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# A simple and constructive proof to a generalization of Lüroth's theorem

François OLLIVIER

LIX, UMR CNRS 7161  
École polytechnique  
91128 Palaiseau CEDEX  
France

francois.ollivier@lix.polytechnique.fr

Brahim SADIK

Département de Mathématiques  
Faculté des Sciences Semlalia  
B.P. 2390, 40000 Marrakech  
Maroc

sadik@ucam.ac.ma

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**Abstract.** A generalization of Lüroth's theorem expresses that every transcendence degree 1 subfield of the rational function field is a simple extension. In this note we show that a classical proof of this theorem also holds to prove this generalization.

**Keywords:** Lüroth's theorem, transcendence degree 1, simple extension.

**Résumé.** Une généralisation du théorème de Lüroth affirme que tout sous-corps de degré de transcendance 1 d'un corps de fractions rationnelles est une extension simple. Dans cette note, nous montrons qu'une preuve classique permet également de prouver cette généralisation.

**Mots-clés :** Th. de Lüroth, degré de transcendance 1, extension simple.

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## Introduction

Lüroth's theorem ([2]) plays an important role in the theory of rational curves. A generalization of this theorem to transcendence degree 1 subfields of rational functions field was proven by Igusa in [1]. A purely field theoretic proof of this generalization was given by Samuel in [6]. In this note we give a simple and constructive proof of this result, based on a classical proof [7, 10.2 p.218].

Let  $k$  be a field and  $k(x)$  be the rational functions field in  $n$  variables  $x_1, \dots, x_n$ . Let  $\mathcal{K}$  be a field extension of  $k$  that is a subfield of  $k(x)$ . To the subfield  $\mathcal{K}$  we associate the prime ideal  $\Delta(\mathcal{K})$  which consists of all polynomials of  $\mathcal{K}[y_1, \dots, y_n]$  that vanish for  $y_1 = x_1, \dots, y_n = x_n$ . When the subfield  $\mathcal{K}$  has transcendence degree 1 over  $k$ , the associated ideal is principal. The idea of our proof relies on a simple relation between coefficients of a generator of the associated ideal  $\Delta(\mathcal{K})$  and a generator of the subfield  $\mathcal{K}$ . When  $\mathcal{K}$  is finitely generated, we can compute a rational fraction  $v$  in  $k(x)$  such that  $\mathcal{K} = k(v)$ . For this, we use some methods developed by the first author in [3] to get a generator of  $\Delta(\mathcal{K})$  by computing a Gröbner basis or a characteristic set.

## Main result

Let  $k$  be a field and  $x_1, \dots, x_n, y_1, \dots, y_n$  be  $2n$  indeterminates over  $k$ . We use the notations  $x$  for  $x_1, \dots, x_n$  and  $y$  for  $y_1, \dots, y_n$ . If  $\mathcal{K}$  is a field extension of  $k$  in  $k(x)$  we define the ideal  $\Delta(\mathcal{K})$  to be the prime ideal of all polynomials in  $\mathcal{K}[y]$  that vanish for  $y_1 = x_1, \dots, y_n = x_n$ .

$$\Delta(\mathcal{K}) = \{P \in \mathcal{K}[y] : P(x_1, \dots, x_n) = 0\}.$$

*Lemma 1.* — *Let  $\mathcal{K}$  be a field extension of  $k$  in  $k(x)$  with transcendence degree 1 over  $k$ .*

