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# Boundary optimal control of a nonsmooth frictionless contact problem

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We consider a mathematical model which describes the equilibrium of an elastic body in contact with a foundation. The contact is frictionless and is modeled with a nonsmooth interface law which involves unilateral constraints and subdifferential conditions. The weak formulation of the model is in the form of an elliptic hemivariational inequality governed by a number of parameters. We prove the unique weak solvability of the problem, then we state and prove a continuous dependence result of the solution with respect to the data and parameters. Next, we formulate a boundary optimal control problem for which we prove the existence of optimal pairs. We also study the dependence of the optimal pairs with respect to the data and parameters and prove a convergence result. The proofs are based on arguments of monotonicity, lower semicontinuity and Clarke subdifferential calculus.

## 1. Introduction

Contact phenomena with deformable bodies arise in a large variety of industrial settings and engineering applications. For this reason, their modeling, analysis and numerical approximation was widely studied, both in the mathematical and engineering literature. A brief research reveals that most of the mathematical models used to describe the contact phenomena are expressed in terms of challenging nonlinear boundary value problems. Their weak formulations are stated in the form of variational or hemivariational inequalities, which have been studied extensively in recent years, e.g., [1–6].

The analysis of various models of contact via the theory of variational inequalities started with the pioneering book of Duvaut and Lions [7]. Then, it was developed in many references, including [8–16]. There, various existence, uniqueness and convergence results have been proved, by using arguments of convexity, monotonicity, lower semicontinuity and fixed point. In contrast, the analysis of hemivariational inequalities uses as main ingredient the properties of the subdifferential in the sense of Clarke, defined for locally Lipschitz functions, which may be nonconvex. Hemivariational inequalities were first introduced in early 1980s by Panagiotopoulos in the context of applications in engineering problems. Studies of hemivariational inequalities with emphasis to mathematical modeling of contact phenomena can be found in several comprehensive references, e.g., [3,4,17] and, more recently, [18,19].

The aim of this paper is to present results on the optimal control of a mathematical model which describes the equilibrium of an elastic body acted upon by body forces and surface tractions in contact with an obstacle. The literature concerning optimal control problems in the study of mathematical models of contact is quite limited. The reason is the

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strong nonlinearities which arise in the boundary conditions included in such models. Results on optimal control for various contact problems with elastic materials can be found in [20–29] and the references therein. The mathematical models presented in these papers were expressed in terms of variational inequalities. In contrast, the model we consider in this current paper is new, involves a large number of parameters, and leads to a weak formulation in terms of hemivariational inequality in which the unknown is the displacement field. This represents the first trait of originality of the paper. The second one consists in the fact that we obtain a very general convergence result for the solution, which shows its continuous dependence with respect to the data and parameters. This result is then used to obtain new results on the existence and convergence of the optimal pairs for an associated optimal control problem.

The paper is structured as follows. In Section 2 we survey some preliminaries of nonsmooth analysis we need in the rest of the paper, including an abstract existence and uniqueness result for elliptic hemivariational inequalities. In Section 3 we introduce the contact model, state its variational formulation and prove its unique weak solvability. Then, in Section 4 we prove a convergence result of the weak solution with respect to the data and parameters. In Sections 5 and 6, we deal with a boundary optimal control problem. We present results on the existence and convergence of the optimal pairs, respectively.

## 2. Preliminaries

Most of the material presented in this section is standard. Therefore, we introduce it without proofs and restrict ourselves to mention that details on the definitions and statements below can be found in the monographs [17,19,30–33] as well as in the papers [34,35].

**Elements of nonsmooth analysis.** For any normed space  $X$  we use  $\|\cdot\|_X$  to denote its norm and  $X^*$  represents its topological dual. Moreover, we denote by  $\langle \cdot, \cdot \rangle$  the duality pairing between  $X^*$  and  $X$ , by  $0_X$  and  $0_{X^*}$  the zero element of  $X$  and  $X^*$ , respectively, and by  $2^{X^*}$  the power set of  $X^*$ . All the limits, upper and lower limits below are considered as  $n \rightarrow \infty$ , even if we do not mention it explicitly. The symbols “ $\rightharpoonup$ ” and “ $\rightarrow$ ” denote the weak and the strong convergence in various spaces which will be specified. Nevertheless, for simplicity, we write  $g_n \rightarrow g$  for the convergence in  $\mathbb{R}$ .

**Definition 2.1.** An operator  $A: X \rightarrow X^*$  is said to be:

- (a) *monotone*, if for all  $u, v \in X$ , we have  $\langle Au - Av, u - v \rangle \geq 0$ ;
- (b) *bounded*, if  $A$  maps bounded sets of  $X$  into bounded sets of  $X^*$ ;
- (c) *Lipschitz continuous*, if there exists a constant  $L_A > 0$  such that

$$\|Au - Av\|_{X^*} \leq L_A \|u - v\|_X \quad \text{for all } u, v \in X;$$

- (d) *pseudomonotone*, if it is bounded and  $u_n \rightharpoonup u$  in  $X$  with

$$\limsup \langle Au_n, u_n - u \rangle \leq 0$$

imply

$$\liminf \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle \quad \text{for all } v \in X.$$

We complete these definitions with the remark that any monotone Lipschitz continuous operator is pseudomonotone. Next, we recall the definition of the Clarke subdifferential for a locally Lipschitz function.

**Definition 2.2.** A function  $h: X \rightarrow \mathbb{R}$  is said to be *locally Lipschitz* if for every  $x \in X$  there exists a neighborhood of  $x$ , denoted  $U_x$ , and a constant  $L_x > 0$ , such that  $|h(y) - h(z)| \leq L_x \|y - z\|_X$  for all  $y, z \in U_x$ .

We note that a convex continuous function  $h: X \rightarrow \mathbb{R}$  is locally Lipschitz. Moreover, if a function  $h: X \rightarrow \mathbb{R}$  is Lipschitz continuous on bounded sets of  $X$ , then it is locally Lipschitz, while the converse does not hold, in general.

**Definition 2.3.** Let  $h: X \rightarrow \mathbb{R}$  be a locally Lipschitz function. The *generalized (Clarke) directional derivative* of  $h$  at the point  $x \in X$  in the direction  $v \in X$  is defined by

$$h^0(x; v) = \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{h(y + \lambda v) - h(y)}{\lambda}.$$

The *generalized gradient (subdifferential)* of  $h$  at  $x$  is a subset of the dual space  $X^*$  given by

$$\partial h(x) = \{ \zeta \in X^* \mid h^0(x; v) \geq \langle \zeta, v \rangle \quad \forall v \in X \}. \quad (2.1)$$

We shall use the following properties of the generalized directional derivative and the generalized gradient.

**Proposition 2.4.** Assume that  $h: X \rightarrow \mathbb{R}$  is a locally Lipschitz function. Then the following hold:

- (i) For every  $x \in X$ , the function  $X \ni v \mapsto h^0(x; v) \in \mathbb{R}$  is positively homogeneous and subadditive, i.e.,  $h^0(x; \lambda v) = \lambda h^0(x; v)$  for all  $\lambda \geq 0$ ,  $v \in X$  and  $h^0(x; v_1 + v_2) \leq h^0(x; v_1) + h^0(x; v_2)$  for all  $v_1, v_2 \in X$ , respectively.
- (ii) The function  $X \times X \ni (x, v) \mapsto h^0(x; v) \in \mathbb{R}$  is upper semicontinuous, i.e., for all  $x, v \in X$ ,  $\{x_n\}, \{v_n\} \subset X$  such that  $x_n \rightarrow x$  and  $v_n \rightarrow v$  in  $X$ , we have  $\limsup h^0(x_n; v_n) \leq h^0(x; v)$ .
- (iii) For every  $v \in X$ , we have  $h^0(x; v) = \max \{ \langle \xi, v \rangle \mid \xi \in \partial h(x) \}$ .

For the convergence result we state and prove in Section 5, we need the notion of Mosco convergence that we recall in what follows.

**Definition 2.5.** Let  $X$  be a normed space,  $\{K_n\}$  a sequence of nonempty subsets of  $X$  and  $K$  a nonempty subset of  $X$ . We say that the sequence  $\{K_n\}$  converges to  $K$  in the Mosco sense if the following conditions hold.

- (M<sub>1</sub>)  $\left\{ \begin{array}{l} \text{For each } v \in K \text{ there exists a sequence } \{v_n\} \text{ such that} \\ v_n \in K_n \text{ for each } n \in \mathbb{N} \text{ and } v_n \rightarrow v \text{ in } X. \end{array} \right.$
- (M<sub>2</sub>)  $\left\{ \begin{array}{l} \text{For each sequence } \{v_n\} \text{ such that} \\ v_n \in K_n \text{ for each } n \in \mathbb{N} \text{ and } v_n \rightharpoonup v \text{ in } X, \text{ we have } v \in K. \end{array} \right.$

We shall denote the convergence in the Mosco sense by  $K_n \xrightarrow{M} K$ . This convergence depends on the topology of the space  $X$ . Therefore, to avoid any ambiguity, sometimes we shall write  $K_n \xrightarrow{M} K$  in  $X$  instead of  $K_n \xrightarrow{M} K$ . More details on the Mosco convergence can be found in [36].

**Hemivariational inequalities.** We now recall a recent result in the study of variational–hemivariational inequalities. The problem under consideration can be formulated as follows.

**Problem 1.** Find an element  $u$  such that

$$u \in K, \quad \langle Au, v - u \rangle + j^0(u; v - u) \geq \langle f, v - u \rangle \quad \text{for all } v \in K.$$

In the study of this problem we consider the following hypotheses on the data.

$$K \text{ is a nonempty closed convex subset of } X. \quad (2.2)$$

$$\left\{ \begin{array}{l} A: X \rightarrow X^* \text{ is such that} \\ \text{(a) it is pseudomonotone,} \\ \text{(b) strongly monotone, i.e., there exists } m_A > 0 \text{ such that} \\ \langle Av_1 - Av_2, v_1 - v_2 \rangle \geq m_A \|v_1 - v_2\|_X^2 \text{ for all } v_1, v_2 \in X. \end{array} \right. \quad (2.3)$$

$$\left\{ \begin{array}{l} j: X \rightarrow \mathbb{R} \text{ is such that} \\ \text{(a) } j \text{ is locally Lipschitz,} \\ \text{(b) } \|\partial j(v)\|_{X^*} \leq c_0 + c_1 \|v\|_X \text{ for all } v \in X \text{ with } c_0, c_1 \geq 0, \\ \text{(c) there exists } \alpha_j > 0 \text{ such that} \\ j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j \|v_1 - v_2\|_X^2 \\ \text{for all } v_1, v_2 \in X. \end{array} \right. \quad (2.4)$$

$$f \in X^*. \quad (2.5)$$

We have the following existence and uniqueness result.

