar{u}_n - u\|_X \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (13.99)$$

We now write

$$\|u_n - u\|_X \leq \|u_n - \tilde{u}_n\|_X + \|\tilde{u}_n - \bar{u}_n\|_X + \|\bar{u}_n - u\|_X,$$

then we use the convergences (13.92), (13.98), and (13.99) to see that

$$\|u_n - u\|_X \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

It follows from here that condition (13.56) is satisfied, which concludes the proof.  $\square$



We now are in a position to provide the proof of Theorem 13.3.8.

*Proof* First, we use Lemma 13.3.10 to see that, under the assumptions of Theorem 13.3.8, condition (13.56) holds. On the other hand, Lemma 13.3.9 shows that the sequence  $\{u_n(f^0)\}$  is bounded in  $X$ , for any  $f^0 \in Y$ . Therefore, condition (13.57) holds, too. Theorem 13.3.8 is now a direct consequence of Theorem 13.3.5.  $\square$

We end this section with the following remarks.

*Remark 13.3.11* In contrast to conditions (13.56) and (13.57), conditions (13.75)–(13.82) are explicit conditions, since they are formulated in terms of the data  $K_n$ ,  $A_n$ , and  $j_n$ . In many applications they are easy to be verified. A concrete example which illustrates this statement will be presented in Sect. 13.4.

*Remark 13.3.12* If Problem  $\mathcal{Q}$  has a unique solution  $(u^*, f^*)$  then, under the assumptions of Theorem 13.3.8 the convergences (13.58) and (13.59) are valid for the whole sequence  $\{(u_n^*, f_n^*)\}$ . This statement is a direct consequence of Remark 13.3.6.

*Remark 13.3.13* The statement of Theorem 13.3.5 still remains valid if we replace the definitions (13.2) and (13.54) with definitions (13.46) and (13.74), respectively,  $U$  being a given nonempty weakly closed subset of  $Y$ .

## 13.4 A Frictional Contact Problem

In this section we use the abstract results presented in Sects. 13.2 and 13.3 in the study of a quasivariational inequality which models the frictional contact of an elastic body with a foundation. We start by introducing the function spaces we need, then we describe the model of contact and prove its unique weak solvability, Theorem 13.4.1. Next, we turn to the optimal control of the problem and prove existence and convergence results, Theorems 13.4.4 and 13.4.6, respectively. Finally, we exemplify our results in the study of a one-dimensional mathematical model which describes the equilibrium of an elastic rod in unilateral contact with a foundation, under the action of a body force.

### 13.4.1 Function Spaces

For the study of mathematical models of contact we need further notation and preliminary material that we introduce in this subsection. Everywhere below we denote by  $\mathbb{S}^d$  the space of second order symmetric tensors on  $\mathbb{R}^d$  or, equivalently, the space of symmetric matrices of order  $d$ . The inner product and norm on  $\mathbb{R}^d$  and

$\mathbb{S}^d$  are defined by

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_i v_i, & \|\mathbf{v}\| &= (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}} & \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\tau} &= \sigma_{ij} \tau_{ij}, & \|\boldsymbol{\tau}\| &= (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} & \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d, \end{aligned}$$

and  $\mathbf{0}$  will denote the zero element of these spaces. Let  $\Omega \subset \mathbb{R}^d$  ( $d = 2, 3$ ) be a bounded domain. We denote by  $\Gamma$  its boundary, assumed to be Lipschitz continuous and divided into three measurable parts  $\Gamma_1, \Gamma_2$ , and  $\Gamma_3$  such that  $meas(\Gamma_1) > 0$ . We use the notation  $\mathbf{x} = (x_i)$  for a typical point in  $\Omega \cup \Gamma$  and we denote by  $\mathbf{v} = (v_i)$  the outward unit normal at  $\Gamma$ . Here and below the indices  $i$  and  $j$  run between 1 and  $d$  and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable  $\mathbf{x}$ , i.e.,  $u_{i,j} = \partial u_i / \partial x_j$ . Moreover,  $\boldsymbol{\varepsilon}$  represents the deformation operator, i.e.,

$$\boldsymbol{\varepsilon}(\mathbf{v}) = (\varepsilon_{ij}(\mathbf{v})), \quad \varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}(v_{i,j} + v_{j,i}).$$

We use the standard notation for Sobolev and Lebesgue spaces associated with  $\Omega$  and  $\Gamma$  and, in addition, we consider the spaces

$$V = \left\{ \mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1 \right\}, \quad Y = L^2(\Omega)^d \times L^2(\Gamma_2)^d.$$

It is well known that  $V$  is a real Hilbert space endowed with the inner product

$$(\mathbf{u}, \mathbf{v})_V = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx,$$

and the associated norm  $\|\cdot\|_V$ . Completeness of the space  $(V, \|\cdot\|_V)$  follows from the assumption  $meas(\Gamma_1) > 0$ , which allows the use of Korn's inequality. For an element  $\mathbf{v} \in V$  we still write  $\mathbf{v}$  for the trace of  $\mathbf{v}$  and we denote by  $v_\nu$  and  $\mathbf{v}_\tau$  the normal and tangential components of  $\mathbf{v}$  on  $\Gamma$  given by  $v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$ ,  $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$ . We also recall that there exists  $d_0 > 0$  which depends on  $\Omega, \Gamma_1$ , and  $\Gamma_3$  such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq d_0 \|\mathbf{v}\|_V \quad \text{for all } \mathbf{v} \in V. \quad (13.100)$$

Inequality (13.100) represents a consequence of the Sobolev trace theorem. The space  $Y$  will be endowed with its canonic inner product and associated norm, denoted by  $(\cdot, \cdot)_Y$  and  $\|\cdot\|$ , respectively.

For a regular function  $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$  we denote by  $\sigma_\nu$  and  $\boldsymbol{\sigma}_\tau$  the normal and tangential stress on  $\Gamma$ , that is,  $\sigma_\nu = (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot \boldsymbol{\nu}$  and  $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ , and we recall that the following Green's formula holds:

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_{\Omega} \text{Div } \boldsymbol{\sigma} \cdot \mathbf{v} \, dx = \int_{\Gamma} \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \mathbf{v} \, da \quad \text{for all } \mathbf{v} \in H^1(\Omega)^d. \quad (13.101)$$

More details on the function spaces used in contact mechanics, including their basic properties, can be found in the books [41, 42].

### 13.4.2 The Model

The physical setting is the following. An elastic body occupies, in its reference configuration, the domain  $\Omega \subset \mathbb{R}^d$ . Its boundary  $\Gamma$  is divided into three measurable disjoint parts  $\Gamma_1, \Gamma_2, \Gamma_3$  such that  $meas(\Gamma_1) > 0$ , as already mentioned. The body is fixed on  $\Gamma_1$ , is acted upon by given surface tractions on  $\Gamma_2$ , and is in potential contact with an obstacle on  $\Gamma_3$ . To construct a mathematical model which corresponds to the equilibrium of the body in this physical setting above we need to prescribe specific interface boundary condition. Here, we assume that the contact is with normal compliance and finite penetration, associated with a version of Coulomb's law of dry friction. Therefore, the classical formulation of the problem is the following.

