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A classic recursive sequence calculus task at the secondary-tertiary level in France

Macarena Flores González^{a*}, Fabrice Vandebrouck^b and Laurent Vivier^c

^{a,b,c}Université de Paris, Univ Paris Est Creteil, CY Cergy Paris Université, Univ. Lille, UNIROUEN, LDAR, F-75013 Paris, France.

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Abstract.

Our work focuses on the transition from high school to university in the field of calculus. In France, recursive sequences are studied as one of the classical exercises in both institutions. Their studies use different theorems and notions, such as functions, convergence, monotonicity, induction, etc. The work expected at this transition requires the development of recognition and control activities on the part of the students. We propose a new task allowing the development of such activities from a dialectic between two paradigms of analysis using the calculator. We highlight that students at the end of high school have difficulties in studying these sequences, and they do not easily develop the activities of recognition and control independently. This could cause problems in understanding recursive sequences at the beginning of university.

Keywords: *Mathematical work, Paradigm, mathematical activities, Activity Theory, control.*

1 Introduction

Many studies have characterized the significant transition between secondary school and university in mathematical education. This transition brings both continuities and discontinuities in teaching and learning (Gueudet, 2008; Gueudet et al., 2016; Winsløw et al., 2018; Gueudet & Thomas, 2020; Monaghan et al., 2019).

In calculus, functions, limits, derivatives and real numbers are mathematical objects that have been widely studied, and are known to be problematic for students (Thomas et al., 2015; Bressoud et al., 2016; Oktaç & Vivier, 2016). However, despite their importance, only a few studies have focused on sequences of the form $u_{n+1} = f(u_n)$ (Boschet, 1982; Weigand, 1991, 2004; Ghedamsi, 2008; Ghedamsi & Fattoum, 2018). These objects are interesting because they draw upon many notions found in calculus that students find difficult. Moreover, in science degrees, recursive sequences are often used to solve problems in science disciplines other than mathematics (Krainer, 2015).

Studies such as Sierpiska (1990), Mamona-Downs (2001) and Przenioslo (2004, 2005) highlight the importance of sequences, and their convergence, in the teaching and learning of

*macarena.flores-gonzalez@cyu.fr, vandebro@univ-paris-diderot.fr, laurent.vivier@univ-paris-diderot.fr

calculus. In Germany, Weigand (1991, 2004) noted the role of representations as high school students discover the properties of recursive sequences. He also emphasized the need to train students in the interpretation of graphical representations with the help of computers.

Our study focuses on the recognition of the properties of recursive sequences using representations given by technological tools, reflecting the transition from high school to university in France. The work we present here is a continuation of a previous study, which confirms that high school teaching of recursive sequences is insufficient in preparing students for higher education. At university, undergraduates are expected to work autonomously and master new knowledge (including formal notions). Although tasks remain in the domain of calculus, they are less clear-cut and give fewer indications. Students are expected to be able to recognize correct methods, but there is no help to check their results. In spite of its potential, and the competencies acquired by students in high school, a calculator is often kept aside or forbidden at university.

In that context, how can we prepare students to recognize and control their work, in using both their theoretical knowledge and their instrumental competencies with calculator?

In section 2, we discuss the importance of the recursive sequences in the curriculum, and outline the main difficulties encountered by students. In Section 3, we present the theoretical tools used in our study, which are mainly drawn from Activity Theory in Didactic of Mathematics (Vandebrouck, 2018). Our methodology is presented in Section 4. Section 5 presents the *a priori* analysis of our experimental exercise, while Section 6 outlines our *a posteriori* analysis.

2 Learning about recursive sequences

2.1 From advanced mathematics to the curriculum

Recursive sequences are of great interest in many mathematical problems at university level. They are used to find solutions to equations $g(x) = 0$, using for instance Newton's numerical method that ask to study the convergence of a sequence (u_n) defined by $u_{n+1} = f(u_n)$ with a new function $f(x) = x + \alpha(x)g(x)$. Examples also include the Cauchy Lipschitz (Picard-Lindelöf) method for solutions of differential equations, the implicit function theorem, multivariable analysis, differential geometry, fractal and dynamical systems theory and so on. Moreover, the structure of \mathbb{N} , and specially the notion of successor, is crucial for the definition of recursive sequences (and induction). Recursive sequences cannot be interpreted only as functions between two sets.

Given their importance in the mathematical universe, in France, recursive sequences appear both at the end of secondary school, and at the start of university. First of all, in high school, sequences constitute an important part of calculus: in 12th grade, it is almost a quarter of the calculus curriculum in the textbook (Barbazo & Barnet, 2020). Calculus starts with two chapters on sequences; one is dedicated to recursiveness. This concerns the students who choose the *spécialité mathématique* option (taken by 41% of the students in 2020¹). In 2019, at the

¹<https://www.education.gouv.fr/la-rentree-2020-les-eleves-de-terminale-precisent-leur-choix-de-parcours-307016>

*macarena.flores-gonzalez@cyu.fr, vandebro@univ-paris-diderot.fr, laurent.vivier@univ-paris-diderot.fr

baccalaureate², there was a study of a recursive sequence with a parameter. High school exercises generally present the task as follows. First, study the function that generates the sequence: determine any fixed points ($f(x) = x$), variations³ and, in some cases, the stability of an interval. Then, study the sequence: look for monotony, bounds (often with a proof by induction), convergence (using the theorem: “any monotone and bounded sequence is convergent”) and the value of the limit. Thus, at high school, the study of recursive sequences is carried out with algebraic techniques and, in some cases, they propose either a graphical representation or the use of the calculator.