**Theorem 2.6.** Assume that  $X$  is a reflexive Banach space, (2.2)–(2.5) hold and, in addition, assume that

$$\alpha_j < m_A. \quad (2.6)$$

Then, Problem 1 has a unique solution  $u \in K$ .

Theorem 2.6 represents a particular case of a result proved in [19,37]. Its proof is based on arguments of surjectivity for pseudomonotone operators and convexity, combined with the properties of the Clarke subdifferential.

**Function spaces.** Let  $d \in \{1, 2, 3\}$  and denote by  $\mathbb{S}^d$  the space of second order symmetric tensors on  $\mathbb{R}^d$  or, equivalently, the space of symmetric matrices of order  $d$ . The zero element of the spaces  $\mathbb{R}^d$  and  $\mathbb{S}^d$  will be denoted by  $\mathbf{0}$ . The inner product and norm on  $\mathbb{R}^d$  and  $\mathbb{S}^d$  are defined by

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_i v_i, & \|\mathbf{v}\| &= (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}} & \forall \mathbf{u} &= (u_i), \mathbf{v} = (v_i) \in \mathbb{R}^d, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\tau} &= \sigma_{ij} \tau_{ij}, & \|\boldsymbol{\tau}\| &= (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} & \forall \boldsymbol{\sigma} &= (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d, \end{aligned}$$

where the indices  $i, j$  run between 1 and  $d$  and, unless stated otherwise, the summation convention over repeated indices is used.

Consider now a bounded domain  $\Omega \subset \mathbb{R}^d$ , with a Lipschitz continuous boundary  $\Gamma$  and let  $\Gamma_1, \Gamma_2, \Gamma_3$  be measurable parts of  $\Gamma$  such that  $\text{meas}(\Gamma_1) > 0$ . Everywhere in this paper we use the standard notation for Sobolev and Lebesgue spaces associated to a bounded domain  $\Omega \subset \mathbb{R}^d$  ( $d = 1, 2, 3$ ), with a Lipschitz continuous boundary  $\Gamma$ . In particular, we use the spaces  $L^2(\Omega)^d, L^2(\Gamma_2)^d, L^2(\Gamma_3), L^2(\Gamma_3)^d$  and  $H^1(\Omega)^d$ , endowed with their canonical inner products and associated norms. Moreover, for an element  $\mathbf{v} \in H^1(\Omega)^d$  we usually write  $\boldsymbol{\gamma} \mathbf{v}$  for the trace  $\boldsymbol{\gamma} \mathbf{v} \in L^2(\Gamma)^d$  of  $\mathbf{v}$  to  $\Gamma$ . In addition, we consider the following spaces:

$$V = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\},$$

$$Q = \{ \boldsymbol{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \}.$$

The spaces  $V$  and  $Q$  are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u}, \mathbf{v})_V = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \, dx. \quad (2.7)$$

Here and below  $\boldsymbol{\varepsilon}$  and  $\text{Div}$  will represent the deformation and the divergence operator, respectively, i.e.,

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad \text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j}),$$

with the index that follows a comma denoting the partial derivative with respect to the corresponding component of the spatial variable  $\mathbf{x}$ , i.e.,  $u_{i,j} = \partial u_i / \partial x_j$ .

The associated norms on these spaces are denoted by  $\|\cdot\|_V$  and  $\|\cdot\|_Q$ , respectively. Recall that the completeness of the space  $V$  follows from the assumption  $\text{meas}(\Gamma_1) > 0$  which allows the use of Korn's inequality. Below in this paper we denote by  $V^*$  and  $\langle \cdot, \cdot \rangle$  the topological dual of  $V$  and the duality pairing between  $V^*$  and  $V$ , respectively. Let  $\mathbf{v} = (v_i)$  be the outward unit normal at  $\Gamma$ . For any element  $\mathbf{v} \in V$ , we denote by  $v_\nu$  and  $\mathbf{v}_\tau$  its normal and tangential components on  $\Gamma$  given by  $v_\nu = \mathbf{v} \cdot \mathbf{v}$  and  $\mathbf{v}_\tau = \mathbf{v} - v_\nu \mathbf{v}$ , respectively. Also, recall that, for a regular stress function  $\boldsymbol{\sigma}$ , the following Green's formula holds:

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_{\Omega} \text{Div } \boldsymbol{\sigma} \cdot \mathbf{v} \, dx = \int_{\Gamma} \boldsymbol{\sigma} \mathbf{v} \cdot \mathbf{v} \, da \quad \text{for all } \mathbf{v} \in H^1(\Omega)^d. \quad (2.8)$$

Finally, we recall that the Sobolev trace theorem yields

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq \|\gamma\| \|\mathbf{v}\|_V \quad \text{for all } \mathbf{v} \in V, \quad (2.9)$$

with  $\|\gamma\|$  being the norm of the trace operator  $\gamma: V \rightarrow L^2(\Gamma_3)^d$ .

### 3. The contact model

The physical setting, already considered in many papers and surveys, can be resumed as follows. A deformable body occupies, in its reference configuration, a bounded domain  $\Omega \subset \mathbb{R}^d$  ( $d = 1, 2, 3$ ), with a Lipschitz continuous boundary  $\Gamma$ , divided into three measurable disjoint parts  $\Gamma_1$ ,  $\Gamma_2$  and  $\Gamma_3$ , such that  $\text{meas}(\Gamma_1) > 0$ . The body is fixed on  $\Gamma_1$ , is acted upon by given surface tractions on  $\Gamma_2$ , and is in contact with an obstacle on  $\Gamma_3$ . The equilibrium of the body in this physical setting can be described by various mathematical models, obtained by using different mechanical assumptions. The model we consider in this paper is based on specific constitutive laws and interface boundary conditions which will be described below in this section. Its statement is as follows.

**Problem 2.** Find a displacement field  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ , a stress field  $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$  and an interface function  $\xi_\nu : \Gamma_3 \rightarrow \mathbb{R}$  such that

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) + \beta(\boldsymbol{\varepsilon}(\mathbf{u}) - P_B\boldsymbol{\varepsilon}(\mathbf{u})) \quad \text{in } \Omega, \quad (3.1)$$

$$\text{Div } \boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega, \quad (3.2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad (3.3)$$

$$\boldsymbol{\sigma} \mathbf{v} = \mathbf{f}_2 \quad \text{on } \Gamma_2, \quad (3.4)$$

$$\left. \begin{array}{l} u_\nu \leq g, \\ \sigma_\nu + \xi_\nu \leq 0, \\ (u_\nu - g)(\sigma_\nu + \xi_\nu) = 0, \\ \xi_\nu \in \omega \partial j_\nu(u_\nu) \end{array} \right\} \quad \text{on } \Gamma_3, \quad (3.5)$$

$$\boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_3. \quad (3.6)$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable  $\mathbf{x} \in \Omega \cup \Gamma$ . Moreover,  $\sigma_\nu$  and  $\boldsymbol{\sigma}_\tau$  denote the normal and tangential stress on  $\Gamma$ , that is  $\sigma_\nu = (\boldsymbol{\sigma} \mathbf{v}) \cdot \mathbf{v}$  and  $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \mathbf{v} - \sigma_\nu \mathbf{v}$ . We now provide a short description of the equations and boundary conditions in Problem 2.

First, Eq. (3.1) represents the constitutive law in which  $\mathcal{A}$  is the elasticity operator, assumed to be nonlinear,  $\beta$  is a given elasticity coefficient which depends on the spatial variable,  $B \subset \mathbb{S}^d$  represents a nonempty convex subset and  $P_B : \mathbb{S}^d \rightarrow B$  denotes the projection operator on  $B$ . Examples of operators  $\mathcal{A}$  which satisfy the conditions presented below in this paper can be found in our books [13,17,19]. For the set  $B$ , a typical example which can be found in the literature is given by

$$B = \{ \boldsymbol{\tau} \in \mathbb{S}^d : \mathcal{F}(\boldsymbol{\tau}) \leq k \},$$

where  $\mathcal{F} : \mathbb{S}^d \rightarrow \mathbb{R}$  is a convex continuous function such that  $\mathcal{F}(\mathbf{0}) = 0$  and  $k$  is a positive constant. It is easy to see that in this case the set  $B$  is a nonempty convex closed subset of  $\mathbb{S}^d$ .

Eq. (3.2) is the equation of equilibrium that we use here since the process is assumed to be static. Conditions (3.3), (3.4) represent the displacement and the traction boundary conditions, respectively.

We now turn on the condition (3.5) which represents the contact condition. There,  $g$  is assumed to be a positive constant,  $\omega$  and  $j_v$  are given functions which will be described below and  $\partial j_v$  denotes the Clarke subdifferential of  $j_v$ . This condition models the contact with a foundation made of a rigid body covered by a deformable layer of thickness  $g$ . The function  $\omega$  represents a parameter which can be interpreted as the stiffness coefficient of the foundation.

Finally, condition (3.6) represents the frictionless contact condition. It shows that the friction force,  $\sigma_\tau$ , vanishes during the process. This is an idealization of the process, since even completely lubricated surfaces generate shear resistance to tangential motion. However, this condition is a sufficiently good approximation of the reality in some situations, especially when the contact surfaces are lubricated.

We mention that Problem 2 was considered in [19] in the case when  $\beta \equiv 0$  and  $\omega \equiv 1$ . There, a unique solvability result was proved. Considering the case when  $\beta \neq 0$  and  $\omega \neq 1$  leads to a new and nonstandard mathematic model which better describes the physical setting. The analysis and optimal control of this problem represent one of the traits of novelty of this paper.

In the study of the mechanical problem (3.1)–(3.6) we assume that the elasticity operator  $\mathcal{A}$  satisfies the following conditions.

