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Quasi-stationary distributions in reducible state spaces

Nicolas Champagnat¹, Denis Villemonais¹

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Abstract

We study quasi-stationary distributions and quasi-limiting behavior of Markov chains in general reducible state spaces with absorption. We propose a set of assumptions dealing with particular situations where the state space can be decomposed into three subsets between which communication is only possible in a single direction. These assumptions allow us to characterize the exponential order of magnitude and the exact polynomial correction, called polynomial convergence parameter, for the leading order term of the semigroup for large time. They also provide explicit convergence speeds to this leading order term. We apply these results to general Markov chains with finitely or denumerably many communication classes using a specific induction over the communication classes of the chain. We are able to explicitly characterize the polynomial convergence parameter, to determine the complete set of quasi-stationary distributions and to provide explicit estimates for the speed of convergence to quasi-limiting distributions in the case of finitely many communication classes. We conclude with an application of these results to the case of denumerable state spaces, where we are able to prove that, in general, there is existence of a quasi-stationary distribution without assuming irreducibility before absorption. This actually holds true assuming only aperiodicity, the existence of a Lyapunov function and the existence of a point in the state space from which the return time is finite with positive probability.

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1 Introduction

Let $(X_n, n \in \mathbb{Z}_+)$ be a Markov chain in $D \cup \{\partial\}$ where D is a measurable space, $\partial \notin D$ and $\mathbb{Z}_+ := \{0, 1, \dots\}$. For all $x \in D \cup \{\partial\}$, we denote as usual by \mathbb{P}_x the law of X given $X_0 = x$ and for any probability measure μ on $D \cup \{\partial\}$, we define $\mathbb{P}_\mu = \int_{D \cup \{\partial\}} \mathbb{P}_x \mu(dx)$. We also denote by \mathbb{E}_x and \mathbb{E}_μ the associated expectations. We assume that ∂ is absorbing, which means that $X_n = \partial$ for all $n \geq \tau_\partial$, \mathbb{P}_x -almost surely, where

$$\tau_\partial = \inf\{n \in \mathbb{Z}_+, X_n = \partial\}.$$

We study the sub-Markovian transition semigroup of X , $(S_n)_{n \in \mathbb{Z}_+}$, defined as

$$S_n f(x) = \mathbb{E}_x(f(X_n) \mathbb{1}_{n < \tau_\partial}), \quad \forall n \in \mathbb{Z}_+,$$

for all bounded or nonnegative measurable function f on D and all $x \in D$. We also define as usual the left-action of P_n on measures as

$$\mu S_n f = \mathbb{E}_\mu(f(X_n) \mathbb{1}_{n < \tau_\partial}) = \int_D S_n f(x) \mu(dx),$$

for all probability measure μ on D and all bounded or nonnegative measurable function $f : D \rightarrow \mathbb{R}$.

The purpose of this article is to provide original and practical criteria allowing to study the quasi-limiting behaviour of absorbed, reducible Markov processes in general state spaces, both in cases of geometric and polynomial convergence in total variation to a quasi-stationary distribution.

We recall that a quasi-stationary distribution (QSD) for X is a probability measure ν_{QS} on D such that

$$\mathbb{P}_{\nu_{QS}}(X_n \in \cdot \mid n < \tau_\partial) = \frac{\nu_{QS} S_n}{\nu_{QS} S_n \mathbb{1}_D} = \nu_{QS}, \quad \forall n \geq 0.$$

It is well known that a probability measure ν_{QS} is a QSD for X if and only if it is a quasi-limiting distribution (see e.g. [12, 23]). By a quasi-limiting distribution ν ,

we mean a probability measure ν such that, for some probability measure μ on D and for any measurable subset $\Gamma \subset D$, the conditional probabilities $\mathbb{P}_\mu(X_n \in \Gamma \mid n < \tau_\partial)$ converges to $\nu(\Gamma)$. To each QSD ν_{QS} is associated an *exponential convergence parameter* $\theta \in (0, 1]$, such that

$$\mathbb{P}_{\nu_{QS}}(\tau_\partial \geq n) = \theta^n, \quad \forall n \geq 0.$$

This parameter is called a *convergence parameter* in [25], and we add the term *exponential* to distinguish it from the *polynomial convergence parameter* that we introduce below.

The study of quasi-limiting behaviour of Markov chains on reducible state spaces started with the work of Mandl [22] (see also [13]). Since then, several works studied cases of finite state spaces [26, 7, 5, 27, 28] or infinite state spaces [17, 8]. Most of these works are devoted to specific processes, while the articles [27, 28] address the general situation in finite state spaces (see also the survey [29]).

In order to obtain results on general state spaces, we make use of results on the principal eigenvalue and eigenvectors of iterates of upper triangular matrices of linear operators over a Banach space (the developments can be found in the Appendix). This allows us to prove sufficient conditions ensuring that a reducible process X satisfies

$$\left\| \theta^{-n} n^{-j(x)} \mathbb{P}_x(X_n \in \cdot) - \eta(x) \nu_{QS} \right\| \xrightarrow{n \rightarrow +\infty} 0, \quad \forall x \in D, \quad (1.1)$$

for some measurable functions $\eta : D \rightarrow [0, +\infty)$ and $j : D \rightarrow \mathbb{Z}_+ = \{0, 1, \dots\}$, and where $\|\cdot\|$ is a weighted total variation norm (see Assumption (A) in Section 2 for more details). We call the function j the *polynomial convergence parameter*. and prove several properties of j , η and ν_{QS} in Section 2.

We emphasize that for many usual irreducible Markov processes, the quasi-limiting behaviour is well understood and it is known that this result holds true with $j \equiv 0$ (see for instance [12, 23, 9, 10]). This is also true for some reducible processes with exponential convergence (see [10, Thm 6.1]). The main novelties of this paper are firstly to consider reducible processes in general state spaces with sub-geometric convergence in (1.1) and with non-uniformly zero j , and secondly to provide a general methodology to obtain precise convergence results as in (1.1). Of course, all the results of the present paper can be easily extended to general unbounded semigroups (i.e. not necessarily sub-Markov) following the same approach as in [11].

The paper is organized as follows. In Section 2, we present our main assumption and its first consequences. In Section 3, we consider reducible sub-Markov processes with two successive sets where this assumption is satisfied. We then consider in Section 4 reducible sub-Markov processes with several communication classes. As an illustration, we prove in Section 5 that, under a mild Lyapunov assumption, processes on discrete state spaces admit quasi-limiting distributions. In the Appendix, we provide a detailed study of iterates of upper triangular matrix of linear operators over a Banach space.

