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# WELL-POSEDNESS OF HISTORY-DEPENDENT SWEEPING PROCESSES

STANISLAW MIGÓRSKI<sup>†</sup>, MIRCEA SOFONEA<sup>‡</sup>, AND SHENGDA ZENG<sup>§</sup>

**Abstract.** This paper is devoted to the study of a class of sweeping processes with history-dependent operators. A well-posedness result is obtained, including the existence, uniqueness, and stability of the solution. Our approach is based on the variable time step-length discrete approximation method combined with a fixed point principle for history-dependent operators. Then, a quasi-static frictional contact problem for viscoelastic materials with unilateral constraints in velocity is considered. The abstract result is applied in the study of this problem in order to provide its unique weak solvability as well as the continuous dependence of the solution with respect to the initial data.

**Key words.** sweeping process, history-dependent operator, existence and uniqueness, variable time step-length method, frictional contact problem, viscoelastic material, unilateral constraints

**AMS subject classifications.** 49J40, 47J20, 47J22, 35L15, 35L86, 35L87, 74Hxx, 74M10

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**1. Introduction.** The notion of sweeping process was initially introduced by Moreau [16, 18, 19]. A sweeping process consists of finding a trajectory function  $u: [0, T] \rightarrow H$  such that  $u(t) \in C(t)$  and

$$(1.1) \quad \begin{cases} -u'(t) \in N_{C(t)}(u(t)) & \text{for a.e. } t \in [0, T], \\ u(0) = u_0, \end{cases}$$

where  $H$  is a Hilbert space,  $C: [0, T] \rightarrow 2^H$  is a set-valued mapping with closed and convex values,  $N_{C(t)}$  stands for the normal cone of  $C(t)$  (see Definition 8 below), and the prime denotes the derivative with respect to the time variable  $t$ . Such problems have been used to study various mathematical models which arise in mechanics and engineering, such as models in unilateral contact in elasticity, perfect plasticity, shape optimization problems, obstacles problems, rigid-body dynamics with friction and impact, etc. References in the field include [2, 15, 17].

The applications of sweeping processes have motivated intensive study in the last 30 years, and many extensions and results have been obtained. Here we restrict ourselves to mentioning the following works: Colombo et al. [8], who addressed a new

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class of optimal control problems governed by the dissipative and discontinuous differential inclusion of the sweeping/Moreau process; Adly, Nacry, and Thibault [4], who studied the well-posedness (in the sense of existence and uniqueness of the solution) of a discontinuous sweeping process involving prox-regular sets in Hilbert spaces, in which the variation of the moving set is controlled by a positive Radon measure and the perturbation is assumed to satisfy a Lipschitz property; Vilches [25], who proved existence results for a class of sweeping processes in Hilbert spaces by using the catching-up algorithm and established a full characterization of nonsmooth Lyapunov pairs; and Tolstonogov [23], who considered a polyhedral sweeping process with a set-valued perturbation in a separable Hilbert space. For other results in this area the reader may consult Adly, Haddad, and Thibault [3], Adly and Le [5], Colombo and Palladino [9], Tolstonogov [24], and the references therein.

The aim of this paper is to present existence, uniqueness, and stability results for a new class of sweeping processes and to illustrate their applications in contact mechanics. We adopt the following functional framework everywhere in this paper. Let  $H$  be a real separable Hilbert space with the inner product  $\langle \cdot, \cdot \rangle$  and the induced norm  $\| \cdot \|$ , let  $T > 0$ , and denote by  $2^H$  and  $C([0, T]; H)$  the set of all subsets of  $H$  and the space of continuous functions defined on  $[0, T]$  with values in  $H$ , respectively. Given the operators  $A: H \rightarrow H$ ,  $B: H \rightarrow H$ , and  $\mathcal{R}: C([0, T]; H) \rightarrow C([0, T]; H)$ , a set-valued mapping  $C: [0, T] \rightarrow 2^H$ , and an element  $u_0 \in H$ , we consider the following abstract sweeping process: find  $u: [0, T] \rightarrow H$  such that

$$(1.2) \quad \begin{cases} -u'(t) \in N_{C(t)}(Au'(t) + Bu(t) + (\mathcal{R}u)(t)) & \text{for a.e. } t \in (0, T), \\ u(0) = u_0. \end{cases}$$

Note that if  $\mathcal{R}u = 0$  for all  $u \in C([0, T]; H)$ ,  $Az = 0$  for all  $z \in H$ , and  $B = I$ , where  $I$  is the identity operator of  $H$ , then problem (1.2) reduces to the classical sweeping process (1.1). In addition, if  $\mathcal{R}u = 0$  for all  $u \in C([0, T]; H)$ , then problem (1.2) consists of finding a function  $u: [0, T] \rightarrow H$  such that

$$(1.3) \quad \begin{cases} -u'(t) \in N_{C(t)}(Au'(t) + Bu(t)) & \text{for a.e. } t \in (0, T), \\ u(0) = u_0. \end{cases}$$

This problem was considered recently by Adly and Haddad [1]. There, its unique solvability was proved by using a time-discretization method.

Besides showing that problem (1.2) is more general than (1.3), our results presented in section 3 extend the results of Adly and Haddad [1, section 3] in the following directions: (a) we remove the regularity condition  $Bu_0 \in C(0)$ ; (b) we assume the operator  $B$  is Lipschitz continuous instead of linear, nonnegative, and symmetric; (c) we use a new variable time step-length discrete approximation instead of the uniform time-discretization method; and (d) we show the dependence of the solution on the initial data, i.e., we provide a stability result, as well.

The outline of the rest of the paper is as follows. In section 2, we recall some preliminary results that we use throughout. In section 3, we present our main existence, uniqueness, and stability result. Finally, in section 4 we use this abstract result in the study of a quasi-static frictional contact problem for viscoelastic materials with long memory and unilateral conditions in velocity, which represents an additional novelty of our paper.

**2. Preliminaries.** In this section, we recall some preliminary material which is needed in the rest of the paper. For more details on the topic, we refer the reader to

[2, 10, 11, 26]. Besides the notation already introduced in the previous section, we use  $\mathcal{L}(H)$  to represent the space of linear continuous operators from  $H$  to  $H$ . Moreover, we recall that for a multivalued operator  $\Lambda: H \rightarrow 2^H$ , its domain  $D(\Lambda)$ , range  $R(\Lambda)$ , and graph  $\text{Gr}(\Lambda)$  are defined by equalities

$$D(\Lambda) = \{u \in H \mid \Lambda u \neq \emptyset\}, \quad R(\Lambda) = \bigcup_{u \in H} \Lambda(u),$$

$$\text{Gr}(\Lambda) = \{(u, u^*) \in H \times H \mid u^* \in \Lambda u\},$$

respectively. We proceed with the following definitions and basic results on multivalued and single-valued operators.

DEFINITION 1. *An operator  $\Lambda: H \rightarrow 2^H$  is said to be*

- (i) *monotone if  $\langle u^* - v^*, u - v \rangle \geq 0$  for all  $(u, u^*), (v, v^*) \in \text{Gr}(\Lambda)$ ;*
- (ii) *maximal monotone if it is monotone and maximal in the sense of inclusion of graphs in the family of monotone operators from  $H$  to  $2^H$ .*

DEFINITION 2. *An operator  $A: H \rightarrow H$  is said to be*

- (i) *monotone if  $\langle Au - Av, u - v \rangle \geq 0$  for all  $u, v \in H$ ;*
- (ii) *strongly monotone with constant  $m_A > 0$  if  $\langle Au - Av, u - v \rangle \geq m_A \|u - v\|^2$  for all  $u, v \in X$ ;*
- (iii) *pseudomonotone if it is bounded and if for every sequence  $\{u_n\} \subset H$  converging weakly to  $u \in H$  such that  $\limsup \langle Au_n, u_n - u \rangle \leq 0$ , we have  $\langle Au, u - v \rangle \leq \liminf \langle Au_n, u_n - v \rangle$  for all  $v \in H$ .*
- (iv)  *$A$  is hemicontinuous if the function  $\lambda \mapsto \langle A(u + \lambda v), w \rangle$  is continuous on  $[0, 1]$  for all  $u, v, w \in H$ .*

It is obvious that  $A: H \rightarrow H$  is pseudomonotone if and only if  $u_n \rightarrow u$  weakly in  $H$  and  $\limsup \langle Au_n, u_n - u \rangle \leq 0$  imply  $\lim \langle Au_n, u_n - u \rangle = 0$  and  $Au_n \rightarrow Au$  weakly in  $H^*$ . Furthermore, if  $A \in \mathcal{L}(H)$  is nonnegative, then it is pseudomonotone.

LEMMA 3 ([6, Corollary 1.1, p. 44]). *Let  $\Lambda: H \rightarrow 2^H$  be a maximal monotone operator, and let  $B: H \rightarrow H$  be a hemicontinuous bounded operator. Then  $\Lambda + B: H \rightarrow 2^H$  is maximal monotone.*

LEMMA 4. *If  $\Lambda: D(\Lambda) \subset H \rightarrow 2^H$  is a maximal monotone operator with bounded domain  $D(\Lambda) \subset H$ , then  $\Lambda$  is surjective, i.e.,  $R(\Lambda) = H$ .*

History-dependent operators will play a crucial role in our paper.

DEFINITION 5. *An operator  $\mathcal{R}: C([0, T]; H) \rightarrow C([0, T]; H)$  is called history-dependent if there exists a constant  $L_{\mathcal{R}} > 0$  such that*

$$(2.1) \quad \|(\mathcal{R}u)(t) - (\mathcal{R}v)(t)\| \leq L_{\mathcal{R}} \int_0^t \|u(s) - v(s)\| ds$$

for all  $u, v \in C([0, T]; H)$ , and  $t \in (0, T)$ .

An important property of history-dependent operators is provided by the following fixed point result [21, Theorem 67].

LEMMA 6. *If  $\mathcal{R}: C([0, T]; H) \rightarrow C([0, T]; H)$  is a history-dependent operator, then there exists a unique function  $u^* \in C([0, T]; H)$  such that  $\mathcal{R}u^* = u^*$ .*

We now move to properties of the functions defined on the space  $H$ . First, we recall that a function  $f: H \rightarrow \overline{\mathbb{R}}$  is called proper, convex, and lower semicontinuous

if it fulfills, respectively, the following conditions:

$$\text{Dom}(f) := \{u \in H \mid f(u) < +\infty\} \neq \emptyset,$$

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v) \quad \text{for all } \lambda \in [0, 1] \quad \text{and } u, v \in H,$$

$$f(u) \leq \liminf_{n \rightarrow \infty} f(u_n) \quad \text{for all sequences } \{u_n\} \subset H \quad \text{with } u_n \rightarrow u \text{ in } H.$$

Note that here and below, we use the notation  $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ . The convex subdifferential of a proper convex function  $f: H \rightarrow \overline{\mathbb{R}}$  is defined by

$$\partial f(u) = \begin{cases} \{\xi \in H \mid f(v) - f(u) \geq \langle \xi, v - u \rangle \quad \text{for all } v \in H\} & \text{if } u \in \text{Dom}(f), \\ \emptyset & \text{otherwise.} \end{cases}$$

Obviously, if  $f$  is Gâteaux differentiable at a point  $u \in H$ , then we have  $\partial f(u) = \{Df(u)\}$ , where  $Df(u)$  is the Gâteaux derivative of  $f$  at  $u$ .

For any proper function  $f: H \rightarrow \overline{\mathbb{R}}$  (not necessarily convex), the Legendre–Fenchel conjugate of  $f$  is defined as follows:

$$f^*: H \rightarrow \overline{\mathbb{R}}, \quad f^*(v) = \sup_{u \in H} \{\langle u, v \rangle - f(u)\} \quad \text{for } v \in H.$$

It is well known that  $f^*$  is always convex and lower semicontinuous. Moreover, by the definition, we obtain the following well-known Fenchel–Young inequality:

$$\langle u^*, u \rangle \leq f(u) + f^*(u^*) \quad \text{for all } u, u^* \in H.$$

LEMMA 7. *Let  $f: H \rightarrow \overline{\mathbb{R}}$  be a proper, convex, and lower semicontinuous function. Then the following statements are equivalent for any  $u, u^* \in H$ :*

- (i)  $\langle u^*, u \rangle = f(u) + f^*(u^*)$ .
- (ii)  $u^* \in \partial f(u)$ .
- (iii)  $u \in \partial f^*(u^*)$ .