**Problem P.** Find a displacement field  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$  and a stress field  $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$  such that

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in } \Omega, \quad (13.102)$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega, \quad (13.103)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad (13.104)$$

$$\boldsymbol{\sigma} \mathbf{v} = \mathbf{f}_2 \quad \text{on } \Gamma_2, \quad (13.105)$$

$$\left. \begin{array}{l} u_\nu \leq k, \quad \sigma_\nu + p(u_\nu) \leq 0, \\ (u_\nu - k)(\sigma_\nu + p(u_\nu)) = 0 \end{array} \right\} \quad \text{on } \Gamma_3, \quad (13.106)$$

$$\left. \begin{array}{l} \|\boldsymbol{\sigma}_\tau\| \leq \mu p(u_\nu), \\ -\boldsymbol{\sigma}_\tau = \mu p(u_\nu) \frac{\mathbf{u}_\tau}{\|\mathbf{u}_\tau\|} \quad \text{if } \mathbf{u}_\tau \neq \mathbf{0} \end{array} \right\} \quad \text{on } \Gamma_3. \quad (13.107)$$

We now provide a description of the equations and boundary conditions in Problem P. First, Eq. (13.102) represents the elastic constitutive law of the material in which  $\mathcal{F}$  is assumed to be a nonlinear constitutive operator. Equation (13.103) is the equation of equilibrium. We use it here since the contact process is assumed to be static and, therefore, the inertial term in the equation of motion is neglected. Conditions (13.104) and (13.105) represent the displacement and traction boundary conditions, respectively.

Condition (13.106) represents the so-called normal compliance condition with unilateral constraint. Here,  $k > 0$  is a given bound which limits the normal

displacement and  $p$  is a given positive function which will be described below. This condition describes the contact with an obstacle made of a rigid body covered by a layer of thickness  $k$  made of deformable material. Condition (13.107) represents a static version of Coulomb's law of dry friction in which  $\mu$  denotes the coefficient of friction and  $\mu p(u_\nu)$  is the friction bound. The coupling of boundary conditions (13.106) and (13.107) was considered for the first time in [5]. Later, it was used in a number of papers, see [42] and the references therein. It describes a contact with normal compliance, as far as the normal displacement satisfies the condition  $u_\nu < k$ , associated with the classical Coulomb's law of dry friction. When  $u_\nu = k$  the contact is with a Signorini-type condition and is associated with the Tresca friction law with the friction bound  $\mu p(k)$ . It follows from here that conditions (13.106), (13.107) describe a natural transition from the Coulomb law of dry friction (which is valid as far as  $0 \leq u_\nu < k$ ) to the Tresca law (which is valid when  $u_\nu = k$ ).

In the study of the mechanical problem (13.102)–(13.107) we assume that the elasticity operator  $\mathcal{F}$  and the normal compliance function  $p$  satisfy the following conditions:

$$\left\{ \begin{array}{l}
 \text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d. \\
 \text{(b) There exists } L_{\mathcal{F}} > 0 \text{ such that} \\
 \quad \|\mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
 \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
 \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
 \quad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
 \text{(d) There exists } m_{\mathcal{F}} > 0 \text{ such that} \\
 \quad (\mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\
 \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\
 \text{(e) } \mathcal{F}(\mathbf{x}, \mathbf{0}) = \mathbf{0}.
 \end{array} \right. \quad (13.108)$$

$$\left\{ \begin{array}{l}
 \text{(a) } p : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+. \\
 \text{(b) There exists } L_p > 0 \text{ such that} \\
 \quad |p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2)| \leq L_p |r_1 - r_2| \\
 \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\
 \text{(c) } (p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2))(r_1 - r_2) \geq 0 \\
 \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\
 \text{(d) The mapping } \mathbf{x} \mapsto p(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \\
 \quad \text{for any } r \in \mathbb{R}. \\
 \text{(e) } p(\mathbf{x}, r) = 0 \text{ for all } r \leq 0, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\
 \text{(f) There exists } p^* \in \mathbb{R} \text{ such that } p(\mathbf{x}, r) \leq p^* \\
 \quad \text{for all } r \geq 0, \text{ a.e. } \mathbf{x} \in \Gamma_3.
 \end{array} \right. \quad (13.109)$$

The coefficient of friction is such that

$$\mu \in L^\infty(\Gamma_3), \quad \mu(\mathbf{x}) \geq 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3. \quad (13.110)$$

Moreover, we assume that

$$d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)} < m_{\mathcal{F}} \quad (13.111)$$

where  $d_0$ ,  $m_{\mathcal{F}}$ , and  $L_p$  are the constants which appear in (13.100), (13.108)(d), and (13.109)(b), respectively. Note that inequality (13.111) could be interpreted as a smallness condition on the coefficient of friction. Such kind of conditions are often used in the variational analysis of frictional contact problems with elastic materials, as explained in [38] and the references therein.

Let  $K$  denote the set defined by

$$K = \{ \mathbf{v} \in V : v_\nu \leq k \quad \text{a.e. on } \Gamma_3 \}, \quad (13.112)$$

and assume that the densities of body forces and tractions are such that  $\mathbf{f}_0 \in L^2(\Omega)^d$ ,  $\mathbf{f}_2 \in L^2(\Gamma_2)^d$ . We now derive the variational formulation of Problem  $P$  and, to this end, we assume that  $(\mathbf{u}, \boldsymbol{\sigma})$  are sufficiently regular functions which satisfy (13.102)–(13.107). Then, using (13.106) and (13.112) it follows that

$$\mathbf{u} \in K. \quad (13.113)$$

Let  $\mathbf{v} \in K$ . We use Green's formula (13.101) and equalities (13.103)–(13.105) to see that

$$\begin{aligned} \int_{\Omega} \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})) \, dx &= \int_{\Omega} \mathbf{f}_0 \cdot (\mathbf{v} - \mathbf{u}) \, dx \\ &+ \int_{\Gamma_2} \mathbf{f}_2 \cdot (\mathbf{v} - \mathbf{u}) \, da + \int_{\Gamma_3} \boldsymbol{\sigma} \mathbf{v} \cdot (\mathbf{v} - \mathbf{u}) \, da. \end{aligned} \quad (13.114)$$

Moreover, using the boundary conditions (13.106) and (13.107) it is easy to see that

$$\begin{aligned} \sigma_\nu(v_\nu - u_\nu) &\geq p(u_\nu)(u_\nu - v_\nu) \quad \text{a.e. on } \Gamma_3, \\ \boldsymbol{\sigma}_\tau(\mathbf{v}_\tau - \mathbf{u}_\tau) &\geq \mu p(u_\nu)(\|\mathbf{u}_\tau\| - \|\mathbf{v}_\tau\|) \quad \text{a.e. on } \Gamma_3. \end{aligned}$$

Therefore, since

$$\boldsymbol{\sigma} \mathbf{v} \cdot (\mathbf{v} - \mathbf{u}) = \sigma_\nu(v_\nu - u_\nu) + \boldsymbol{\sigma}_\tau(\mathbf{v}_\tau - \mathbf{u}_\tau) \quad \text{a.e. on } \Gamma_3,$$

we deduce that

$$\begin{aligned} & \int_{\Gamma_3} \boldsymbol{\sigma} \mathbf{v} \cdot (\mathbf{v} - \mathbf{u}) \, da \\ & \geq \int_{\Gamma_3} p(u_\nu)(u_\nu - v_\nu) \, da + \int_{\Gamma_3} \mu p(u_\nu)(\|\mathbf{u}_\tau\| - \|\mathbf{v}_\tau\|) \, da. \end{aligned} \quad (13.115)$$

Next, we combine equality (13.114) with inequality (13.115), then we use the constitutive law (13.102) and the regularity (13.113). As a result we find the following variational formulation of Problem  $P$ .

**Problem  $P^V$ .** Given  $\mathbf{f} = (\mathbf{f}_0, \mathbf{f}_2) \in Y$ , find  $\mathbf{u}$  such that

$$\begin{aligned} \mathbf{u} \in K, \quad & \int_{\Omega} \mathcal{F} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})) \, dx \\ & + \int_{\Gamma_3} p(u_\nu)(v_\nu - u_\nu) \, da + \int_{\Gamma_3} \mu p(u_\nu)(\|\mathbf{v}_\tau\| - \|\mathbf{u}_\tau\|) \, da \\ & \geq \int_{\Omega} \mathbf{f}_0 \cdot (\mathbf{v} - \mathbf{u}) \, dx + \int_{\Gamma_2} \mathbf{f}_2 \cdot (\mathbf{v} - \mathbf{u}) \, da \quad \forall \mathbf{v} \in K. \end{aligned} \quad (13.116)$$

Note that Problem  $P^V$  is formulated in terms of the displacement field. Once the displacement field is known, the stress field can be easily obtained by using the constitutive law (13.102). A couple  $(\mathbf{u}, \boldsymbol{\sigma})$  which satisfies (13.102) and (13.116) is called a *weak solution* to the contact problem  $P$ .