Notation. The set $\mathcal{M}(D)$ is the Banach space of finite signed measures over D , endowed with the total variation norm. We denote by $\mathcal{M}_+(D) \subset \mathcal{M}(D)$ the set of non-negative finite measures over D . Given a positive measurable function W , the set $\mathcal{M}(W)$ is the Banach space of signed measures μ such that $|\mu|(W) < +\infty$, endowed with the norm

$$\|\mu\|_W := |\mu|(W).$$

We extend the operator S_n to $\mathcal{M}(D)$ by $\mu S_n = \int_D \delta_x S_n \mu(dx)$. The set $L^\infty(W)$ is the Banach space of measurable functions f such that $\|f/W\|_\infty < +\infty$, endowed with norm

$$\|f\|_W := \|f/W\|_\infty.$$

Because of the nature of our problem, we will often consider the extensions to $D \cup \{\partial\}$ of functions defined on a subset of $D \cup \{\partial\}$. Systematically and without further notice, all functions are extended by the value 0 outside of their domain of definition. In all the sequel, C will denote a finite constant that may change from line to line.

2 Exponential and polynomial convergence parameter

We consider a discrete time Markov process $(X_n, n \in \mathbb{Z}_+)$ evolving in a measurable set $D \cup \{\partial\}$ with absorption at $\partial \notin D$ at time τ_∂ , and sub-Markovian semigroup $(S_n)_{n \in \mathbb{Z}_+}$. We recall that the *exponential convergence parameter* of the semigroup $(S_n)_{n \in \mathbb{N}}$ is given by

$$\theta_S(\mu) := \inf \left\{ \theta \geq 0, \liminf_{n \rightarrow +\infty} \theta^{-n} \mu S_n \mathbb{1}_D = 0 \right\}, \quad \forall \mu \in \mathcal{M}_+(D).$$

We also set $\theta_{0,S} = \sup_{x \in D} \theta_S(x)$, where $\theta_S(x) = \theta_S(\delta_x)$. We define the *polynomial convergence parameter* of the semigroup $(S_n)_{n \in \mathbb{N}}$ as

$$j_S(\mu) := \inf\{\ell \geq 0, \liminf_{n \rightarrow +\infty} n^{-\ell} \theta_{0,S}^{-n} \mu S_n \mathbb{1}_D = 0\}, \quad \forall \mu \in \mathcal{M}_+(D) \quad (2.1)$$

with the convention $\inf \emptyset = 0$. We also set $j_{0,S} = \sup_x j_S(x)$, where $j_S(x) := j_S(\delta_x)$. Note that if $\theta_S(\mu) < \theta_{0,S}$, then $j_S(\mu) = 0$. We will see in Proposition 2.1 below that the converse inequality $\theta_S(\mu) > \theta_{0,S}$ never happens.

In this section, we are interested in the implications of the following assumption (A) on j_S and on the existence and convergence toward a quasi-stationary distribution for X . In the following sections, we will study sufficient properties implying that X satisfies this condition.

Assumption (A). We have $\theta_{0,S} \in (0, 1]$, j_S is integer valued and there exist a measurable function $W_S : D \rightarrow [1, +\infty)$, a finite or countable set I_S and some probability measures $\nu_{S,i} \in \mathcal{M}(W_S)$ and non-identically zero non-negative $\eta_{S,i} \in L^\infty(W_S)$ for each $i \in I_S$, such that

$$\sum_{i \in I_S} \eta_{S,i} \nu_{S,i}(W_S) \in L^\infty(W_S) \quad (2.2)$$

and such that, for all $f \in L^\infty(W_S)$, all $n \geq 1$ and all $x \in D$,

$$\left| \theta_{0,S}^{-n} n^{-j_S(x)} \mathbb{E}_x(f(X_n) \mathbb{1}_{n < \tau_\partial}) - \sum_{i \in I_S} \eta_{S,i}(x) \nu_{S,i}(f) \right| \leq \alpha_{S,n} W_S(x) \|f\|_{W_S}, \quad (2.3)$$

where $\alpha_{S,n}$ goes to 0 when $n \rightarrow +\infty$.

When Assumption (A) holds true, we define

$$\eta_S := \sum_{i \in I_S} \eta_{S,i} \in L^\infty(W_S). \quad (2.4)$$

Note that (2.3) only gives an equivalent of $\delta_x S_n \mathbb{1}_D$ when $\eta_{S,i}(x) > 0$ for at least one $i \in I_S$. In particular, for all x such that $\theta_S(x) < \theta_{S,0}$, (2.3) implies that $\eta_{S,i}(x) = 0$ for all $i \in I_S$.

We also emphasize that, for all $x \in D$ such that $\eta_S(x) > 0$, (2.3) entails that $\mathbb{P}_x(X_n \in \cdot \mid n < \tau_\partial)$ converges in $\mathcal{M}(W_S)$ toward $\frac{1}{\eta_S(x)} \sum_{i \in I_S} \eta_{S,i}(x) \nu_{S,i}$, which is thus a quasi-limiting distribution and hence a quasi-stationary distribution. The rest of this section is dedicated to the exposition and proofs of finer properties on j_S and on the quasi-stationary distributions of X under Assumption (A).

Remark 1. The results of this section, after minor modifications, hold true without assuming that j_S is integer-valued in Assumption (A). However, assuming j_S integer valued simplifies several expressions and allows us to obtain explicit speed of convergence toward a quasi-stationary distribution (see in particular Remark 6 below). In addition, in the next sections, we prove that j_S is integer valued for the class of reducible processes we are interested in. We refer to [10, Section 3.2] where similar arguments are developed in a more complicated context, but with $j_S \equiv 0$. \triangle

Remark 2. For simplicity, we use a countable set I_S in Assumption (A). Most of the results below can be extended without too much effort to a non-countable measured set $(I_S, \mathcal{I}_S, \xi)$, but the exposition of the assumptions and results become a little bit more technical. In particular, $\sum_{i \in I_S} \eta_{S,i} \nu_{S,i}$ would be replaced by

$$\int_{I_S} \eta_{S,i} \nu_{S,i} \, d\xi,$$

with appropriate measurability conditions. Since our setting is complicated enough, we leave the details of the adaptation to the interested reader. \triangle

Remark 3. The results of this paper are stated in the discrete-time setting. The adaptation to the continuous time setting can be obtained by considering Assumption (A) for the included Markov chain and by assuming in addition that, for all $t \in [0, 1]$, $\mathbb{E}_x(W_S(X_t)) \leq CW_S(x)$ for some constant $C > 0$. \triangle

We start our study with simple properties on the polynomial convergence parameter j_S .

Proposition 2.1. *For all $\mu \in \mathcal{M}_+(D)$,*

$$\theta_S(\mu) \geq \sup \left\{ \theta \geq 0, \mu \{x, \theta_S(x) \geq \theta\} > 0 \right\} \quad (2.5)$$

and

$$j_S(\mu) \geq \sup \left\{ \ell \geq 0, \mu \{x, j_S(x) \geq \ell\} > 0 \right\} \quad (2.6)$$

If Assumption (A) holds true, then j_S is lower semi-continuous on $\mathcal{M}_+(W_S)$ and, for all $\mu \in \mathcal{M}_+(W_S)$,

$$\theta_S(\mu) = \sup \left\{ \theta \geq 0, \mu \{x, \theta_S(x) \geq \theta\} > 0 \right\} \quad (2.7)$$

and

$$j_S(\mu) = \sup \left\{ \ell \geq 0, \mu\{x, j_S(x) \geq \ell\} > 0 \right\}. \quad (2.8)$$

In addition,

$$j_S(\mu) = j_S(\mu S_1). \quad (2.9)$$

In particular, $(j_S(X_n))_{n \geq 0}$ is \mathbb{P}_x -almost surely non-increasing, for all $x \in D$.