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad \|\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(d) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\ \quad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) } \mathcal{A}(\mathbf{x}, \mathbf{0}) = \mathbf{0} \text{ a.e. } \mathbf{x} \in \Omega. \end{array} \right. \quad (3.7)$$

We also assume that the set  $B$ , the elasticity coefficient, the densities of body forces and tractions, the stiffness coefficient and the bound of the normal displacement are such that

$$B \text{ is a closed convex subset of } \mathbb{S}^d \text{ such that } \mathbf{0} \in B. \quad (3.8)$$

$$\beta \in L^\infty(\Omega), \quad \beta(\mathbf{x}) \geq 0 \quad \text{a.e. } \mathbf{x} \in \Omega. \quad (3.9)$$

$$\mathbf{f}_0 \in L^2(\Omega)^d. \quad (3.10)$$

$$\mathbf{f}_2 \in L^2(\Gamma_2)^d. \quad (3.11)$$

$$\omega \in L^\infty(\Gamma_3), \quad \omega(\mathbf{x}) \geq 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3. \quad (3.12)$$

$$g > 0. \quad (3.13)$$

Finally, the normal compliance function  $j_v$  satisfies the following condition.

$$\left\{ \begin{array}{l} j_v : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R} \text{ is such that} \\ \text{(a) } j_v(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \quad \text{exists } \bar{e} \in L^1(\Gamma_3) \text{ such that } j_v(\cdot, \bar{e}(\cdot)) \in L^1(\Gamma_3). \\ \text{(b) } j_v(\mathbf{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c) } |\partial j_v(\mathbf{x}, r)| \leq \bar{c}_0 + \bar{c}_1 |r| \text{ for a.e. } \mathbf{x} \in \Gamma_3, \\ \quad \text{for all } r \in \mathbb{R} \text{ with } \bar{c}_0, \bar{c}_1 \geq 0. \\ \text{(d) } j_v^0(\mathbf{x}, r_1; r_2 - r_1) + j_v^0(\mathbf{x}, r_2; r_1 - r_2) \leq \alpha_{j_v} |r_1 - r_2|^2 \\ \quad \text{for a.e. } \mathbf{x} \in \Gamma_3, \text{ all } r_1, r_2 \in \mathbb{R} \text{ with } \alpha_{j_v} \geq 0. \end{array} \right. \quad (3.14)$$

Consider the set  $U$  defined by

$$U = \{ \mathbf{v} \in V : v_v \leq g \text{ a.e. on } \Gamma_3 \}. \quad (3.15)$$

Then, using standard arguments, we obtain the following variational formulation of Problem 2.

**Problem 3.** Find a displacement field  $\mathbf{u} \in U$  such that

$$\begin{aligned} & \int_{\Omega} \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})) dx + \int_{\Omega} \beta(\boldsymbol{\varepsilon}(\mathbf{u}) - P_B \boldsymbol{\varepsilon}(\mathbf{u})) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})) dx \\ & + \int_{\Gamma_3} \omega j_v^0(u_v; v_v - u_v) da \geq \int_{\Omega} \mathbf{f}_0 \cdot (\mathbf{v} - \mathbf{u}) dx + \int_{\Gamma_2} \mathbf{f}_2 \cdot (\mathbf{v} - \mathbf{u}) da \\ & \text{for all } \mathbf{v} \in U. \end{aligned} \quad (3.16)$$

In the study of Problem 3 we have the following existence and uniqueness result.

**Theorem 3.1.** Assume that (3.7)–(3.14) hold and, in addition, assume the smallness condition

$$\alpha_{j_v} \|\omega\|_{L^\infty(\Gamma_3)} \|\gamma\|^2 < m_{\mathcal{A}}. \quad (3.17)$$

Then Problem 3 has a unique solution  $\mathbf{u} \in U$ .

**Proof.** The proof of Theorem 3.1 is carried out in three steps, based on arguments similar to those used in [19,37]. For this reason, in order to avoid repetitions, we restrict ourselves to sketch the main steps of the proof.

(i) *An intermediate hemivariational inequality.* We consider the operator  $A: V \rightarrow V^*$ , the function  $j: V \rightarrow \mathbb{R}$  and the element  $\mathbf{f} \in V^*$  defined by

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_{\Omega} \beta(\boldsymbol{\varepsilon}(\mathbf{u}) - P_B \boldsymbol{\varepsilon}(\mathbf{u})) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx, \quad (3.18)$$

$$j(\mathbf{v}) = \int_{\Gamma_3} \omega j_v(v_v) \, da, \quad (3.19)$$

$$\langle \mathbf{f}, \mathbf{v} \rangle = \int_{\Omega} \mathbf{f}_0 \cdot \mathbf{v} \, dx + \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v} \, da, \quad (3.20)$$

for all  $\mathbf{u}, \mathbf{v} \in V$ . Then, we consider the problem of finding an element  $\mathbf{u}$  such that

$$\mathbf{u} \in U, \quad \langle A\mathbf{u}, \mathbf{v} - \mathbf{u} \rangle + j^0(\mathbf{u}; \mathbf{v} - \mathbf{u}) \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle \quad \text{for all } \mathbf{v} \in U. \quad (3.21)$$

We claim that the hemivariational inequality (3.21) has a unique solution. To this end we apply Theorem 2.6 with  $X = V$  and  $K = U$ . First, we use the definition (3.15) and assumption (3.13) to see that  $U$  is a closed convex subset of  $V$  such that  $\mathbf{0}_V \in U$  and, therefore, condition (2.2) is satisfied. Next, we use the definition (3.18), assumptions (3.7)–(3.9) and the properties of the projection operator to see that

$$(A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_V \geq m_{\mathcal{A}} \|\mathbf{u} - \mathbf{v}\|_V^2 \quad \forall \mathbf{u}, \mathbf{v} \in V, \quad (3.22)$$

$$\|A\mathbf{u} - A\mathbf{v}\|_{V^*} \leq (L_{\mathcal{A}} + 2\|\beta\|_{L^\infty(\Omega)}) \|\mathbf{u} - \mathbf{v}\|_V \quad \forall \mathbf{u}, \mathbf{v} \in V. \quad (3.23)$$

We conclude from here that  $A$  is a strongly monotone Lipschitz continuous operator on the space  $V$  and, therefore, it satisfies condition (2.3).

Next, using definition (3.19), assumptions (3.14), (3.12) and Lemma 8 in [19, p. 126] we deduce that  $j$  is a locally Lipschitz function on  $V$ , satisfies condition (2.4) with  $c_0 = \sqrt{2 \operatorname{meas}(\Gamma_3)} \bar{c}_0 \|\omega\|_{L^\infty(\Gamma_3)} \|\gamma\|$ ,  $c_1 = \sqrt{2} \bar{c}_1 \|\omega\|_{L^\infty(\Gamma_3)} \|\gamma\|^2$  and  $\alpha_j = \alpha_{j_v} \|\omega\|_{L^\infty(\Gamma_3)} \|\gamma\|^2$  and, moreover,

$$j^0(\mathbf{u}; \mathbf{v}) \leq \int_{\Gamma_3} \omega j_v^0(u_v; v_v) \, da \quad \text{for all } \mathbf{u}, \mathbf{v} \in V. \quad (3.24)$$

In addition, assumptions (3.10) and (3.11) imply (2.5) for  $\mathbf{f}$ . Finally, assumption (3.17) implies the smallness condition (2.6). Therefore, we are in a position to use Theorem 2.6. In this way we deduce that there exists a unique element  $\mathbf{u}$  which solves (3.21), as claimed.

(ii) *Existence.* We combine now (3.21) with inequality (3.24) and notation (3.18), (3.20) to see that the solution  $\mathbf{u}$  of (3.21) satisfies (3.16), which proves the existence part of the theorem.

(iii) *Uniqueness.* Let  $\mathbf{u}_1, \mathbf{u}_2 \in U$  be solutions to inequality (3.16). Then, using notation (3.18)–(3.20) we deduce that

$$\langle A\mathbf{u}_1, \mathbf{v} - \mathbf{u}_1 \rangle + \int_{\Gamma_3} \omega j_v^0(u_{1v}; v_v - u_{1v}) \, da \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u}_1 \rangle,$$

$$\langle A\mathbf{u}_2, \mathbf{v} - \mathbf{u}_2 \rangle + \int_{\Gamma_3} \omega j_v^0(u_{2v}; v_v - u_{2v}) \, da \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u}_1 \rangle$$

for all  $\mathbf{v} \in U$ . We take  $\mathbf{v} = \mathbf{u}_2$  in the first inequality and  $\mathbf{v} = \mathbf{u}_1$  in the second one, and add the resulting inequalities. Then, by using the strong monotonicity of the operator  $A$ , (3.22), hypotheses (3.14), (3.12) and the trace inequality (2.9) we obtain that

$$(m_{\mathcal{A}} - \alpha_{j_v} \|\omega\|_{L^\infty(\Gamma_3)} \|\gamma\|^2) \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2 \leq 0.$$

Finally, we use the smallness condition (3.17) to deduce that  $\mathbf{u}_1 = \mathbf{u}_2$ , which concludes the proof.  $\square$

#### 4. A convergence result

In this section we state and prove a convergence result which shows the continuous dependence of the weak solution of Problem 2 with respect to the data and parameters. To this end, we assume in what follows that (3.7)–(3.14) and (3.17)

hold. Therefore, using Theorem 3.1 we deduce that Problem 3 has a unique solution  $\mathbf{u} \in U$ . The solution depends on the data and, therefore, we shall denote it by  $\mathbf{u} = \mathbf{u}(\beta, \mathbf{f}_0, \mathbf{f}_2, \omega, g)$ . The proof of Theorem 3.1 also shows that the solution satisfies the variational–hemivariational inequality (3.21).