We now recall some definitions and properties related to convex subsets of a Hilbert space.

DEFINITION 8. *Let  $K$  and  $C$  be nonempty, closed, and convex subsets of  $H$ .*

- (i) *The metric projection of  $H$  onto  $K$  is defined by*

$$P_K: H \rightarrow H, \quad P_K(u) = \arg \min_{v \in K} \|u - v\| \quad \text{for all } u \in H.$$

- (ii) *The indicator function of  $K$  is defined by*

$$I_K: H \rightarrow \overline{\mathbb{R}}, \quad I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

- (iii) *The support function of  $K$  is defined by*

$$\sigma_K: H \rightarrow \overline{\mathbb{R}}, \quad \sigma_K(u^*) = \sup_{u \in K} \langle u^*, u \rangle.$$

- (iv) *The normal cone of  $K$  is defined by*

$$N_K(u) = \begin{cases} \{u^* \in H \mid \langle u^*, v - u \rangle \leq 0 \quad \text{for all } v \in K\} & \text{if } u \in K, \\ \emptyset & \text{otherwise.} \end{cases}$$

(v) The distance from a point  $u \in H$  to  $K$  is defined by

$$d_K(u) = \inf_{v \in K} \|u - v\|.$$

(vi) The Hausdorff distance between the sets  $K$  and  $C$  is defined by

$$\mathcal{H}(K, C) = \max \left\{ \sup_{u \in K} d_C(u), \sup_{u \in C} d_K(u) \right\}.$$

*Remark 9.* It is obvious that the support function  $\sigma_K$  of  $K$  coincides with the Legendre–Fenchel conjugate of the indicator function of  $K$ , that is,  $\sigma_K = (I_K)^*$ .

**DEFINITION 10.** A set-valued mapping  $C: [0, T] \rightarrow 2^H$  is called absolutely continuous if there exist a nondecreasing absolutely continuous function  $\eta: [0, T] \rightarrow \mathbb{R}_+$  with  $\eta(0) = 0$  and a constant  $c_0 > 0$  such that

$$\mathcal{H}(C(t), C(s)) \leq c_0 |\eta(t) - \eta(s)| \quad \text{for all } t, s \in [0, T].$$

Below, given  $u \in H$  and  $s > 0$ , we denote by  $O(u, s)$  (or  $\bar{O}(u, s)$ ) the open ball (or closed ball) centered at  $u$  with radius  $s > 0$ . The following result will be used to prove the unique solvability of the sweeping process, problem (1.2).

**LEMMA 11** ([1, Lemma 2.3]). *Let  $C: [0, T] \rightarrow 2^H$  be a set-valued absolutely continuous mapping such that  $C(t)$  is a nonempty, closed, and convex subset of  $H$  for all  $t \in [0, T]$ . Then, there exists  $n_0 \in \mathbb{N}$  such that*

$$C_n(t) := C(t) \cap \bar{O}(0, n) \neq \emptyset \quad \text{for all } n \geq n_0, \quad t \in [0, T],$$

and

$$\mathcal{H}(C_n(t), C_n(s)) \leq c_0 \mathcal{H}(C(t), C(s)) \leq c_0 |\eta(t) - \eta(s)| \quad \text{for all } s, t \in [0, T],$$

where  $c_0$  is the positive constant in Definition 10.

**3. A well-posedness result.** In this section, we state and prove our main results in the study of the abstract sweeping process (1.2). They concern the existence, uniqueness, and continuous dependence of the solution with respect to the initial data. Our approach is based on a variable time step-length discrete approximation algorithm combined with a fixed point theorem for history-dependent operators. To introduce the algorithm, we consider the following hypotheses on its data.

$H(A)$ :  $A: H \rightarrow H$  is linear, bounded, symmetric, and coercive with constant  $\alpha > 0$ , i.e.,

$$\langle Au, u \rangle \geq \alpha \|u\|^2 \quad \text{for all } u \in H.$$

$H(B)$ :  $B: H \rightarrow H$  is a Lipschitz continuous operator; that is, there exists a constant  $L_B > 0$  such that

$$\|Bu - Bv\| \leq L_B \|u - v\| \quad \text{for all } u, v \in H.$$

$H(\mathcal{R})$ :  $\mathcal{R}: C([0, T]; H) \rightarrow C([0, T]; H)$  is a history-dependent operator, i.e., it satisfies inequality (2.1).

$H(C)$ : The set-valued mapping  $C: [0, T] \rightarrow 2^H$  is such that  $C(t)$  is nonempty, closed, and convex for all  $t \in [0, T]$ , and the function  $t \mapsto C(t)$  is absolutely continuous.

Everywhere below, we use the standard notation for the Bochner Lebesgue spaces  $L^p(0; T; H)$  as well as the Sobolev spaces  $W^{k,p}(0, T; H)$ . Recall that  $W^{1,1}(0, T; H)$  coincides with the space of absolutely continuous functions defined on  $[0, T]$  with values in  $H$ , and  $W^{1,\infty}(0, T; H)$  coincides with the space of Lipschitz continuous functions defined on  $[0, T]$  with values in  $H$ . The space  $W^{1,\infty}(0, T; H)$  is a Banach space endowed with the norm

$$(3.1) \quad \|u\|_{W^{1,\infty}(0,T;H)} = \|u\|_{L^\infty(0,T;H)} + \|u'\|_{L^\infty(0,T;H)}.$$

Our main result in this section is the following.

**THEOREM 12.** *Assume that  $H(A)$ ,  $H(B)$ ,  $H(\mathcal{R})$ , and  $H(C)$  hold. Then, for any initial data  $u_0 \in H$ , there exists a unique solution  $u = u(u_0) \in W^{1,\infty}(0, T; H)$  of the sweeping process (1.2). Moreover, the map  $u_0 \mapsto u(u_0): H \rightarrow W^{1,\infty}(0, T; H)$  is Lipschitz continuous.*

To prove the theorem, everywhere in the rest of this section we assume that  $H(A)$ ,  $H(B)$ ,  $H(\mathcal{R})$ , and  $H(C)$  hold. We start by introducing the following intermediate problem: given  $w \in C([0, T]; H)$ , find a function  $u: [0, T] \rightarrow H$  such that

$$(3.2) \quad \begin{cases} -u'(t) \in N_{C(t)}(Au'(t) + Bw(t) + (\mathcal{R}w)(t)) & \text{for a.e. } t \in (0, T), \\ u(0) = u_0. \end{cases}$$

To solve this problem we introduce a variable time step-length grid indexed by  $n \in \mathbb{N}$ ,

$$\mathcal{T}_n = \{0 = t_n^0 < t_n^1 < \dots < t_n^n = T\}.$$

Moreover, let  $\tau_n^k = t_n^k - t_n^{k-1}$  for  $k \in \{1, 2, \dots, n\}$  denote

$$\tau_n^{\max} = \max_{k=1,2,\dots,n} \tau_n^k \quad \text{and} \quad \tau_n^{\min} = \min_{k=1,2,\dots,n} \tau_n^k,$$

and consider the following assumption.

$H(t)$ : The sequence of time grids satisfies

(i)  $\overline{\lim}_{n \rightarrow \infty} \tau_n^{\max} = 0$ ,

(ii) there exists a constant  $M_0 > 0$  such that  $\tau_n^{\max} \leq M_0 \tau_n^{\min}$  for all  $n \in \mathbb{N}$ .

Next, we introduce the operator  $\pi_n^{2,H}: L^2(0, T; H) \rightarrow L^2(0, T; H)$  defined by

$$(\pi_n^{2,H}(v))(t) = \frac{1}{\tau_n^k} \int_{t_n^{k-1}}^{t_n^k} v(s) ds \quad \text{for all } t \in (t_n^{k-1}, t_n^k], \quad v \in L^2(0, T; H).$$

*Remark 13.* We mention that a sequence of time grids that satisfies  $H(t)$  is called regular; see [7]. Moreover, by invoking [7, Lemma 3.3], we can see that if the regularity condition  $H(t)$  is satisfied, then

$$\pi_n^{2,H}(v) \rightarrow v \quad \text{strongly in } L^2(0, T; H) \quad \text{as } n \rightarrow \infty$$

for every  $v \in L^2(0, T; H)$ .

In what follows, we define the auxiliary quantities

$$\mathcal{R}_n^k = \frac{1}{\tau_n^k} \int_{t_n^{k-1}}^{t_n^k} (\mathcal{R}w)(s) ds \quad \text{and} \quad \mathcal{B}_n^k = \frac{1}{\tau_n^k} \int_{t_n^{k-1}}^{t_n^k} Bw(s) ds$$

for  $k = 1, 2, \dots, n$ . In addition, we consider the following variable time step-length discrete problem corresponding to the sweeping process (3.2):

$$(3.3) \quad -v_n^{k+1} \in N_{C_n(t_n^{k+1})} \left( Av_n^{k+1} + \mathcal{B}_n^{k+1} + \mathcal{R}_n^{k+1} \right)$$

for  $k = 0, \dots, n-1$ , where  $C_n(t)$  is defined by  $C_n(t) := C(t) \cap \overline{O}(0, n)$  for all  $t \in [0, T]$ , and

$$v_n^{k+1} = \frac{u_n^{k+1} - u_n^k}{\tau_n^{k+1}} \quad \text{with } u_n^0 = u_0.$$

The following lemma shows that problem (3.3) has a unique solution.

LEMMA 14. *There exists  $N_0 \in \mathbb{N}$  large enough such that problem (3.3) has a unique solution for all  $n \geq N_0$ .*

*Proof.* First, Lemma 11 ensures that there exists  $N_0 \in \mathbb{N}$  such that

$$C_n(t) = C(t) \cap \overline{O}(0, n) \neq \emptyset \quad \text{for all } t \in [0, T]$$

and for all  $n \geq N_0$ . Next, we will use induction with respect to  $k$  to show the existence of solutions to problem (3.3).

Let  $k = 0$ . Our goal is to find an element  $v_n^1$  such that

$$(3.4) \quad -v_n^1 \in N_{C_n(t_n^1)} (Av_n^1 + \mathcal{B}_n^1 + \mathcal{R}_n^1).$$

Recall assumption  $H(A)$  which, in particular, shows that the operator  $A: H \rightarrow H$  is symmetric and coercive, and thus it is strongly monotone with constant  $\alpha > 0$ . This implies that its inverse  $A^{-1}: H \rightarrow H$  is coercive and Lipschitz continuous with Lipschitz constant  $\frac{1}{\alpha}$ . Obviously, the above inclusion can be rewritten, equivalently, as follows:

$$A^{-1}(\mathcal{B}_n^1 + \mathcal{R}_n^1) \in (A^{-1} + N_{C_n(t_n^1)}) \circ (Av_n^1 + \mathcal{B}_n^1 + \mathcal{R}_n^1).$$

So, to prove that there exists an element  $v_n^1$  such that (3.4) holds, it is enough to verify that the mappings  $A^{-1} + N_{C_n(t_n^1)}: H \rightarrow 2^H$  and  $v \mapsto Av + \mathcal{B}_n^1 + \mathcal{R}_n^1: H \rightarrow H$  are both surjective. Using the continuity and boundedness of operator  $A^{-1}$  and the maximal monotonicity of operator  $N_{C_n(t_n^1)}$ , by applying Lemma 3, we get that operator

$$A^{-1} + N_{C_n(t_n^1)}: D(A^{-1} + N_{C_n(t_n^1)}) \rightarrow R(A^{-1} + N_{C_n(t_n^1)})$$

is maximal monotone, too. Furthermore, since  $D(A^{-1} + N_{C_n(t_n^1)}) = C_n(t_n^1)$  and the set  $C_n(t_n^1)$  is bounded, one can apply the surjectivity result in Lemma 4 to conclude that there exists an element  $z \in C_n(t_n^1)$  such that

$$A^{-1}(\mathcal{B}_n^1 + \mathcal{R}_n^1) \in (A^{-1} + N_{C_n(t_n^1)})(z).$$

On the other hand, assumption  $H(A)$  implies that  $A$  is a pseudomonotone operator. Combining this with the coercivity of  $A$ , and applying [13, Theorem 3.74, p. 88], we can find an element  $v_n^1 \in H$  such that

$$Av_n^1 = z - \mathcal{B}_n^1 - \mathcal{R}_n^1.$$



We conclude that  $v_n^1$  is also a solution to problem (3.4).