### 13.4.3 Weak Solvability

Our main result in this section, which represents a continuation of our previous results in [5, 39], is the following.

**Theorem 13.4.1** *Assume that (13.108)–(13.111) hold. Then, for each  $\mathbf{f} = (\mathbf{f}_0, \mathbf{f}_2) \in Y$  there exists a unique solution  $\mathbf{u} = \mathbf{u}(\mathbf{f})$  to the variational inequality (13.116). Moreover, if  $\mathbf{f}_n = (\mathbf{f}_{0n}, \mathbf{f}_{2n}) \in Y$ ,  $\mathbf{f} = (\mathbf{f}_0, \mathbf{f}_2) \in Y$ , and  $\mathbf{f}_{0n} \rightharpoonup \mathbf{f}_0$  in  $L^2(\Omega)^d$ ,  $\mathbf{f}_{2n} \rightharpoonup \mathbf{f}_2$  in  $L^2(\Gamma_2)^d$ , as  $n \rightarrow \infty$ , then  $\mathbf{u}_n(\mathbf{f}_n) \rightarrow \mathbf{u}(\mathbf{f})$  in  $X$ , as  $n \rightarrow \infty$ .*

Note that Theorem 13.4.1 provides the existence of a unique weak solution to the frictional contact Problem  $P$  as well as its continuous dependence with respect to the density of body forces and tractions.

The proof of Theorem 13.4.1 will be carried out in several steps, based on the abstract existence and convergence results in Sect. 13.2. To present it assume in what

follows that (13.108)–(13.111) hold and we consider the operator  $A : V \rightarrow V$ , the function  $j : V \times V \rightarrow \mathbb{R}$ , and the operator  $\pi : V \rightarrow Y$  defined by

$$(Au, v)_V = \int_{\Omega} \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_{\Gamma_3} p(u_\nu)v_\nu \, da \quad \forall \mathbf{u}, \mathbf{v} \in V, \quad (13.117)$$

$$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \mu p(u_\nu) \|\mathbf{v}_\tau\| \, da \quad \forall \mathbf{u} \in V, \mathbf{v} \in V, \quad (13.118)$$

$$\pi \mathbf{v} = (\iota \mathbf{v}, \gamma_2 \mathbf{v}) \quad \forall \mathbf{v} \in V. \quad (13.119)$$

Here  $\iota : V \rightarrow L^2(\Omega)^d$  is the canonic embedding and  $\gamma_2 : V \rightarrow L^2(\Gamma_2)^d$  is the restriction to the trace map to  $\Gamma_2$ . The first step of the proof is the following.

**Lemma 13.4.2** *Given  $\mathbf{f} = (\mathbf{f}_0, \mathbf{f}_2) \in Y$ , an element  $\mathbf{u} \in V$  is solution to the variational inequality (13.116) if and only if*

$$\mathbf{u} \in K, \quad (Au, \mathbf{v} - \mathbf{u})_V + j(\mathbf{u}, \mathbf{v}) - j(\mathbf{u}, \mathbf{u}) \geq (\mathbf{f}, \pi \mathbf{v} - \pi \mathbf{u})_Y \quad \forall \mathbf{v} \in K. \quad (13.120)$$

*Proof* The statement of Lemma 13.4.2 is a direct consequence of the notation (13.117)–(13.119).  $\square$

**Lemma 13.4.3** *The function  $j$  defined by (13.118) satisfies conditions (13.11), (13.14), and (13.19) on the space  $X = V$ .*

*Proof* Condition (13.11)(a) is obviously satisfied. On the other hand, an elementary calculation based on the definition (13.118) and assumptions (13.109), (13.110) yields

$$\begin{aligned} & j(\mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2) \\ & \leq L_p \|\mu\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} \|\mathbf{u}_1 - \mathbf{u}_2\| \|\mathbf{v}_1 - \mathbf{v}_2\| \, da \end{aligned}$$

for all  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$ . Therefore, the trace inequality (13.100) shows that condition (13.11)(b) holds with  $\alpha = d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}$ . Next, using assumptions (13.109)(b), (e) and (13.110) it is easy to see that

$$\begin{aligned} j(\boldsymbol{\eta}, \mathbf{v}_1) - j(\boldsymbol{\eta}, \mathbf{v}_2) & \leq \int_{\Gamma_3} \mu p(\eta_\nu) \|\mathbf{v}_1 - \mathbf{v}_2\| \, da \\ & \leq L_p \|\mu\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} \|\boldsymbol{\eta}\| \|\mathbf{v}_1 - \mathbf{v}_2\| \, da \end{aligned}$$

for all  $\boldsymbol{\eta}, \mathbf{v}_1, \mathbf{v}_2 \in V$ . Therefore, the trace inequality (13.100) shows that condition (13.14)(b) is satisfied with  $\beta = 0$  and  $\gamma = d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}$ . Finally, note that

condition (13.19) holds from assumption (13.109) and the compactness of the trace operator, since  $\mathbf{u}_k \rightharpoonup \mathbf{u}$  in  $V$  implies that  $p(u_{k\nu}) \rightarrow p(u_\nu)$  and  $\|\mathbf{u}_{k\tau}\| \rightarrow \|\mathbf{u}_\tau\|$  in  $L^2(\Gamma_3)$ .  $\square$

We now have all the ingredients to provide the proof of Theorem 13.4.1.

*Proof* The set  $K$  is obviously a convex nonempty subset of  $V$ . Moreover, using the properties of the trace map we deduce that  $K$  is closed and, therefore, (13.9) holds. Next, we use assumptions (13.108) and (13.109) and the trace inequality (13.100) to see that

$$(\mathbf{A}\mathbf{u} - \mathbf{A}\mathbf{v}, \mathbf{u} - \mathbf{v})_V \geq m_{\mathcal{F}}\|\mathbf{u} - \mathbf{v}\|_V^2,$$

$$\|\mathbf{A}\mathbf{u} - \mathbf{A}\mathbf{v}\|_V \leq \left( L_{\mathcal{F}} + d_0^2 L_p \right) \|\mathbf{u} - \mathbf{v}\|_V$$

for all  $\mathbf{u}, \mathbf{v} \in V$ . Therefore, condition (13.10) holds with  $X = V$  and  $m = m_{\mathcal{F}}$ . On the other hand, Lemma 13.4.3 guarantees that the functional (13.118) satisfies conditions (13.11), (13.14), and (13.19) on the space  $X = V$ , with  $\alpha = d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}$ ,  $\beta = 0$ , and  $\gamma = d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}$ . Therefore, using (13.111) it follows that the smallness assumption (13.12) is satisfied and, moreover, (13.15) holds, too. Finally, we note that conditions (13.13) and (13.20) are a direct consequence of definition (13.119) combined with the properties of the operators  $\iota$  and  $\gamma_2$ .

It follows from above that we are in a position to apply Theorem 13.2.12 on the space  $X = V$ . As a result we deduce the unique solvability of the variational inequality (13.120), for each  $\mathbf{f} = (\mathbf{f}_0, \mathbf{f}_2) \in Y$ . This result combined with Lemma 13.4.2 proves the existence of a unique solution to the variational inequality (13.116), for each  $\mathbf{f} = (\mathbf{f}_0, \mathbf{f}_2) \in Y$ .