- i) The ideal  $\Delta(\mathcal{K})$  is principal in  $\mathcal{K}[y]$ .*
- ii) If  $\mathcal{K}_1 \subset \mathcal{K}_2$  and  $\Delta(\mathcal{K}_i) = \mathcal{K}_i[y]G$ , for  $i = 1, 2$ , then  $\mathcal{K}_1 = \mathcal{K}_2$ .*
- iii)  $\Delta(\mathcal{K}) = \tilde{\Delta}(\mathcal{K}) := (p(y) - p(x)/q(x)q(y) | p/q \in \mathcal{K})$ .*
- iv) The ideal  $\hat{\Delta}(\mathcal{K}) := k[x]\Delta(\mathcal{K}) \cap k[x, y]$  is a radical ideal, which is equal to  $(q(x)p(y) - p(x)q(y) | p/q \in \mathcal{K})$ .*

v) Let  $G$  be such that  $\Delta(\mathcal{K}) = (G)$ , with  $G = \sum_{j=0}^d p_j(x)/q_j(x)y^j$  and  $\text{GCD}(p_j, q_j) = 1$ , for  $0 \leq j \leq d$ . Let  $Q := \text{PPCM}(q_j \mid 0 \leq j \leq d)$ , then  $\hat{G} := QG$  is such that  $G(y, x) = -G(x, y)$  and  $\deg_x \hat{G} = \deg_y \hat{G} = d$ .

PROOF. — i) In the unique factorization domain  $\mathcal{K}[y]$ , the prime ideal  $\Delta(\mathcal{K})$  has codimension 1. Hence, it is principal.

ii) Assume that  $\mathcal{K}_1 \neq \mathcal{K}_2$ . There exists  $p(x)/q(x) \in \mathcal{K}_2$  a reduced fraction, with  $p(x)/q(x) \notin \mathcal{K}_1$ . The set  $\{1, p(x)/q(x)\}$  may be completed to form a basis  $\{e_1 = 1, e_2 = p/q, \dots, e_s\}$  of  $\mathcal{K}_2$  as a  $\mathcal{K}_1$ -vector space. Then,  $e$  is also a basis of  $\mathcal{K}_2[y] = \mathcal{K}_2\mathcal{K}_1[y]$  as a  $\mathcal{K}_1[y]$ -module and  $Ge$  is a basis of  $\Delta(\mathcal{K}_2) = \mathcal{K}_2\Delta(\mathcal{K}_1)$  as a  $\mathcal{K}_1[y]$ -module. So,  $p(y) - p(x)/q(x)q(y) \in \Delta(\mathcal{K}_2)$  is equal to  $p(y)e_1 - q(y)e_2$ , which implies that  $G$  divides  $p$  and  $q$ , a contradiction.

iii) We remark that  $\tilde{\Delta}(\mathcal{K})$  does not define any prime component containing polynomials  $k[y]$ , so that  $\tilde{\Delta}(\mathcal{K}) : k[y] = \tilde{\Delta}(\mathcal{K})$ . The inclusion  $\supset$  is immediate. Let  $P \in \Delta(\mathcal{K})$  with  $P(x, y) = \sum_{j=0}^s p_j(x)/q_j(x)y^j$ . We have  $P(x, x) = 0$  and by symmetry  $P(y, y) = 0$ , so  $P = P(x, y) - P(y, y) = \sum_{j=0}^s (p_j(x)/q_j(x) - p_j(y)/q_j(y))y^j$ . So, throwing away denominators in  $k[y]$ ,  $\prod_{j=1}^s q_j(y)P \in \tilde{\Delta}(\mathcal{K})$ , so that  $P \in \tilde{\Delta}(\mathcal{K}) : k[y] = \tilde{\Delta}(\mathcal{K})$ , hence the result.

iv) The ideal  $\Delta(\mathcal{K})$  is prime, so that  $k(x)\Delta(\mathcal{K})$  and  $\hat{\Delta}(\mathcal{K})$  are radical. We remark that  $\hat{\Delta}(\mathcal{K})$  does not define any prime component containing polynomials  $k[x]$  or in  $k[y]$ , so that  $\hat{\Delta}(\mathcal{K}) : (k[x]k[y]) = \hat{\Delta}(\mathcal{K})$ . The inclusion  $\supset$  is immediate. Using the generators  $p(y) - p(x)/q(x)q(y)$ ,  $p/q \in \mathcal{K}$ , a finite set of fractions  $\Sigma$  is enough by Noetherianity, so that  $\prod_{p/q \in \Sigma} q(x)\delta(\mathcal{K}) \subset (p(y) - p(x)/q(x)q(y) \mid p/q \in \mathcal{K})$ , which provides the reverse inclusion, using the previous remark.