Next, we assume that  $u_n^0, v_n^1, v_n^2, \dots, v_n^k$  have been obtained. Then, from the equality  $v_n^l = \frac{u_n^l - u_n^{l-1}}{\tau_n^l}$  for all  $1 \leq l \leq n$ , we can directly calculate  $u_n^1, u_n^2, \dots, u_n^k$ . We further reformulate (3.3) to obtain

$$A^{-1}(\mathcal{B}_n^{k+1} + \mathcal{R}_n^{k+1}) \in (A^{-1} + N_{C_n(t_n^{k+1})}) \circ (Av_n^{k+1} + \mathcal{B}u_n^{k+1} + \mathcal{R}_n^{k+1}).$$

Note that the mapping  $A^{-1} + N_{C_n(t_n^{k+1})}: D(A^{-1} + N_{C_n(t_n^{k+1})}) \rightarrow R(A^{-1} + N_{C_n(t_n^{k+1})})$  is surjective since it is maximal monotone with bounded domain  $C_n(t_n^{k+1})$ ; see Lemma 4. Therefore, we are able to find  $z \in C_n(t_n^{k+1})$  such that

$$A^{-1}(\mathcal{B}_n^{k+1} + \mathcal{R}_n^{k+1}) \in (A^{-1} + N_{C_n(t_n^{k+1})})(z).$$

In the meantime, there is also  $v_n^{k+1} \in H$  such that

$$Av_n^{k+1} = z - \mathcal{B}_n^{k+1} - \mathcal{R}_n^{k+1}$$

since  $A$  is onto. Clearly,  $v_n^{k+1} \in H$  is a solution to problem (3.3).

Finally, we will prove the uniqueness of solution to problem (3.3). Let  $\widehat{v_n^{k+1}}, \widetilde{v_n^{k+1}}$  be two solutions to problem (3.3), namely,

$$\langle \widehat{v_n^{k+1}}, v - Av_n^{k+1} - \mathcal{B}_n^{k+1} - \mathcal{R}_n^{k+1} \rangle \geq 0 \quad \text{for all } v \in C_n(t_n^{k+1})$$

and

$$\langle \widetilde{v_n^{k+1}}, v - Av_n^{k+1} - \mathcal{B}_n^{k+1} - \mathcal{R}_n^{k+1} \rangle \geq 0 \quad \text{for all } v \in C_n(t_n^{k+1}).$$

Bearing in mind that both  $\widehat{Av_n^{k+1}} + \mathcal{B}_n^{k+1} + \mathcal{R}_n^{k+1}$  and  $\widetilde{Av_n^{k+1}} + \mathcal{B}_n^{k+1} + \mathcal{R}_n^{k+1}$  belong to  $C_n(t_n^{k+1})$ , we now insert  $v = \widetilde{Av_n^{k+1}} + \mathcal{B}_n^{k+1} + \mathcal{R}_n^{k+1}$  into the first inequality and  $v = \widehat{Av_n^{k+1}} + \mathcal{B}_n^{k+1} + \mathcal{R}_n^{k+1}$  into the second, and then we sum the resulting inequalities to find that

$$\alpha \|\widehat{v_n^{k+1}} - \widetilde{v_n^{k+1}}\|^2 \leq \langle \widehat{v_n^{k+1}} - \widetilde{v_n^{k+1}}, \widehat{Av_n^{k+1}} - \widetilde{Av_n^{k+1}} \rangle \leq 0.$$

This implies that  $\widehat{v_n^{k+1}} = \widetilde{v_n^{k+1}}$ , which completes the proof of the lemma.  $\square$

The next result provides an a priori estimate for solution to problem (3.3).

LEMMA 15. *There exists a constant  $L > 0$ , which is independent of  $n$ , such that*

$$(3.5) \quad \max_{1 \leq k \leq n} \|v_n^k\| \leq L.$$

*Proof.* For any  $k \in \{0, 1, 2, \dots, n-1\}$  fixed, we can reformulate the inclusion (3.3) as follows:

$$(3.6) \quad \langle Av_n^{k+1} + \mathcal{B}_n^{k+1} + \mathcal{R}_n^{k+1} - v, v_n^{k+1} \rangle \leq 0$$

for all  $v \in C_n(t_n^{k+1})$ . From the coercivity of  $A$  and the above inequality, we deduce

that

$$\begin{aligned}
\alpha \|v_n^{k+1}\|^2 &\leq \langle Av_n^{k+1}, v_n^{k+1} \rangle \\
&= \langle Av_n^{k+1} + \mathcal{B}_n^{k+1} + \mathcal{R}_n^{k+1} - v, v_n^{k+1} \rangle + \langle v - \mathcal{B}_n^{k+1} - \mathcal{R}_n^{k+1}, v_n^{k+1} \rangle \\
&\leq \langle v - \mathcal{B}_n^{k+1} - \mathcal{R}_n^{k+1}, v_n^{k+1} \rangle \\
(3.7) \quad &\leq \|v - \mathcal{B}_n^{k+1} - \mathcal{R}_n^{k+1}\| \|v_n^{k+1}\| \quad \text{for all } v \in C_n(t_n^{k+1}).
\end{aligned}$$

Now, for any  $v_0 \in C(0)$  fixed, we have

$$(3.8) \quad \|v - \mathcal{B}_n^{k+1} - \mathcal{R}_n^{k+1}\| \leq \|v - v_0\| + \|v_0\| + \|\mathcal{B}_n^{k+1}\| + \|\mathcal{R}_n^{k+1}\|.$$

Therefore, inserting (3.8) into (3.7) yields

$$(3.9) \quad \alpha \|v_n^{k+1}\| \leq \|v - v_0\| + \|v_0\| + \|\mathcal{B}_n^{k+1}\| + \|\mathcal{R}_n^{k+1}\|.$$

Next, since  $v \in C_n(t_n^{k+1})$  and  $v_0 \in C(0)$ , using Lemma 11 it follows that for  $n$  large enough we have  $v_0 \in C_n(0)$  and

$$\begin{aligned}
\inf_{v \in C_n(t_n^{k+1})} \|v - v_0\| &= d(v_0, C_n(t_n^{k+1})) \leq \mathcal{H}(C_n(0), C_n(t_n^{k+1})) \\
&\leq c_0 \mathcal{H}(C(0), C(t_n^{k+1})) \leq c_0 (\eta(t_n^{k+1}) - \eta(0)) = c_0 \eta(t_n^{k+1}) \leq c_0 \eta(T).
\end{aligned}$$

Taking into account the above inequality and (3.9) we obtain that

$$\|v_n^{k+1}\| \leq \frac{1}{\alpha} (c_0 \eta(T) + \|v_0\| + \|\mathcal{B}_n^{k+1}\| + \|\mathcal{R}_n^{k+1}\|).$$

Recall that  $w, \mathcal{R}w \in C([0, T]; H)$ . This implies that

$$\begin{aligned}
\|\mathcal{R}_n^{k+1}\| &= \left\| \frac{1}{\tau_n^{k+1}} \int_{t_n^k}^{t_n^{k+1}} (\mathcal{R}w)(s) ds \right\| \leq M_w, \\
\|\mathcal{B}_n^{k+1}\| &= \left\| \frac{1}{\tau_n^{k+1}} \int_{t_n^k}^{t_n^{k+1}} Bw(s) ds \right\| \leq M_w,
\end{aligned}$$

where  $M_w > 0$  is given by equality

$$M_w = \max \left\{ \max_{t \in [0, T]} \|Bw(t)\|, \max_{t \in [0, T]} \|(\mathcal{R}w)(t)\| \right\}.$$

Therefore, one has

$$\|v_n^{k+1}\| \leq \frac{1}{\alpha} (c_0 \eta(T) + 2M_w + \|v_0\|).$$

Consequently, there exists a constant  $L := \frac{1}{\alpha} (c_0 \eta(T) + 2M_w + \|v_0\|)$ , which is independent of  $n$  and  $k$ , such that

$$\|v_n^k\| \leq L$$

for all  $k \in \{1, 2, \dots, n\}$ . In conclusion, inequality (3.5) holds, which completes the proof of the lemma.  $\square$

Subsequently, for given  $n \in \mathbb{N}$ , we define the piecewise affine function  $u_n$  and the piecewise constant interpolant functions  $\bar{u}_n$ ,  $\mathcal{B}_n$ , and  $\mathcal{R}_n$  by

$$\begin{aligned} u_n(t) &= \begin{cases} u_n^0 & \text{if } t = 0, \\ u_n^k + \frac{t - t_n^{k-1}}{\tau_n^k} (u_n^k - u_n^{k-1}) & \text{if } t \in (t_n^{k-1}, t_n^k], \end{cases} \\ \bar{u}_n(t) &= \begin{cases} u_n^0 & \text{if } t = 0, \\ u_n^{k-1} & \text{if } t \in (t_n^{k-1}, t_n^k], \end{cases} \\ \mathcal{B}_n(t) &= \begin{cases} \mathcal{B}_n^k & \text{if } t \in (t_n^{k-1}, t_n^k], \\ 0 & \text{if } t = 0, \end{cases} \\ \mathcal{R}_n(t) &= \begin{cases} \mathcal{R}_n^k & \text{if } t \in (t_n^{k-1}, t_n^k], \\ 0 & \text{if } t = 0. \end{cases} \end{aligned}$$

Then, problem (3.3) can be equivalently rewritten as

$$(3.10) \quad -u'_n(t) \in N_{C_n(\delta_n(t))} (Au'_n(t) + \mathcal{B}_n(t) + \mathcal{R}_n(t)) \quad \text{for all } t \in [0, T],$$

where the function  $\delta_n$  is defined by

$$\delta_n(t) = \begin{cases} 0 & \text{if } t = 0, \\ t_n^k & \text{if } t \in (t_n^{k-1}, t_n^k]. \end{cases}$$

We are now in position to explore the convergence of the sequences  $\{u_n\}$  and  $\{\bar{u}_n\}$ .

LEMMA 16. *There exist a Lipschitz continuous function  $u: [0, T] \rightarrow H$  and two subsequences of the sequences  $\{\bar{u}_n\}$  and  $\{u_n\}$ , still denoted by  $\{\bar{u}_n\}$  and  $\{u_n\}$ , respectively, such that*

$$(3.11) \quad \bar{u}_n \rightarrow u \quad \text{weakly in } L^2(0, T; H),$$

$$(3.12) \quad \begin{cases} u_n \rightarrow u \quad \text{weakly in } L^2(0, T; H), \\ u_n(t) \rightarrow u(t) \quad \text{weakly in } H \text{ for all } t \in [0, T], \end{cases}$$

$$(3.13) \quad u'_n \rightarrow u' \quad \text{weakly in } L^2(0, T; H).$$

*Proof.* The estimate (3.5) and equality  $u_n^i = u_n^0 + \sum_{j=1}^i \tau_n^j v_n^j$  for  $1 \leq i \leq n$  indicate that

$$\|u_n^i\| \leq \|u_n^0\| + \sum_{j=1}^i \tau_n^j \|v_n^j\| \leq \|u_0\| + \sum_{j=1}^n \tau_n^j L \leq \|u_0\| + LT$$

and

$$\begin{aligned} \|\bar{u}_n\|_{L^2(0, T; H)}^2 &= \int_0^T \|\bar{u}_n(s)\|^2 ds = \sum_{i=1}^n \int_{t_n^{i-1}}^{t_n^i} \|u_n^{i-1}\|^2 ds = \sum_{i=1}^n \tau_n^i \|u_n^{i-1}\|^2 \\ &\leq \sum_{i=1}^n \tau_n^i (\|u_0\| + LT)^2 = (\|u_0\| + LT)^2 T. \end{aligned}$$

This implies that the sequence  $\{\bar{u}_n\}$  is bounded in  $L^2(0, T; H)$ . Therefore, without loss of generality, we may assume that there exists a function  $u \in L^2(0, T; H)$  such that the convergence (3.11) holds.