Next, for each  $n \in \mathbb{N}$ , consider a perturbation  $\beta_n, \mathbf{f}_{0n}, \mathbf{f}_{2n}, \omega_n, g_n$  of  $\beta, \mathbf{f}_0, \mathbf{f}_2, \omega$  and  $g$ , respectively, such that

$$\beta_n \in L^\infty(\Omega), \quad \beta_n(\mathbf{x}) \geq 0 \quad \text{a.e. } \mathbf{x} \in \Omega. \quad (4.1)$$

$$\mathbf{f}_{0n} \in L^2(\Omega)^d. \quad (4.2)$$

$$\mathbf{f}_{2n} \in L^2(\Gamma_2)^d. \quad (4.3)$$

$$\omega_n \in L^\infty(\Gamma_3), \quad \omega_n(\mathbf{x}) \geq 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3. \quad (4.4)$$

$$g_n > 0. \quad (4.5)$$

$$\begin{cases} \text{There exists } m_0 > 0 \text{ such that} \\ \|\alpha_{j_v} \|\omega_n\|_{L^\infty(\Gamma_3)}\|\gamma\|^2 < m_{\mathcal{A}} - m_0 \quad \forall n \in \mathbb{N}. \end{cases} \quad (4.6)$$

Let

$$U_n = \{\mathbf{v} \in V : v_v \leq g_n \quad \text{a.e. on } \Gamma_3\}. \quad (4.7)$$

With these data we consider the following perturbed version of Problem 3.

**Problem 4.** Find a displacement field  $\mathbf{u}_n \in U_n$  such that

$$\begin{aligned} & \int_{\Omega} \mathcal{A}\varepsilon(\mathbf{u}_n) \cdot (\varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}_n)) dx + \int_{\Omega} \beta_n (\varepsilon(\mathbf{u}_n) - P_B \varepsilon(\mathbf{u}_n)) \cdot (\varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}_n)) dx \\ & + \int_{\Gamma_3} \omega_n j_v^0(u_{nv}; v_v - u_{nv}) da \geq \int_{\Omega} \mathbf{f}_{0n} \cdot (\mathbf{v} - \mathbf{u}_n) dx + \int_{\Gamma_2} \mathbf{f}_{2n} \cdot (\mathbf{v} - \mathbf{u}_n) da \\ & \text{for all } \mathbf{v} \in U_n. \end{aligned} \quad (4.8)$$

Using Theorem 3.1 we deduce that Problem 4 has a unique solution  $\mathbf{u}_n \in U_n$ . The solution depends on the perturbed data and, therefore, sometimes we shall write  $\mathbf{u}_n = \mathbf{u}(\beta_n, \mathbf{f}_{0n}, \mathbf{f}_{2n}, \omega_n, g_n)$ . The proof of Theorem 3.1 also shows that the solution satisfies the variational–hemivariational inequality

$$\mathbf{u}_n \in U_n, \quad \langle A_n \mathbf{u}_n, \mathbf{v} - \mathbf{u}_n \rangle + j_n^0(\mathbf{u}_n; \mathbf{v} - \mathbf{u}_n) \geq \langle \mathbf{f}_n, \mathbf{v} - \mathbf{u}_n \rangle, \quad \forall \mathbf{v} \in U_n, \quad (4.9)$$

where, here and below, the operator  $A_n: V \rightarrow V^*$ , the function  $j_n: V \rightarrow \mathbb{R}$  and the element  $\mathbf{f}_n \in V^*$  are defined by

$$\langle A_n \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mathcal{A}\varepsilon(\mathbf{u}) \cdot \varepsilon(\mathbf{v}) dx + \int_{\Omega} \beta_n (\varepsilon(\mathbf{u}) - P_B \varepsilon(\mathbf{u})) \cdot \varepsilon(\mathbf{v}) dx, \quad (4.10)$$

$$j_n(\mathbf{v}) = \int_{\Gamma_3} \omega_n j_v(v_v) da, \quad (4.11)$$

$$\langle \mathbf{f}_n, \mathbf{v} \rangle = \int_{\Omega} \mathbf{f}_{0n} \cdot \mathbf{v} dx + \int_{\Gamma_2} \mathbf{f}_{2n} \cdot \mathbf{v} da \quad (4.12)$$

for all  $\mathbf{u}, \mathbf{v} \in V$ . Our main result in this section is as follows.

**Theorem 4.1.** Assume that (3.7)–(3.14), (3.17), (4.1)–(4.6) hold. Moreover, assume that

$$\beta_n \rightarrow \beta \quad \text{in } L^\infty(\Omega), \quad (4.13)$$

$$\mathbf{f}_{0n} \rightharpoonup \mathbf{f}_0 \quad \text{in } L^2(\Omega)^d, \quad (4.14)$$

$$\mathbf{f}_{2n} \rightharpoonup \mathbf{f}_2 \quad \text{in } L^2(\Gamma_2)^d, \quad (4.15)$$

$$\omega_n \rightarrow \omega \quad \text{in } L^\infty(\Gamma_3), \quad (4.16)$$

$$g_n \rightarrow g. \quad (4.17)$$

Then, the solution  $\mathbf{u}_n$  of Problem 4 converges to the solution  $\mathbf{u}$  of Problem 3, i.e.,

$$\mathbf{u}_n \rightarrow \mathbf{u} \quad \text{in } V. \quad (4.18)$$

In order to present the proof of Theorem 4.1 we need the following intermediate result which will be used in several places below.

**Lemma 4.2.** Under the assumptions of Theorem 4.1, for any sequences  $\{\mathbf{u}_n\}, \{\mathbf{v}_n\} \subset V$  such that  $\mathbf{u}_n \rightharpoonup \mathbf{u}$  in  $V$ ,  $\mathbf{v}_n \rightarrow \mathbf{v}$  in  $V$ , the statements below hold.

$$A_n \mathbf{u}_n - A \mathbf{u}_n \rightarrow \mathbf{0}_{V^*} \quad \text{in } V^*, \quad (4.19)$$

$$\limsup j_n^0(\mathbf{u}_n; \mathbf{v}_n - \mathbf{u}_n) \leq \int_{\Gamma_3} \omega j_v^0(u_v; v_v - u_v) da, \quad (4.20)$$

$$\langle \mathbf{f}_n, \mathbf{u}_n - \mathbf{v}_n \rangle \rightarrow \langle \mathbf{f}, \mathbf{u} - \mathbf{v} \rangle. \quad (4.21)$$

**Proof.** Let  $n \in \mathbb{N}$  and let  $\mathbf{v} \in V$ . We use definition (3.18), assumption (3.8) and the property of the projection operator to see that

$$\begin{aligned} \langle A_n \mathbf{u}_n - A \mathbf{u}_n, \mathbf{v} \rangle &= \int_{\Omega} (\beta_n - \beta) (\boldsymbol{\varepsilon}(\mathbf{u}_n) - P_B \boldsymbol{\varepsilon}(\mathbf{u}_n)) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) dx \\ &\leq \|\beta_n - \beta\|_{L^\infty(\Omega)} \|\boldsymbol{\varepsilon}(\mathbf{u}_n) - P_B \boldsymbol{\varepsilon}(\mathbf{u}_n)\|_Q \|\boldsymbol{\varepsilon}(\mathbf{v})\|_Q \\ &\leq 2 \|\beta_n - \beta\|_{L^\infty(\Omega)} \|\mathbf{u}_n\|_V \|\mathbf{v}\|_V, \end{aligned}$$

which implies that

$$\|A_n \mathbf{u} - A \mathbf{u}_n\|_{V^*} \leq 2 \|\beta_n - \beta\|_{L^\infty(\Omega)} \|\mathbf{u}_n\|_V.$$

The convergence (4.19) is now a consequence of assumption (4.13), since the sequence  $\{\mathbf{u}_n\}$  is bounded in  $V$  due to its weak convergence.

Next, we use inequality (3.24) to see that

$$j_n^0(\mathbf{u}_n; \mathbf{v}_n - \mathbf{u}_n) \leq \int_{\Gamma_3} \omega_n j_v^0(u_{nv}; v_{nv} - u_{nv}) da \quad \forall n \in \mathbb{N}.$$

We now use assumption (4.16), the compactness of the trace operator and the upper semicontinuity of  $j_v^0$ , guaranteed by Proposition 2.4(ii), to see that (4.20) holds.

Finally, the convergence (4.21) follows from assumptions (4.14), (4.15), the compactness of the embedding  $V \subset L^2(\Omega)^d$  and the compactness of the trace operator.  $\square$

We now have all the ingredients to provide the proof of Theorem 4.1 which is structured in several steps.

**Proof.** (i) *A uniform bound.* We claim that the sequence  $\{\mathbf{u}_n\} \subset V$  is bounded. To prove this claim we fix  $n \in \mathbb{N}$ . Since  $\mathbf{0}_V \in U_n$  we may take  $\mathbf{v} = \mathbf{0}_V$  in (4.9) to see that

$$\langle A \mathbf{u}_n, \mathbf{u}_n \rangle \leq j_n^0(\mathbf{u}_n; -\mathbf{u}_n) + \langle \mathbf{f}_n, \mathbf{u}_n \rangle. \quad (4.22)$$

Moreover, note that assumptions (3.7)(e) and (3.8) imply that  $A \mathbf{0} = \mathbf{0}_{V^*}$  and, therefore inequality (3.22) yields

$$m_{\mathcal{A}} \|\mathbf{u}_n\|_V^2 \leq \langle A \mathbf{u}_n, \mathbf{u}_n \rangle. \quad (4.23)$$

In addition, using arguments similar to those used in the proof of Theorem 3.1 it follows that  $j_n$  satisfies condition (2.4) on the space  $V$  with the constants  $c_0 = \sqrt{2 \text{meas}(\Gamma_3)} \bar{c}_0 \|\omega_n\|_{L^\infty(\Gamma_3)} \|\gamma\|$ ,  $c_1 = \sqrt{2} \bar{c}_1 \|\omega_n\|_{L^\infty(\Gamma_3)} \|\gamma\|^2$  and  $\alpha_j = \alpha_{j_v} \|\omega_n\|_{L^\infty(\Gamma_3)} \|\gamma\|^2$ . Therefore,

$$j_n^0(\mathbf{u}_n; -\mathbf{u}_n) \leq \alpha_{j_v} \|\omega_n\|_{L^\infty(\Gamma_3)} \|\gamma\|^2 \|\mathbf{u}_n\|_V^2 - j_n^0(\mathbf{0}_V; \mathbf{u}_n) \quad (4.24)$$

and, using Proposition 2.4(iii) we have

$$-j_n^0(\mathbf{0}_V; \mathbf{u}_n) \leq |j_n^0(\mathbf{0}_V; \mathbf{u}_n)| \leq c_0 \|\mathbf{u}_n\|_V. \quad (4.25)$$