On the other hand, in the first year at university, students begin with some preliminary exercises (limits, continuity and differentiability of functions), before moving to tasks based on recursive sequences of the form $u_{n+1} = f(u_n)$. As an example, on the exercise sheet about real numbers and sequences for the first semester at the University, one fifth of the exercises directly or indirectly concern such recursive sequences, without any graphical representation and without the use of the calculator. At this stage, students are expected to use a wider range of theorems and methods. In addition to the methods and theorems studied in high school, students work with a formal definition of convergence, monotonic, adjacent and extracted sequences, the Bolzano-Weierstrass theorem, the extension of broad inequalities, and the two theorems: 1) if $\lim_{x \rightarrow k} f(x) = l$ and if $\lim u_n = k \Rightarrow \lim f(u_n) = l$; 2) if for all sequences (u_n) convergent to k one has $\lim f(u_n) \rightarrow l$, hence $\lim_{x \rightarrow k} f(x) \rightarrow l$. Moreover, neither the proof by induction nor the study of the recursive sequences are explicitly addressed in the lecture course.

2.2 Students’ difficulties in the high school-university transition

From a didactic perspective, recursive sequences, together with the concepts of convergence and limit, bring difficulties for students. These topics combine two objects: sequences and functions, which are not yet mastered by students at the end of high school and the beginning of university (Rousse, 2018). Concerning semiotic representations, representations such as the “cobweb diagram” (Figure 1), frequently used in teaching, are not understood by students and are not at all helpful (Weigand, 1991).

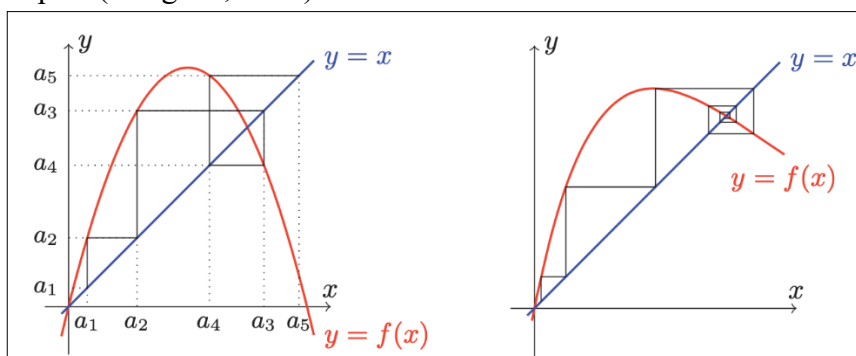


Figure 1: Examples of cobweb diagrams (Krainer, 2015, p. 301).

² The final exam of high school required to enter at university.

³ With respect to functions, it is usual to summarize variations using a table with arrows on each interval where the function is increasing or decreasing.

In our previous study (Flores González et al., 2020), we compared some assessment tasks at the end of high school and in the first year of university in France about recursive sequences, and we analysed students' responses at both levels of education. Success rate was low, especially at the university level. Students easily confuse exercises that are based on techniques from the domain of sequences and those from the domain of functions. In spite of few similarities in both domains, overall for functional sequences of type $f(n)$ and sometimes with subtlety like in “if f is increasing then $u_{n+1} = f(u_n)$ is monotone”, most calculus techniques are different; derivative computation or proof by induction are specific to, respectively, functions and sequences. There is no link between the limits of f and the limit of recursive sequence (u_n) , and the existence of a recursive sequence is a specific problem, even if it remains implicit in teaching. Some students did not recognize recursive sequences as specific sequences, and treated them as $u_n = f(n)$ (see Figure 2). Thus, many students were unclear about the distinction between sequences and functions and were unable to produce coherent mathematical work. Finally, the task did not use a numerical tool to explore the recursive sequence in either the last year of high school or the first year of university.

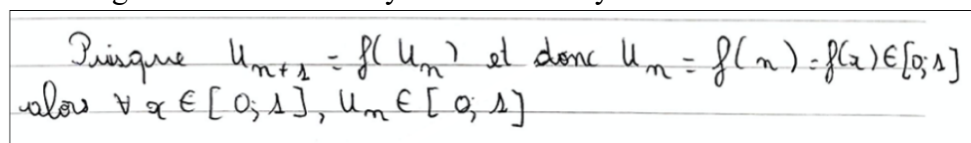


Figure 2: First year university student who does not recognize the correct sequence.

In this paper, we design and experiment a task at high school level aiming at preparing future students for the recognition and mathematical control activities in the field of recursive sequences.

3 Theoretical tools

3.1 Activity Theory in Didactic of Mathematics

Adapted from Activity Theory, Activity Theory in Didactic of Mathematics (ATDM) was developed about 20 years ago (Vandebrouck, 2018). ATDM models epistemological and cognitive aspects by differentiating between *task* and *activity* (Rogalski, 2013). Drawing on the tools that ATDM uses to describe and interpret the student's *activity*, we identify both the mathematical knowledge needed to solve the *task* and the way it must be used in terms of *adaptations* of knowledge. Examples of *adaptations* of knowledge are: mixtures of knowledge; the use of intermediaries; a change of mathematical domain; the introduction of steps; the introduction of results from previous questions, etc. (Horoks & Robert, 2007). The direct application of a theorem does not need any adaptation. *Adaptations* of knowledge leads us to the concept of individual *mathematical activities* among students. This emphasis on the cognitive dimension of the student's activity can explain the process of internalisation (Vygotsky, 1978) from a socio-constructivist point of view, providing, as Simon et al. (2018, p. 2) note, “a complement to sociocultural theory”.

Our study expands on complex tasks (those that require *adaptations* of knowledge), their context, the mediations, and the student's activities. We focus especially on two critical mathematical activities that are characteristic of complex tasks (Robert & Vandebrout, 2014):

- *Recognizing activities* mainly occur in a context where students have to recognize the mathematical knowledge they can use to solve the task they are given. They may also be asked to recognize how they can apply or adapt this knowledge. Students can also recognize a method and that various steps in their reasoning can be connected.
- *Control activities* are found when students must highlight that their mathematical reasoning is coherent, by introducing several check points. They also ensure that the answer produced corresponds to the intended goal of the activity.

Tasks of recursive sequences are complex, due to the mix of knowledge about functions and sequences and the need for students to adapt this knowledge. These tasks lead specifically to recognizing and control activities⁴.