Remark 4. The proof of this proposition does not use the fact that j_S is assumed integer-valued in Assumption (A). In particular, this result implies that, under Assumption (A) without the property that j_S is integer valued, the function j_S is integer-valued over $\mathcal{M}_+(W_S)$ if and only if $j_S(x)$ is an integer for all $x \in D$. \triangle

Proof of Proposition 2.1. We prove (2.6), (2.8) and (2.9) in this order. The proof of (2.5), (2.7) are similar and thus omitted.

Proof of (2.6). Fix a positive measure μ on D (the result is trivial if $\mu = 0$). For all $\varepsilon > 0$ and for all $x \in D$ such that $j_S(\mu) + \varepsilon < j_S(x)$, we have by definition of $j_S(x)$ and the fact that $(j_S(\mu) + \varepsilon + j_S(x))/2 < j_S(x)$,

$$\liminf_{n \rightarrow +\infty} \theta_{0,S}^{-n} n^{-(j_S(\mu) + \varepsilon + j_S(x))/2} \delta_x S^n \mathbb{1}_D > 0$$

and hence, since $(j_S(\mu) + \varepsilon + j_S(x))/2 > j_S(\mu) + \varepsilon$,

$$\liminf_{n \rightarrow +\infty} \theta_{0,S}^{-n} n^{-j_S(\mu) - \varepsilon} \delta_x S^n \mathbb{1}_D = +\infty.$$

Using Fatou's Lemma, we obtain

$$\begin{aligned} 0 = \liminf_{n \rightarrow +\infty} \theta_{0,S}^{-n} n^{-j_S(\mu) - \varepsilon} \mu S^n \mathbb{1}_D &\geq \mu \left(\liminf_{n \rightarrow +\infty} \theta_{0,S}^{-n} n^{-j_S(\mu) - \varepsilon} S^n \mathbb{1}_D \right) \\ &\geq \mu \left(+\infty \mathbb{1}_{j_S(\cdot) > j_S(\mu) + \varepsilon} \right). \end{aligned}$$

This implies that, for all $\varepsilon > 0$, $\mu\{x, j_S(x) > j_S(\mu) + \varepsilon\} = 0$, and hence that $\mu\{x, j_S(x) > j_S(\mu)\} = 0$. In particular, any $\ell \geq 0$ such that $\mu\{x, j_S(x) \geq \ell\} > 0$ satisfies $\ell \leq j_S(\mu)$.

We thus proved that

$$j_S(\mu) \geq \ell_\mu := \sup \left\{ \ell \geq 0, \mu\{x, j_S(x) \geq \ell\} > 0 \right\}. \quad (2.10)$$

Proof of (2.8) and the fact that j_S is lower semi-continuous. We assume that $\mu \in \mathcal{M}_+(W_S)$ is a positive measure and that Assumption (A) holds true, and we

prove $j_S(\mu) \leq \ell_\mu$, where ℓ_μ is defined in (2.10). Fix $\ell > \ell_\mu$, so $j_S(x) < \ell$ $\mu(dx)$ -almost everywhere. Then, by Assumption (A),

$$\begin{aligned} \left| \theta_{0,S}^{-n} n^{-\ell} \delta_x S^n \mathbb{1}_D \right| &\leq n^{-(\ell-j_S(x))} (\alpha_{S,n} W_S(x) + \eta_S(x)) \\ &\leq n^{-(\ell-j_S(x))} C W_S(x) \\ &\xrightarrow[n \rightarrow +\infty]{\mu(dx)\text{-a.e.}} 0 \end{aligned}$$

for some constant $C > 0$. This also implies that $\left| \theta_{0,S}^{-n} n^{-\ell} \delta_x S^n \mathbb{1}_D \right|$ is bounded, up to a multiplicative constant, by the μ -integrable function W_S , and hence, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n \rightarrow +\infty} \theta_{0,S}^{-n} n^{-\ell} \mu S^n \mathbb{1}_D = 0.$$

This entails that $\ell \geq j_S(\mu)$. Since $\ell > \ell_\mu$ was arbitrary, we deduce that $\ell_\mu \geq j_S(\mu)$. This concludes the proof of (2.8).

Now let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of elements of $\mathcal{M}_+(W_S)$ converging toward μ in $\mathcal{M}_+(W_S)$. Then for all measurable subset $A \subset D$, we have $\mu_n(A) \rightarrow \mu(A)$ and hence, for all $\ell \geq 0$ such that $\mu\{x, j_S(x) \geq \ell\} > 0$,

$$\liminf_{n \geq +\infty} \mu_n\{x, j_S(x) \geq \ell\} > 0,$$

so that

$$\liminf_{n \geq +\infty} j_S(\mu_n) \geq \ell.$$

This holds true for all $\ell < j_S(\mu)$, so

$$\liminf_{n \geq +\infty} j_S(\mu_n) \geq j_S(\mu),$$

which concludes the proof of the fact that j_S is lower semi-continuous.

Proof of (2.9) and that $j_S(X_n)$ is a.s. non-increasing. We still assume that $\mu \in \mathcal{M}_+(W_S)$ and that Assumption (A) holds true. Let us first prove that $j_S(\mu) = j_S(\mu S_1)$. We have, for all $\ell \geq 0$,

$$\begin{aligned} \theta_0^{-n} n^{-\ell} (\mu S_1) S_n &= \left(\frac{n}{n+1} \right)^\ell \theta_{0,S} \theta_{0,S}^{-(n+1)} (n+1)^{-\ell} \mu S_{n+1} \\ &\sim_{n \rightarrow +\infty} \theta_{0,S} \theta_{0,S}^{-(n+1)} (n+1)^{-\ell} \mu S_{n+1}. \end{aligned}$$

This implies that the \liminf of $\theta_0^{-n} n^{-\ell} (\mu S_1) S_n$ equals 0 if and only if the \liminf of $\theta_{0,S}^{-n} n^{-\ell} \mu S_n$ equals 0, and hence that $j_S(\mu S_1) = j_S(\mu)$.

We conclude by proving the last assertion of the proposition. We have

$$j_S(\mu S_1) = \sup \left\{ \ell \geq 0, \mu S_1 \{x, j_S(x) \geq \ell\} > 0 \right\},$$

hence

$$\mu S_1 \{x, j_S(x) > j_S(\mu S_1)\} = 0.$$

Using the equality $j_S(\mu S_1) = j_S(\mu)$ and the fact that $\mu S_1 = \mathbb{P}_\mu(X_1 \in \cdot, X_1 \neq \partial)$, we deduce that

$$\mathbb{P}_\mu(j_S(X_1) > j_S(\mu)) = 0,$$

where we used $j_S(\partial) = 0$ due to our notational convention about extension of functions. This and a straightforward application of the Markov property concludes the proof of the proposition. \square

We now turn our attention to the implications of Assumption (A) for quasi-stationary distributions. The following proposition considers quasi-stationary distributions $\nu \in \mathcal{M}_+(W_S)$ such that $\nu(\eta_S) > 0$.