In addition

$$\langle \mathbf{f}_n, \mathbf{u}_n \rangle \leq (\|\mathbf{f}_{0n}\|_{L^2(\Omega)^d} + \|\mathbf{f}_{2n}\|_{L^2(\Gamma_2)^d} \|\gamma\|) \|\mathbf{u}_n\|_V. \quad (4.26)$$

We now combine inequalities (4.22)–(4.26) to see that

$$(m_{\mathcal{A}} - \alpha_{j_v} \|\omega_n\|_{L^\infty(\Gamma_3)} \|\gamma\|^2) \|\mathbf{u}_n\|_V \leq (c_0 + \|\mathbf{f}_{0n}\|_{L^2(\Omega)^d} + \|\mathbf{f}_{2n}\|_{L^2(\Gamma_2)^d} \|\gamma\|),$$

then we use assumption (4.6) to deduce that

$$\|\mathbf{u}_n\|_V \leq \frac{1}{m_0} (c_0 + \|\mathbf{f}_{0n}\|_{L^2(\Omega)^d} + \|\mathbf{f}_{2n}\|_{L^2(\Gamma_2)^d} \|\gamma\|). \quad (4.27)$$

We now use the convergences (4.14) and (4.15) to see that the sequences  $\{\mathbf{f}_{0n}\}$  and  $\{\mathbf{f}_{2n}\}$  are bounded in  $L^2(\Omega)^d$  and  $L^2(\Gamma_2)^d$ , respectively. Therefore, inequality (4.27) shows that there exists a constant  $C > 0$ , which does not depend on  $n$ , such that  $\|\mathbf{u}_n\|_V \leq C$ , which concludes the proof of the claim.

(ii) *Weak convergence.* We now claim that the sequence  $\{\mathbf{u}_n\}$  converges weakly to  $\mathbf{u}$ , i.e.,

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } V. \quad (4.28)$$

To this end we note that step (i) and the reflexivity of the space  $V$  imply that there exist a subsequence, denoted again by  $\{\mathbf{u}_n\}$ , and an element  $\tilde{\mathbf{u}} \in V$  such that

$$\mathbf{u}_n \rightharpoonup \tilde{\mathbf{u}} \quad \text{in } V. \quad (4.29)$$

This convergence combined with (4.17) implies that  $\frac{g}{g_n} \mathbf{u}_n \rightharpoonup \tilde{\mathbf{u}}$  in  $V$  and, since  $\frac{g}{g_n} \mathbf{u}_n \in U$  we find that

$$\tilde{\mathbf{u}} \in U. \quad (4.30)$$

Let  $n \in \mathbb{N}$ , and denote  $\mathbf{v}_n = \frac{g_n}{g} \tilde{\mathbf{u}}$ . Then

$$\mathbf{v}_n \rightarrow \tilde{\mathbf{u}} \quad \text{in } V. \quad (4.31)$$

Moreover,  $\mathbf{v}_n \in U_n$  and, using (4.9) we obtain that

$$\langle A_n \mathbf{u}_n, \mathbf{u}_n - \mathbf{v}_n \rangle \leq j_n^0(\mathbf{u}_n; \mathbf{v}_n - \mathbf{u}_n) + \langle \mathbf{f}_n, \mathbf{u}_n - \mathbf{v}_n \rangle. \quad (4.32)$$

Therefore,

$$\begin{aligned} \langle A \mathbf{u}_n, \mathbf{u}_n - \tilde{\mathbf{u}} \rangle &\leq \langle A \mathbf{u}_n, \mathbf{v}_n - \tilde{\mathbf{u}} \rangle \\ &+ \langle A \mathbf{u}_n - A_n \mathbf{u}_n, \mathbf{u}_n - \mathbf{v}_n \rangle + j_n^0(\mathbf{u}_n; \mathbf{v}_n - \mathbf{u}_n) + \langle \mathbf{f}_n, \mathbf{u}_n - \mathbf{v}_n \rangle. \end{aligned} \quad (4.33)$$

We now use the boundedness of the operator  $A$  and the strong convergence (4.31) to see that

$$\langle A \mathbf{u}_n, \mathbf{v}_n - \tilde{\mathbf{u}} \rangle \rightarrow 0. \quad (4.34)$$

Moreover, the convergences (4.29) and (4.31), Lemma 4.2, and Proposition 2.4(ii) imply that

$$\langle A \mathbf{u}_n - A_n \mathbf{u}_n, \mathbf{u}_n - \mathbf{v}_n \rangle \rightarrow 0, \quad (4.35)$$

$$\limsup j_n^0(\mathbf{u}_n; \mathbf{v}_n - \mathbf{u}_n) \leq 0, \quad (4.36)$$

$$\langle \mathbf{f}_n, \mathbf{u}_n - \mathbf{v}_n \rangle \rightarrow 0. \quad (4.37)$$

We now pass to the upper limit in (4.33) and use (4.34)–(4.37) to deduce that

$$\limsup \langle A \mathbf{u}_n, \mathbf{u}_n - \tilde{\mathbf{u}} \rangle \leq 0. \quad (4.38)$$

Therefore, using the pseudomonotonicity of the operator  $A$  we deduce that

$$\langle A \tilde{\mathbf{u}}, \tilde{\mathbf{u}} - \mathbf{v} \rangle \leq \liminf \langle A \mathbf{u}_n, \mathbf{u}_n - \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in V. \quad (4.39)$$

Let  $n \in \mathbb{N}$  and  $\mathbf{v} \in U$ . We let  $\mathbf{v}_n = \frac{g_n}{g} \mathbf{v}$  in (4.9), then we use the convergence  $\mathbf{v}_n \rightarrow \mathbf{v}$  in  $V$ , Lemma 4.2 and arguments similar as above to obtain that

$$\limsup \langle A \mathbf{u}_n, \mathbf{u}_n - \mathbf{v} \rangle \leq \int_{\Gamma_3} \omega j_v^0(\tilde{\mathbf{u}}_v; v_v - \tilde{\mathbf{u}}_v) da + \langle \mathbf{f}, \tilde{\mathbf{u}} - \mathbf{v} \rangle. \quad (4.40)$$

Inequalities (4.39) and (4.40) imply that

$$\langle A \tilde{\mathbf{u}}, \tilde{\mathbf{u}} - \mathbf{v} \rangle \leq \int_{\Gamma_3} \omega j_v^0(\tilde{\mathbf{u}}_v; v_v - \tilde{\mathbf{u}}_v) da + \langle \mathbf{f}, \tilde{\mathbf{u}} - \mathbf{v} \rangle. \quad (4.41)$$

This inequality shows that  $\tilde{\mathbf{u}}$  is a solution of Problem 3 and, by the uniqueness of its solution, guaranteed by Theorem 3.1, we obtain that  $\tilde{\mathbf{u}} = \mathbf{u}$ . This implies that the whole sequence  $\{\mathbf{u}_n\}$  converges weakly to  $\mathbf{u}$  as  $n \rightarrow \infty$ , which concludes the proof of the claim.

(iii) *Strong convergence.* Let  $n \in \mathbb{N}$  and  $\tilde{\mathbf{u}}_n = \frac{g_n}{g} \mathbf{u}$ . Since  $\mathbf{u} \in U$  it follows that  $\tilde{\mathbf{u}}_n \in U_n$  and, moreover, (4.17) implies that

$$\tilde{\mathbf{u}}_n \rightarrow \mathbf{u} \quad \text{in } V. \quad (4.42)$$

We now write inequality (4.9) with  $\mathbf{v} = \tilde{\mathbf{u}}_n$  to deduce that

$$- \langle A_n \mathbf{u}_n, \tilde{\mathbf{u}}_n - \mathbf{u}_n \rangle \leq j_n^0(\mathbf{u}_n; \tilde{\mathbf{u}}_n - \mathbf{u}_n) + \langle \mathbf{f}_n, \mathbf{u}_n - \tilde{\mathbf{u}}_n \rangle. \quad (4.43)$$

Next, using the strong monotonicity of the operator  $A_n$  we find that

$$\begin{aligned} m_A \|\tilde{\mathbf{u}}_n - \mathbf{u}_n\|_V^2 &\leq \langle A_n \tilde{\mathbf{u}}_n - A_n \mathbf{u}_n, \tilde{\mathbf{u}}_n - \mathbf{u}_n \rangle \\ &= \langle A_n \tilde{\mathbf{u}}_n, \tilde{\mathbf{u}}_n - \mathbf{u}_n \rangle - \langle A_n \mathbf{u}_n, \tilde{\mathbf{u}}_n - \mathbf{u}_n \rangle \\ &= \langle A_n \tilde{\mathbf{u}}_n - A \tilde{\mathbf{u}}_n, \tilde{\mathbf{u}}_n - \mathbf{u}_n \rangle + \langle A \tilde{\mathbf{u}}_n - A_n \mathbf{u}_n, \tilde{\mathbf{u}}_n - \mathbf{u}_n \rangle \end{aligned} \quad (4.44)$$

and, therefore, (4.43) implies that

$$\begin{aligned} m_A \|\tilde{\mathbf{u}}_n - \mathbf{u}_n\|_V^2 &\leq \langle A_n \tilde{\mathbf{u}}_n - A \tilde{\mathbf{u}}_n, \tilde{\mathbf{u}}_n - \mathbf{u}_n \rangle + \langle A \tilde{\mathbf{u}}_n, \tilde{\mathbf{u}}_n - \mathbf{u}_n \rangle \\ &\quad + j_n^0(\mathbf{u}_n; \tilde{\mathbf{u}}_n - \mathbf{u}_n) + \langle \mathbf{f}_n, \mathbf{u}_n - \tilde{\mathbf{u}}_n \rangle. \end{aligned} \quad (4.45)$$

We now use the convergences (4.28), (4.42) and the continuity of the operator  $A$  to find that

$$\lim \langle A \tilde{\mathbf{u}}_n, \tilde{\mathbf{u}}_n - \mathbf{u}_n \rangle \rightarrow 0. \quad (4.46)$$