Recognizing activities are developed when students autonomously choose the method and the associate knowledge that will help to solve the task: how to determine the variation of a sequence, to calculate the sign of $u_{n+1} - u_n$ or compare $\frac{u_{n+1}}{u_n}$ with 1, to use a proof by induction or not, to find out whether the sequence is bounded above or below, to select a possible theorem among all the theorem about sequences, to recognize the need of the results of the previous questions etc. Of course, mediations can help students recognize the method to be applied. Their form depends on the context (i.e., it may come from the problem statement or from the teacher). However, mediation often reduces the cognitive demand. On the other hand, control activities are seen when students relate different elements and properties of the recursive sequence to their results.

3.2 Paradigms of analysis

The transition from school to university is marked by significant differences in mathematical work. For instance, calculators are widely used in high school, but are often not allowed in French universities.

In the following, we refer to the MWS notion of paradigms (Houdement & Kuzniak, 2006). This notion allows us to identify different types of work that are internally coherent. In the domain of analysis, Montoya Delgadillo and Vivier (2016, pp. 742–743) distinguish three paradigms:

- Arithmetic/geometric analysis [A1]. This supports interpretations that draw upon implicit assumptions based on geometry, arithmetic calculations or the real world. Although argumentation plays a role, work is grounded on the visualization of signs, possibly produced by software or a calculator. Visualizing the curve of a function or a table of values is admissible as a proof in [A1].

⁴ Note that sometimes we omit the word “activity”, and simply use “control” or “recognize”.

- Calculation analysis [A2]. The rules of the calculation are defined more-or-less explicitly, and are applied independently of any reflection on the existence and nature of the objects in question. Calculations are often based on an algorithmic approach, along with formal expressions that have a representative role. These routines are executed without being aware of the nature of the mathematical objects. Work is oriented toward the production of proofs and demonstrations, using the properties of the objects and theorems, in the mathematical tradition. Visualizing the curve of a function or a table of values is not admissible in [A2]. However, tools, such as computer algebra system software may be used (to compute a derivative, for instance).

The third paradigm appears at university level, especially when students begin to manipulate the epsilon definitions. Our study on control activities focuses on paradigms [A1] and [A2], leaving the third paradigm for further investigations.

In the context of a recursive sequence of the form $u_{n+1} = f(u_n)$, paradigm [A1] consists in working with the representation of the first terms of the sequence, for example in a table of values or graphically, and visualizing, for instance, monotony or convergence. The work can be controlled, for example, by calculating more terms or by zooming in or out on a graph. This is similar to how a mathematician studies a new object for which no procedure is available. On the other hand, work in the [A2] paradigm consists in working with algebraic calculations and theorems. For example, showing by induction that the sequence is bounded by 1, that $u_{n+1} - u_n$ is positive, therefore the sequence is increasing, to conclude, by theorem, that it is convergent.

It can be difficult to evaluate a student's work in terms of a single paradigm, and the paradigms given above should not be seen as a way to categorize students. Rather, they offer a way to understand how the work is expected by the institution and the work actually done. Moreover, paradigms are not isolated from each other and the links between them can promote mathematical control. For example, paradigm [A1] can be used to develop a conjecture and, thus, to guide the work in paradigm [A2] which helps to prevent errors and promote a mathematical coherence (control). Consequently, mathematical work is based on a dialectic between paradigms; the same idea is seen in geometry, where few would attempt to make a demonstration without preparing a figure.

However, in the French context, the teaching of recursive sequences at the transition from high school to university focuses on the [A2] paradigm which guides the expected work. Work in the [A1] paradigm appears mainly at the beginning of the teaching in high school to introduce new notions and to make (the correct) conjectures before switching to a work in [A2], without going back to an [A1] work. Only a little part of the [A1]/[A2] dialectic is used and there is almost nothing about control, which causes difficulties for students at university (Flores González, 2021).

3.3 Hypotheses and research question

The study aims to deepen the understanding of the transition from high school to university in the domain of calculus. In France, recursive sequences play an important role in the curriculum

of both institutions with specific difficulties for students at university level. In this sense, the research is based on the following two hypotheses: (1) in order to be better prepared for the mathematical work which awaits them at university, students need to develop recognizing and control activities for recursive sequences (Flores González et al., 2020); (2) the dialectic between [A1] and [A2] paradigms seems to have a good potential for the development of these two activities in the field of calculus.

Then, focusing on high school students, we wonder to what extent does a [A1]/[A2] dialectic paradigms support the development of the recognizing and control activities expected in the study of recursive sequences?

4 Method: the design of an experimental task

In order to answer our research question, taking into account our previous study, we design an experimental task devoted to promote a dialectic between [A1] and [A2] paradigms aiming at improving the students' recognizing and control activities. We pay particular attention to the calculator, an emblematic tool of the transition that enables students to produce numerical or graphical representations on which a work in [A1] is possible.

In our previous study (Flores González et al., 2020), we studied an exercise of first year of a university in the Paris region, on the recursive sequence $u_{n+1} = \frac{1}{2-u_n}$ with $u_0 = \alpha$ in $[0,1]$. This exercise was complex, above all because of the parameter α . In this case, the sequence and the function $f(x) = \frac{1}{2-x}$ were interlinked and required students to be able to recognize, and control, the mathematical objects they are working with. In a baccalaureate exercise, this kind of task was divided into two parts: the first related to the function, and the second to the recursive sequence. With these two exercises, we designed the experimental complex task in adopting the structure of the baccalaureate, in two parts, with the sequence of the university exam. Finally, and consistent with our research question, we introduced a part that ask for a work in the [A1] paradigm to promote a dialectic between the [A1] and [A2] paradigms. Then the exercise was divided into three parts:

- Part A concerned the function $f(x) = \frac{1}{2-x}$ (Figure 3, below) which properties are needed to study the sequence. Technical questions are stated like in the baccalaureate.
- Part B was directly related to the study of the sequence $u_{n+1} = f(u_n)$ in the [A1] paradigm. After using their calculator to calculate the first terms of the sequence, we asked students to develop conjectures about the properties of the sequence.
- Part C was dedicated to proving the properties in the [A2] paradigm.

Hence, part B (paradigm [A1]) prepares students for part C (paradigm [A2]). Part C is designed to reflect undergraduate exercises; the questions are more open than in the baccalaureate task, and require undergraduates to develop recognizing activities. For instance, proof by induction is not explicitly stated, and the question about the variation of the sequence is less directive than in the baccalaureate – students are asked to study it once they have worked on part B.