Assume now that  $\mathbf{f}_n = (\mathbf{f}_{0n}, \mathbf{f}_{2n}) \in Y$ ,  $\mathbf{f} = (\mathbf{f}_0, \mathbf{f}_2) \in Y$ , and  $\mathbf{f}_{0n} \rightharpoonup \mathbf{f}_0$  in  $L^2(\Omega)^d$ ,  $\mathbf{f}_{2n} \rightharpoonup \mathbf{f}_2$  in  $L^2(\Gamma_2)^d$ , as  $n \rightarrow \infty$ . Then  $\mathbf{f}_n \rightharpoonup \mathbf{f}$  in  $Y$ , as  $n \rightarrow \infty$ . Therefore, using Theorem 13.2.13 and Lemma 13.4.2 we deduce that  $\mathbf{u}(\mathbf{f}_n) \rightarrow \mathbf{u}(\mathbf{f})$  in  $V$ , as  $n \rightarrow \infty$ , which concludes the proof.  $\square$

### 13.4.4 Optimal Control

We now associate to Problem  $P^V$  the set of admissible pairs  $\mathcal{V}_{ad}$  and the cost function  $\mathcal{L}$  given by

$$\mathcal{V}_{ad} = \{ (\mathbf{u}, \mathbf{f}) \in K \times Y \text{ such that } \mathbf{f} = (\mathbf{f}_0, \mathbf{f}_2) \text{ and (13.116) holds} \}, \quad (13.121)$$

$$\mathcal{L}(\mathbf{u}, \mathbf{f}) = a_0 \int_{\Omega} \|\mathbf{f}_0\|^2 dx + a_2 \int_{\Gamma_2} \|\mathbf{f}_2\|^2 da + a_3 \int_{\Gamma_3} |u_\nu - \theta|^2 da \quad (13.122)$$



for all  $\mathbf{u} \in V$ ,  $\mathbf{f} = (\mathbf{f}_0, \mathbf{f}_2) \in Y$ . Here  $\theta$  is a given element in  $L^2(\Gamma_3)$  and  $a_0, a_2, a_3$  are strictly positive constants. Moreover, we consider the following optimal control problem.

**Problem**  $Q^V$ . Find  $(\mathbf{u}^*, \mathbf{f}^*) \in \mathcal{V}_{ad}$  such that

$$\mathcal{L}(\mathbf{u}^*, \mathbf{f}^*) = \min_{(\mathbf{u}, \mathbf{f}) \in \mathcal{V}_{ad}} \mathcal{L}(\mathbf{u}, \mathbf{f}). \quad (13.123)$$

Our first result in this subsection is the following.

**Theorem 13.4.4** *Assume that (13.108)–(13.111) hold. Then, the optimal control problem  $Q^V$  has at least one solution  $(\mathbf{u}^*, \mathbf{f}^*)$ .*

*Proof* It is easy to see that the function  $\mathcal{L}$  defined by (13.122) satisfy conditions (13.32)–(13.34) on the spaces  $X = V$ ,  $Y = L^2(\Omega)^d \times L^2(\Gamma_3)^d$  with

$$g(\mathbf{v}) = a_3 \int_{\Gamma_3} |v_v - \theta|^2 da, \quad h(\mathbf{f}) = a_0 \int_{\Omega} \|\mathbf{f}_0\|^2 dx + a_2 \int_{\Gamma_2} \|\mathbf{f}_2\|^2 da$$

for all  $\mathbf{v} \in V$ ,  $\mathbf{f} = (\mathbf{f}_0, \mathbf{f}_2) \in Y$ . Therefore, the solvability of the optimal control problem  $Q^V$  is a direct consequence of Lemma 13.4.2 and Theorem 13.3.1.  $\square$

Next, besides conditions (13.108)–(13.111), we assume that

$$B \text{ is a closed convex set of } \mathbb{S}^d \text{ such that } \mathbf{0} \in B \quad (13.124)$$

and we denote by  $P_B : \mathbb{S}^d \rightarrow B$  the projection operator. We also consider the sequences  $\{\omega_n\}$ ,  $\{k_n\}$ , and  $\{\varepsilon_n\}$  such that

$$\omega_n \geq 0, \quad k_n > 0, \quad \varepsilon_n \geq 0 \quad \forall n \in \mathbb{N} \quad (13.125)$$

and, for each  $n \in \mathbb{N}$ , we define the set  $K_n$  by

$$K_n = \{ \mathbf{v} \in V : v_v \leq k_n \text{ a.e. on } \Gamma_3 \}. \quad (13.126)$$

With these data, we consider the following perturbation of Problem  $P^V$ .

**Problem**  $P_n^V$ . Given  $\mathbf{f}_n = (\mathbf{f}_{0n}, \mathbf{f}_{2n}) \in Y$ , find  $\mathbf{u}_n \in V$  such that

$$\begin{aligned} \mathbf{u}_n \in K_n, \quad & \int_{\Omega} \mathcal{F} \boldsymbol{\varepsilon}(\mathbf{u}_n) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}_n)) dx \\ & + \omega_n \int_{\Omega} (\boldsymbol{\varepsilon}(\mathbf{u}_n) - P_B \boldsymbol{\varepsilon}(\mathbf{u}_n)) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}_n)) dx + \int_{\Gamma_3} p(u_{nv})(v_v - u_{nv}) da \end{aligned} \quad (13.127)$$

$$\begin{aligned}
& + \int_{\Gamma_3} \mu p(u_{nv}) \left( \sqrt{\|\mathbf{v}_\tau\|^2 + \varepsilon_n^2} - \sqrt{\|\mathbf{u}_{n\tau}\|^2 + \varepsilon_n^2} \right) da \\
& \geq \int_{\Omega} \mathbf{f}_{0n} \cdot (\mathbf{v} - \mathbf{u}_n) dx + \int_{\Gamma_2} \mathbf{f}_{2n} \cdot (\mathbf{v} - \mathbf{u}_n) da \quad \forall \mathbf{v} \in K_n.
\end{aligned}$$

Note that Problem  $P_n^V$  represents the variational formulation of an elastic contact problem of the form (13.102)–(13.107) in which the following changes have been operated in the model:

- The elastic constitutive law (13.102) was replaced with the constitutive law  $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}) + \omega_n(\boldsymbol{\varepsilon}(\mathbf{u}) - P_B\boldsymbol{\varepsilon}(\mathbf{u}))$ . Such kind of constitutive law has been used by many authors, see [38, 41] and the references therein.
- The bound  $k$  in (13.106) was replaced by a perturbation, denoted  $k_n$ .
- The Coulomb law of dry friction (13.107) was replaced with its regularization

$$-\boldsymbol{\sigma}_\tau = \mu p(u_v) \frac{\mathbf{u}_\tau}{\sqrt{\|\mathbf{u}_\tau\|^2 + \varepsilon_n^2}}.$$

- The densities  $\mathbf{f}_0$  and  $\mathbf{f}_2$  of body forces and tractions, respectively, were replaced by their perturbations  $\mathbf{f}_{0n}$  and  $\mathbf{f}_{2n}$ , respectively.

For Problem  $P_n^V$  we have the following existence and uniqueness result.

