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A simple and constructive proof to a generalization of Lüroth's theorem

François OLLIVIER

LIX, UMR CNRS 7161
École polytechnique
91128 Palaiseau CEDEX
France

francois.ollivier@lix.polytechnique.fr

Brahim SADIK

Département de Mathématiques
Faculté des Sciences Semlalia
B.P. 2390, 40000 Marrakech
Maroc

sadik@ucam.ac.ma

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Abstract. A generalization of Lüroth's theorem expresses that every transcendence degree 1 subfield of the rational function field is a simple extension. In this note we show that a classical proof of this theorem also holds to prove this generalization.

Keywords: Lüroth's theorem, transcendence degree 1, simple extension.

Résumé. Une généralisation du théorème de Lüroth affirme que tout sous-corps de degré de transcendance 1 d'un corps de fractions rationnelles est une extension simple. Dans cette note, nous montrons qu'une preuve classique permet également de prouver cette généralisation.

Mots-clés : Th. de Lüroth, degré de transcendance 1, extension simple.

Introduction

Lüroth's theorem ([2]) plays an important role in the theory of rational curves. A generalization of this theorem to transcendence degree 1 subfields of rational functions field was proven by Igusa in [1]. A purely field theoretic proof of this generalization was given by Samuel in [6]. In this note we give a simple and constructive proof of this result, based on a classical proof [7, 10.2 p.218].

Let k be a field and $k(x)$ be the rational functions field in n variables x_1, \dots, x_n . Let \mathcal{K} be a field extension of k that is a subfield of $k(x)$. To the subfield \mathcal{K} we associate the prime ideal $\Delta(\mathcal{K})$ which consists of all polynomials of $\mathcal{K}[y_1, \dots, y_n]$ that vanish for $y_1 = x_1, \dots, y_n = x_n$. When the subfield \mathcal{K} has transcendence degree 1 over k , the associated ideal is principal. The idea of our proof relies on a simple relation between coefficients of a generator of the associated ideal $\Delta(\mathcal{K})$ and a generator of the subfield \mathcal{K} . When \mathcal{K} is finitely generated, we can compute a rational fraction v in $k(x)$ such that $\mathcal{K} = k(v)$. For this, we use some methods developed by the first author in [3] to get a generator of $\Delta(\mathcal{K})$ by computing a Gröbner basis or a characteristic set.

Main result

Let k be a field and $x_1, \dots, x_n, y_1, \dots, y_n$ be $2n$ indeterminates over k . We use the notations x for x_1, \dots, x_n and y for y_1, \dots, y_n . If \mathcal{K} is a field extension of k in $k(x)$ we define the ideal $\Delta(\mathcal{K})$ to be the prime ideal of all polynomials in $\mathcal{K}[y]$ that vanish for $y_1 = x_1, \dots, y_n = x_n$.

$$\Delta(\mathcal{K}) = \{P \in \mathcal{K}[y] : P(x_1, \dots, x_n) = 0\}.$$

Lemma 1. — *Let \mathcal{K} be a field extension of k in $k(x)$ with transcendence degree 1 over k . Then the ideal $\Delta(\mathcal{K})$ is principal in $\mathcal{K}[y]$.*

PROOF. — In the unique factorization domain $\mathcal{K}[y]$ the prime ideal $\Delta(\mathcal{K})$ has codimension 1. Hence, it is principal. ■

Theorem 2. — *Let \mathcal{K} be a field extension of k in $k(x)$ with transcendence degree 1 over k . Then, there exists v in $k(x)$ such that $\mathcal{K} = k(v)$.*

PROOF. — By the last lemma the prime ideal $\Delta(\mathcal{K})$ of $\mathcal{K}[y]$ is principal. Let G be a monic polynomial such that $\Delta(\mathcal{K}) = (G)$ in $\mathcal{K}[y]$. We arrange

G with respect to a term order on y and we multiply by a suitable element $A \in k[x]$ so that $F = AG$ is primitive in $k[x][y]$. Let $A_0(x), \dots, A_r(x)$ be the coefficients of F as a polynomial in $k[x][y]$ then all the ratios $\frac{A_i(x)}{A_r(x)}$ lie in \mathcal{K} .

Since x_1, \dots, x_n are transcendental over k there must be a ratio $v = \frac{A_{i_0}(x)}{A_r(x)}$ that lies in $\mathcal{K} \setminus k$. Write $v = \frac{f(x)}{g(x)}$ where f and g are relatively prime in $k[x]$ and let $D = f(y)g(x) - f(x)g(y)$. The polynomial $f(y) - vg(y)$ lies in $\Delta(\mathcal{K})[y]$, so G divides $f(y) - vg(y)$ in $\mathcal{K}[y]$. Therefore F divides D in $k(x)[y]$. But F is primitive in $k[x][y]$, so that F divides D in $k[x][y]$. Since $\deg_{x_i}(D) \leq \deg_{x_i}(F)$ and $\deg_{y_i}(D) \leq \deg_{y_i}(F)$ for $i = 1, \dots, n$ there must be $c \in k$ such that $D = cF$. We have now $\Delta(\mathcal{K}) = \Delta(k(v))$. Hence $\mathcal{K} = k(v)$. ■

The following result, given by the first author in [3, prop. 4 p. 35] and [4, th. 1] in a differential setting that includes the algebraic case, permits to compute a basis for the ideal $\Delta(\mathcal{K})$.

PROPOSITION 3. — *Let $\mathcal{K} = k(f_1, \dots, f_r)$ where the $f_i = \frac{P_i}{Q_i}$ are elements of $k(x)$. Let u be a new indeterminate and consider the ideal*

$$\mathcal{J} = \left(P_1(y) - f_1 Q_1(y), \dots, P_r(y) - f_r Q_r(y), u \left(\prod_{i=1}^r Q_i(y) - 1 \right) \right)$$

in $\mathcal{K}[y, u]$. Then

$$\Delta(\mathcal{K}) = \mathcal{J} \cap \mathcal{K}[y].$$

Conclusion

A generalization of Lüroth's theorem to differential algebra has been proven by J. Ritt in [5]. One can use the theory of characteristic sets to compute a generator of a finitely generated differential subfield of the differential field $\mathcal{F}\langle y \rangle$ where \mathcal{F} is an ordinary differential field and y is a differential indeterminate. In a forthcoming work we will show that Lüroth's theorem can be generalized to one differential transcendence degree subfields of the differential field $\mathcal{F}\langle y_1, \dots, y_n \rangle$.

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