Our aim is to identify students' ability to recognize the sequence as a recursive one $u_{n+1} = f(u_n)$, especially the clear distinction between the properties of the function and the properties of the sequence. For instance, f is increasing on $[0,1]$ and this property only implies that the sequence is monotone. From the point of view of control activities, our aim is to identify examples of such activities or, on the other hand, inconsistencies indicative of a lack of control in part C. For this, part B includes an explicit instruction for using a calculator. By experimenting in this order (paradigm [A1] in part B for conjectures, and paradigm [A2] in part C for proving), we want to promote control and recognizing activities relying on the [A1]/[A2] dialectic. This order is known to high school students, but only at the beginning of the teaching and not, as in our experimentation, after the teaching where [A2] is usually predominant.

Our experiment was run with a 12th grade mathematics class in a general public high school in Paris. It took place at the beginning of the school year, after the teaching of limits of sequences and functions. Thus, the students had already worked on proof by induction, finite and infinite limits of a sequence, general theorems on the limits of sequences and functions (operations and comparisons), and limits of usual functions (polynomial, exponential). The teacher regularly used the scientific calculator with the students.

The class was composed of 30 students, named E1 to E30, who worked individually with their calculators (Texas Instruments TI 83 and CASIO graph) without any interaction with the teacher. A researcher was in the classroom and took some photos when she noticed interesting work. We analysed the students' written answers.

5 A priori analysis of the designed task and students awaited activities

In this section, we present the three parts of the designed tasks, detailing the possible (and expected) recognizing and control activities both in paradigm [A1] (part B) and paradigm [A2] (part C) as well as the control activities based on the [A1]/[A2] dialectic (going back to part B when working in part C).

In part A (Figure 3), work can be done in either of the two paradigms. But we do not present an analysis of part A, as we focus only on parts B and C.

Part A: Let f be the function defined on $\mathbb{R} - \{2\}$, by $f(x) = \frac{1}{2-x}$.

- 1) Solve in $\mathbb{R} - \{2\}$ the equation: $f(x) = x$.
- 2) What is the variation of f ?
- 3) Show that for all real x in $[0,1]$, $f(x)$ is in $[0,1]$.

Figure 3: Part A devoted to the study of f .

5.1 Part B

Part B begins (Figure 4) with three tables to be filled in, by computing values of u_1 to u_6 for three values of α , 0, 0.5 and 0.9 (Table 1).

Let α be a real number belonging to $[0,1]$. Consider the sequence (u_n) defined by $u_0 = \alpha$ and, for all natural numbers n , $u_{n+1} = f(u_n)$, where f is the function defined on $\mathbb{R} - \{2\}$, given by $f(x) = \frac{1}{2-x}$.

B1) Using your calculator, fill in the table below for three values of α (with an accuracy of 10^{-4}).

Figure 4: First question in part B.

Table 1: Expected answers to question B1, in decimal register.

Case $\alpha_1 = 0$		Case $\alpha_2 = 0,5$		Case $\alpha_3 = 0,9$	
u_0	0	u_0	0,5	u_0	0,9
u_1	0,5000	u_1	0,6666	u_1	0,9090
u_2	0,6666	u_2	0,7500	u_2	0,9166
u_3	0,7500	u_3	0,8000	u_3	0,9230
u_4	0,8000	u_4	0,8333	u_4	0,9285
u_5	0,8333	u_5	0,8571	u_5	0,9333
u_6	0,8571	u_6	0,8750	u_6	0,9375

The purpose of question B1 was to identify and become familiar with the sequence using explicit examples for three values of α . In theoretical terms, we expected students to recognize the recursive sequence $u_{n+1} = \frac{1}{2-u_n}$ (not explicitly given in the problem statement). This recognition was expected to lead to a control of the terms of the sequence using the calculator which here was essential: computing $f(n)$ leads to a problem of definition of the sequence for $n = 2$ and the three tables, for different value of α , are the same.

Note that the calculator could give two numerical representations: decimal (Table 1) and fractional. The former (the one we were hoping to see) is more helpful in answering the following questions (B2 and B3). Although the latter is less useful, it does help in conjecturing the algebraic expression of the sequence $u_n = \frac{n}{n+1}$. The choice of representation can either be conscious, or ‘decided’ by the calculator’s configuration. If conscious, it could help students to either develop a control activity (using the expression of u_n or the calculator).

The following two questions (Figure 5) asked students to develop conjectures based on the table of values.

B2) Look at your answers to question B1 and develop a conjecture for the variation, convergence, and a possible limit of the sequence.

How did you use your calculator?

B3) Is it possible to develop a conjecture for the variation, convergence and a possible limit of (u_n) for all values of α in $[0,1]$?

Could a calculator be used? If so, how?

Figure 5: Questions B2 and B3, [A1] paradigm.

Questions B2 and B3 asked the student to develop conjectures for the variation, convergence and limit of the sequence based on their observation of its first six terms, for each given value of α . While the monotony of the sequence is quite clear from the first terms (possibly supported by a control using the calculator to compute more terms), this is not so easy for the convergence and the limit, which are a local property at infinity and cannot be clearly determined from the information of B1⁵. Thus, some non-responses to the question could come from a good understanding of the values given in the table and the notion of convergence.

We anticipated that students would recognize (increasing) monotonicity, convergence, and that the limit of the sequence is 1. The recognizing activities initially rely on the interpretation of signs, by visualization. Then, students develop their mathematical discourse and the conjecture that it is an increasing, convergent sequence, and that its limit is possibly 1.

In order to stimulate control of the mathematical object and successful recognition, we added the question “How did you use the calculator?”. The aim of this question was to make the student aware of how they used their calculator (and of the possibility to use it!) and stimulate them to recognize the coherence (or non-coherence) of their approach. These elements were expected to encourage them to control their mathematical work by, for example, programming an algorithm, developing a graphical representation, or calculating more terms of the sequence.