Further, using equality  $u_n(t) = u_n^0 + \int_0^t u'_n(s) ds$  and the bound  $\|u'_n(t)\| \leq L$  for a.e.  $t \in [0, T]$ , we have

$$\|u_n(t) - u_n(s)\| \leq \left\| \int_s^t u'_n(r) dr \right\| \leq L|t - s| \quad \text{for all } s, t \in [0, T].$$

Moreover, for any division  $0 = b_0 < b_1 < \dots < b_N = T$  of interval  $[0, T]$ , one has

$$\sum_{i=1}^N \|u_n(b_i) - u_n(b_{i-1})\| \leq L \sum_{i=1}^N (b_i - b_{i-1}) = LT,$$

which implies that the variation of  $u_n$  on  $[0, T]$ , denoted  $\text{var}(u_n, [0, T])$ , satisfies the inequality

$$\text{var}(u_n, [0, T]) \leq LT.$$

It follows from above that the sequence  $\{u_n\}$  is uniformly bounded in norm and variation. This property, combined with the inequality  $\|u'_n(t)\| \leq L$  for a.e.  $t \in [0, T]$  and [14, Theorem 2.1, p. 10], allows us to deduce that there exist a subsequence of  $\{u_n\}$ , still denoted  $\{u_n\}$ , and a bounded variation function  $\tilde{u}: [0, T] \rightarrow H$  such that

$$\begin{cases} u_n \rightarrow \tilde{u} \quad \text{weakly in } L^2(0, T; H), \\ u_n(t) \rightarrow \tilde{u}(t) \quad \text{weakly in } H \text{ for all } t \in [0, T]. \end{cases}$$

On the other hand, from the estimates

$$\begin{aligned} \|u_n(t) - \bar{u}_n(t)\| &= \left\| u_n^k - u_n^{k-1} + \frac{t - t_n^k}{\tau_n^k} (u_n^k - u_n^{k-1}) \right\| \\ &\leq \tau_n^k \left\| \frac{u_n^k - u_n^{k-1}}{\tau_n^k} \right\| + (t_n^k - t) \left\| \frac{u_n^k - u_n^{k-1}}{\tau_n^k} \right\| \\ &\leq 2L\tau_n^k \quad \text{for all } t \in (t_n^{k-1}, t_n^k], \end{aligned}$$

it follows that

$$\begin{aligned} \|u_n - \bar{u}_n\|_{L^2(0, T; H)}^2 &= \int_0^T \|u_n(t) - \bar{u}_n(t)\|^2 dt \\ &= \sum_{k=1}^n \int_{t_n^{k-1}}^{t_n^k} \|u_n(t) - \bar{u}_n(t)\|^2 dt \leq \sum_{k=1}^n 4L^2(\tau_n^k)^3 \leq 4L^2T(\tau_n^{\max})^2. \end{aligned}$$

The regularity condition  $H(t)$ (i) shows that  $u_n - \bar{u}_n \rightarrow 0$  strongly in  $L^2(0, T; H)$ , as  $n \rightarrow \infty$ , and therefore (3.11) ensures that  $u = \tilde{u}$ . Hence, the convergences (3.12) hold.

We now show that  $u: [0, T] \rightarrow H$  is a Lipschitz continuous function. Indeed, for any  $t, s \in [0, T]$ , using the weak convergences  $u_n(t) \rightarrow u(t)$ ,  $u_n(s) \rightarrow u(s)$ , both in  $H$ , and the weak lower semicontinuity of the norm, we find that

$$\|u(t) - u(s)\| \leq \liminf_{n \rightarrow \infty} \|u_n(t) - u_n(s)\| = \liminf_{n \rightarrow \infty} \left\| \int_s^t u'_n(r) dr \right\| \leq L|t - s|.$$

So,  $u$  is Lipschitz continuous, i.e.,  $u \in W^{1,\infty}(0, T; H)$  and, moreover,

$$u(t) = u_0 + \int_0^t u'(s) ds \quad \text{for all } t \in [0, T].$$

It remains to verify convergence (3.13). To this end, since  $\{u'_n\}$  is bounded in  $L^2(0, T; H)$ , by reflexivity of  $L^2(0, T; H)$ , passing to a relabeled subsequence, we may assume that

$$(3.14) \quad u'_n \rightarrow u^* \quad \text{weakly in } L^2(0, T; H)$$

for some  $u^* \in L^2(0, T; H)$ . Moreover, for any  $z \in H$ , we have

$$\begin{aligned} \langle z, u_n(t) \rangle &= \left\langle z, u_n^0 + \int_0^t u'_n(s) ds \right\rangle = \langle z, u_n^0 \rangle + \int_0^t \langle z, u'_n(s) \rangle ds \\ &= \langle z, u_n^0 \rangle + \int_0^T \langle z \chi_{[0,t]}, u'_n(s) \rangle ds \quad \text{for all } t \in [0, T]. \end{aligned}$$

Now, using the convergence  $u_n(t) \rightarrow u(t)$  weakly in  $H$  for all  $t \in [0, T]$  and (3.14), one has

$$\begin{cases} \langle z, u_n(t) \rangle \rightarrow \langle z, u(t) \rangle = \left\langle z, u_0 + \int_0^t u'(s) ds \right\rangle, \\ \langle z, u_n^0 \rangle + \int_0^T \langle z \chi_{[0,t]}, u'_n(s) \rangle ds \rightarrow \langle z, u_0 \rangle + \int_0^T \langle z \chi_{[0,t]}, u^*(s) \rangle ds. \end{cases}$$

Hence, it easily follows that

$$\langle z, u_0 \rangle + \int_0^t \langle z, u^*(s) \rangle ds = \langle z, u(t) \rangle = \langle z, u_0 \rangle + \int_0^t \langle z, u'(s) \rangle ds$$

for all  $z \in H$  and  $t \in [0, T]$ , that is,

$$\int_0^t u^*(s) ds = \int_0^t u'(s) ds \quad \text{for all } t \in [0, T].$$

This equality implies that  $u^* = u'$  a.e.  $t \in [0, T]$ . So, the convergence (3.13) holds, which completes the proof of the lemma.  $\square$

In what follows, we use the above results, Lemmas 14–16, to show that the function  $u$  obtained in Lemma 16 is the unique solution to problem (3.2).

LEMMA 17. *For any  $u_0 \in H$ , there exists a unique solution  $u \in W^{1,\infty}(0, T; H)$  to problem (3.2).*

*Proof.* First, we claim that

$$Au'(t) + Bw(t) + (\mathcal{R}w)(t) \in D(N_{C(t)}) \quad \text{for a.e. } t \in [0, T]$$

or, equivalently,

$$(3.15) \quad Au'(t) + Bw(t) + (\mathcal{R}w)(t) \in C(t) \quad \text{for a.e. } t \in [0, T].$$

To prove this inclusion, let  $t \in [0, T]$  and  $z \in H$ . Note that relations

$$\begin{aligned} & \langle Au'(t) + Bw(t) + (\mathcal{R}w)(t), z \rangle = \langle Au'_n(t) + \mathcal{B}_n(t) + \mathcal{R}_n(t), z \rangle \\ & \quad + \langle Au'(t) + Bw(t) + (\mathcal{R}w)(t) - Au'_n(t) - \mathcal{B}_n(t) - \mathcal{R}_n(t), z \rangle, \\ & Au'_n(t) + \mathcal{B}_n(t) + \mathcal{R}_n(t) \in C_n(\delta_n(t)) \subset C(\delta_n(t)), \end{aligned}$$

imply that

$$\begin{aligned} & \langle Au'(t) + Bw(t) + (\mathcal{R}w)(t), z \rangle \leq \sup_{y \in C(\delta_n(t))} \langle y, z \rangle \\ & \quad + \langle Au'(t) + Bw(t) + (\mathcal{R}w)(t) - Au'_n(t) - \mathcal{B}_n(t) - \mathcal{R}_n(t), z \rangle = \sigma_{C(\delta_n(t))}(z) \\ (3.16) \quad & + \langle Au'(t) + Bw(t) + (\mathcal{R}w)(t) - Au'_n(t) - \mathcal{B}_n(t) - \mathcal{R}_n(t), z \rangle. \end{aligned}$$

On the other hand, by [1, Lemma 2.1], we have

$$\sigma_{C(\delta_n(t))}(z) \leq \sigma_{C(t)}(z) + \|z\| |\eta(\delta_n(t)) - \eta(t)|.$$

So, we can reformulate (3.16) as follows:

$$\begin{aligned} & \langle Au'(t) + Bw(t) + (\mathcal{R}w)(t), z \rangle \leq \sigma_{C(t)}(z) + \|z\| |\eta(\delta_n(t)) - \eta(t)| \\ & \quad + \langle Au'(t) - Au'_n(t), z \rangle + \langle Bw(t) - \mathcal{B}_n(t), z \rangle + \langle (\mathcal{R}w)(t) - \mathcal{R}_n(t), z \rangle. \end{aligned}$$

Next, given  $\varepsilon > 0$ , we integrate the above inequality on  $[t - \varepsilon, t + \varepsilon]$  to obtain

$$\begin{aligned} & \int_{t-\varepsilon}^{t+\varepsilon} \langle Au'(s) + Bw(s) + (\mathcal{R}w)(s), z \rangle ds \leq \int_{t-\varepsilon}^{t+\varepsilon} \sigma_{C(s)}(z) ds \\ & \quad + \|z\| \int_{t-\varepsilon}^{t+\varepsilon} |\eta(\delta_n(s)) - \eta(s)| ds + \int_{t-\varepsilon}^{t+\varepsilon} \langle Au'(s) - Au'_n(s), z \rangle ds \\ (3.17) \quad & + \int_{t-\varepsilon}^{t+\varepsilon} \langle Bw(s) - \mathcal{B}_n(s), z \rangle ds + \int_{t-\varepsilon}^{t+\varepsilon} \langle (\mathcal{R}w)(s) - \mathcal{R}_n(s), z \rangle ds. \end{aligned}$$

Since  $\eta$  is absolutely continuous and  $\delta_n(s) \rightarrow s$  as  $n \rightarrow \infty$ , using Lebesgue's dominated convergence theorem, we find that

$$(3.18) \quad \int_{t-\varepsilon}^{t+\varepsilon} |\eta(\delta_n(s)) - \eta(s)| ds \leq \int_0^T |\eta(\delta_n(s)) - \eta(s)| ds \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Further, the convergence (3.13) and hypotheses  $H(A)$  imply that

$$\begin{aligned} & \int_{t-\varepsilon}^{t+\varepsilon} \langle Au'(s) - Au'_n(s), z \rangle ds = \int_0^T \langle Au'(s) - Au'_n(s), z \chi_{[t-\varepsilon, t+\varepsilon]}(s) \rangle ds \\ (3.19) \quad & = \int_0^T \langle u'(s) - u'_n(s), Az \chi_{[t-\varepsilon, t+\varepsilon]}(s) \rangle ds \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Finally, Remark 13 guarantees that

$$\begin{aligned} & \int_{t-\varepsilon}^{t+\varepsilon} \langle Bw(s) - \mathcal{B}_n(s), z \rangle ds \leq \|z\| \int_{t-\varepsilon}^{t+\varepsilon} \|Bw(s) - \mathcal{B}_n(s)\| ds \\ (3.20) \quad & \leq \|z\| \int_0^T \|Bw(s) - \mathcal{B}_n(s)\| ds \leq \|z\| T^{\frac{1}{2}} \|Bw - \mathcal{B}_n\|_{L^2(0, T; H)} \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned}
& \int_{t-\varepsilon}^{t+\varepsilon} \langle (\mathcal{R}w)(s) - \mathcal{R}_n(s), z \rangle ds \leq \|z\| \int_{t-\varepsilon}^{t+\varepsilon} \|(\mathcal{R}w)(s) - \mathcal{R}_n(s)\| ds \\
(3.21) \quad & \leq \|z\| \int_0^T \|(\mathcal{R}w)(s) - \mathcal{R}_n(s)\| ds \leq \|z\| T^{\frac{1}{2}} \|(\mathcal{R}w) - \mathcal{R}_n\|_{L^2(0,T;H)} \rightarrow 0
\end{aligned}$$

as  $n \rightarrow \infty$ . Therefore, from inequality (3.17), letting  $n$  tend to infinity and then taking into account (3.18)–(3.21) yields

$$\int_{t-\varepsilon}^{t+\varepsilon} \langle Au'(s) + Bw(s) + (\mathcal{R}w)(s), z \rangle ds \leq \int_{t-\varepsilon}^{t+\varepsilon} \sigma_{C(s)}(z) ds \quad \text{for all } z \in H.$$

Through dividing by  $2\varepsilon$ , passing to the limit as  $\varepsilon \rightarrow 0$  in the above inequality, and using the Lebesgue differentiation theorem, we obtain

$$\langle Au'(t) + Bw(t) + (\mathcal{R}w)(t), z \rangle \leq \sigma_{C(t)}(z) \quad \text{for all } z \in H.$$

Since  $\sigma_{C(t)}(z) = \sup_{y \in C(t)} \langle y, z \rangle$  for all  $z \in H$ , the latter inequality implies

$$Au'(t) + Bw(t) + (\mathcal{R}w)(t) \in C(t).$$

Hence, we conclude that (3.15) holds, which proves the claim.