**Theorem 13.4.5** *Assume (13.108)–(13.111) and, moreover, assume that (13.124)–(13.125) hold. Then, for each  $\mathbf{f}_n = (\mathbf{f}_{0n}, \mathbf{f}_{2n}) \in Y$  there exists a unique solution  $\mathbf{u}_n = \mathbf{u}_n(\mathbf{f}_n)$  to the variational inequality (13.127).*

*Proof* The proof of Theorem 13.4.5 is based on arguments similar to those used in the proof of Theorem 13.4.1 and, for this reason, we skip the details. The steps of the proof are the following.

i) *The quasivariational inequality.* Besides the operator  $\pi$  defined in (13.119), for each  $n \in \mathbb{N}$  we consider the operator  $A_n : V \rightarrow V$  and the function  $j_n : V \times V \rightarrow \mathbb{R}$  given by

$$\begin{aligned}
(A_n \mathbf{u}, \mathbf{v})_V &= \int_{\Omega} \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx + \int_{\Gamma_3} p(u_v) v_v da \\
&+ \omega_n \int_{\Omega} (\boldsymbol{\varepsilon}(\mathbf{u}) - P_B\boldsymbol{\varepsilon}(\mathbf{u})) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx \quad \forall \mathbf{u}, \mathbf{v} \in V.
\end{aligned} \tag{13.128}$$

$$j_n(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \mu p(u_v) \left( \sqrt{\|\mathbf{v}_\tau\|^2 + \varepsilon_n^2} - \varepsilon_n \right) da \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{13.129}$$

Then, it is easy to see that, given  $\mathbf{f}_n = (\mathbf{f}_{0n}, \mathbf{f}_{2n}) \in Y$ , an element  $\mathbf{u}_n \in V$  is a solution to inequality (13.127) if and only if

$$\begin{aligned} \mathbf{u}_n \in K_n, \quad & (A_n \mathbf{u}_n, \mathbf{v} - \mathbf{u}_n)_V + j_n(\mathbf{u}_n, \mathbf{v}) - j_n(\mathbf{u}_n, \mathbf{u}_n) \\ & \geq (\mathbf{f}_n, \pi \mathbf{v} - \pi \mathbf{u}_n)_Y \quad \forall \mathbf{v} \in K_n. \end{aligned} \quad (13.130)$$

ii) *The operator  $A_n$ .* First, we recall that the projector operator  $P_B : \mathbb{S}^d \rightarrow B$  is nonexpansive, i.e.,

$$\|P_B \boldsymbol{\tau}_1 - P_B \boldsymbol{\tau}_2\| \leq \|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\| \quad (13.131)$$

for all  $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{S}^d$ . This inequality implies that

$$((\boldsymbol{\tau}_1 - P_B \boldsymbol{\tau}_1) - (\boldsymbol{\tau}_2 - P_B \boldsymbol{\tau}_2)) \cdot (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2) \geq 0, \quad (13.132)$$

for all  $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \mathbb{S}^d$ . Therefore, using assumptions (13.108) and (13.109), the trace inequality (13.100), and estimates (13.131), (13.132) we deduce that

$$\begin{aligned} (A_n \mathbf{u} - A_n \mathbf{v}, \mathbf{u} - \mathbf{v})_V & \geq m_{\mathcal{F}} \|\mathbf{u} - \mathbf{v}\|_V^2, \\ \|A_n \mathbf{u} - A_n \mathbf{v}\|_V & \leq (L_{\mathcal{F}} + d_0^2 L_p + 2\omega_n) \|\mathbf{u} - \mathbf{v}\|_V \end{aligned}$$

for all  $\mathbf{u}, \mathbf{v} \in X$ . It follows from here that condition (13.48) holds with  $X = V$ ,  $m_n = m_{\mathcal{F}}$ , and  $M_n = L_{\mathcal{F}} + d_0^2 L_p + 2\omega_n$ .

iii) *The function  $j_n$ .* We claim that the function  $j_n$  defined by (13.129) satisfies conditions (13.11), (13.14), and (13.19) on the space  $X = V$ .

First, condition (13.11)(a) is obviously satisfied. On the other hand, an elementary calculation based on the definition (13.129) and assumptions (13.109), (13.110), combined with inequality

$$\left| \sqrt{a^2 + \varepsilon^2} - \sqrt{b^2 + \varepsilon^2} \right| \leq |a - b| \quad \forall a, b, \varepsilon > 0,$$

implies that

$$\begin{aligned} & j_n(\mathbf{u}_1, \mathbf{v}_2) - j_n(\mathbf{u}_1, \mathbf{v}_1) + j_n(\mathbf{u}_2, \mathbf{v}_1) - j_n(\mathbf{u}_2, \mathbf{v}_2) \\ & \leq L_p \|\mu\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} \|\mathbf{u}_1 - \mathbf{u}_2\| \|\mathbf{v}_1 - \mathbf{v}_2\| da \end{aligned}$$

for all  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$ . Therefore, the trace inequality (13.100) shows that condition (13.11)(b) holds with  $\alpha = d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}$ . A similar argument shows that condition (13.14) is satisfied with  $\beta = 0$  and  $\gamma = d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}$ . Finally, note that condition (13.19) holds from assumption (13.109) and the compactness of the

trace operator, since  $\mathbf{u}_k \rightharpoonup \mathbf{u}$  in  $V$  implies that  $p(u_{k\nu}) \rightarrow p(u_\nu)$  and  $\|\mathbf{u}_{k\tau}\| \rightarrow \|\mathbf{u}_\tau\|$  in  $L^2(\Gamma_3)$ .

iv) *End of proof.* The set  $K_n$  is obviously a convex nonempty subset of  $V$ . Moreover, recall that step ii) shows that condition (13.48) holds with  $X = V$  and  $m = m_{\mathcal{F}}$ . On the other hand, step iii) guarantees that the functional (13.118) satisfies conditions (13.11), (13.14), and (13.19) on the space  $X = V$ , with  $\alpha = \gamma = d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}$ . Therefore, using (13.111) it follows that the smallness assumption (13.12) is satisfied and, moreover, (13.15) holds, too. Finally, we note that conditions (13.13) and (13.20) are a direct consequence of definition (13.119) combined with the properties of the operators  $\iota$  and  $\gamma_2$ .

It follows from above that we are in a position to apply Theorem 13.2.12 on the space  $X = V$ . In this way we deduce the unique solvability of the quasivariational inequality (13.130), for each  $\mathbf{f}_n = (\mathbf{f}_{0n}, \mathbf{f}_{2n}) \in Y$ . This result, combined with step i), leads to the existence of a unique solution to the quasivariational inequality (13.127), for each  $\mathbf{f}_n = (\mathbf{f}_{0n}, \mathbf{f}_{2n}) \in Y$ .  $\square$

We now move to the control of Problem  $P_n^V$ . The set of admissible pairs for this problem is given by

$$\mathcal{V}_{ad}^n = \{(\mathbf{u}_n, \mathbf{f}_n) \in K_n \times Y \text{ s. t. } \mathbf{f}_n = (\mathbf{f}_{0n}, \mathbf{f}_{2n}) \text{ and (13.127) holds}\}. \quad (13.133)$$

Moreover, the corresponding optimal control problem is the following.

**Problem**  $Q_n^V$ . Find  $(\mathbf{u}_n^*, \mathbf{f}_n^*) \in \mathcal{V}_{ad}^n$  such that

$$\mathcal{L}(\mathbf{u}_n^*, \mathbf{f}_n^*) = \min_{(\mathbf{u}_n, \mathbf{f}_n) \in \mathcal{V}_{ad}^n} \mathcal{L}(\mathbf{u}_n, \mathbf{f}_n). \quad (13.134)$$

Our main result in this section is the following.