v) By construction,  $\hat{G}$  is a generator of  $\hat{\Delta}(\mathcal{K})$ . All the generators of  $\hat{\Delta}(\mathcal{K})$  in iv) being antisymmetric,  $\hat{G}$  is antisymmetric, which also implies that  $\deg_x \hat{G} = \deg_y \hat{G} = d$ . ■

**THEOREM 2.** — *Let  $\mathcal{K}$  be a field extension of  $k$  in  $k(x)$  with transcendence degree 1 over  $k$ . Then, there exists  $v$  in  $k(x)$  such that  $\mathcal{K} = k(v)$ .*

PROOF. — By lem. 1 i), the prime ideal  $\Delta(\mathcal{K})$  of  $\mathcal{K}[y]$  is principal. Let  $G$  be a monic polynomial such that  $\Delta(\mathcal{K}) = (G)$  in  $\mathcal{K}[y]$ . Let  $c_0(x), \dots, c_r(x)$  be the coefficients of  $F$  as a polynomial in  $\mathcal{K}[y]$ . Since  $x_1, \dots, x_n$  are transcendental over  $k$  there must be a coefficient  $v := c_i$  that lies in  $\mathcal{K} \setminus k$ .

Write  $v = \frac{f(x)}{g(x)}$  where  $f$  and  $g$  are relatively prime in  $k[x]$ . By lem. 1 v),  $\max(\deg_x f, \deg_x g) \leq d := \deg_x G$ . As  $g(x)f(y) - f(x)g(y)$  is a multiple of  $\hat{G}$ ,  $\max(\deg_x f, \deg_x g) = d$ . Let  $D := f(y) - vg(y)$ . As  $D \in \Delta(\mathcal{K})$ , the remainder of the Euclidean division of  $G$  by  $D$  is also in  $\Delta(\mathcal{K})$  and of degree less than the degree of  $G$ . It must then be 0. Therefore  $D$  is a generator of  $\Delta(k(v))$  and of  $\Delta(\mathcal{K})$ , with  $k(v) \subset \mathcal{K}$ , and by lem. 1 ii), we need have  $\Delta(\mathcal{K}) = \Delta(k(v))$  and  $\mathcal{K} = k(v)$ . ■

The following result, given by the first author in [3, prop. 4 p. 35] and [4, th. 1] in a differential setting that includes the algebraic case, permits to compute a basis for the ideal  $\Delta(\mathcal{K})$ .

PROPOSITION 3. — *Let  $\mathcal{K} = k(f_1, \dots, f_r)$  where the  $f_i = \frac{P_i}{Q_i}$  are elements of  $k(x)$ . Let  $u$  be a new indeterminate and consider the ideal*

$$\mathcal{J} = \left( P_1(y) - f_1 Q_1(y), \dots, P_r(y) - f_r Q_r(y), u \left( \prod_{i=1}^r Q_i(y) - 1 \right) \right)$$

in  $\mathcal{K}[y, u]$ . Then

$$\Delta(\mathcal{K}) = \mathcal{J} \cap \mathcal{K}[y].$$

## Conclusion

A generalization of Lüroth's theorem to differential algebra has been proven by J. Ritt in [5]. One can use the theory of characteristic sets to compute a generator of a finitely generated differential subfield of the differential field  $\mathcal{F}\langle y \rangle$  where  $\mathcal{F}$  is an ordinary differential field and  $y$  is a differential indeterminate. In a forthcoming work we will show that Lüroth's theorem can be generalized to one differential transcendence degree subfields of the differential field  $\mathcal{F}\langle y_1, \dots, y_n \rangle$ .

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