Moreover, the convergences (4.28), (4.42) combined with the statements (4.19)–(4.21) in Lemma 4.2 and Proposition 2.4(iii) imply that

$$\langle A_n \tilde{\mathbf{u}}_n - A \tilde{\mathbf{u}}_n, \tilde{\mathbf{u}}_n - \mathbf{u}_n \rangle \rightarrow 0, \quad (4.47)$$

$$\limsup j_n^0(\mathbf{u}_n; \tilde{\mathbf{u}}_n - \mathbf{u}_n) \leq 0, \quad (4.48)$$

$$\langle \mathbf{f}_n, \mathbf{u}_n - \tilde{\mathbf{u}}_n \rangle \rightarrow 0. \quad (4.49)$$

Therefore, passing to the upper limit in (4.45) and using (4.46)–(4.49) we deduce that

$$\|\mathbf{u}_n - \tilde{\mathbf{u}}_n\|_V \rightarrow 0. \quad (4.50)$$

Finally, we write

$$0 \leq \|\mathbf{u}_n - \mathbf{u}\|_V \leq \|\mathbf{u}_n - \tilde{\mathbf{u}}_n\|_V + \|\tilde{\mathbf{u}}_n - \mathbf{u}\|_V$$

and use the convergences (4.42), (4.50) to see that  $\mathbf{u}_n \rightarrow \mathbf{u}$  in  $V$  which concludes the proof.  $\square$

We end this section with the remark that Theorem 4.1 shows the convergence of the solution of Problem 2 with respect to the data and parameters. Indeed, recall that  $\mathbf{u}_n = \mathbf{u}(\beta_n, \mathbf{f}_{0n}, \mathbf{f}_{2n}, \omega_n, \mathbf{g}_n)$  and  $\mathbf{u} = \mathbf{u}(\beta, \mathbf{f}_0, \mathbf{f}_2, \omega, \mathbf{g})$ . Then (4.18) shows that

$$\mathbf{u}(\beta_n, \mathbf{f}_{0n}, \mathbf{f}_{2n}, \omega_n, \mathbf{g}_n) \rightarrow \mathbf{u}(\beta, \mathbf{f}_0, \mathbf{f}_2, \omega, \mathbf{g}) \quad \text{in } V, \quad (4.51)$$

provided that (4.13)–(4.17) hold. In addition to the mathematical interest in this convergence result it is important from mechanical point of view, since it shows that the weak solution of the contact Problem 2 depends continuously on the elasticity coefficient, the densities of body forces and tractions, the stiffness coefficient and the thickness of the deformable layer.

## 5. An optimal control problem

In this section we study a boundary control problem associated to Problem 3. Everywhere below we use the notation  $V \times L^2(\Gamma_2)^d$  for the product of the Hilbert spaces  $V$  and  $L^2(\Gamma_2)^d$ , equipped with the canonical topology product. Also,  $U \times W$  will represent the Cartesian product of the sets  $U$  and  $W$ .

Let  $W \subset L^2(\Gamma_3)^d$  and let  $\beta, \mathbf{f}_0, \omega$ , and  $\mathbf{g}$  be given. We define the set of admissible pairs for Problem 3 by equality

$$\mathcal{V}_{ad} = \{(\mathbf{u}, \mathbf{f}_2) \in U \times W : \mathbf{u} = \mathbf{u}(\beta, \mathbf{f}_0, \mathbf{f}_2, \omega, \mathbf{g})\}. \quad (5.1)$$

In other words, a pair  $(\mathbf{u}, \mathbf{f}_2)$  belongs to  $\mathcal{V}_{ad}$  if and only if  $\mathbf{f}_2 \in W$  and, moreover,  $\mathbf{u}$  is the solution of Problem 3 or, equivalently, of the variational–hemivariational inequality (3.21). Consider also a cost functional  $\mathcal{L} : V \times L^2(\Gamma_3)^d \rightarrow \mathbb{R}$ . Then, the optimal control problem we are interested in is as follows.

**Problem 5.** Find  $(\mathbf{u}^*, \mathbf{f}_2^*) \in \mathcal{V}_{ad}$  such that

$$\mathcal{L}(\mathbf{u}^*, \mathbf{f}_2^*) = \min_{(\mathbf{u}, \mathbf{f}_2) \in \mathcal{V}_{ad}} \mathcal{L}(\mathbf{u}, \mathbf{f}_2). \quad (5.2)$$

To solve Problem 5 we consider the following assumptions.

$$\begin{cases} W \text{ is a nonempty weakly closed subset of } L^2(\Gamma_3)^d, \text{ i.e.,} \\ \{\mathbf{f}_{2n}\} \subset W, \mathbf{f}_{2n} \rightharpoonup \mathbf{f}_2 \text{ in } L^2(\Gamma_3)^d \implies \mathbf{f}_2 \in W. \end{cases} \quad (5.3)$$

$$\begin{cases} \text{For all sequences } \{\mathbf{u}_n\} \subset V \text{ and } \{\mathbf{f}_{2n}\} \subset L^2(\Gamma_3)^d \text{ such that} \\ \mathbf{u}_n \rightarrow \mathbf{u} \text{ in } V, \mathbf{f}_{2n} \rightharpoonup \mathbf{f}_2 \text{ in } L^2(\Gamma_3)^d, \text{ we have} \\ \liminf \mathcal{L}(\mathbf{u}_n, \mathbf{f}_{2n}) \geq \mathcal{L}(\mathbf{u}, \mathbf{f}_2). \end{cases} \quad (5.4)$$

$$\begin{cases} \text{There exists } h : W \rightarrow \mathbb{R} \text{ such that} \\ \text{(a) } \mathcal{L}(\mathbf{u}, \mathbf{f}_2) \geq h(\mathbf{f}_2) \quad \forall \mathbf{u} \in V, \mathbf{f}_2 \in L^2(\Gamma_3)^d. \\ \text{(b) } \|\mathbf{f}_{2n}\|_{L^2(\Gamma_3)^d} \rightarrow +\infty \implies h(\mathbf{f}_{2n}) \rightarrow \infty. \end{cases} \quad (5.5)$$

$$W \text{ is a bounded subset of } L^2(\Gamma_3)^d. \quad (5.6)$$

Our main result in this section is as follows.

**Theorem 5.1.** Assume that (3.7)–(3.10), (3.12)–(3.14), (3.17), (5.3)–(5.4) hold. Moreover, assume that either one of the conditions (5.5) or (5.6) holds. Then, there exists at least one solution  $(\mathbf{u}^*, \mathbf{f}_2^*) \in \mathcal{V}_{ad}$  of Problem 5.

**Proof.** Let

$$\theta = \inf_{(\mathbf{u}, \mathbf{f}_2) \in \mathcal{V}_{ad}} \mathcal{L}(\mathbf{u}, \mathbf{f}_2) \in [-\infty, +\infty) \quad (5.7)$$

and let  $\{(\mathbf{u}_n, \mathbf{f}_{2n})\} \subset \mathcal{V}_{ad}$  be a minimizing sequence for the functional  $\mathcal{L}$ , i.e.

$$\lim \mathcal{L}(\mathbf{u}_n, \mathbf{f}_{2n}) = \theta. \quad (5.8)$$

We claim that the sequence  $\{\mathbf{f}_{2n}\}$  is bounded in  $L^2(\Gamma_3)^d$ . Indeed, the claim is clearly valid if (5.6) holds, i.e., if  $W$  is a bounded subset of  $L^2(\Gamma_3)^d$ . Assume in what follows that (5.5) holds. Arguing by contradiction, assume that  $\{\mathbf{f}_{2n}\}$  is not bounded in  $L^2(\Gamma_3)^d$ . Then, passing to a subsequence still denoted by  $\{\mathbf{f}_{2n}\}$ , we have

$$\|\mathbf{f}_{2n}\|_{L^2(\Gamma_3)^d} \rightarrow +\infty \quad \text{as } n \rightarrow +\infty. \quad (5.9)$$

We now use assumption (5.5)(a) to see that

$$\mathcal{L}(\mathbf{u}_n, \mathbf{f}_{2n}) \geq h(\mathbf{f}_{2n}).$$

Therefore, passing to the limit as  $n \rightarrow +\infty$ , we deduce by (5.5) and (5.9)(b) that

$$\lim \mathcal{L}(\mathbf{u}_n, \mathbf{f}_{2n}) = +\infty. \quad (5.10)$$

Equalities (5.8) and (5.10) imply that  $\theta = +\infty$  which is in contradiction with (5.7).

We conclude from above that the sequence  $\{\mathbf{f}_{2n}\}$  is bounded in  $L^2(\Gamma_3)^d$  as claimed. Therefore, there exists  $\mathbf{f}_2^* \in L^2(\Gamma_3)^d$  such that, passing to a subsequence still denoted  $\{\mathbf{f}_{2n}\}$ , we have

$$\mathbf{f}_{2n} \rightharpoonup \mathbf{f}_2^* \quad \text{in } L^2(\Gamma_3)^d \quad \text{as } n \rightarrow +\infty. \quad (5.11)$$

This convergence combined with assumption (5.3) implies that  $\mathbf{f}_2^* \in W$ .

Let  $\mathbf{u}^*$  be the solution of the variational inequality (3.21) for  $\mathbf{f}_2 = \mathbf{f}_2^*$ , i.e.,  $\mathbf{u}^* = \mathbf{u}(\beta, \mathbf{f}_0, \mathbf{f}_2^*, \omega, g)$ . Then, using the definition (5.1) of the set  $\mathcal{V}_{ad}$  we have

$$(\mathbf{u}^*, \mathbf{f}_2^*) \in \mathcal{V}_{ad}. \quad (5.12)$$

Moreover, since  $\mathbf{u}_n = \mathbf{u}(\beta, \mathbf{f}_0, \mathbf{f}_{2n}, \omega, g)$ , it follows from (4.51) and (5.11) that

$$\mathbf{u}_n \rightarrow \mathbf{u}^* \quad \text{in } V \quad \text{as } n \rightarrow +\infty. \quad (5.13)$$

We now use the convergences (5.11), (5.13) and assumptions (5.4) to deduce that

$$\liminf \mathcal{L}(\mathbf{u}_n, \mathbf{f}_{2n}) \geq \mathcal{L}(\mathbf{u}^*, \mathbf{f}_2^*). \quad (5.14)$$

It follows now from (5.8) and (5.14) that

$$\theta \geq \mathcal{L}(\mathbf{u}^*, \mathbf{f}_2^*). \quad (5.15)$$

In addition, (5.7) and (5.12) yield

$$\theta \leq \mathcal{L}(\mathbf{u}^*, \mathbf{f}_2^*). \quad (5.16)$$

We now combine (5.12) with inequalities (5.15) and (5.16) to see that (5.2) holds, which concludes the proof.  $\square$

We end this section with two examples of optimal control problems for which the result provided by Theorem 5.1 holds.