**Theorem 13.4.6** *Assume (13.108)–(13.111), (13.124), and (13.125). Then the following statements hold.*

- i) *For each  $n \in \mathbb{N}$  the optimal control problem  $Q_n^V$  has at least one solution  $(\mathbf{u}_n^*, \mathbf{f}_n^*)$ .*
- ii) *If  $\omega_n \rightarrow 0$ ,  $k_n \rightarrow k$ , and  $\varepsilon_n \rightarrow \varepsilon$  as  $n \rightarrow \infty$ , then for any sequence  $\{(\mathbf{u}_n^*, \mathbf{f}_n^*)\}$  of solutions of Problem  $Q_n^V$  there exists a subsequence, again denoted  $\{(\mathbf{u}_n^*, \mathbf{f}_n^*)\}$ , and an element  $(\mathbf{u}^*, \mathbf{f}^*) \in X \times Y$ , such that*

$$\mathbf{f}_n^* \rightharpoonup \mathbf{f}^* \quad \text{in } Y, \quad \text{as } n \rightarrow \infty, \quad (13.135)$$

$$\mathbf{u}_n^* \rightarrow \mathbf{u}^* \quad \text{in } X \quad \text{as } n \rightarrow \infty, \quad (13.136)$$

$$(\mathbf{u}^*, \mathbf{f}^*) \quad \text{is a solution of Problem } Q^V. \quad (13.137)$$

*Proof* i) First, it is easy to see that condition (13.75) holds with  $c_n = \frac{k_n}{k} > 0$ . Next, we use (13.131), (13.132) to see that condition (13.76) is satisfied with

$$(T_n \mathbf{u}, \mathbf{v})_V = \omega_n \int_{\Omega} (\boldsymbol{\varepsilon}(\mathbf{u}) - P_B \boldsymbol{\varepsilon}(\mathbf{u})) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx \quad \forall \mathbf{u}, \mathbf{v} \in V. \quad (13.138)$$

On the other hand, it follows from step iii) in the proof of Theorem 13.4.5 that the function  $j_n$  satisfies condition (13.77) with  $\alpha_n = d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}$ ,  $\beta_n = 0$ , and  $\gamma_n = d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}$ . In addition, recall that the function  $\mathcal{L}$  defined by (13.122) satisfies conditions (13.32)–(13.34) on the spaces  $X = V$ ,  $Y = L^2(\Omega)^d \times L^2(\Gamma_3)^d$  with

$$g(\mathbf{v}) = a_3 \int_{\Gamma_3} |v_\nu - \theta|^2 \, da, \quad h(\mathbf{f}) = a_0 \int_{\Omega} \|\mathbf{f}_0\|^2 \, dx + a_2 \int_{\Gamma_2} \|\mathbf{f}_2\|^2 \, da.$$

Therefore, as already explained in page 463, we are in a position to apply Theorem 13.3.5 to deduce that the optimal control problem  $Q_n^V$  has at least one solution  $(\mathbf{u}_n^*, \mathbf{f}_n^*)$ , for each  $n \in \mathbb{N}$ .

ii) Assume now that  $\omega_n \rightarrow 0$ ,  $k_n \rightarrow k$ , and  $\varepsilon_n \rightarrow 0$ , as  $n \rightarrow \infty$ . In order to prove (13.135)–(13.137) we use Theorem 13.3.8 and, to this end, we start by checking that conditions (13.78)–(13.82) are satisfied.

First, since  $c_n = \frac{k_n}{k}$  we deduce that condition (13.78) holds. Moreover, it is easy to see that condition (13.79) holds, too. On the other hand, assumption  $\mathbf{0} \in B$  combined with inequality (13.131) shows that  $\|\boldsymbol{\tau} - P_B \boldsymbol{\tau}\| \leq 2 \|\boldsymbol{\tau}\|$  for all  $\boldsymbol{\tau} \in \mathbb{S}^d$ . Therefore, definition (13.138) implies that

$$(T_n \mathbf{v}, \mathbf{w})_V = \omega_n \int_{\Omega} \|\boldsymbol{\varepsilon}(\mathbf{v}) - P_B \boldsymbol{\varepsilon}(\mathbf{v})\| \|\boldsymbol{\varepsilon}(\mathbf{w})\| \, dx \leq 2\omega_n \|\mathbf{v}\|_V \|\mathbf{w}\|_V$$

for all  $\mathbf{u}, \mathbf{v} \in V$ ,  $n \in \mathbb{N}$ , which shows that

$$\|T_n \mathbf{v}\|_V \leq 2\omega_n \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V, n \in \mathbb{N}.$$

We conclude from here that condition (13.80) holds with  $F_n = 2\omega_n$  and  $\delta_n = 0$ .

Assume now that  $n \in \mathbb{N}$  is fixed and  $\mathbf{v}_1, \mathbf{v}_2 \in V$ . We use definitions (13.129) and (13.118) to see that

$$\begin{aligned} & j_n(\mathbf{v}_1, \mathbf{v}_2) - j_n(\mathbf{v}_1, \mathbf{v}_1) + j(\mathbf{v}_2, \mathbf{v}_1) - j(\mathbf{v}_2, \mathbf{v}_2) \\ &= \int_{\Gamma_3} \mu p(v_{1\nu}) (\sqrt{\|\mathbf{v}_{2\tau}\|^2 + \varepsilon_n^2} - \varepsilon_n - \|\mathbf{v}_{2\tau}\|) \, da \end{aligned}$$

$$\begin{aligned}
& + \int_{\Gamma_3} \mu p(v_{1\nu})(\|\mathbf{v}_{1\tau}\| - \sqrt{\|\mathbf{v}_{1\tau}\|^2 + \varepsilon_n^2} + \varepsilon_n) da \\
& + \int_{\Gamma_3} \mu (p(v_{1\nu}) - p(v_{2\nu}))(\|\mathbf{v}_{2\tau}\| - \|\mathbf{v}_{1\tau}\|) da.
\end{aligned}$$

Therefore, using assumptions (13.109)(b) and (13.110) combined with the inequality

$$\left| \sqrt{a^2 + \varepsilon^2} - a - \varepsilon \right| \leq \varepsilon \quad \forall a, \varepsilon > 0,$$

we find that

$$\begin{aligned}
& j_n(\mathbf{v}_1, \mathbf{v}_2) - j_n(\mathbf{v}_1, \mathbf{v}_1) + j(\mathbf{v}_2, \mathbf{v}_1) - j(\mathbf{v}_2, \mathbf{v}_2) \\
& \leq 2\varepsilon_n \|\mu\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} p(v_{1\nu}) da + L_p \|\mu\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} \|\mathbf{v}_{1\tau} - \mathbf{v}_{2\tau}\|^2 da.
\end{aligned}$$

Next, we use assumption (13.109)(f) and the trace inequality (13.100) to deduce that

$$\begin{aligned}
& j_n(\mathbf{v}_1, \mathbf{v}_2) - j_n(\mathbf{v}_1, \mathbf{v}_1) + j(\mathbf{v}_2, \mathbf{v}_1) - j(\mathbf{v}_2, \mathbf{v}_2) \\
& \leq 2\varepsilon_n p^* \|\mu\|_{L^\infty(\Gamma_3)} \text{meas}(\Gamma_3) + d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)} \|\mathbf{v}_1 - \mathbf{v}_2\|_V^2.
\end{aligned}$$

It follows from here that condition (13.81) holds with  $G_n = 2\varepsilon_n p^* \|\mu\|_{L^\infty(\Gamma_3)} \text{meas}(\Gamma_3)$  and  $H_n = 0$ .

Recall now that  $m_n = m_{\mathcal{F}}$ ,  $\alpha = \gamma_n = d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)}$ , and  $\beta_n = 0$ , for each  $n \in \mathbb{N}$ . Moreover,  $d_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)} < m_{\mathcal{F}}$ , as assumed in (13.111). We conclude from here that condition (13.82) is satisfied.

Finally, note that the rest of the conditions in Theorem 13.3.8 are satisfied, as it follows from the previous results proved in this section. Theorem 13.4.6 is now a direct consequence of Theorem 13.3.8.  $\square$

*Remark 13.4.7* As a consequence of Remarks 13.3.2 and 13.3.7 we deduce that the statements of Theorems 13.4.4 and 13.4.6 still remain valid if we replace the definition (13.121) and (13.133) with the following ones:

$$\mathcal{V}_{ad} = \{(\mathbf{u}, \mathbf{f}) \in K \times U \text{ s.t. } \mathbf{f} = (\mathbf{f}_0, \mathbf{f}_2) \text{ and (13.116) holds}\}, \quad (13.139)$$

$$\mathcal{V}_{ad}^n = \{(\mathbf{u}_n, \mathbf{f}_n) \in K_n \times U \text{ s.t. } \mathbf{f} = (\mathbf{f}_{0n}, \mathbf{f}_{2n}) \text{ and (13.127) holds}\}, \quad (13.140)$$

$U$  being a given nonempty weakly closed subset of  $Y$ . The proof of this statement is based on the property (13.45) of the set  $U$ .