Next, we verify that  $u$  solves problem (3.2). For each  $n \in \mathbb{N}$ , inclusion (3.10) can be rewritten as

$$\langle -u'_n(t), z - Au'_n(t) - \mathcal{B}_n(t) - \mathcal{R}_n(t) \rangle \leq 0 \quad \text{for all } z \in C_n(\delta_n(t))$$

for a.e.  $t \in [0, T]$ . Let  $z \in C_n(\delta_n(t))$  and  $v \in C_n(t)$ . We use the previous inequality to obtain that

$$\begin{aligned}
& \langle -u'_n(t), v - Au'_n(t) - \mathcal{B}_n(t) - \mathcal{R}_n(t) \rangle \\
& = \langle -u'_n(t), v - z + z - Au'_n(t) - \mathcal{B}_n(t) - \mathcal{R}_n(t) \rangle \\
& = \langle -u'_n(t), z - Au'_n(t) - \mathcal{B}_n(t) - \mathcal{R}_n(t) \rangle + \langle -u'_n(t), v - z \rangle \\
& \leq \langle -u'_n(t), v - z \rangle \leq \|u'_n(t)\| \|v - z\|.
\end{aligned}$$

Recall that  $\|v - z\| \leq c_0 |\eta(\delta_n(t)) - \eta(t)|$  due to inclusion  $C_n(t) \subset C_n(\delta_n(t)) + c_0 |\eta(\delta_n(t)) - \eta(t)| \overline{O}(0, 1)$ . Hence, by estimate (3.5), we have

$$(3.22) \quad \langle -u'_n(t), v - Au'_n(t) - \mathcal{B}_n(t) - \mathcal{R}_n(t) \rangle \leq c_0 L |\eta(\delta_n(t)) - \eta(t)|$$

for all  $v \in C_n(t)$  and a.e.  $t \in [0, T]$ .

Let  $t^* \in [0, T]$  and  $v^* \in C(t^*)$  be given. For any  $\varepsilon > 0$ , we define a function  $v: [t^* - \varepsilon, t^* + \varepsilon] \rightarrow H$  by

$$v(t) = P_{C(t)}(v^*) \quad \text{for all } t \in [t^* - \varepsilon, t^* + \varepsilon],$$

where  $P_{C(t)}$  denotes the projector operator on  $C(t)$ . The absolute continuity of  $t \mapsto C(t)$  indicates that the function  $v$  is also absolutely continuous on  $[t^* - \varepsilon, t^* + \varepsilon]$ . On

the other hand, the continuity of  $v$  and compactness of  $[t^* - \varepsilon, t^* + \varepsilon]$  imply that  $v$  is bounded on  $[t^* - \varepsilon, t^* + \varepsilon]$ . This ensures that there exists  $N_1 > 0$  such that

$$v(t) \in C_n(t) \quad \text{for all } t \in [t^* - \varepsilon, t^* + \varepsilon] \quad \text{and } n \geq N_1.$$

Putting  $v = v(t)$  into (3.22) and then integrating the resulting inequality on interval  $[t^* - \varepsilon, t^* + \varepsilon]$ , we deduce that

$$\begin{aligned} & \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle -u'_n(s), v(s) - Au'_n(s) - \mathcal{B}_n(s) - \mathcal{R}_n(s) \rangle ds \\ & \leq c_0 L \int_{t^* - \varepsilon}^{t^* + \varepsilon} |\eta(\delta_n(s)) - \eta(s)| ds \end{aligned}$$

and, therefore,

$$\begin{aligned} & \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle u'_n(s), Au'_n(s) \rangle ds + \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle u'_n(s), \mathcal{B}_n(s) \rangle ds + \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle u'_n(s), \mathcal{R}_n(s) \rangle ds \\ (3.23) \quad & - \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle u'_n(s), v(s) \rangle ds \leq c_0 L \int_0^T |\eta(\delta_n(s)) - \eta(s)| ds. \end{aligned}$$

Recall that  $\eta$  is absolutely continuous and  $\delta_n(s) \rightarrow s$  as  $n \rightarrow \infty$ , which gives that

$$(3.24) \quad c_0 L \int_0^T |\eta(\delta_n(s)) - \eta(s)| ds \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Moreover, due to the convergence  $u'_n \rightarrow u'$  weakly in  $L^2(0, T; H)$ , one has  $u_n \chi_{[t^* - \varepsilon, t^* + \varepsilon]} \rightarrow u' \chi_{[t^* - \varepsilon, t^* + \varepsilon]}$  weakly in  $L^2(0, T; H)$  as  $n \rightarrow \infty$ . Next, recalling that  $A$  is linear, bounded, and coercive, we get that the function

$$L^2(0, T; H) \ni u \mapsto \int_0^T \langle u(t), Au(t) \rangle dt \in \mathbb{R}$$

is convex on  $L^2(0, T; H)$ . This entails

$$\begin{aligned} & \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle u'(s), Au'(s) \rangle ds = \int_0^T \langle (u' \chi_{[t^* - \varepsilon, t^* + \varepsilon]})(s), A(u' \chi_{[t^* - \varepsilon, t^* + \varepsilon]})(s) \rangle ds \\ & \leq \liminf_{n \rightarrow \infty} \int_0^T \langle (u'_n \chi_{[t^* - \varepsilon, t^* + \varepsilon]})(s), A(u'_n \chi_{[t^* - \varepsilon, t^* + \varepsilon]})(s) \rangle ds \\ (3.25) \quad & = \liminf_{n \rightarrow \infty} \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle u'_n(s), Au'_n(s) \rangle ds \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Moreover, Remark 13 and the convergence  $u'_n \rightarrow u'$  weakly in  $L^2(0, T; H)$ , guaranteed by Lemma 16, show that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle u'_n(t), \mathcal{B}_n(s) \rangle ds = \lim_{n \rightarrow \infty} \int_0^T \langle u'_n(t), \mathcal{B}_n(s) \chi_{[t^* - \varepsilon, t^* + \varepsilon]}(s) \rangle ds \\ (3.26) \quad & = \int_0^T \langle u'(t), Bw(s) \chi_{[t^* - \varepsilon, t^* + \varepsilon]}(s) \rangle ds = \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle u'(t), Bw(s) \rangle ds \end{aligned}$$



and

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle u'_n(t), \mathcal{R}_n(s) \rangle ds = \lim_{n \rightarrow \infty} \int_0^T \langle u'_n(t), \mathcal{R}_n(s) \chi_{[t^* - \varepsilon, t^* + \varepsilon]}(s) \rangle ds \\
(3.27) \quad & = \int_0^T \langle u'(t), (\mathcal{R}w)(s) \chi_{[t^* - \varepsilon, t^* + \varepsilon]}(s) \rangle ds = \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle u'(t), (\mathcal{R}w)(s) \rangle ds.
\end{aligned}$$

In addition, (3.13) implies that the following convergence holds:

$$\begin{aligned}
& \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle u'_n(s), v(s) \rangle ds = \int_0^T \langle u'_n(s), (v \chi_{[t^* - \varepsilon, t^* + \varepsilon]})(s) \rangle ds \\
(3.28) \quad & \rightarrow \int_0^T \langle u'(s), (v \chi_{[t^* - \varepsilon, t^* + \varepsilon]})(s) \rangle ds = \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle u'(s), v(s) \rangle ds.
\end{aligned}$$

Now, taking into account (3.23)–(3.28), we obtain

$$\int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle -u'(s), v(s) - Au'(s) - Bw(s) - (\mathcal{R}w)(s) \rangle ds \leq 0.$$

Subsequently, we divide the above inequality by  $2\varepsilon$  to get

$$\frac{1}{2\varepsilon} \int_{t^* - \varepsilon}^{t^* + \varepsilon} \langle -u'(s), v(s) - Au'(s) - Bw(s) - (\mathcal{R}w)(s) \rangle ds \leq 0.$$

Passing to the limit as  $\varepsilon \rightarrow 0$  in this inequality, by the Lebesgue differentiation theorem we conclude that

$$\langle -u'(t^*), v(t^*) - Au'(t^*) - Bw(t^*) - (\mathcal{R}w)(t^*) \rangle \leq 0.$$

On the other hand, bearing in mind that  $v^* \in C(t^*)$  and  $v(t^*) = P_{C(t^*)}(v^*) = v^*$ , we have

$$\langle -u'(t^*), v^* - Au'(t^*) - Bw(t^*) - (\mathcal{R}w)(t^*) \rangle \leq 0.$$

Now, since  $v^* \in C(t^*)$  and  $t^* \in [0, T]$  are both arbitrary, the previous inequality shows that

$$-u'(t^*) \in N_{C(t^*)}(Au'(t^*) + Bw(t^*) + (\mathcal{R}w)(t^*)) \quad \text{for a.e. } t^* \in [0, T].$$

Consequently,  $u$  is a solution to problem (3.2), which concludes the existence part of the lemma.

To prove the uniqueness part, we assume in what follows that  $u_1, u_2$  are two Lipschitz continuous functions which solve problem (3.2). Then,

$$\langle -u'_i(t), v - Au'_i(t) - Bw(t) - (\mathcal{R}w)(t) \rangle \leq 0 \quad \text{for all } v \in C(t)$$

a.e.  $t \in [0, T]$  for  $i = 1, 2$ . Note that  $Au'_i(t) + Bw(t) + (\mathcal{R}w)(t) \in C(t)$  for a.e.  $t \in [0, T]$  and  $i = 1, 2$ . We substitute  $v = Au'_2(t) + Bw(t) + (\mathcal{R}w)(t)$  and  $v = Au'_1(t) + Bw(t) + (\mathcal{R}w)(t)$  in the above inequality for  $i = 1$  and  $i = 2$ , respectively, and then we sum the resulting inequalities to get

$$\langle Au'_1(t) - Au'_2(t), u'_1(t) - u'_2(t) \rangle \leq 0 \quad \text{for a.e. } t \in [0, T].$$

Therefore,

$$\alpha \|u'_1(t) - u'_2(t)\|^2 \leq 0 \quad \text{for a.e. } t \in [0, T],$$

which shows that  $u'_1 = u'_2$ . Next, since  $u_i(t) = u_0 + \int_0^t u'_i(s) ds$  for all  $t \in [0, T]$ , we deduce that  $u_1(t) = u_2(t)$  for all  $t \in [0, T]$ , which concludes the proof.  $\square$

We now proceed with the following convergence result.

LEMMA 18. *The whole sequence  $\{u_n\}$  converges to  $u$  strongly in  $L^2(0, T; H)$ .*

*Proof.* It follows from Lemmas 16 and 17 that any weakly convergent subsequence of  $\{u_n\}$  has the same limit, which is, in fact, the unique solution  $u$  of problem (3.2). This conclusion, combined with boundedness of  $\{u_n\}$ , implies that the whole sequence  $\{u_n\}$  converges to  $u$  weakly in  $L^2(0, T; H)$ .