**Example 5.2.** Let

$$\begin{aligned} W &= \{\mathbf{f}_2 \in L^2(\Gamma_3)^d : \mathbf{f}_2(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \leq 0 \text{ a.e. } \mathbf{x} \in \Gamma_2\}, \\ &\text{and for all } (\mathbf{u}, \mathbf{f}_2) \in V \times L^2(\Gamma_3)^d, \\ \mathcal{L}(\mathbf{u}, \mathbf{f}_2) &= \alpha \int_{\Gamma_3} \|u_v - \phi\|^2 da + \delta \int_{\Gamma_2} \|\mathbf{f}_2\|^2 da, \end{aligned} \quad (5.17)$$

where  $\alpha$  and  $\delta$  are strictly positive constants and  $\phi \in L^2(\Gamma_3)$  is given. With this choice, the mechanical interpretation of Problem 5 is the following: we are looking for a surface pressure  $\mathbf{f}_2$  acting on  $\Gamma_2$  such that the corresponding normal displacement  $u_v$  is as close as possible to the “desired displacement”  $\phi$ . Furthermore, this choice has to fulfill a minimum expenditure condition which is taken into account by the last term in (5.17). Note that in this case conditions (5.3)–(5.5) are satisfied and, therefore, using Theorem 5.1 we deduce that Problem 5 has at least one solution.

**Example 5.3.** Let

$$W = \{\mathbf{f}_2 \in L^2(\Gamma_3)^d : \|\mathbf{f}_2\|_{L^2(\Gamma_3)^d} \leq M \text{ a.e. } \mathbf{x} \in \Gamma_2\},$$

$$\mathcal{L}(\mathbf{u}, \mathbf{f}_2) = \int_{\Omega} \|\boldsymbol{\varepsilon}(\mathbf{u})\|^2 dx \quad \text{for all } (\mathbf{u}, \mathbf{f}_2) \in V \times L^2(\Gamma_3)^d,$$

where  $M > 0$ . With this choice, the mechanical interpretation of Problem 5 is the following: we are looking for a bounded surface pressure or traction  $\mathbf{f}_2 \in W$  acting on  $\Gamma_2$  such that the corresponding deformation in the body is as small as possible. Note that in this case conditions (5.3)–(5.4) and (5.6) are satisfied and, therefore, Theorem 5.1 guarantees the existence of at least one solution of the corresponding optimal control problem.

## 6. Convergence of optimal pairs

In this section we focus on the dependence of the optimal pairs of Problem 5 with respect to the data  $\beta, \mathbf{f}_0, \omega, g$  and the set  $W$ . To this end, we assume in what follows that (3.7)–(3.10), (3.12)–(3.14), (3.17), (5.3)–(5.5) hold and, for each  $n \in \mathbb{N}$ , let  $\beta_n, \mathbf{f}_{0n}, \omega_n, g_n$  be a perturbation of  $\beta, \mathbf{f}_0, \omega, g$  which satisfies (4.1), (4.2), (4.4)–(4.6). Moreover, assume that  $W_n$  is a given set which satisfies condition (5.3). Then, we define the set of admissible pairs for inequality Problem 4 by equality

$$\mathcal{V}_{ad}^n = \{(\mathbf{u}_n, \mathbf{f}_{2n}) \in U_n \times W_n : \mathbf{u}_n = \mathbf{u}(\beta_n, \mathbf{f}_{0n}, \mathbf{f}_{2n}, \omega_n, g_n)\}. \quad (6.1)$$

In other words, a pair  $(\mathbf{u}_n, \mathbf{f}_{2n})$  belongs to  $\mathcal{V}_{ad}^n$  if and only if  $\mathbf{f}_{2n} \in W_n$  and, moreover,  $\mathbf{u}_n$  is the solution of Problem 4 or, equivalently, of the variational–hemivariational inequality (4.9), i.e.  $\mathbf{u}_n = \mathbf{u}(\beta_n, \mathbf{f}_{0n}, \mathbf{f}_{2n}, \omega_n, g_n)$ . Consider also a cost functional  $\mathcal{L} : V \times L^2(\Gamma_3)^d \rightarrow \mathbb{R}$ . Then, the optimal control problem we are interested in is as follows.

**Problem 6.** Find  $(\mathbf{u}_n^*, \mathbf{f}_{2n}^*) \in \mathcal{V}_{ad}^n$  such that

$$\mathcal{L}(\mathbf{u}_n^*, \mathbf{f}_{2n}^*) = \min_{(\mathbf{u}_n, \mathbf{f}_{2n}) \in \mathcal{V}_{ad}^n} \mathcal{L}(\mathbf{u}_n, \mathbf{f}_{2n}). \quad (6.2)$$

Using Theorem 5.1 it follows that for each  $n \in \mathbb{N}$  there exists at least one solution  $(\mathbf{u}_n^*, \mathbf{f}_{2n}^*) \in \mathcal{V}_{ad}^n$  of Problem 6. Next, besides assumptions (5.4), (5.5) on  $\mathcal{L}$  we assume that  $\mathcal{L} : V \times L^2(\Gamma_3)^d \rightarrow \mathbb{R}$  is continuous, i.e.,

$$\begin{cases} \text{For all sequences } \{\mathbf{u}_n\} \subset V \text{ and } \{\mathbf{f}_{2n}\} \subset L^2(\Gamma_3)^d \text{ such that} \\ \mathbf{u}_n \rightarrow \mathbf{u} \text{ in } V, \mathbf{f}_{2n} \rightarrow \mathbf{f}_2 \text{ in } L^2(\Gamma_3)^d \text{ we have} \\ \lim \mathcal{L}(\mathbf{u}_n, \mathbf{f}_{2n}) = \mathcal{L}(\mathbf{u}, \mathbf{f}_2). \end{cases} \quad (6.3)$$

Moreover, we assume that

$$W_n \xrightarrow{M} W \text{ in } L^2(\Gamma_3)^d \quad (6.4)$$

where notation  $\xrightarrow{M}$  denotes the Mosco convergence of sets, see Definition 2.5.

Our main result in this section is as follows.

**Theorem 6.1.** Assume that (3.7)–(3.10), (3.12)–(3.14), (3.17), (4.1), (4.2), (4.4)–(4.6), (5.3)–(5.5) hold and let  $\{(\mathbf{u}_n^*, \mathbf{f}_{2n}^*)\}$  be a sequence of solutions of Problem 6. In addition, assume that (4.13), (4.14), (4.16)–(4.17), (6.3), (6.4) hold. Then, there exists a subsequence of the sequence  $\{(\mathbf{u}_n^*, \mathbf{f}_{2n}^*)\}$ , again denoted by  $\{(\mathbf{u}_n^*, \mathbf{f}_{2n}^*)\}$ , such that

$$\mathbf{u}_n^* \rightarrow \mathbf{u}^* \text{ in } V \text{ and } \mathbf{f}_{2n}^* \rightharpoonup \mathbf{f}_2^* \text{ in } L^2(\Gamma_3)^d. \quad (6.5)$$

Moreover,  $(\mathbf{u}^*, \mathbf{f}_2^*)$  is a solution of Problem 5.

**Proof.** We claim that the sequence  $\{\mathbf{f}_{2n}^*\}$  is bounded in  $L^2(\Gamma_3)^d$ . Arguing by contradiction, assume that  $\{\mathbf{f}_{2n}^*\}$  is not bounded in  $L^2(\Gamma_3)^d$ . Then, passing to a subsequence still denoted by  $\{\mathbf{f}_{2n}^*\}$ , we have

$$\|\mathbf{f}_{2n}^*\|_{L^2(\Gamma_3)^d} \rightarrow +\infty \text{ as } n \rightarrow +\infty. \quad (6.6)$$

We use assumption (5.5) and (6.6) to see that

$$\lim \mathcal{L}(\mathbf{u}_n^*, \mathbf{f}_{2n}^*) = +\infty. \quad (6.7)$$

Let  $(\mathbf{u}, \mathbf{f}_2) \in \mathcal{V}_{ad}$  be given where, recall,  $\mathcal{V}_{ad}$  is defined by (5.1). Then  $\mathbf{f}_2 \in W$  and, using assumption (6.4) and property  $(M_1)$  in Definition 2.5 we deduce that there exists a sequence  $\{\mathbf{f}_{2n}\}$  such that  $\mathbf{f}_{2n} \in W_n \forall n \in \mathbb{N}$  and, moreover,

$$\mathbf{f}_{2n} \rightarrow \mathbf{f}_2 \text{ in } L^2(\Gamma_3)^d. \quad (6.8)$$

This implies that (4.15) holds. Denote  $\mathbf{u}_n = \mathbf{u}(\beta_n, \mathbf{f}_{0n}, \mathbf{f}_{2n}, \omega_n, g_n)$ . Then  $(\mathbf{u}_n, \mathbf{f}_{2n}) \in \mathcal{V}_{ad}^n$  and, in addition, convergences (4.13)–(4.17), (4.51) imply that

$$\mathbf{u}_n \rightarrow \mathbf{u} \text{ in } V. \quad (6.9)$$

We now use convergences (6.9), (6.8) and assumption (6.3) to see that

$$\mathcal{L}(\mathbf{u}_n, \mathbf{f}_{2n}) \rightarrow \mathcal{L}(\mathbf{u}, \mathbf{f}_2). \quad (6.10)$$

On the other hand, by the optimality of the pair  $(\mathbf{u}_n^*, \mathbf{f}_{2n}^*)$  we have that

$$\mathcal{L}(\mathbf{u}_n^*, \mathbf{f}_{2n}^*) \leq \mathcal{L}(\mathbf{u}_n, \mathbf{f}_{2n}) \quad \forall n \in \mathbb{N}.$$

We now pass to the limit in this inequality and use convergences (6.7), (6.10) to deduce that  $\mathcal{L}(\mathbf{u}, \mathbf{f}_2) = +\infty$  which represents a contradiction.