5.2 Part C

In part C, we expected students to recognize theorems, procedures and proofs related to each of the questions. The task brings many recognizing activities which are characteristic of the university level. Mathematical work was anchored in the [A2] paradigm in each of the questions with possible controls grounded in their work in part B. Moreover, some students may work in paradigm [A1] based on their answers to part B.

For the sequence (u_n) , defined in part B.

C1) Show that for all natural numbers n , u_n is in $[0,1]$.

C2) What is the variation of the sequence (u_n) ?

⁵ It is difficult to move from a punctual perspective based on a few values, to a local perspective: can it be established that the sequence $\ln(n)$ converges to 3, given the 20 first values? Or that the harmonic series converges to 4 from the 30 first values?

- C3) Show that the sequence (u_n) is convergent.
- C4) We denote l as a limit of the sequence (u_n) , explain why $l = f(l)$.
- C5) Given $l = f(l)$, compute the value of l .

Figure 6: Questions in part C, [A2] paradigm. For each question, students were asked if the calculator could be used and, if so, to explain how.

In question C1, we expected to see a recognition activity, with a demonstration by induction of the steps to be taken (as in the university exam). We interpret that students have a control activity if they explicitly relate their proofs to the conjectures of part B.

In question C2, we expected to see recognition of a method to determine the increasing monotonicity of the sequence, as in the original university exam. There are various options. Students can either continue to use a proof by induction (showing that for every n , $u_n \leq u_{n+1}$), using directly - or not - the growing property of the function f from part A, or analyse the variation of the sequence with algebraic expressions $u_{n+1} - u_n$ or $\frac{u_{n+1}}{u_n}$. For the former strategy,

we expect them to recognize the identity $(u_n - 1)^2$ in the numerator of $u_{n+1} - u_n = \frac{u_n^2 - 2u_n + 1}{2 - u_n}$, to conclude, using $u_n \leq 1$, that $u_{n+1} - u_n$ is positive and, then, that the sequence is increasing. In the latter case, the student must be able to recognize the quadratic polynomial in the numerator and use their answer to question C1 (i.e. that u_n is in $[0,1]$). A lack of coherence with the part B answers is seen as a lack of, or absence of, controls (notably when the students develop an interpretation without having recognized the method).

Finally, question C3 only required the students to recognize the monotonic convergence theorem. Here, students can use their results from previous questions. From their answer to C1 they can interpret that the sequence (u_n) is bounded by 1; from C2 they can interpret that the sequence (u_n) is increasing; therefore, by using the theorem, it is convergent. We also expected students to check their work (control activities) by linking their answer to question C3 with the conjectures developed in part B.

C4 and C5 are quite technical and we do not analyse them, except the value of limit l with respect to conjecture of part B.

In this way, the design of the task and its a priori analysis are directly related to the hypotheses outlined in section 3.3.

6 A posteriori analyses of the part B and C

We gave students' work one of the three statuses: 1) **correct** mathematical work (the mathematical activities were the expected ones); 2) **incomplete** or not totally correct mathematical work (discourse and written assertions that were mathematically true, but only partially justified); and 3) **non-correct** mathematical work (when the discourse and written assertions were false or did not correspond to what was expected). In the analyses below, this status of work is combined with an analysis in terms of mathematical activities and paradigms.

As previously mentioned, we do not analyse part A. We give a report about part B, the original part of our study, then we focus on part C and the influence of work in part B.

6.1 Part B

6.1.1 Question B1: recognizing of the recursive sequence

First, we distinguished between students who correctly recognized the recursive sequence as it is defined, $u_{n+1} = f(u_n) = \frac{1}{2-u_n}$, and those who did not. In particular, we identify the students who incorrectly recognized the sequence as $u_{n+1} = f(n) = \frac{1}{2-n}$. About half of the students succeeded in recognizing the correct recursive sequence (17 out of 30 students): fourteen students correctly used the calculator, which led them to produce correct tables of values, and the other three produced incomplete tables. For the other thirteen students: nine confused $f(u_n)$ with $f(n)$, three did not answer the question, and one filled the table with zeros.

The answers given by student E22 (Figure 7) illustrate the typical confusion between $f(u_n)$ and $f(n)$, with three identical tables, together with an explanation of the shift $n - 1$ (however, the student calculates $\frac{1}{2+n}$).

Cas $\alpha_1 = 0$		Cas $\alpha_2 = 0, 1$		Cas $\alpha_3 = 0, 5$	
u_0	0	u_0	0,1	u_0	0,5
u_1	0,3	u_1	0,5 0,3	u_1	0,3
u_2	0,15	u_2	0,25	u_2	0,25
u_3	0,0	u_3	0,2	u_3	0,2
u_4	0,17	u_4	0,17	u_4	0,16
u_5	0,14	u_5	0,14	u_5	0,14
u_6	0,125	u_6	0,125	u_6	0,125
La suite semble croissante puis décroissante Elle semble être convergente, et sa limite serait 0.			Comment avez-vous utilisé la calculatrice ? On a rentré la suite $\frac{1}{2-(n-1)}$ on a tapé table et on a noté les solutions.		
The sequence seems increasing and then decreasing. It seems to be convergent, and its limit would be 0.			I put in the sequence $\frac{1}{2-(n-1)}$, pressed on table, and wrote down the solutions.		

Figure 7: E22's answers to B1.

This work in paradigm [A1] shows that many students found it difficult to correctly distinguish between recursive and functional sequences. As this distinction requires theoretical knowledge about recursive versus functional sequences, it is unlikely that they will produce *correct* work in paradigm [A2]. Consequently, in the following analysis, we focus only on the seventeen students who correctly recognized the sequence even if their work was *non-correct*.

6.1.2 An example of control, possible with the use of the calculator

The following figures exemplify one type of control. The student E8 uses their calculator to calculate the values of the sequence. E8 is confronted with a systematic error message (Figure 8 shows the message “ERREUR” on the calculator’s screen). We can make the hypothesis that E8 has a sufficient level of control to be able to understand the lack of coherence. E8 reviews its calculations. The whole process of E8 is not clear, but E8 may start by computing $u_1 = f(1) = 1$ and wrote “ERROR” on the sheet as on the calculator screen (Figure 8). Hence, starting with a $f(n)$ interpretation, the calculator helped E8 by giving feedback that allowed E8 to produce the correct table (Figure 9). Moreover, it should be noted that this student is the one who performed best during the whole task.