It remains to show that the convergence is strong. Since  $u$  is Lipschitz continuous with Lipschitz constant  $L > 0$ , using the regularity  $w \in C([0, T]; H)$  and hypotheses  $H(B)$ ,  $H(\mathcal{R})$ , we can see that  $\{Au'(t) + Bw(t) + (\mathcal{R}w)(t)\}_{t \in [0, T]}$  is bounded in  $H$ . Lemma 11 guarantees that there is  $N_2 > 0$  such that  $Au'(t) + Bw(t) + (\mathcal{R}w)(t) \in C_n(t)$  for a.e.  $t \in [0, T]$  for all  $n \geq N_2$ . On the other hand, since  $u$  is a solution of problem (3.2), we also have

$$\begin{aligned} \langle -u'(t), v - Au'(t) - Bw(t) - (\mathcal{R}w)(t) \rangle &= \langle -u'(t), v - z \rangle \\ &+ \langle -u'(t), z - Au'(t) - Bw(t) - (\mathcal{R}w)(t) \rangle \leq \langle -u'(t), v - z \rangle \\ &\leq L \|v - z\| \quad \text{for all } z \in C_n(t) \subset C(t), v \in C_n(\delta_n(t)) \end{aligned}$$

and a.e.  $t \in [0, T]$ . Applying the inequality  $\|v - z\| \leq c_0 |\eta(\delta_n(t)) - \eta(t)|$  (valid since  $C_n(t) \subset C_n(\delta_n(t)) + c_0 |\eta(\delta_n(t)) - \eta(t)| \overline{O}(0, 1)$ ), we deduce that

$$\langle -u'(t), v - Au'(t) - Bw(t) - (\mathcal{R}w)(t) \rangle \leq c_0 L |\eta(\delta_n(t)) - \eta(t)|$$

for all  $v \in C_n(\delta_n(t))$  and a.e.  $t \in [0, T]$ . We now choose  $v = Au'_n(t) + \mathcal{B}_n(t) + \mathcal{R}_n(t) \in C_n(\delta_n(t))$  in the above inequality and  $v = Au'(t) + Bw(t) + (\mathcal{R}w)(t) \in C_n(t) \subset C(t)$  in (3.22), and then we add the resulting inequalities to find that

$$\begin{aligned} \langle u'(t) - u'_n(t), Au'(t) - Au'_n(t) \rangle &+ \langle u'(t) - u'_n(t), Bw(t) - \mathcal{B}_n(t) \rangle \\ &+ \langle u'(t) - u'_n(t), (\mathcal{R}w)(t) - \mathcal{R}_n(t) \rangle \leq 2c_0 L |\eta(\delta_n(t)) - \eta(t)| \end{aligned}$$

for a.e.  $t \in [0, T]$ . Hence, we have

$$\begin{aligned} \alpha \|u'(t) - u'_n(t)\|^2 &\leq 2c_0 L |\eta(\delta_n(t)) - \eta(t)| + \|Bw(t) - \mathcal{B}_n(t)\| (\|u'(t)\| + \|u'_n(t)\|) \\ &+ \|(\mathcal{R}w)(t) - \mathcal{R}_n(t)\| (\|u'(t)\| + \|u'_n(t)\|) \quad \text{for a.e. } t \in [0, T]. \end{aligned}$$

Integrating the above inequality on  $[0, T]$  and using the bounds  $\|u'(t)\| \leq L$ ,  $\|u'_n(t)\| \leq L$  for a.e.  $t \in [0, T]$ , we infer that

$$\begin{aligned} \|u'_n - u'\|_{L^2(0, T; H)}^2 &= \int_0^T \|u'(t) - u'_n(t)\|^2 dt \leq \frac{2c_0}{\alpha} \int_0^T |\eta(\delta_n(t)) - \eta(t)| dt \\ &+ \frac{2L}{\alpha} \int_0^T \|Bw(t) - \mathcal{B}_n(t)\| dt + \frac{2L}{\alpha} \int_0^T \|(\mathcal{R}w)(t) - \mathcal{R}_n(t)\| dt. \end{aligned}$$

Passing to the limit, as  $n \rightarrow \infty$ , in the above inequality and using Remark 13, we deduce that

$$\|u'_n - u'\|_{L^2(0,T;H)} \rightarrow 0,$$

i.e.,  $u'_n \rightarrow u'$  strongly in  $L^2(0,T;H)$ . Finally, by equalities  $u(t) = u_0 + \int_0^t u'(s) ds$  and  $u_n(t) = u_0 + \int_0^t u'_n(s) ds$ , valid for all  $t \in [0, T]$ , we deduce that  $u_n \rightarrow u$  strongly in  $L^2(0, T; H)$ , which completes the proof.  $\square$

We now have all the ingredients to provide the proof of our main existence, uniqueness, and stability results.

*Proof of Theorem 12.* For any  $w \in C([0, T]; H)$ , it follows from Lemma 17 that there exists a unique Lipschitz continuous solution  $u = u(w)$  of problem (3.2). We now introduce the operator  $\mathcal{S}: C([0, T]; H) \rightarrow W^{1, \infty}(0, T; H) \subset C([0, T]; H)$  defined by

$$(3.29) \quad \mathcal{S}w = u(w).$$

We claim that  $\mathcal{S}$  has a unique fixed point. To this end, let  $w_i \in C([0, T]; H)$ , and denote  $u_i = u(w_i)$  for  $i = 1, 2$ . We have

$$\begin{cases} \langle -u'_i(t), v - Au'_i(t) - Bw_i(t) - (\mathcal{R}w_i)(t) \rangle \leq 0 & \text{for all } v \in C(t), \\ u_i(0) = u_0 \end{cases}$$

for a.e.  $t \in [0, T]$ . We test with  $v = Au'_2(t) - Bw_2(t) - (\mathcal{R}w_2)(t)$  and  $v = Au'_1(t) - Bw_1(t) - (\mathcal{R}w_1)(t)$  in the above inequality for  $i = 1$  and  $i = 2$ , respectively, and then we add the resulting inequalities to obtain that

$$\begin{aligned} \langle Au'_1(t) - Au'_2(t), u'_1(t) - u'_2(t) \rangle &\leq \langle Bw_1(t) - Bw_2(t), u'_2(t) - u'_1(t) \rangle \\ &+ \langle (\mathcal{R}w_1)(t) - (\mathcal{R}w_2)(t), u'_2(t) - u'_1(t) \rangle \end{aligned}$$

for a.e.  $t \in [0, T]$ . Therefore, using hypotheses  $H(A)$  we find that

$$\alpha \|u'_1(t) - u'_2(t)\| \leq \|Bw_1(t) - Bw_2(t)\| + \|(\mathcal{R}w_1)(t) - (\mathcal{R}w_2)(t)\|$$

for a.e.  $t \in [0, T]$ . Now using hypotheses  $H(B)$  and  $H(\mathcal{R})$ , we have

$$(3.30) \quad \|u'_1(t) - u'_2(t)\| \leq \frac{L_B}{\alpha} \|w_1(t) - w_2(t)\| + \frac{L_{\mathcal{R}}}{\alpha} \int_0^t \|w_1(s) - w_2(s)\| ds$$

for a.e.  $t \in [0, T]$ . Next, using the definition of the operator  $\mathcal{S}$ , the initial conditions  $u_1(0) = u_2(0) = u_0$ , and inequality (3.30), we have that

$$\begin{aligned} \|(\mathcal{S}w_1)(t) - (\mathcal{S}w_2)(t)\| &= \|u_1(t) - u_2(t)\| \leq \int_0^t \|u'_1(s) - u'_2(s)\| ds \\ &\leq \frac{L_B}{\alpha} \int_0^t \|w_1(s) - w_2(s)\| ds + \frac{L_{\mathcal{R}}}{\alpha} \int_0^t \int_0^s \|w_1(\xi) - w_2(\xi)\| d\xi ds \\ &\leq \frac{L_B + TL_{\mathcal{R}}}{\alpha} \int_0^t \|w_1(s) - w_2(s)\| ds \quad \text{for all } t \in [0, T]. \end{aligned}$$

This inequality shows that  $\mathcal{S}$  is a history-dependent operator. Now using Lemma 6, we deduce that there exists a unique function  $w^*$  such that  $w^* = \mathcal{S}w^*$ , and since  $\mathcal{S}$

takes values in the space  $W^{1,\infty}(0, T; H)$ , we deduce that  $w^* \in W^{1,\infty}(0, T; H)$ . Then, writing (3.2) for  $w = w^*$  and using (3.29), it is easy to see that  $w^* = \mathcal{S}w^* = u(w^*)$  is a solution to problem (1.2) with regularity  $W^{1,\infty}(0, T; H)$ , which proves the existence part in Theorem 12. The uniqueness part follows from the uniqueness of the fixed point of operator  $\mathcal{S}$ , guaranteed by Lemma 6.

We now turn to the dependence of the solution with respect to the initial data, and to this end, we let  $u_0^1, u_0^2 \in H$  be arbitrary, and for  $i = 1, 2$  denote by  $u_i \in W^{1,\infty}(0, T; H)$  the solution of problem (1.2) corresponding to the initial data  $u_0^i$ . Using arguments similar to those in the proof of inequality (3.30) we deduce that

$$(3.31) \quad \|u_1'(s) - u_2'(s)\| \leq \frac{L_B}{\alpha} \|u_1(s) - u_2(s)\| + \frac{L_{\mathcal{R}}}{\alpha} \int_0^s \|u_1(r) - u_2(r)\| dr$$

for a.e.  $s \in [0, T]$ . On the other hand, since  $u_i(t) = \int_0^t u_i'(s) ds + u_0^i$  for all  $t \in [0, T]$ , we find that

$$(3.32) \quad \|u_1(t) - u_2(t)\| \leq \int_0^t \|u_1'(s) - u_2'(s)\| ds + \|u_0^1 - u_0^2\|.$$

We now substitute inequality (3.31) in (3.32) to obtain that

$$\begin{aligned} \|u_1(t) - u_2(t)\| &\leq \frac{L_B + TL_{\mathcal{R}}}{\alpha} \int_0^t \|u_1(s) - u_2(s)\| ds \\ &\quad + \|u_0^1 - u_0^2\| \text{ for all } t \in [0, T], \end{aligned}$$

and using the Gronwall argument, we find that

$$(3.33) \quad \|u_1(t) - u_2(t)\| \leq L_0 \|u_0^1 - u_0^2\| \text{ for all } t \in [0, T],$$

with some  $L_0 > 0$ . Inequalities (3.31) and (3.33) yield

$$(3.34) \quad \|u_1'(t) - u_2'(t)\| \leq L_1 \|u_0^1 - u_0^2\| \text{ for a.e. } t \in [0, T],$$

with a constant  $L_1 > 0$ . We now combine inequalities (3.33) and (3.34) and use definition (3.1) to see that

$$\|u_1 - u_2\|_{W^{1,\infty}(0, T; H)} \leq (L_0 + L_1) \|u_0^1 - u_0^2\|,$$

which completes the proof of the theorem.  $\square$

We end this section with the following consequence of Theorem 12, obtained for  $\mathcal{R} \equiv 0$ .

**COROLLARY 19.** *Assume that  $H(A)$ ,  $H(B)$ , and  $H(C)$  hold. Then, for any  $u_0 \in H$ , there exists a unique function  $u = u(u_0) \in W^{1,\infty}(0, T, H)$  such that*

$$\begin{cases} -u'(t) \in N_{C(t)}(Au'(t) + Bu(t)) \text{ for a.e. } t \in [0, T], \\ u(0) = u_0. \end{cases}$$

Moreover, the map  $u_0 \mapsto u(u_0): H \rightarrow W^{1,\infty}(0, T; H)$  is Lipschitz continuous.

Note that Corollary 19 extends Theorem 3.1 in [1] (obtained under the assumption that  $B: H \rightarrow H$  is a linear continuous positive operator) and completes it with a continuous dependence result of the solution with respect to the initial data.

**4. Application to a history-dependent viscoelastic contact problem.** In this section we will illustrate the applicability of the results obtained in section 3 to a model of a quasi-static contact problem for viscoelastic material with unilateral constraints in velocity.