### 13.4.5 A One-Dimensional Example

In this subsection we illustrate our results in the study of a one-dimensional example. Thus, we consider Problem  $P$  in the particular case when  $\Omega = (0, 1)$ ,  $\Gamma_1 = \{0\}$ ,  $\Gamma_2 = \emptyset$ ,  $\Gamma_3 = \{1\}$ . Note that in this case the linearized strain field is given by  $\varepsilon = u'$ , where, here and below, the prime denotes the derivative with respect to the spatial variable  $x \in [0, 1]$ . Moreover, we assume that the material is homogeneous and behaves linearly elastic. Therefore, the elasticity operator is  $\mathcal{F}\varepsilon = E\varepsilon$  where  $E > 0$  is the Young modulus of the material. In addition, we assume that the density of the body force does not depend on the spatial variable and we denote it by  $f \in \mathbb{R}$ . Then, the statement of the problem is the following.

**Problem  $P^{1d}$ .** Find a displacement field  $u: [0, 1] \rightarrow \mathbb{R}$  and a stress field  $\sigma: [0, 1] \rightarrow \mathbb{R}$  such that

$$\sigma(x) = E u'(x) \quad \text{for } x \in (0, 1), \quad (13.141)$$

$$\sigma'(x) + f = 0 \quad \text{for } x \in (0, 1), \quad (13.142)$$

$$u(0) = 0, \quad (13.143)$$

$$\left. \begin{aligned} u(1) \leq k, \quad \sigma(1) + p(u(1)) \leq 0, \\ (u(1) - k)(\sigma(1) + p(u(1))) = 0 \end{aligned} \right\}. \quad (13.144)$$

Note that Problem  $P^{1d}$  models the contact of an elastic rod of length  $l = 1$ . The rod occupies the domain  $[0, 1]$  on the  $Ox$  axis, is fixed at its end  $x = 0$ , as acted by a body force, and its extremity  $x = 1$  is in contact with a foundation made of a deformable material of thickness  $k > 0$ , which covers a rigid body. The reaction of the deformable material is described with the function  $p: \mathbb{R} \rightarrow \mathbb{R}$  which is positive, monotone, and vanishes for a negative argument. This physical setting is depicted in Fig. 13.1.

For the analysis of Problem  $P^{1d}$  we use the space

$$V = \{v \in H^1(0, 1) : v(0) = 0\}$$

and the set of admissible displacement field defined by

$$K = \{u \in V \mid u(1) \leq k\}.$$

The variational formulation of Problem  $P^{1d}$ , obtained using integration by parts, is the following.

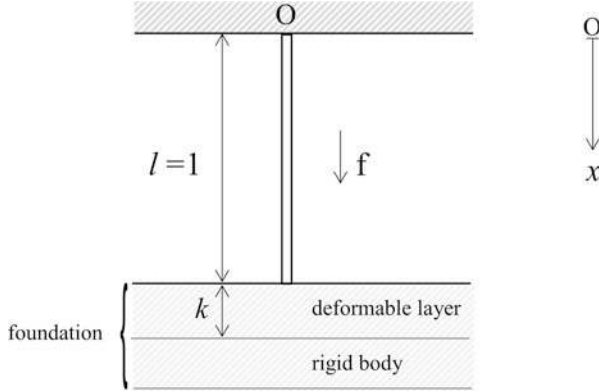


Fig. 13.1 Physical setting

**Problem**  $P_V^{1d}$ . Find a displacement field  $u \in K$  such that

$$\int_0^1 Eu'(v' - u') dx + p(u(1))(v(1) - u(1)) \geq \int_0^1 f(v - u) dx \quad \forall v \in K. \quad (13.145)$$

The existence of a unique solution to Problem  $P_V^{1d}$  follows from Theorem 13.4.1. Consider now the case when

$$p(r) = r^+ = \begin{cases} 0 & \text{if } r < 0, \\ r & \text{if } r \geq 0. \end{cases} \quad (13.146)$$

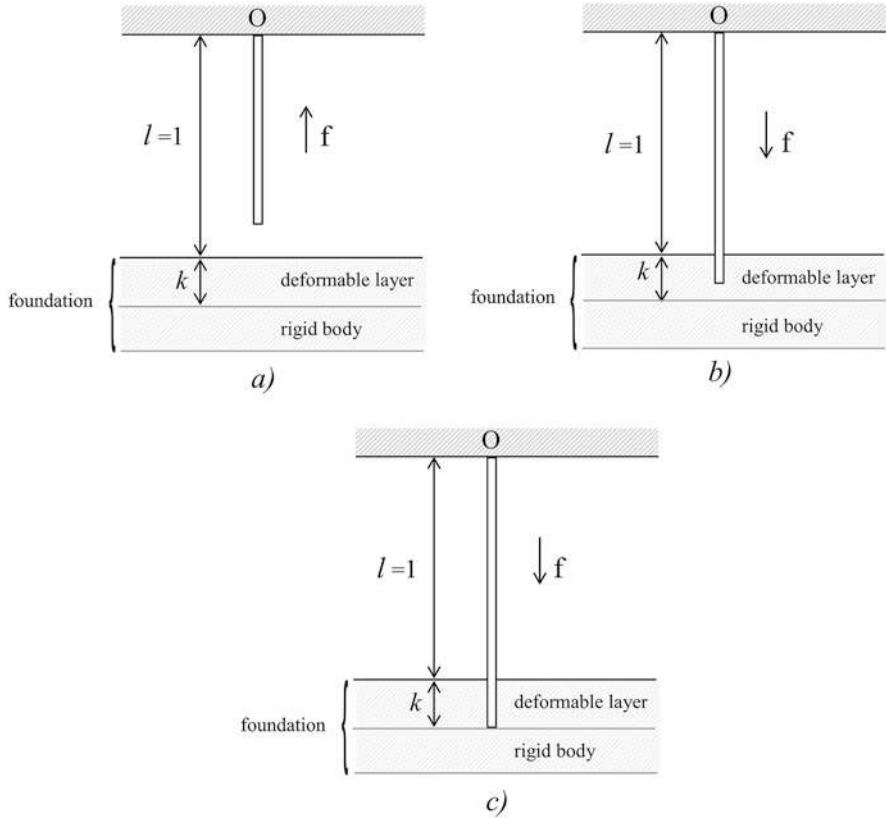
Then, a simple calculation allows us to solve Problem  $P^{1d}$ . Three cases are possible, described below, together with the corresponding mechanical interpretations.

a) **The case**  $f < 0$ . In this case the body force acts in the opposite direction of the foundation and the solution of Problem  $P^{1d}$  is given by

$$\begin{cases} \sigma(x) = -fx + f, \\ u(x) = -\frac{f}{2E}x^2 + \frac{f}{E}x \end{cases} \quad \forall x \in [0, 1]. \quad (13.147)$$

We have  $u(1) < 0$  and  $\sigma(1) = 0$  which shows that there is separation between the rod and the foundation and, therefore, there is no reaction on the point  $x = 1$ . This case corresponds to Fig. 13.2a.





**Fig. 13.2** The rod in contact with a foundation. (a) The case  $f < 0$ ; (b) The case  $0 \leq f < 2k(E + 1)$ ; (c) The case  $f \geq 2k(E + 1)$

- b) **The case  $0 \leq f < 2k(E + 1)$ .** In this case the body force pushes the rod towards the foundation and the solution of Problem  $P^{1d}$  is given by

$$\begin{cases} \sigma(x) = -fx + \frac{f(2E+1)}{2(E+1)}, \\ u(x) = -\frac{f}{2E}x^2 + \frac{f(2E+1)}{2E(E+1)}x \end{cases} \quad \forall x \in [0, 1]. \quad (13.148)$$

We have  $0 \leq u(1) < k$  and  $\sigma(1) \leq 0$  which shows that there is penetration into the deformable layer and the reaction of the foundation is towards the rod. Nevertheless, the penetration is partial, since  $u(1) < k$ . This case corresponds to Fig. 13.2b.