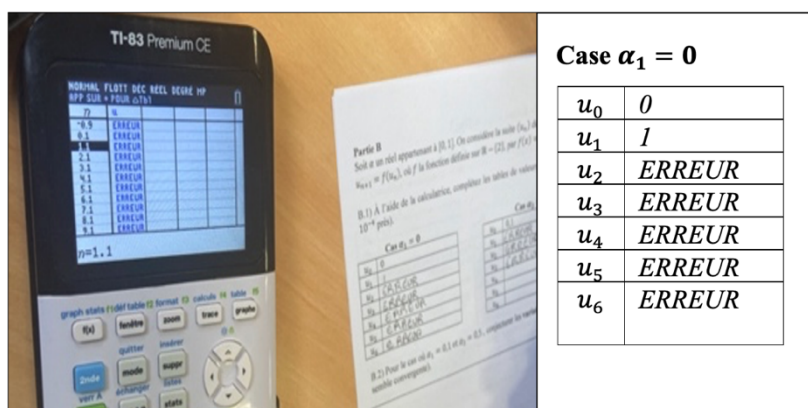


Figure 8: E8’s answers before control.

Cas $\alpha_1 = 0$		Cas $\alpha_2 = 0, 1$	
u_0	0	u_0	0,1
u_1	0,5	u_1	0,5263
u_2	0,6667	u_2	0,6786
u_3	0,7500	u_3	0,7568
u_4	0,8	u_4	0,8044
u_5	0,8333	u_5	0,8364
u_6	0,8571	u_6	0,8594

Figure 9: E8’s answers given after control.

This raises the question of whether some students correctly recognize the sequence because they are explicitly asked to use a calculator to compute the first terms, and thus receive the ERROR feedback of the calculator for a $f(n)$ interpretation.

6.1.3 Questions B2 and B3: recognizing growth, convergence and limit

Among the seventeen students who successfully recognize the recursive sequence in B1, fifteen answered B2 (Table 2).

Table 2: Summary of the work done by fifteen students at B2 who recognized the recursive sequence

Class number in B2	Tables for B1	Recognition in B2			Paradigm	Correctness		
		Growth	Convergence	Limit		C	I	NC

15 students	12-C	11	10	7	[A1]	7	3	2
	3-I	2	2				2	1

For the twelve students (12-C) who had produced correct tables in B1: seven recognized the growth, the convergence and the limit of the sequence (7-C); one did not recognize convergence, one found that the limit was 0.9 and one only recognized the growth (3-I); finally, two found that the limit was 0 (2-NC). Students are better able to recognize growth compared to convergence or the value of the limit.

For the three students who had produced incomplete tables of values in B1 (3-I): two recognized that the sequence was growing, and one of them also recognized a convergence towards 0.9 (2-I); and one recognized both convergence and divergence (1-NC).

For the two students who consider the limit as 0.9, there is a lack of control. Indeed, calculating additional terms would enable to identify the mistake. Maybe there is a lack of control also for students who answer “limit is 1”, but there is no evidence.

Among these fifteen students, ten gave an answer to B3, and only five with *correct* work. Indeed, recognition of growth, convergence and the limit are much more successful for the three fixed values of α (B2) than for the generalisation (B3). As this question asks the student to develop a conjecture (or the generalisation of another conjecture) based on their answer to B2, the falling success rate from B2 to B3 is unsurprising.

Note that only one student worked with the fractional representation of the terms of the sequence given by the calculator: their work was *correct* for B2 and *incomplete* for B3.

For the following, we decided to focus on these fifteen students who answered B2 (since very few students were able to answer B3) and recognized, even partially, the properties of growth, convergence and the limit of the sequence.

6.2 Part C

6.2.1 Question C1: the sequence is bounded

Table 3 presents a summary of the work in question C1 done by the fifteen previous students. This table indicates the paradigms, the recognition of the proof by induction and the correctness of the work. We expected that these students would check the coherence of their discourse in part C with their answers in part B.

From the students of table 2, we analyze separately: the seven students who produced a correct solution to B1 and B2; the three who produced correct tables in B1 and an incomplete solution to B2; the two students who produced *incomplete* work both in B1 and B2; and finally, we regroup the three students who did not produce a correct solution to B2 (two B1-C and one B1-I).

Table 3: Summary of the work done by fifteen students at C1.

Categories from B1-B2	NR	Paradigm			Induction	Correctness		
		[A1]	[A2]	?		C	I	NC
B1-C and B2-C: 7 students	1	4	2		2 [A2]	1 [A2]	4	1
B1-C and B2-I (recognition of growth but not convergence): 3 students	1		2		2 [A2]		2	
B1-I and B2-I: 2 students		1		1			2	
B2-NC: 3 students	3							

Of the fifteen students who answered B2, five could not give any answer to C1 (NR), and only one of these five have produced a *correct* work in B2. *Correct* work in paradigm [A1] (part B) therefore seems to be a good precursor for the work that is expected in paradigm [A2] (part C). We can hypothesize that the more students are able to work in part B, the more they are prepared to address question C1.

It should be noted, however, that among the seven students who produced *correct* work in B2 (and who answered question C1), four spontaneously continued to work in the [A1] paradigm. Although working in the [A1] paradigm seems to be good preparation for switching to [A2], making the switch appears to be an obstacle for some students.

Five students worked in the [A1] paradigm: four produced *incomplete* work but they were able to explicitly control their work in relation to part B. For example, student E15 argues that the sequence converges to 1 but does not reach it, so it is in $[0,1]$. Then, they say that they can use the calculator's table function.

One student (E4), who had produced an *incomplete* work in B1 and B2, gives an intermediate work between [A1] and [A2]. They base their arguments on the graph of the function f (Figure 10). Here again, although their answer is not in line with expectations, there is a control based on the graph of f and they produce an *incomplete* work.