The physical setting is as follows. A viscoelastic body occupies an open, bounded, and connected domain  $\Omega$  in  $\mathbb{R}^d$ ,  $d = 2, 3$ , with Lipschitz boundary  $\partial\Omega = \Gamma$ . The boundary is partitioned into four disjoint measurable parts  $\Gamma_D$ ,  $\Gamma_N$ ,  $\Gamma_{C_1}$ , and  $\Gamma_{C_2}$  such that  $\text{meas}(\Gamma_D) > 0$ . The body is fixed on  $\Gamma_D$ , is acted on by body forces on  $\Omega$  and surface traction on  $\Gamma_N$ , and is in contact with two obstacles on  $\Gamma_{C_1}$  and  $\Gamma_{C_2}$ , respectively. As a result its mechanical state evolves in the time interval  $[0, T]$  with  $T > 0$ . To describe its evolution, we need to introduce the following notation. For a vector field  $\boldsymbol{\xi}$  we denote by  $\xi_\nu$  and  $\boldsymbol{\xi}_\tau$  its normal and tangential components on the boundary, i.e.,  $\xi_\nu = \boldsymbol{\xi} \cdot \boldsymbol{\nu}$  and  $\boldsymbol{\xi}_\tau = \boldsymbol{\xi} - \xi_\nu \boldsymbol{\nu}$ , respectively, where  $\boldsymbol{\nu} = (\nu_i)$  denotes the outward unit normal at  $\Gamma$ . The notation  $\sigma_\nu$  and  $\boldsymbol{\sigma}_\tau$  represents the normal and tangential components of the stress field  $\boldsymbol{\sigma}$  on the boundary, that is,  $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$  and  $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ . Moreover,  $\mathbb{S}^d$  denotes the space of symmetric  $d \times d$  matrices. On  $\mathbb{R}^d$  and  $\mathbb{S}^d$  we use inner products and norms defined by

$$\begin{aligned} \boldsymbol{\xi} \cdot \boldsymbol{\eta} &= \xi_i \eta_i, \quad \|\boldsymbol{\xi}\|_{\mathbb{R}^d} = (\boldsymbol{\xi} \cdot \boldsymbol{\xi})^{1/2} \quad \text{for } \boldsymbol{\xi} = (\xi_i), \quad \boldsymbol{\eta} = (\eta_i) \in \mathbb{R}^d, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\tau} &= \sigma_{ij} \tau_{ij}, \quad \|\boldsymbol{\sigma}\|_{\mathbb{S}^d} = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{1/2} \quad \text{for } \boldsymbol{\sigma} = (\sigma_{ij}), \quad \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d. \end{aligned}$$

The indices  $i, j, k, l$  run between 1 and  $d$ , and the summation convention over repeated indices is applied. We also set  $\mathcal{Q} = \Omega \times (0, T)$ ,  $\Sigma = \Gamma \times (0, T)$ ,  $\Sigma_D = \Gamma_D \times (0, T)$ ,  $\Sigma_N = \Gamma_N \times (0, T)$ ,  $\Sigma_{C_1} = \Gamma_{C_1} \times (0, T)$ , and  $\Sigma_{C_2} = \Gamma_{C_2} \times (0, T)$ .

With these preliminaries, the classical formulation of the mathematical model of contact we consider in this section is the following.

**PROBLEM 20.** *Find a displacement field  $\mathbf{u}: \mathcal{Q} \rightarrow \mathbb{R}^d$  and a stress field  $\boldsymbol{\sigma}: \mathcal{Q} \rightarrow \mathbb{S}^d$  such that*

$$(4.1) \quad \boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}'(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\mathbf{u}(s)) ds \quad \text{in } \mathcal{Q},$$

$$(4.2) \quad \text{Div } \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \text{in } \mathcal{Q},$$

$$(4.3) \quad \mathbf{u}(t) = \mathbf{0} \quad \text{on } \Sigma_D,$$

$$(4.4) \quad \boldsymbol{\sigma}(t)\boldsymbol{\nu} = \mathbf{f}_N(t) \quad \text{on } \Sigma_N,$$

$$(4.5) \quad \begin{cases} u'_\nu(t) \leq 0, \quad \sigma_\nu(t) \leq 0, \quad \sigma_\nu(t)u'_\nu(t) = 0 \\ \boldsymbol{\sigma}_\tau(t) = \mathbf{0} \end{cases} \quad \text{on } \Sigma_{C_1},$$

$$(4.6) \quad \begin{cases} -\sigma_\nu(t) = F \\ \|\boldsymbol{\sigma}_\tau(t)\|_{\mathbb{S}^d} \leq \mu|\sigma_\nu(t)|, \quad -\boldsymbol{\sigma}_\tau(t) = \mu|\sigma_\nu(t)| \frac{\boldsymbol{u}'_\tau(t)}{\|\boldsymbol{u}'_\tau(t)\|_{\mathbb{R}^d}} \quad \text{if } \boldsymbol{u}'_\tau(t) \neq \mathbf{0} \end{cases} \quad \text{on } \Sigma_{C_2},$$

$$(4.7) \quad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega.$$

A brief comment on the equations and conditions in Problem 20 reads as follows. First, (4.1) is a general constitutive law for viscoelastic materials with long memory

in which  $\mathcal{A}$  is the viscosity operator,  $\mathcal{B}$  represents the elasticity operator, and  $\mathcal{R}$  is the relaxation tensor. We recall the strain-displacement relation

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \mathcal{Q},$$

which defines the linearized strain tensor. For more details on the general viscoelastic constitutive law with long memory of the form (4.1), we refer the reader to [13, 21] and the references therein. Equation (4.2) is the equilibrium equation, in which  $\mathbf{f}_0$  denotes the density of the body forces. Equation (4.3) is the displacement condition, which describes the fact that the body is fixed on the boundary  $\Gamma_D$ . Equation (4.4) represents the traction boundary condition, where  $\mathbf{f}_N$  is the density of traction acting on  $\Gamma_N$ . Relation (4.5) describes the frictionless contact with unilateral constraints in velocity (see, for instance, [12, 22]), and (4.6) represents a version of Coulomb's law of dry friction, in which the normal stress on the contact boundary is assumed to be given (see, for instance, [20] and the references therein). Here,  $F$  is a positive function,  $\mu \geq 0$  denotes the coefficient of friction, and therefore  $\mu F$  represents the friction bound. Finally, condition (4.7) is the initial condition in which  $\mathbf{u}_0$  stands for the initial displacement field.

To establish the variational formulation of the contact problem (4.1)–(4.7), we introduce the function spaces

$$(4.8) \quad H = \{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^d) \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \}, \quad \mathcal{V} = L^2(\Omega; \mathbb{S}^d).$$

Recall that the space  $\mathcal{V}$  is a Hilbert space with the inner product

$$\langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_{\mathcal{V}} = \int_{\Omega} \sigma_{ij}(\mathbf{x}) \tau_{ij}(\mathbf{x}) d\mathbf{x} \quad \text{for } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{V},$$

and the associated norm is denoted by  $\| \cdot \|_{\mathcal{V}}$ . Moreover, since  $\text{meas}(\Gamma_D) > 0$ , the space  $H$  is a Hilbert space with the inner product given by  $\langle \mathbf{u}, \mathbf{v} \rangle_H = \langle \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}) \rangle_{\mathcal{V}}$  for  $\mathbf{u}, \mathbf{v} \in H$ , and the associated norm  $\| \cdot \|_H$ .

Subsequently, the trace of an element  $\mathbf{v} \in H$  is also denoted by  $\mathbf{v}$ . Also, we need the set of admissible velocity fields given by

$$K = \{ \mathbf{v} \in H \mid \mathbf{v}_{\nu} \leq 0 \text{ for a.e. on } \Gamma_{C_1} \}.$$

It follows from the continuity of the trace operator that

$$\| \mathbf{v} \|_{L^2(\Gamma_C; \mathbb{R}^d)} \leq \| \gamma \| \| \mathbf{v} \|_H$$

for all  $\mathbf{v} \in H$ , where  $\Gamma_C = \Gamma_{C_1} \cup \Gamma_{C_2}$ , and  $\| \gamma \|$  denotes the norm of the trace operator  $\gamma: H \rightarrow L^2(\Gamma_C; \mathbb{R}^d)$ . In addition, we consider the space of the fourth-order tensor fields defined by

$$\mathbf{Q}_{\infty} = \{ \mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^{\infty}(\Omega), \quad 1 \leq i, j, k, l \leq d \}.$$

It is well known that  $\mathbf{Q}_{\infty}$  is a real Banach space with the usual norm

$$\| \mathcal{E} \|_{\mathbf{Q}_{\infty}} = \max_{1 \leq i, j, k, l \leq d} \| \mathcal{E}_{ijkl} \|_{L^{\infty}(\Omega)} \quad \text{for all } \mathcal{E} \in \mathbf{Q}_{\infty}.$$

In the study of Problem 20, we need the following assumptions on the data.

$H(\mathcal{A})$ :  $\mathcal{A} \in \mathbf{Q}_\infty$  and satisfies

$$\mathcal{A}\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} \geq L_{\mathcal{A}} \|\boldsymbol{\varepsilon}\|_{\mathbb{S}^d}^2 \quad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ with some } L_{\mathcal{A}} > 0.$$

$H(\mathcal{B})$ :  $\mathcal{B}: \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$  is such that

$$\left\{ \begin{array}{l} \text{(a) there exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \quad \|\mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\|_{\mathbb{S}^d} \leq L_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d} \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ a.e. } \mathbf{x} \in \Omega, \\ \text{(b) the mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\ \quad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \\ \text{(c) the mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{0}) \text{ belongs to } \mathcal{V}. \end{array} \right.$$

$H(\mathcal{R})$ :  $\mathcal{R} \in C([0, T]; \mathbf{Q}_\infty)$ .

$$\underline{H(f)}: \left\{ \begin{array}{l} \mathbf{f}_0 \in W^{1,1}(0, T; L^2(\Omega; \mathbb{R}^d)), \\ \mathbf{f}_N \in W^{1,1}(0, T; L^2(\Gamma_N; \mathbb{R}^d)), \\ F \in L^2(\Gamma_{C_2}), \quad F(\mathbf{x}) \geq 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_{C_2}, \\ \mu \in L^\infty(\Gamma_{C_2}), \quad \mu(\mathbf{x}) \geq 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_{C_2}. \end{array} \right.$$

The variational formulation of Problem 20 is obtained by standard arguments that we present in what follows. Assume that  $(\mathbf{u}, \boldsymbol{\sigma})$  is a pair of smooth functions which satisfies (4.1)–(4.7). Let  $\mathbf{v} \in K$  and  $t \in [0, T]$ . We multiply the equilibrium equation (4.2) by  $\mathbf{v} - \mathbf{u}'(t)$  and use the Green formula to obtain that

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}'(t)) \rangle_{\mathcal{V}} = \langle \mathbf{f}_0(t), \mathbf{v} - \mathbf{u}'(t) \rangle_V + \int_{\Gamma} \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot (\mathbf{v} - \mathbf{u}'(t)) \, d\Gamma.$$

Subsequently, we use the equality

$$\int_{\Gamma} \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot \mathbf{v} \, d\Gamma = \int_{\Gamma_D} \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot \mathbf{v} \, d\Gamma + \int_{\Gamma_N} \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot \mathbf{v} \, d\Gamma + \int_{\Gamma_C} \boldsymbol{\sigma}(t) \boldsymbol{\nu} \cdot \mathbf{v} \, d\Gamma$$

and the boundary conditions (4.3), (4.4) to find that

$$\begin{aligned} \langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}'(t)) \rangle_{\mathcal{V}} &= \langle \mathbf{f}(t), \mathbf{v} - \mathbf{u}'(t) \rangle_H \\ &+ \int_{\Gamma_{C_1}} (\sigma_\nu(t)(v_\nu - u'_\nu(t)) + \boldsymbol{\sigma}_\tau(t) \cdot (\mathbf{v}_\tau - \mathbf{u}'_\tau(t))) \, d\Gamma \\ (4.9) \quad &+ \int_{\Gamma_{C_2}} (\sigma_\nu(t)(v_\nu - u'_\nu(t)) + \boldsymbol{\sigma}_\tau(t) \cdot (\mathbf{v}_\tau - \mathbf{u}'_\tau(t))) \, d\Gamma, \end{aligned}$$

where the element  $\mathbf{f} \in W^{1,1}(0, T; H)$  is defined by

$$(4.10) \quad \langle \mathbf{f}(t), \mathbf{v} \rangle_H = \langle \mathbf{f}_0(t), \mathbf{v} \rangle_{L^2(\Omega; \mathbb{R}^d)} + \langle \mathbf{f}_N(t), \mathbf{v} \rangle_{L^2(\Gamma_N; \mathbb{R}^d)}$$

for all  $\mathbf{v} \in H$  and  $t \in [0, T]$ . The boundary condition (4.5) reveals that

$$\begin{aligned} &\int_{\Gamma_{C_1}} (\sigma_\nu(t)(v_\nu - u'_\nu(t)) + \boldsymbol{\sigma}_\tau(t) \cdot (\mathbf{v}_\tau - \mathbf{u}'_\tau(t))) \, d\Gamma \\ (4.11) \quad &= \int_{\Gamma_{C_1}} \sigma_\nu(t)(v_\nu - u'_\nu(t)) \, d\Gamma \geq 0. \end{aligned}$$

On the other hand, by virtue of (4.6), we have

$$(4.12) \quad \int_{\Gamma_{C_2}} (\sigma_\nu(t)(v_\nu - u'_\nu(t)) + \boldsymbol{\sigma}_\tau(t) \cdot (\mathbf{v}_\tau - \mathbf{u}'_\tau(t))) d\Gamma \geq \int_{\Gamma_{C_2}} F(u'_\nu(t) - v_\nu) d\Gamma \\ + \int_{\Gamma_{C_2}} \mu F \|\mathbf{u}'_\tau(t)\|_{\mathbb{R}^d} d\Gamma - \int_{\Gamma_{C_2}} \mu F \|\mathbf{v}_\tau\|_{\mathbb{R}^d} d\Gamma.$$

We now substitute (4.11) and (4.12) into (4.9) and combine the resulting inequality with the constitutive law (4.1), the initial condition (4.7), and the regularity  $\mathbf{u}(t) \in H, \mathbf{u}'(t) \in K$ . As a result we obtain the following variational formulation for Problem 20.