Proposition 2.2. *Assume that Assumption (A) holds true and let $\nu \in \mathcal{M}_+(W_S)$ be a quasi-stationary distribution such that $\nu(\eta_S) > 0$ and such that $\nu\{j_S(\cdot) \leq \ell\} = 1$ for some $\ell \geq 0$. Then the exponential absorption parameter of ν is $\theta_{0,S}$.*

Proof. According to (2.3), we have for all $x \in D$

$$\theta_{0,S}^{-n} n^{-j_S(x)} \mathbb{E}_x(\mathbb{1}_D(X_n) \mathbb{1}_{n < \tau_\partial}) \xrightarrow{n \rightarrow +\infty} \eta_S(x), \quad (2.11)$$

and

$$\left| \theta_{0,S}^{-n} n^{-j_S(x)} \mathbb{E}_x(\mathbb{1}_D(X_n) \mathbb{1}_{n < \tau_\partial}) \right| \leq |\eta_S(x)| + \alpha_{S,n} |W_S(x)| \|\mathbb{1}_D\|_{W_S} \leq C W_S(x). \quad (2.12)$$

Denote by θ_ν the absorption parameter of ν . Assume that $\theta_\nu > \theta_{0,S}$, then, for any $\ell \geq 1$ such that $\nu\{j_S(\cdot) \leq \ell\} = 1$, and $n \geq 1$ large enough so that $\theta_\nu^{-n} \leq \theta_{0,S}^{-n} n^{-\ell-1}$,

$$\begin{aligned} \nu(D) &= \theta_\nu^{-n} \mathbb{E}_\nu(\mathbb{1}_D(X_n) \mathbb{1}_{n < \tau_\partial}) \\ &\leq n^{-1} \nu \left(\theta_{0,S}^{-n} n^{-j_S(\cdot)} \mathbb{E}_\nu(\mathbb{1}_D(X_n) \mathbb{1}_{n < \tau_\partial}) \right) \xrightarrow{n \rightarrow +\infty} 0, \end{aligned}$$

using (2.12). This is a contradiction and hence $\theta_\nu \leq \theta_{0,S}$.

Assume now that $\theta_\nu < \theta_{0,S}$, then Fatou's lemma entails that

$$\begin{aligned} 1 &= \nu(\mathbb{1}_D) = \theta_\nu^{-n} \mathbb{E}_\nu(\mathbb{1}_D(X_n) \mathbb{1}_{n < \tau_\delta}) \\ &\geq \nu \left(\theta_\nu^{-n} n^{-j_S(\cdot)} \mathbb{E}_\nu(\mathbb{1}_D(X_n) \mathbb{1}_{n < \tau_\delta}) \right) \xrightarrow{n \rightarrow +\infty} +\infty \end{aligned}$$

by (2.11) and since $\nu(\eta_S) > 0$. Hence, we have proved that $\theta_\nu = \theta_{0,S}$. \square

The following proposition shows that all quasi-stationary distributions in $\mathcal{M}_+(W_S)$ with parameter $\theta_{0,S}$ are convex combinations of the $\nu_{S,i}$.

Proposition 2.3. *Assume that there exists a QSD ν with exponential absorption parameter $\theta_{0,S}$. Then*

$$j_S(\nu) = 0 \quad \text{and} \quad \nu \{j_S(\cdot) > 0\} = 0.$$

If in addition Assumption (A) holds true and $\nu \in \mathcal{M}_+(W_S)$, then $\nu(\eta_S) = 1$ and $\nu = \sum_{i \in I_S} \nu(\eta_{S,i}) \nu_{S,i}$.

Proof. The property $j_S(\nu) = 0$ is immediate, while the second equality derives immediately from (2.6) in Proposition 2.1.

If in addition Assumption (A) holds true and $\nu \in \mathcal{M}_+(W_S)$, then (2.11) and (2.12) are satisfied. Using the fact that $\nu(W_S) < +\infty$, we deduce from Lebesgue's dominated convergence theorem that

$$\begin{aligned} 1 &= \nu(\mathbb{1}_D) = \theta_{0,S}^{-n} \mathbb{E}_\nu(\mathbb{1}_D(X_n) \mathbb{1}_{n < \tau_\delta}) \\ &= \nu \left(\theta_{0,S}^{-n} n^{-j_S(\cdot)} \mathbb{E}_\nu(\mathbb{1}_D(X_n) \mathbb{1}_{n < \tau_\delta}) \right) \xrightarrow{n \rightarrow +\infty} \nu(\eta_S), \end{aligned}$$

which shows that $\nu(\eta_S) = 1$.

Finally, integrating (2.3) with respect to ν and using the fact that $j_S(x) = 0$ $\nu(dx)$ -almost surely, we deduce that, for all $f \in L^\infty(W_S)$,

$$\left| \theta_{0,S}^{-n} \mathbb{E}_\nu(f(X_n) \mathbb{1}_{n < \tau_\delta}) - \sum_{i \in I_S} \nu(\eta_{S,i}) \nu_{S,i}(f) \right| \leq \alpha_{S,n} \nu(W_S) \|f\|_{W_S}.$$

Since $\theta_{0,S}^{-n} \mathbb{E}_\nu(f(X_n) \mathbb{1}_{n < \tau_\delta}) = \nu(f)$ and $\nu(W_S) < +\infty$, we deduce that $\nu = \sum_{i \in I_S} \nu(\eta_{S,i}) \nu_{S,i}$. This concludes the proof of the proposition. \square

In Assumption (A), the $\nu_{S,i}$ do not need to be quasi-stationary distributions. However, we will see in the results of Section 4 that this is typically the case if the set I_S and the measures $\nu_{S,i}$ are defined correctly. In the following corollary, we consider a special situation where this holds true.