$f(0) = \frac{1}{2}$ $f(1) = 1$ on a pas de variation anormale dans l'encodage $\forall n \in [0,1]$ et avoir une courbe croissante alors pour tout entier n , u_n appartient à $[0,1]$	$f(0) = \frac{1}{2}$ $f(1) = 1$ With no anormal variation in the bound $[0,1]$ and as the curve is increasing, then for any whole number n , u_n is in $[0,1]$.
---	--

Figure 10: Solution provided by E4 to C1.

Four students worked in the expected [A2] paradigm. While two provided *correct* work to B1 and B2, the remaining two correctly recognized the growth of the sequence but did not give *correct* work to the convergence question (notably, a value of 0.9 for the limit). These four students started with a proof by induction: one provided a *correct* work, two provided an *incomplete* work, and the fourth (E10) switched to the model $u_n = f(n)$.

Finally, among the fifteen students, only one student produced a *non-correct* work (E14) and only one produced fully *correct* work (E8). Eight students did *incomplete* work, in paradigms [A1] or [A2] (4 with an attempt of proof by induction).

6.2.2 Question C2: the sequence is increasing

All of the ten students who answered C1 also answered C2. This is consistent with the idea that working in paradigm [A1], even in question C1, supports students' work up to question C2.

Table 4 presents a summary to question C2.

Table 4: Summary of the work done by 10 students at C2.

Class number: Categories from C1	Paradigm			$u_{n+1} - u_n$	f increasing $\Rightarrow (u_n)$ increasing	Correctness		
	[A1]	[A2]	?			C	I	NC
[A2]: 4 students	1	3		2	1	1	2	1
[A1]: 5 students	4	1		1	1		2	3
Other: 1 student			1		1		1	

Data only refers to the ten students who answered both questions C1 and C2 (and who recognized the recursive sequence in B1 and answered B2). It shows the paradigms, the two most-used strategies⁶ and the correctness of the work.

Four out of ten students worked in the [A2] paradigm. Three of these had already worked in [A2] to answer C1 and one after working in [A1] in question C1 (E14). Among the first three, one produced a *correct* work and one produced an *incomplete* work (both by using the strategy $u_{n+1} - u_n$). One student (E25, Figure 11) produced a *non-correct* work without control (because of the confusion $u_n = f(n)$) – while an incomplete proof by induction in C1 was accompanied by the correct expression of the recursive sequence.

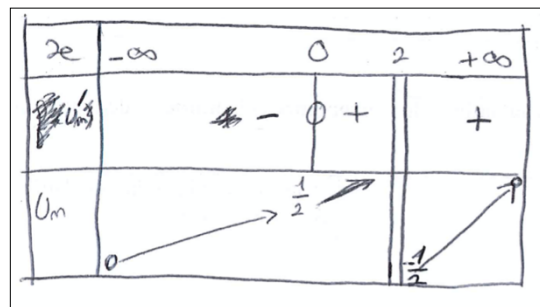


Figure 11: E25's answer to C2.

⁶ None of the students proposed the calculation $\frac{u_{n+1}}{u_n}$.

Student E14 (Figure 12) shifted from the paradigm [A1] in C1 to [A2] in C2. In C2, they calculated $u_{n+1} - u_n$ but with $u_n = f(n)$ (*non-correct* work; they had already switched to the $u_n = f(n)$ model in C1), without referring to part B.

$$\begin{aligned}
 u_{n+1} - u_n &= \frac{1}{2(n+2)} - \frac{1}{2n} \\
 &= \frac{1}{-n} - \frac{1}{2-n} \\
 &= \frac{1}{(2-n)-n} - \frac{1(1-n)}{(2-n)-n} \\
 u_{n+1} - u_n &= \frac{1+1}{-2n+n^2} = \frac{2+n}{-2n+n^2} \quad \nearrow
 \end{aligned}$$

Figure 12: E14's answer to C2.

Five students worked in the [A1] paradigm. Four had worked also in [A1] to solve question C1: two produced an *incomplete* work based on tables of values or the graph given by the calculator, and two produced a *non-correct* work. The fifth (E29) switched from [A2] in question C1 (where they had worked with the $u_n = f(n)$ model) to [A1] in question C2, this time with a generalisation, saying that “the sequence seems to be increasing because $u_0 < u_1 < u_2 \dots u_{n+1}$ ”. They probably relied on their correct tables of values produced in part B, however, their work is *incomplete* in C2.

For the tenth student considered (E4), it is difficult to identify a paradigm: they argue that the sequence is increasing by referring to the increasing curve of the function. The theorem-in-act (Vergnaud, 1991) “ f is increasing therefore (u_n) is increasing” that they seem to use may either hide a confusion between sequences and functions, suggesting a model of the type $u_n = f(n)$, or indicating a transfer of the properties of f to the sequence in the case of the model $u_{n+1} = f(u_n)$. Four of the students in class applied this theorem-in-act.

6.2.3 Question C3: the sequence is convergent

We recall that fifteen students provided at least partially correct answers to questions B1 and B2. Ten of these provided answers to questions C1 and C2. Of these, only seven managed to answer to C3.

Table 5 summarizes answers to C3 given by the seven students: paradigms, recognition of the theorem and the correctness of the work.

Table 5: Summary of the work of seven students to C3.

Categories from C2	Paradigm			Theorem	Correctness		
	[A1]	[A2]	?		C	I	NC
[A1] in C1 and C2: 3 students	3					3	
[A2] in C1 and C2: 2 students		2		1	1		1
Different paradigms in C1 and C2: 2 students		1	1				2

Three students worked in [A1] paradigm and provided *incomplete* work. It is remarkable that all of them worked in [A1] to answer all of the questions in part C. In particular, one of them worked with both references to the tables of values and the use of a calculator (Figure 13).

d'après la calculatrice on voit que la suite converge vers 1 mais ne l'atteint pas	Pourrait-on utiliser la calculatrice ? Si oui, comment ? oui avec le tableau
According to the calculator, the sequence converges towards 1 but doesn't reach it	Yes, with the table

Figure 13: E15's answer to C3.