- c) **The case  $f \geq 2k(E + 1)$ .** In this case the solution of Problem  $P^{1d}$  is given by

$$\begin{cases} \sigma(x) = -fx + \frac{f}{2} + kE, \\ u(x) = -\frac{f}{2E}x^2 + \left(\frac{f}{2E} + k\right)x \end{cases} \quad \forall x \in [0, 1]. \quad (13.149)$$

We have  $u(1) = k$  which shows that the rigid-plastic layer is completely penetrated and the point  $x = 1$  reaches the rigid body. This case corresponds to Fig. 13.2c).

We now formulate the optimal control problem  $Q^V$  in the one-dimensional case of Problem  $P^{1d}$ . In this particular setting  $Y = L^2(0, 1)$  and we choose

$$U = \{ f \in Y : f \text{ is a constant} \}.$$

We use (13.139) to see that in this case

$$\mathcal{V}_{ad} = \{ (u, f) \in K \times U : (13.145) \text{ holds} \} \quad (13.150)$$

and

$$\mathcal{L}(u, f) = a_0 |f|^2 + a_3 |u(1) - \theta|^2, \quad (13.151)$$

where  $\theta \in \mathbb{R}$ ,  $a_0 > 0$ ,  $a_3 > 0$ . Then, using (13.123) we see that the problem can be formulated as follows.

**Problem**  $Q^{1d}$ . Find  $(u^*, f^*) \in \mathcal{V}_{ad}$  such that

$$\mathcal{L}(u^*, f^*) = \min_{(u, g) \in \mathcal{V}_{ad}} \mathcal{L}(u, f). \quad (13.152)$$

We now take  $E = 1$ . Then, it is easy to see that if  $0 \leq f < 4k$ , then  $0 \leq f < 2k(E + 1)$  and, if  $f \geq 4k$ , then  $f \geq 2k(E + 1)$ . Therefore, using (13.147)–(13.149) we have

$$u(x) = \begin{cases} -\frac{f}{2}x^2 + fx & \text{if } f < 0, \\ -\frac{f}{2}x^2 + \frac{3f}{4}x & \text{if } 0 \leq f < 4k \\ -\frac{f}{2}x^2 + (\frac{f}{2} + k)x & \text{if } f \geq 4k \end{cases} \quad \forall x \in [0, 1]. \quad (13.153)$$

So,

$$u(1) = \begin{cases} \frac{f}{2} & \text{if } f < 0, \\ \frac{f}{4} & \text{if } 0 \leq f < 4k, \\ k & \text{if } f \geq 4k \end{cases}$$

and, using (13.151) with  $\theta = 1$ ,  $a_0 = 1$ ,  $a_3 = 16$ , we find that

$$\mathcal{L}(u, f) = \begin{cases} 5f^2 - 16f + 16 & \text{if } f < 0, \\ 2f^2 - 8f + 16 & \text{if } 0 \leq f < 4k, \\ f^2 + 16 & \text{if } f \geq 4k. \end{cases} \quad (13.154)$$

To conclude, problem (13.152) consists to minimize the function (13.154) when  $f \in \mathbb{R}$ , for a given  $k > 0$ . For this reason, in what follows we denote by  $J_k$  the function defined by (13.154), i.e.,

$$J_k(f) = \begin{cases} 5f^2 - 16f + 16 & \text{if } f < 0, \\ 2f^2 - 8f + 16 & \text{if } 0 \leq f < 4k, \\ f^2 + 16(k-1)^2 & \text{if } f \geq 4k. \end{cases} \quad (13.155)$$

It is easy to see that this function is not a convex function. Nevertheless, it has a unique point of minimum given by

$$f^*(k) = \begin{cases} 4k & \text{if } 0 < k \leq \frac{1}{2}, \\ 2 & \text{if } k > \frac{1}{2}. \end{cases} \quad (13.156)$$

Then, using (13.153) we find that the optimal control problem  $Q^{1d}$  has a unique solution  $(u^*(k), f^*(k))$ , given by

$$u^*(k) = \begin{cases} -\frac{f^*(k)}{2}x^2 + \left(\frac{f^*(k)}{2} + k\right)x & \forall x \in [0, 1], & \text{if } 0 \leq k \leq \frac{1}{2}, \\ -\frac{f^*(k)}{2}x^2 + \frac{3f^*(k)}{4}x & \forall x \in [0, 1], & \text{if } k > \frac{1}{2} \end{cases}$$

where, recall,  $f^*(k)$  is given by (13.156). It is easy to see that when  $k_n \rightarrow k$ , then  $f^*(k_n) \rightarrow f^*(k)$  and, therefore,  $u^*(k_n) \rightarrow u^*(k)$ . This represents a validation of the abstract convergence result in Theorem 13.3.8.

## 13.5 Conclusion

In this chapter we studied an optimal control problem for elliptic quasivariational inequalities in Hilbert spaces. We provided the existence of optimal pairs and proved a convergence result. The proofs were based on arguments of monotonicity and lower semicontinuity. Then, we applied these abstract results in the study of a mathematical model which describes the equilibrium of an elastic body in frictional contact with an obstacle, the so-called foundation. We presented various mechanical interpretations of these results and we exemplified them in the particular case of an elastic rod in contact with a rigid body covered by a layer of soft material.

The study presented in this chapter gives rise to several open problems that we describe in what follows. Any progress in these directions will complete our work and will open the way for new advances and ideas.

First, it would be interesting to derive necessary optimality conditions in the study of Problem  $\mathcal{Q}$  introduced on page 447. Due to the nonsmooth and nonconvex feature of the functional  $\mathcal{L}$ , the treatment of this problem requires the use of its approximation by smooth optimization problems. And, in this matter, the abstract convergence results for the optimal pairs in this chapter could be a crucial tool. Next, it would be useful to establish an optimality condition for the optimal control problem  $\mathcal{Q}^V$  stated on page 476. We are convinced that such conditions could be established for a regularization of this problem, by using arguments similar to those used in [27]. There, a boundary optimal control problem for a frictional contact problem with normal compliance has been considered.

Another interesting continuation of the results presented in this chapter would be their extension to evolutionary variational inequalities. For such inequalities both the data and the unknown depend on time variable and, moreover, the time derivative of the unknown appears in the statement of the problem. In addition, an initial condition is needed. Such kind of inequalities model quasistatic process of contact for elastic, viscoelastic, and viscoplastic materials. The optimal control of a quasistatic model of contact with linearly elastic materials was studied in [3]. There, besides the existence of the optimal pairs, necessary optimality conditions for a regularization problem have been established.

An interesting continuation of the results presented in this chapter would be their extension to variational–hemivariational inequalities. These inequalities represent a generalization of variational inequalities, in which both convex and nonconvex functions are involved. Besides arguments of convexity and monotonicity, the theory of variational–hemivariational inequalities was built based on the properties of Clarke subdifferential, defined for locally Lipschitz function. The details can be found in the books [32, 42] and the edited volume [18]. Some preliminary results in the study of optimal control for variational–hemivariational inequalities can be found in [37].

We end this section by recalling that the control of mathematical models of contact, as well as their optimal shape design, deserves to make the object of important studies in the future. These topics are of considerable theoretical and applied interest. Indeed, in most applications this is the main interest of the design engineer and any result in this direction will illustrate the cross fertilization between models and applications, in one hand, and the nonlinear functional analysis, on the other hand. The related issues are the observability properties of the models and parameter identification. Using reliable parameter identification procedures will help in establishing the validity of various mathematical models of contact with deformable bodies. This, in turn, will help in the construction of effective and efficient numerical algorithms for the problems with established convergence. As better models for specific applications are obtained, improved mathematical models and numerical simulations will be possible.

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