Corollary 2.4. *Assume that Assumption (A) holds true and that there exists a measurable partition $D = N \cup (\bigcup_{i \in I_S} M_i)$ such that for all $i \in I_S$, and all $x \in M_i$, we have $\nu_{S,i}(M_i) = 1$, $j_S(x) = 0$, $\eta_{S,i}(x) > 0$, and, for all $i \neq j \in I_S$ and $y \in M_j$, we have $\eta_{S,i}(y) = 0$. Then the quasi-stationary distributions in $\mathcal{M}_+(W_S)$ with absorption parameter $\theta_{0,S}$ are exactly the convex combinations of the probability measures $\nu_{S,i}$. Similarly, the quasi-stationary distributions ν in $\mathcal{M}_+(W_S)$ such that $\nu(D \setminus N) > 0$ are exactly the convex combinations of the probability measures $\nu_{S,i}$.*

Remark 5. Observe that the set N is $\sum_{i \in I_S} \nu_{S,i}$ -negligible, but is typically non-empty since it contains all the points with $j_S > 0$ (as we will see in the next sections, j_S is usually non-identically zero in reducible state spaces). \triangle

Proof. Proposition 2.3 entails that any quasi-stationary distribution in $\mathcal{M}_+(W_S)$ with absorption parameter $\theta_{0,S}$ is a convex combination of the probability measures $\nu_{S,i}$. Reciprocally, for any $x \in M_i$, we have $\eta_S(x) = \eta_{S,i}(x) > 0$ and hence, according to (2.3),

$$\frac{1}{\eta_S(x)} \sum_{i \in I_S} \eta_{S,i}(x) \nu_{S,i} = \nu_{S,i}$$

is the limit when $n \rightarrow +\infty$ of the conditional distribution $\mathbb{P}_x(X_n \in \cdot \mid n < \tau_\partial)$, hence it is a quasi-limiting distribution and thus a quasi-stationary distribution in $\mathcal{M}_+(W_S)$. Moreover $\nu_{S,i}$ satisfies $\nu_{S,i}(M_i) = 1$ and, since $\eta_{S,i} > 0$ on M_i , $\nu_{S,i}(\eta_S) > 0$. Proposition 2.2 entails that the absorption parameter of $\nu_{S,i}$ is $\theta_{0,S}$. In particular, for any convex combination $\nu = \sum_{i \in I_S} \lambda_i \nu_{S,i}$, we have $\nu \in \mathcal{M}_+(W_S)$ and

$$\mathbb{P}_\nu(X_n \in \cdot) = \sum_{i \in I_S} \lambda_i \mathbb{P}_{\nu_{S,i}}(X_n \in \cdot) = \sum_{i \in I_S} \lambda_i \theta_{0,S}^n \nu_{S,i} = \theta_{0,S}^n \nu,$$

which implies that ν is a quasi-stationary distribution with absorption parameter $\theta_{0,S}$.

To conclude the proof, we simply observe that any quasi-stationary distribution ν in $\mathcal{M}_+(W_S)$ such that $\nu(D \setminus N) > 0$ satisfies $\nu(\eta_S) > 0$ and hence, according to Proposition 2.2, its absorption parameter is $\theta_{0,S}$. \square

We conclude this section with properties on the measures $\sum_i \eta_{S,i}(x) \nu_{S,i}$ and on η_S .

Proposition 2.5. *Under Assumption (A):*

(i) For all $x \in D$ such that $\eta_S(x) > 0$, $\frac{1}{\eta_S(x)} \sum_{i \in I_S} \eta_{S,i}(x) \nu_{S,i}$ is a quasi-stationary distribution with absorption parameter $\theta_{0,S}$. In addition, there exists $x \in D$ such that $\eta_S(x) > 0$ and $j_S(x) = 0$.

(ii) The function η_S satisfies, for all $x \in D$,

$$\mathbb{E}_x [\eta_S(X_n) \mathbb{1}_{j_S(X_n)=j_S(x)}] = \theta_{0,S}^n \eta_S(x).$$

(iii) For all $n \geq 0$ and all positive measure $\mu \in \mathcal{M}_+(W_S)$ such that $\mu(\eta_S) > 0$ and $\mu(n^{j_S(\cdot)} W_S) < +\infty$, we have

$$\left\| \mathbb{P}_\mu(X_n \in \cdot \mid n < \tau_\partial) - \frac{\sum_{i \in I_S} \mu(n^{j_S(\cdot)} \eta_{S,i}) \nu_{S,i}}{\sum_{i \in I_S} \mu(n^{j_S(\cdot)} \eta_{S,i})} \right\|_{TV} \leq 2\alpha_{S,n} \frac{\mu(n^{j_S(\cdot)} W_S)}{\mu(n^{j_S(\cdot)} \eta_S)}. \quad (2.13)$$

(iv) For all measure $\mu \in \mathcal{M}(W_S)$ and all $f \in L^\infty(W_S)$, we have

$$\begin{aligned} & \left| \theta_{0,S}^{-n} n^{-j_S(|\mu|)} \mathbb{E}_\mu(f(X_n) \mathbb{1}_{n < \tau_\partial}) - \sum_{i \in I_S} \mu(\mathbb{1}_{j_S(\cdot)=j_S(|\mu|)} \eta_{S,i}) \nu_{S,i}(f) \right| \\ & \leq \left(\alpha_{S,n} + \frac{1}{n} \left\| \sum_{i \in I_S} \eta_{S,i} \nu_{S,i}(W_S) \right\|_{W_S} \mathbb{1}_{j_S(|\mu|) \geq 1} \right) \|\mu\|_{W_S} \|f\|_{W_S}, \quad (2.14) \end{aligned}$$

for all $n \geq 1$.

Remark 6. If j_S is not assumed integer valued, then Proposition 2.5 remains true, except that the right hand term in (2.14) has a different form (as will clearly appear in the proof, the term $\frac{1}{n}$ is inherited from the fact that different values taken by j_S are spaced by a distance at least one). \triangle

We start with a preliminary lemma. Under Assumption (A), we have, because of (2.8) in Proposition 2.1, for any $\ell \geq 0$,

$$G_\ell := \{\mu \in \mathcal{M}(W_S), j_S(|\mu|) \leq \ell\} = \{\mu \in \mathcal{M}(W_S), j_S(x) \leq \ell \mid \mu|(dx)\text{-ae}\}.$$

Under Assumption (A), the vector space G_ℓ , endowed with the norm $\|\cdot\|_{W_S}$, is a Banach space.

Lemma 2.6. Assume that Assumption (A) holds true. For all $\ell \geq 0$, the operator $\mathfrak{S} : G_\ell \rightarrow G_\ell$ defined by $\mathfrak{S}\mu = \theta_{0,S}^{-1} \mu S_1$ satisfies Assumption (H) of the Appendix with

$$\mathfrak{J}_\mathfrak{S} = \ell, E_\mathfrak{S}\mu = \sum_{i \in I_S} \mu(\mathbb{1}_{j_S(\cdot)=\mathfrak{J}_\mathfrak{S}} \eta_{S,i}) \nu_{S,i} \text{ and } \alpha_{\mathfrak{S},n} = \alpha_{S,n} + \frac{1}{n} \left\| \sum_{i \in I_S} \eta_{S,i} \nu_{S,i}(W_S) \right\|_{W_S} \mathbb{1}_{\ell \geq 1}.$$