Three students worked in the [A2] paradigm: only E8 correctly recognized the theorem to be used. All of their work in part C was in the [A2] paradigm. There is only one evidence of controls: this student explains, at the end of their work in part C, that the sequence converges towards 1 and not 0.9, which they had mistakenly conjectured in part B. The two other students provided *non-correct* work: one argued that the sequence converges because it was increasing and bounded (E21); another tried to calculate the limit of the sequence directly and did so erroneously (E14).

The last student (E25) moved from a work in the [A2] paradigm, in C1 and C2, to work between [A1] and [A2] in C3; after a *correct* work in question C1 and *non-correct* work in C2, they conclude from their erroneous table of variation (Figure 11) that the sequence converges towards 0.

What emerges from this analysis of answers is that constant reliance on paradigm [A1] makes it possible to continue to attempt to work up to question C3, even if the work is *incomplete*. On the other hand, work in paradigm [A2] does not necessarily lead to the expected work. It seems that students who want to make their work conform to the (expected) [A2] paradigm ultimately produce incorrect mathematics, for instance E14 (Figure 14).

$$\lim_{n \rightarrow \infty} (u_n) = \frac{1+n}{-2n+n^2} = \frac{+\infty^2}{-\infty} = -\infty$$

$$-\frac{1}{n} + \frac{1}{n^2} = -\infty$$

Figure 14: E14's answer to C3.

6.2.4 Non-correct work in [A1] (part B) does not produce a correct work in [A2] (part C)

Among the nine students who used $u_n = f(n)$ in B1, 5 do not answer Part C. Of the remaining four, with respect to C1: two work with induction ([A2] paradigm), but they do not necessarily follow the rules of the proof (1 NC and 1 I). One still works in the [A1] paradigm (I) and one only wrote “For $n \in \mathbb{N}$, $0 \leq u_n \leq 1$ ” (I) ([A2] paradigm). In C2, two of these four students make the table of variation to show the growth of the sequence (NC); only one states the strategy

$u_{n+1} - u_n$ (I), and one student writes that the sequence is growing but provides no proof (I). For the answers to C3, 2 do not answer, one writes that the sequence is diverging (NC), and the last one that the sequence converges to 0 (NC) (relative to their working paradigm which we cannot tell). These observations are consistent with our choice in section 6.2 to focus only on the fifteen students who answered B2 and recognized, even partially, the properties of growth, convergence and the limit of the sequence.

7 Conclusion

Our work aims to deepen the understanding of the transition from high school to university with respect to recursive sequences $u_{n+1} = f(u_n)$.

Considering our hypotheses and our research question, the task sought to determine the extent to which the [A1]/[A2] dialectic could be used to develop recognition and control activities in students. We start by summarizing the results for parts B and C. Then we answer, partially, our research question.

7.1 Recognizing and control activities in [A1] and [A2] paradigms

In part B, fewer than half of the students correctly recognized the recursive sequence, its growth and limit. Hence, the earlier practical work in the [A1] paradigm did not help these students to better recognize the form of the sequence: this recognizing activity is not obvious for many students. This pleads for more work in the [A1] paradigm.

However, some students have already used some control activities in [A1], like E8 in B1 (section 6.1.2) whose control allows him to avoid the error of considering $f(n)$... and to do very good work up to C3.

In part C, some students continued to work in paradigm [A1] partly based on their work in part B, with controls activities, and their work was *incomplete*. It seems that working in paradigm [A1] was helpful for these students.

Students who worked in paradigm [A2] recognized the need for a proof by induction; this recognition seems to be a strong indicator of the switch to paradigm [A2] to study recursive sequences. However, some students who worked in the [A2] paradigm produced incorrect solutions and did not relate their work to part B (no control).

We saw paradigm shifts between questions C1 and C2 (from [A1] to [A2] and from [A2] to [A1]). A work in the [A2] paradigm required students to recognize an adequate method, but these students were unable to produce *correct* work. Moreover, it seems that they cannot control their work.

Only one student answered all the questions correctly. They are the only one to answer C1 correctly and also the only one to recognize the correct theorem in C3. This student appeared to be able to control their work based on part B, stating that “*the sequence converges towards 1 and not 0.9*” (his conjecture).

7.2 What can be said about the research question

In our previous study (Flores González et al., 2020), we concluded that a dialectic between paradigms could be interesting to help students improve their understanding of sequences $u_{n+1} = f(u_n)$. In this new research, we wonder to what extent does a [A1]/[A2] dialectic paradigm support the development of the recognizing and control activities expected in the study of recursive sequences. With this aim, we designed a task that was given to students of one class at the end of high school.

Obviously, the results are limited to the sample, and further investigations would be required if we wanted to extend them. However, we want to point out the following conclusions:

- Some students recognized the adequate knowledge, whether in [A1] or in [A2] paradigms, sometimes with control;
- The use of calculator favors control, even if it is not automatic;
- A lot of students did not make any connection between paradigms [A1] and [A2], therefore blocking any fruitful control relying on the dialectic [A1]/[A2].

If the two first points prove the potential of the task with respect to the development of recognizing and control activities, it did not succeed in engaging the majority of students in this way. The third point is clearly a constraint, but it is worth mentioning that developing such activities is an ongoing work in progress.

7.3 Other important findings

In French high school, working in paradigm [A1] is asked mainly at the beginning of a chapter when new notions are introduced. There is an implicit hypothesis that students naturally connect ‘elementary’ work in [A1] and ‘serious’ work in [A2]. But this is not so easy. Our study clearly shows that [A1] and [A2] are disconnected for most of the students. Are they aware that they are dealing with the same mathematical object?

This has a consequence on the understanding that students might develop. Indeed, the recursive sequence is not recognized by a lot of students even for ‘elementary’ computations. What representations therefore of such a mathematical object do these students have? If they cannot make this recognition in [A1] paradigm, what is the meaning for them when working in the expected [A2] paradigm? It is paradoxical to think that such students would be able to correctly work in paradigm [A2], which is more abstract, when it appears that they are not even able to use the relevant recognition and control activities in paradigm [A1].

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