We conclude from above that the sequence  $\{\mathbf{f}_{2n}^*\}$  is bounded in  $L^2(\Gamma_3)^d$ , as claimed. Therefore we can find a subsequence, again denoted by  $\{\mathbf{f}_{2n}^*\}$ , and an element  $\mathbf{f}_2^* \in L^2(\Gamma_3)^d$  such that

$$\mathbf{f}_{2n}^* \rightharpoonup \mathbf{f}_2^* \quad \text{in } L^2(\Gamma_3)^d \quad \text{as } n \rightarrow \infty. \quad (6.11)$$

Recall now that  $\mathbf{f}_{2n} \in W_n \quad \forall n \in \mathbb{N}$ . Thus, using assumption (6.4) and property  $(M_2)$  in Definition 2.5 we deduce that

$$\mathbf{f}_2^* \in W. \quad (6.12)$$

Denote  $\mathbf{u}^* = \mathbf{u}(\beta, \mathbf{f}_0, \mathbf{f}_2^*, \omega, g)$  and  $\mathbf{u}_n^* = \mathbf{u}(\beta_n, \mathbf{f}_{0n}, \mathbf{f}_{2n}^*, \omega_n, g_n)$ . Then, we have

$$(\mathbf{u}^*, \mathbf{f}_2^*) \in \mathcal{V}_{ad}. \quad (6.13)$$

Moreover, using (4.51) yields

$$\mathbf{u}_n^* \rightarrow \mathbf{u}^* \quad \text{in } V \quad \text{as } n \rightarrow \infty. \quad (6.14)$$

The convergences (6.11) and (6.14) show that (6.5) holds.

We now prove that  $(\mathbf{u}^*, \mathbf{f}_2^*)$  is a solution of Problem 5. To this end we consider an arbitrary element  $(\mathbf{u}, \mathbf{f}_2) \in \mathcal{V}_{ad}$ . Then  $\mathbf{f}_2 \in W$  and, using assumption (6.4) we deduce that there exists a sequence  $\{\mathbf{f}_{2n}\}$  such that  $\mathbf{f}_{2n} \in W_n \quad \forall n \in \mathbb{N}$  and, moreover, (6.8) holds. This implies that (4.15) holds, too. Let  $\mathbf{u}_n = \mathbf{u}(\beta_n, \mathbf{f}_{0n}, \mathbf{f}_{2n}, \omega_n, g_n)$ . Then  $(\mathbf{u}_n, \mathbf{f}_{2n}) \in \mathcal{V}_{ad}^n$  and, in addition, convergences (4.13)–(4.17), (4.51) imply that

$$\mathbf{u}_n \rightarrow \mathbf{u} \quad \text{in } V. \quad (6.15)$$

On the other hand, by the optimality of the pair  $(\mathbf{u}_n^*, \mathbf{f}_{2n}^*)$  we have that

$$\mathcal{L}(\mathbf{u}_n^*, \mathbf{f}_{2n}^*) \leq \mathcal{L}(\mathbf{u}_n, \mathbf{f}_{2n}) \quad \forall n \in \mathbb{N}.$$

We now pass to the limit in this inequality and use the convergences (6.14), (6.11) and assumption (5.4) to see that

$$\mathcal{L}(\mathbf{u}^*, \mathbf{f}_2^*) \leq \liminf \mathcal{L}(\mathbf{u}_n^*, \mathbf{f}_{2n}^*) \leq \liminf \mathcal{L}(\mathbf{u}_n, \mathbf{f}_{2n}).$$

Moreover, convergences (6.15), (6.8) combined with assumption (6.3) show that

$$\lim \mathcal{L}(\mathbf{u}_n, \mathbf{f}_{2n}) = \mathcal{L}(\mathbf{u}, \mathbf{f}_2).$$

It follows from here that

$$\mathcal{L}(\mathbf{u}^*, \mathbf{f}_2^*) \leq \mathcal{L}(\mathbf{u}, \mathbf{f}_2). \quad (6.16)$$

Inclusion (6.13) and inequality (6.16) show that  $(\mathbf{u}^*, \mathbf{f}_2^*)$  is a solution of Problem 5, which concludes the proof.  $\square$

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## References

- [1] R. Hu, et al., Equivalence results of well-posedness for split variational-hemivariational inequalities, *J Nonlinear Convex Anal* (2019) (In press).
- [2] J. Lu, Y.B. Xiao, N.J. Huang, A Stackelberg quasi-equilibrium problem via quasi-variational inequalities, *Carpathian J. Math.* 34 (2018) 355–362.
- [3] Z. Naniwicz, P.D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1995.
- [4] P.D. Panagiotopoulos, *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer-Verlag, Berlin, 1993.
- [5] Q.Y. Shu, R. Hu, Y.B. Xiao, Metric characterizations for well-posedness of split hemivariational inequalities, *J. Inequal. Appl.* 2018 (190) (2018) <http://dx.doi.org/10.1186/s13660-018-1761-4>.
- [6] Y.B. Xiao, M. Sofonea, Generalized penalty method for elliptic variational- hemivariational inequalities, submitted for publication.
- [7] G. Duvaut, J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.
- [8] C. Eck, J. Jarušek, M. Krbeč, *Unilateral Contact Problems: Variational Methods and Existence Theorems*, in: *Pure and Applied Mathematics*, vol. 270, Chapman/CRC Press, New York, 2005.

- [9] W. Han, M. Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, in: *Studies in Advanced Mathematics*, vol. 30, American Mathematical Society, Providence, RI-International Press, Somerville, MA, 2002.
- [10] W. Li, et al., A class of differential inverse quasi-variational inequalities in finite dimensional spaces, *J. Nonlinear Sci. Appl.* 10 (2017) 4532–4543.
- [11] S. Migórski, S.D. Zeng, Penalty and regularization method for variational-hemivariational inequalities with application to frictional contact, *Z. Angew. Math. Mech.* 98 (2018) 1503–1520.
- [12] P.D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, 1985.
- [13] M. Sofonea, A. Matei, *Mathematical Models in Contact Mechanics*, in: *London Mathematical Society Lecture Note Series*, vol. 398, Cambridge University Press, 2012.
- [14] M. Sofonea, Y.B. Xiao, Fully history-dependent quasivariational inequalities in contact mechanics, *Appl. Anal.* 95 (2016) 2464–2484.
- [15] Y.B. Xiao, N.J. Huang, Y.J. Cho, A class of generalized evolution variational inequalities in Banach space, *Appl. Math. Lett.* 25 (2012) 914–920.
- [16] S.D. Zeng, S. Migórski, Noncoercive hyperbolic variational inequalities with applications to contact mechanics, *J. Math. Anal. Appl.* 455 (2017) 619–637.
- [17] S. Migórski, A. Ochal, M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, in: *Advances in Mechanics and Mathematics*, vol. 26, Springer, New York, 2013.
- [18] S. Migórski, S.D. Zeng, A class of differential hemivariational inequalities in Banach spaces, *J. Global Optim.* 72 (2018) 761–779.
- [19] M. Sofonea, S. Migórski, *Variational-Hemivariational Inequalities with Applications*, in: *Pure and Applied Mathematics*, Chapman & Hall/CRC Press, Boca Raton-London, 2018.
- [20] A. Amassad, D. Chenais, C. Fabre, Optimal control of an elastic contact problem involving Tresca friction law, *Nonlinear Anal. TMA* 48 (2002) 1107–1135.
- [21] A. Capatina, Optimal control of Signorini problem, *Numer. Funct. Anal. Optim.* 21 (2000) 817–828.
- [22] A. Capatina, *Variational Inequalities Frictional Contact Problems*, in: *Advances in Mechanics and Mathematics*, vol. 31, Springer, New York, 2014.
- [23] A. Matei, S. Micu, Boundary optimal control for nonlinear antiplane problems, *Nonlinear Anal. TMA* 74 (2011) 1641–1652.
- [24] A. Matei, S. Micu, **Boundary optimal control for a frictional contact problem with normal compliance**, *Appl Math Optim.* <https://dx.doi.org/10.1007/s00245-017-9410-8>.
- [25] A. Matei, S. Micu, C. Niță, Optimal control for antiplane frictional contact problems involving nonlinearly elastic materials of Hencky type, *Math. Mech. Solids* 23 (2018) 308–328.
- [26] M. Sofonea, A. Matei, Y.B. Xiao, **Optimal control for a class of mixed variational problems**, submitted for publication.
- [27] M. Sofonea, Y.B. Xiao, M. Couderc, Optimization problems for elastic contact models with unilateral constraints, *Z. Angew. Math. Phys.* 70 (1) (2019) <http://dx.doi.org/10.1007/s00033-018-1046-2>.
- [28] A. Touzaline, Optimal control of a frictional contact problem, *Acta Math. Appl. Sin. Engl. Ser.* 31 (2015) 991–1000.
- [29] Y.B. Xiao, M. Sofonea, **On the optimal control of variational-hemivariational inequalities**, *J. Math. Anal. Appl.* (2019) <http://dx.doi.org/10.1016/j.jmaa.2019.02.046>.
- [30] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, Interscience, New York, 1983.
- [31] Z. Denkowski, S. Migórski, N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.
- [32] Z. Denkowski, S. Migórski, N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Applications*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.
- [33] E. Zeidler, *Nonlinear Functional Analysis and Applications II A/B*, Springer, New York, 1990.
- [34] Y.M. Wang, et al., Equivalence of well-posedness between systems of hemivariational inequalities and inclusion problems, *J. Nonlinear Sci. Appl.* 9 (2016) 1178–1192.
- [35] Y.B. Xiao, N.J. Huang, M.M. Wong, Well-posedness of hemivariational inequalities and inclusion problems, *Taiwanese J. Math.* 15 (2011) 1261–1276.
- [36] U. Mosco, Convergence of convex sets and of solutions of variational inequalities, *Adv. Math.* 3 (1968) 510–585.
- [37] S. Migórski, A. Ochal, M. Sofonea, A class of variational-hemivariational inequalities in reflexive Banach spaces, *J. Elasticity* 127 (2017) 151–178.