Proof of Lemma 2.6. We have $\sum_{i \in I_S} \eta_{S,i} \nu_{S,i}(W_S) \in L^\infty(W_S)$ by assumption. It follows from (2.3) with $n = 1$ that $\mathcal{M}(W_S)$ is stable under \mathfrak{S} . Therefore, the stability of G_ℓ under \mathfrak{S} is a consequence of (2.9) in Proposition 2.1. Then, for all $\mu \in G_\ell$ and $f \in L^\infty(W_S)$ such that $\|f\|_{W_S} \leq 1$, we have, using Assumption (A),

$$\begin{aligned} \left| n^{-\tilde{\mathfrak{J}}_\mathfrak{S}} (\mathfrak{S}^n \mu)(f) - (E_\mathfrak{S} \mu)(f) \right| &= \left| n^{-\ell} \theta_{0,S}^{-n} \mu S_n f - \sum_{i \in I_S} \mu(\mathbb{1}_{j_S(\cdot)=\ell} \eta_{S,i}) \nu_{S,i}(f) \right| \\ &\leq \left| \theta_{0,S}^{-n} \mu \left(n^{-j_S(\cdot)} \mathbb{1}_{j_S(\cdot)=\ell} S_n f \right) - \sum_{i \in I_S} \mu(\mathbb{1}_{j_S(\cdot)=\ell} \eta_{S,i}) \nu_{S,i}(f) \right| \\ &\quad + \frac{1}{n} \theta_{0,S}^{-n} |\mu| \left(n^{-j_S(\cdot)} \mathbb{1}_{j_S(\cdot) \leq \ell-1} S_n W_S \right) \\ &\leq \alpha_{S,n} |\mu| (\mathbb{1}_{j_S(\cdot)=\ell} W_S) + \frac{1}{n} \left[\sum_{i \in I_S} |\mu| (\mathbb{1}_{j_S(\cdot) \leq \ell-1} \eta_{S,i}) \nu_{S,i}(W_S) \right. \\ &\quad \left. + \alpha_{S,n} |\mu| (\mathbb{1}_{j_S(\cdot) \leq \ell-1} W_S) \right]. \end{aligned}$$

This implies that $\|n^{-\tilde{\mathfrak{J}}_\mathfrak{S}} \mathfrak{S}^n \mu - E_\mathfrak{S} \mu\|_{W_S} \rightarrow 0$ when $n \rightarrow +\infty$. Since G_ℓ is a closed subset of the Banach space $\mathcal{M}(W_S)$, we deduce that $E_\mathfrak{S} \mu \in G_\ell$ for all $\mu \in G_\ell$. Hence (H) is proved. \square

Proof. Proof of (i). Fix $x_* \in D$ such that $\eta_S(x_*) > 0$ and let $\ell_* = j_S(x_*)$. It follows from Lemma 2.6 that $\mathfrak{S} \mu = \theta_{0,S}^{-1} \mu S_1$ satisfies (H) on the Banach space G_{ℓ_*} . Using Proposition A.1 of Appendix A.1, we deduce that the operator

$$E_\mathfrak{S} \mu = \sum_{i \in I_S} \mu(\mathbb{1}_{j_S(\cdot)=\tilde{\mathfrak{J}}_\mathfrak{S}} \eta_{S,i}) \nu_{S,i}$$

on G_{ℓ_*} , with $\tilde{\mathfrak{J}}_\mathfrak{S} = \ell_*$, satisfies

$$(E_\mathfrak{S} \mathfrak{S}) \delta_x = (\mathfrak{S} E_\mathfrak{S}) \delta_x = E_\mathfrak{S} \delta_x, \text{ for all } x \in D \text{ such that } j_S(x) \leq \ell_*.$$

This means that, for all $x \in D$ such that $j_S(x) \leq \ell_*$,

$$\theta_{0,S}^{-1} \sum_{i \in I_S} \delta_x S_1 (\mathbb{1}_{j_S(\cdot)=\tilde{\mathfrak{J}}_\mathfrak{S}} \eta_{S,i}) \nu_{S,i} = \mathbb{1}_{j_S(x)=\ell_*} \theta_{0,S}^{-1} \sum_{i \in I_S} \eta_{S,i}(x) \nu_{S,i} S_1 \quad (2.15)$$

$$= \mathbb{1}_{j_S(x)=\ell_*} \sum_{i \in I_S} \eta_{S,i}(x) \nu_{S,i}. \quad (2.16)$$

Since $j_S(x_*) = \ell_*$, we deduce from the second equality that $\nu := \frac{1}{\eta_S(x_*)} \sum_{i \in I_S} \eta_{S,i}(x_*) \nu_{S,i} \in \mathcal{M}_+(W_S)$ is a quasi-stationary distribution for S_1 with absorption parameter

$\theta_{0,S}$. In addition, according to Proposition 2.3, we have $\nu(\eta_S) = 1$ and $\nu(\mathbb{1}_{j_S(\cdot)=0}) = 1$, so that $\nu(\mathbb{1}_{j_S(\cdot)=0}\eta_S) = 1$. Therefore, there exists $x \in D$ such that $\eta_S(x) > 0$ and $j_S(x) = 0$.

Proof of (ii). Fix $x \in D$. Then, applying as above Proposition A.1 of Appendix A.1 on the Banach space $G_{j_S(x)}$, we obtain (2.15) and (2.16) with ℓ_* replaced by $j_S(x)$. Integrating on both sides the test function $\mathbb{1}_D$, this implies that

$$\eta_S(x) = \sum_{i \in I_S} \eta_{S,i}(x) = \theta_{0,S}^{-1} \sum_{i \in I_S} \delta_x S_1(\mathbb{1}_{j_S(\cdot)=j_S(x)} \eta_{S,i}) = \theta_{0,S}^{-1} \delta_x S_1(\mathbb{1}_{j_S(\cdot)=j_S(x)} \eta_S).$$

Hence

$$\mathbb{E}_x[\eta_S(X_1) \mathbb{1}_{j_S(X_1)=j_S(x)}] = \theta_{0,S} \eta_S(x).$$

We deduce that, for all $n \geq 1$, \mathbb{P}_x -almost surely,

$$\mathbb{E}_{X_{n-1}}[\eta_S(X_1) \mathbb{1}_{j_S(X_1)=j_S(x)}] \mathbb{1}_{j_S(X_{n-1})=j_S(x)} = \theta_{0,S} \eta_S(X_{n-1}) \mathbb{1}_{j_S(X_{n-1})=j_S(x)}.$$

Taking the expectation and using the Markov property, we deduce that

$$\mathbb{E}_x[\eta_S(X_n) \mathbb{1}_{j_S(X_n)=j_S(x)} \mathbb{1}_{j_S(X_{n-1})=j_S(x)}] = \theta_{0,S} \mathbb{E}_x[\eta_S(X_{n-1}) \mathbb{1}_{j_S(X_{n-1})=j_S(x)}].$$

Because of the last assertion of Proposition 2.1, we deduce that $\{j_S(X_n) = j_S(x)\} = \{j_S(X_n) = j_S(X_{n-1}) = \dots = j_S(x)\}$ up to a \mathbb{P}_x -negligible event, so that

$$\mathbb{E}_x[\eta_S(X_n) \mathbb{1}_{j_S(X_n)=j_S(x)}] = \theta_{0,S} \mathbb{E}_x[\eta_S(X_{n-1}) \mathbb{1}_{j_S(X_{n-1})=j_S(x)}].$$

Then the property (ii) follows by induction.