PROBLEM 21. *Find a displacement field  $\mathbf{u}: [0, T] \rightarrow H$  such that*

$$\mathbf{u}'(t) \in K, \quad \langle \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}'(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\mathbf{u}(s)) ds, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}'(t)) \rangle_{\mathcal{V}} \\ + \int_{\Gamma_{C_2}} F(v_\nu - u'_\nu(t)) d\Gamma + \int_{\Gamma_{C_2}} \mu F \|\mathbf{v}_\tau\|_{\mathbb{R}^d} d\Gamma - \int_{\Gamma_{C_2}} \mu F \|\mathbf{u}'_\tau(t)\|_{\mathbb{R}^d} d\Gamma \\ \geq \langle \mathbf{f}(t), \mathbf{v} - \mathbf{u}'(t) \rangle_H \quad \text{for all } \mathbf{v} \in K \text{ and a.e. } t \in (0, T),$$

$$\mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega.$$

Our main result in the study Problem 21 reads as follows.

THEOREM 22. *Assume  $H(\mathcal{A}), H(\mathcal{B}), H(\mathcal{R}),$  and  $H(f)$  are fulfilled. Then, for any initial displacement  $\mathbf{u}_0 \in H$  there exists a unique solution  $\mathbf{u} = \mathbf{u}(\mathbf{u}_0) \in W^{1,\infty}(0, T; H)$  of Problem 21. Moreover, the map  $\mathbf{u}_0 \mapsto \mathbf{u}(\mathbf{u}_0): H \rightarrow W^{1,\infty}(0, T; H)$  is Lipschitz continuous.*

*Proof.* The proof is based on Theorem 12 and is carried out in several steps. The idea is to reformulate Problem 21 as a sweeping process of the form (1.2) for which the conditions in Theorem 12 are satisfied. The steps of the proof are the following.

(i) *Equivalence with an evolutionary inclusion problem.* We introduce the operators  $A: H \rightarrow H, B: H \rightarrow H,$  and  $\mathcal{R}: C([0, T]; H) \rightarrow C([0, T]; H)$  defined by

$$(4.13) \quad \langle A\mathbf{u}, \mathbf{v} \rangle_H = \langle \mathcal{A}(\boldsymbol{\varepsilon}(\mathbf{u})), \boldsymbol{\varepsilon}(\mathbf{v}) \rangle_{\mathcal{V}},$$

$$(4.14) \quad \langle B\mathbf{u}, \mathbf{v} \rangle_H = \langle \mathcal{B}(\boldsymbol{\varepsilon}(\mathbf{u})), \boldsymbol{\varepsilon}(\mathbf{v}) \rangle_{\mathcal{V}},$$

$$(4.15) \quad \langle (\mathcal{R}\mathbf{w})(t), \mathbf{v} \rangle_H = \left\langle \int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\mathbf{w}(s)) ds, \boldsymbol{\varepsilon}(\mathbf{v}) \right\rangle_{\mathcal{V}}$$

for all  $\mathbf{v}, \mathbf{u} \in H,$  and  $\mathbf{w} \in C([0, T]; H)$ . Also, we consider the function  $\varphi: H \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$  given by



$$\varphi(\mathbf{v}) = \begin{cases} \int_{\Gamma_{C_2}} \mu F(v_\nu + \|\mathbf{v}_\tau\|_{\mathbb{R}^d}) d\Gamma & \text{if } \mathbf{v} \in K, \\ +\infty & \text{otherwise} \end{cases}$$

for all  $\mathbf{v} \in H$ . Under the above notation, it is easy to see that Problem 21 is equivalent to the problem of finding a displacement field  $\mathbf{u}: [0, T] \rightarrow H$  such that

$$\mathbf{u}'(t) \in K, \quad \langle A\mathbf{u}'(t) + B\mathbf{u}(t) + (\mathcal{R}\mathbf{u})(t) - \mathbf{f}(t), \mathbf{v} - \mathbf{u}'(t) \rangle_H + \varphi(\mathbf{v}) - \varphi(\mathbf{u}'(t)) \geq 0$$

for all  $\mathbf{v} \in H$  and a.e.  $t \in (0, T)$ ,

$$\mathbf{u}(0) = \mathbf{u}_0,$$

which, in turn, is equivalent to the following evolutionary inclusion problem: find a displacement field  $\mathbf{u}: [0, T] \rightarrow H$  such that

$$(4.16) \quad \begin{cases} \mathbf{f}(t) - A\mathbf{u}'(t) - B\mathbf{u}(t) - (\mathcal{R}\mathbf{u})(t) \in \partial\varphi(\mathbf{u}'(t)) & \text{for a.e. } t \in (0, T), \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases}$$

(ii) *Equivalence with a sweeping process.* By virtue of the definitions of  $K$  and  $\varphi$ , we can see that  $K$  is a nonempty, closed, convex cone, and  $\varphi$  is proper, convex, and positively homogeneous of degree 1 (i.e.,  $\varphi(\lambda\mathbf{u}) = \lambda\varphi(\mathbf{u})$  for all  $\lambda > 0$  and  $\mathbf{u} \in H$ ). Moreover,  $\varphi$  is Lipschitz continuous on  $K$ , and, in addition,  $\varphi(\mathbf{0}) = 0$ . We now define a closed convex subset  $C$  of  $H$  by

$$C := \partial\varphi(\mathbf{0}) = \{ \boldsymbol{\xi} \in H \mid \varphi(\mathbf{v}) \geq \langle \boldsymbol{\xi}, \mathbf{v} \rangle_H \text{ for all } \mathbf{v} \in H \}.$$

Using this definition and the properties of  $\varphi$  we have

$$\varphi(\mathbf{v}) = \sup_{\boldsymbol{\xi} \in C} \langle \boldsymbol{\xi}, \mathbf{v} \rangle_H = \sigma_C(\mathbf{v}) = I_C^*(\mathbf{v}) \quad \text{for all } \mathbf{v} \in H,$$

where  $I_C^*$  is the Legendre–Fenchel conjugate of the indicator function  $I_C$ . Moreover, since  $I_C$  is proper, convex, and lower semicontinuous, it has the following properties:

$$\begin{cases} \partial\varphi(\mathbf{v}) = \partial I_C^*(\mathbf{v}), \\ \varphi^*(\mathbf{v}) = I_C^{**}(\mathbf{v}) = I_C(\mathbf{v}). \end{cases}$$

On the other hand, by the definitions of the Legendre–Fenchel conjugate and the convex subdifferential operator, the following equivalence holds:

$$\boldsymbol{\xi} \in \partial\varphi(\mathbf{v}) \iff \mathbf{v} \in \partial\varphi^*(\boldsymbol{\xi}).$$

From the above relations, it follows that

$$\boldsymbol{\xi} \in \partial\varphi(\mathbf{v}) \iff \mathbf{v} \in \partial I_C(\boldsymbol{\xi}) \iff \mathbf{v} \in N_C(\boldsymbol{\xi}).$$

Therefore, we have

$$\begin{aligned} \mathbf{f}(t) - A\mathbf{u}'(t) - B\mathbf{u}(t) - (\mathcal{R}\mathbf{u})(t) &\in \partial\varphi(\mathbf{u}'(t)) \\ \iff \mathbf{u}'(t) &\in N_C(\mathbf{f}(t) - A\mathbf{u}'(t) - B\mathbf{u}(t) - (\mathcal{R}\mathbf{u})(t)) \\ \iff \mathbf{u}'(t) &\in N_{C - \mathbf{f}(t)}(-A\mathbf{u}'(t) - B\mathbf{u}(t) - (\mathcal{R}\mathbf{u})(t)) \\ \iff -\mathbf{u}'(t) &\in N_{C(t)}(A\mathbf{u}'(t) + B\mathbf{u}(t) + (\mathcal{R}\mathbf{u})(t)) \end{aligned}$$

for a.e.  $t \in [0, T]$ , where  $C(t) := \mathbf{f}(t) - C = \mathbf{f}(t) - \partial\varphi(\mathbf{0})$ . Now using (4.16), it follows that Problem 21 is equivalent to the following sweeping process: find  $\mathbf{u}: [0, T] \rightarrow H$  such that

$$\begin{cases} -\mathbf{u}'(t) \in N_{C(t)}(\mathbf{A}\mathbf{u}'(t) + \mathbf{B}\mathbf{u}(t) + (\mathcal{R}\mathbf{u})(t)) & \text{for a.e. } t \in (0, T), \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases}$$

(iii) *Validity of the assumptions of Theorem 12.* We now verify that all conditions in Theorem 12 are fulfilled. To this end, from hypotheses  $H(\mathcal{A})$ ,  $H(\mathcal{B})$  and definitions (4.13), (4.14) of the operators  $A$  and  $B$ , respectively, we obtain that  $A$  and  $B$  satisfy conditions  $H(A)$  and  $H(B)$  with  $\alpha = L_{\mathcal{A}}$  and  $L_B = L_{\mathcal{B}}$ , respectively. Moreover, since  $\mathcal{R} \in C([0, T]; \mathbf{Q}_{\infty})$ , a simple calculation based on the elementary inequality

$$\|\mathcal{E}\boldsymbol{\tau}\|_{\mathcal{V}} \leq d \|\mathcal{E}\|_{\mathbf{Q}_{\infty}} \|\boldsymbol{\tau}\|_{\mathcal{V}} \quad \text{for all } \mathcal{E} \in \mathbf{Q}_{\infty}, \boldsymbol{\tau} \in \mathcal{V},$$

shows that the operator  $\mathcal{R}$  defined by (4.15) is a history-dependent operator, i.e., it satisfies condition  $H(\mathcal{R})$  with  $L_{\mathcal{R}} = d \max_{t \in [0, T]} \|\mathcal{R}(t)\|_{\mathbf{Q}_{\infty}}$ .

Next, let  $s, t \in [0, T]$  be such that  $s \leq t$ . Using the definition of  $C(t)$ , we have

$$\begin{aligned} \mathcal{H}(C(t), C(s)) &\leq \|\mathbf{f}(t) - \mathbf{f}(s)\|_H = \left\| \int_s^t \mathbf{f}'(\theta) d\theta \right\|_H \leq \int_s^t \|\mathbf{f}'(\theta)\|_H d\theta \\ &= \int_0^t \|\mathbf{f}'(\theta)\|_H d\theta - \int_0^s \|\mathbf{f}'(\theta)\|_H d\theta = \eta(t) - \eta(s), \end{aligned}$$

where  $\eta: [0, T] \rightarrow \mathbb{R}_+$  is the function defined by

$$\eta(t) = \int_0^t \|\mathbf{f}'(\theta)\| d\theta \quad \text{for all } t \in [0, T].$$

Recall that  $\mathbf{f} \in W^{1,1}(0, T; H)$ , and therefore we conclude from the above that  $\eta$  is an absolutely continuous and nondecreasing function. This implies that the multivalued mapping  $t \mapsto C(t): [0, T] \rightarrow 2^H$  fulfills condition  $H(C)$ .

(iv) *End of proof.* We are now in a position to apply Theorem 12 to conclude that Problem 21 has a unique solution  $\mathbf{u} \in W^{1,\infty}(0, T; H)$  which depends Lipschitz continuously on the initial data  $\mathbf{u}_0 \in H$ .  $\square$

A pair of functions  $(\mathbf{u}, \boldsymbol{\sigma})$  such that  $\mathbf{u}$  is a solution of the contact Problem 21 and  $\boldsymbol{\sigma}$  is given by the constitutive law (4.1) is called a weak solution to Problem 20. Theorem 22 states the unique weak solvability of Problem 20 as well as its continuous dependence with respect to the initial data, i.e., its stability. Note that the weak solution has the regularity  $\mathbf{u} \in W^{1,\infty}(0, T; H)$  and  $\boldsymbol{\sigma} \in L^{\infty}(0, T; \mathcal{V})$ .

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