Proof of (iii). Fix $n \geq 1$ and let $\mu \in \mathcal{M}_+(W_S)$ such that $\mu(\eta_S) > 0$ and $\mu(n^{j_S(\cdot)} W_S) < +\infty$. Integrating (2.3) with respect to the measure $n^{j_S(x)} \mu(dx)$ we obtain, for all $f \in L^\infty(W_S)$ such that $\|f\|_{W_S} \leq 1$,

$$\left| \theta_{0,S}^{-n} \mathbb{E}_\mu(f(X_n) \mathbb{1}_{n < \tau_\partial}) - \sum_{i \in I_S} \mu(n^{j_S(\cdot)} \eta_{S,i}) \nu_{S,i}(f) \right| \leq \alpha_{S,n} \mu(n^{j_S(\cdot)} W_S). \quad (2.17)$$

Then we have for all f measurable bounded by 1

$$\begin{aligned}
& \left| \frac{\mathbb{E}_\mu f(X_n) \mathbb{1}_{n < \tau_\partial}}{\mathbb{P}_\mu(n < \tau_\partial)} - \frac{\sum_{i \in I_S} \mu(n^{j_S(\cdot)} \eta_{S,i}) \nu_{S,i}(f)}{\sum_{i \in I_S} \mu(n^{j_S(\cdot)} \eta_{S,i})} \right| \\
& \leq \frac{\mathbb{E}_\mu f(X_n) \mathbb{1}_{n < \tau_\partial}}{\mathbb{P}_\mu(n < \tau_\partial)} \frac{|\sum_{i \in I_S} \mu(n^{j_S(\cdot)} \eta_{S,i}) - \theta_{0,S}^{-n} \mathbb{P}_\mu(n < \tau_\partial)|}{\sum_{i \in I_S} \mu(n^{j_S(\cdot)} \eta_{S,i})} \\
& \quad + \frac{|\theta_{0,S}^{-n} \mathbb{E}_\mu f(X_n) \mathbb{1}_{n < \tau_\partial} - \sum_{i \in I_S} \mu(n^{j_S(\cdot)} \eta_{S,i}) \nu_{S,i}(f)|}{\sum_{i \in I_S} \mu(n^{j_S(\cdot)} \eta_{S,i})} \\
& \leq 2\alpha_{S,n} \frac{\mu(n^{j_S(\cdot)} W_S)}{\mu(n^{j_S(\cdot)} \eta_S)},
\end{aligned}$$

which concludes the proof.

Proof of (iv). Let μ be a measure in $\mathcal{M}(W_S)$. The property (iv) is an immediate consequence of Lemma 2.6 with $\ell = j_S(|\mu|)$. \square

Remark 7. A natural alternative to Assumption (A) is the following, where j_S is replaced by an arbitrary function.

Assumption (A'). We have $\theta_{0,S} \in (0, 1]$, there exists a measurable, bounded, integer-valued function $j : D \rightarrow \mathbb{Z}_+$, a measurable function $W_S : D \rightarrow [1, +\infty)$, a finite or countable set I_S , some probability measures $\nu_{S,i} \in \mathcal{M}(W_S)$ and non-identically zero non-negative $\eta_{S,i} \in L^\infty(W_S)$ for each $i \in I_S$, such that

$$\sum_{i \in I_S} \eta_{S,i} \nu_{S,i}(W_S) \in L^\infty(W_S) \tag{2.18}$$

and such that, for all $f \in L^\infty(W_S)$, all $n \geq 1$ and all $x \in D$,

$$\left| \theta_{0,S}^{-n} n^{-j(x)} \mathbb{E}_x(f(X_n) \mathbb{1}_{n < \tau_\partial}) - \sum_{i \in I_S} \eta_{S,i}(x) \nu_{S,i}(f) \right| \leq \alpha_{S,n} W_S(x) \|f\|_{W_S}, \tag{2.19}$$

where $\alpha_{S,n}$ goes to 0 when $n \rightarrow +\infty$.

It is clear that, if (A') is satisfied, then $j_S \leq j$ and, if $x \in D$ is such that $\eta_S(x) > 0$, then $j_S(x) = j(x)$. In particular, Assumption (A') is less demanding than Assumption (A) and may be deduced under less stringent conditions (see next section). However, j lacks most of the properties proved for j_S above and, in order to adapt the proof of most of our results, one needs in addition to impose the following condition on j :

Assumption (A''). Assumption (A') holds true and, in addition, $(j(X_n))_{n \in \mathbb{N}}$ is \mathbb{P}_x -almost surely decreasing for all $x \in D$.

Since this last property is verified by j_S , as proved in Proposition 2.1, Assumption (A) implies Assumption (A''). Most of the results of the next sections can be extended using Assumption (A'') instead of (A). However, it seems difficult to find situations where Assumption (A'') is satisfied but Assumption (A) fails, and the sufficient criteria for Assumption (A) obtained in the next section allow the explicit computation of j_S , which would of course not be the case for a similar study with Assumption (A'') instead. This is why we mainly focus on Assumption (A) in this paper.

To conclude this remark, we comment on the extension of the results of this section under the weaker Assumption (A') or (A''). Proposition 2.2 and its proof holds true under the weaker Assumption (A'), for quasi-stationary distributions ν such that $\nu\{j(\cdot) \leq \ell\} = 1$ for some $\ell \geq 0$. Concerning Proposition 2.3, under the weaker Assumption (A'), we have $\nu\{j(\cdot) > 0 \text{ and } \eta_S(\cdot) > 0\} = 0$, and, under Assumptions (A') or (A''), one may have $\nu(\eta_S) = 0$. Corollary 2.4 does not hold true under Assumption (A') or (A''), however, under Assumption (A'), for any $x \in D$ such that $\mu := \sum_{i \in I_S} \eta_{S,i}(x) \nu_{S,i} \neq 0$, the probability measure $\mu/\mu(E)$ is a quasi-limiting distribution, and thus a quasi-stationary distribution. Finally, Proposition 2.5 remains partially true under the weaker Assumption (A''), replacing j_S by j in the statements: the first part of (i) remains true, the second part of (i) does not stand true, (ii) and (iii) remain true, and (iv) holds true with $j_S(|\mu|)$ replaced by $j(|\mu|) := \inf\{\ell \geq 0, j(x) \leq \ell \mid \mu|(dx)\text{-ae}\}$. \triangle

3 Quasi-stationary distributions in reducible state spaces with two successive sets

We start our study of the quasi-stationary distribution for reducible processes by focusing on cases where the state space can be separated into two successive classes. This is the generic situation that can be used iteratively to treat more complicated cases (see Section 4).

We consider a discrete time Markov process $(X_n, n \in \mathbb{Z}_+)$ evolving in a measurable set $D \cup \{\partial\}$ with absorption at $\partial \notin D$ at time τ_∂ , and sub-Markovian semigroup $(S_n)_{n \in \mathbb{Z}_+}$. We assume that the transition probabilities of X satisfy the structure displayed in Figure 1: there is a partition $\{D_1, D_2\}$ of D such that the process starting from D_1 can access $D_1 \cup D_2 \cup \{\partial\}$ and the process starting from

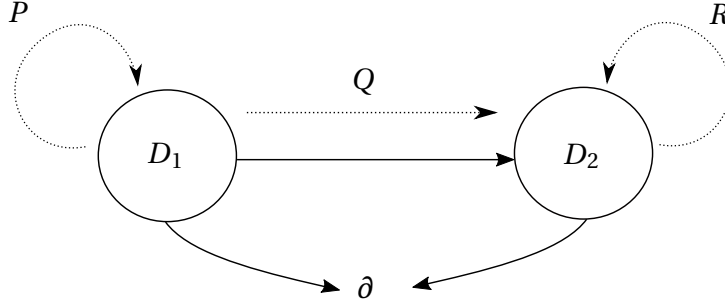


Figure 1: Transition graph displaying the relations between the sets D_1 , D_2 and ∂ . The dashed lines indicate the domains and co-domains of the sub-Markov kernels P, Q, R .

D_2 can only access $D_2 \cup \{\partial\}$. More formally, we assume that $\mathbb{P}_x(T_{D_1} = +\infty) = 1$ for all $x \in D_2$, where we denote, for any measurable set $A \subset D$, $T_A = \inf\{n \in \mathbb{Z}_+, X_n \in A\}$.

We denote by (P_n) the sub-Markovian semigroup of the process X restricted to D_1 , by (R_n) the sub-Markovian semigroup of the processes X restricted to D_2 and by Q the transition kernel from D_1 to D_2 for X . More formally, for all measurable $f : D_1 \rightarrow [0, +\infty)$ and $g : D_2 \rightarrow [0, +\infty)$, for all $x \in D_1$ and $y \in D_2$, we define

$$P_n f(x) = \mathbb{E}_x(f(X_n)), R_n g(y) = \mathbb{E}_y(g(X_n)) \text{ and } Qg(x) = \mathbb{E}_x(g(X_1)).$$

Note that, due to our notational convention about extensions of functions by 0 outside of their domain, the previous definitions mean

$$P_n f(x) = \mathbb{E}_x(f(X_n) \mathbb{1}_{n < T_{D_2 \cup \partial}}), R_n g(y) = \mathbb{E}_y(g(X_n) \mathbb{1}_{n < \tau_\partial}) \text{ and } Qg(x) = \mathbb{E}_x(g(X_1) \mathbb{1}_{X_1 \in D_2}).$$

We consider now three different situations (which are in correspondence with the three cases studied in Appendix A). The constants $\theta_{0,P}$ and $\theta_{0,R}$ denote respectively the exponential convergence parameters of the semigroups $(P_n)_{n \geq 0}$ and $(R_n)_{n \geq 0}$.

Assumption (A1) We have $j_{0,P} < +\infty$, the process X restricted to D_1 satisfies Assumption (A) and there exists a measurable function $W_R : D_2 \rightarrow [1, +\infty)$ such that, for some constants $\gamma \in [0, \theta_{0,P})$ and $c_1 > 0$, for all $x \in D_1$ and $y \in D_2$,

$$\mathbb{E}_x(W_R(X_1)) \leq W_P(x) \text{ and } \mathbb{E}_y(W_R(X_n)) \leq c_1 \gamma^n W_R(y), \forall n \geq 0. \quad (3.1)$$

Assumption (A2) We have $j_{0,R} < +\infty$, the process X restricted to D_2 satisfies Assumption (A) and there exists a measurable function $W_P : D_1 \rightarrow [1, +\infty)$ such that, for some constants $\gamma \in [0, \theta_{0,R})$ and $c_2 > 0$, for all $x \in D_1$,

$$\mathbb{E}_x(W_R(X_1)) \leq W_P(x) \text{ and } \mathbb{E}_x(W_P(X_n)) \leq c_2 \gamma^n W_P(x), \forall n \geq 0. \quad (3.2)$$

Assumption (A3) We have $j_{0,P} = 0$, $j_{0,R} < +\infty$ and $\theta_{0,R} = \theta_{0,P}$. In addition, the process X restricted to D_1 satisfies Assumption (A) with $\eta_P > 0$, and the process X restricted to D_2 also satisfies Assumption (A). Finally,

$$\mathbb{E}_x(W_R(X_1)) \leq W_P(x), \forall x \in D_1, \quad (3.3)$$

and there exists $\ell_* \in \mathbb{Z}_+$ such that, for all $x \in D_1$ and all $i \in I_P$,

$$\mathbb{P}_x \left(j_R(X_{T_{D_2}}) \leq \ell_* \text{ and } T_{D_2} < +\infty \right) = \mathbb{P}_x(T_{D_2} < +\infty) \quad (3.4)$$

$$\text{and } \mathbb{P}_{\nu_{Pi}} \left(j_R(X_{T_{D_2}}) = \ell_* \text{ and } \eta_R(X_{T_{D_2}}) > 0 \text{ and } T_{D_2} < +\infty \right) > 0, \quad (3.5)$$

where we recall that $\eta_R = \sum_{k \in I_R} \eta_{R,k}$. Note that $\ell_* \leq j_{0,R}$.

Remark 8. Note that Assumption (A) remains valid if the function W_S is multiplied by a positive constant. Hence, in the above assumptions (A1), (A2) and (A3), the requirement $\mathbb{E}_x(W_R(X_1)) \leq W_P(x)$ is actually equivalent to $\mathbb{E}_x(W_R(X_1)) \leq C W_P(x)$ for some positive constant $C > 0$. \triangle

Remark 9. In (A3), the assumptions $j_{0,P} = 0$ and the fact that (3.4) is satisfied for all $x \in D_1$ may seem restrictive conditions. However, we will see in Section 4 that, applying this property inductively in a precise order, this is sufficient to obtain a convergence result similar to (2.3) with non-zero $j_{0,S}$ (in cases with a finite or denumerable number of communication classes under mild assumptions). \triangle

Remark 10. Possible candidates for $W_R \geq 1$ in Assumption (A1) (resp. $W_P \geq 1$ in Assumption (A2)) are the exponential moment of exit times from D_2 (resp. from D_1). Indeed, if $W_R(y) = \mathbb{E}_y(\gamma^{-\tau_\partial})$ is finite for all $y \in D_2$, then $\mathbb{E}_x(W_R(X_1)) \leq \gamma W_R(y)$ for all $y \in D_2$. Similarly, if $W_P(x) = \mathbb{E}_x(\gamma^{-T_{D_2 \cup \{\partial\}}})$ is finite for all $x \in D_1$, then $\mathbb{E}_x(W_P(X_1)) \leq \gamma W_P(x)$ for all $x \in D_1$. Proofs of these classical results can be found e.g. in [24]. \triangle

Under each of the above assumptions, we prove that Assumption (A) actually holds true for the process X evolving on $D = D_1 \cup D_2$ before absorption, with formal expressions of the parameters W_S , j_S , ν_S and $\alpha_{S,n}$. In the following result, we set $\alpha_{P,0} = \alpha_{R,0} = 1$.

Theorem 3.1. Assume that either Assumption (A1), or (A2), or (A3) holds true. Then X satisfies Assumption (A) with $W_S = W_P + W_R$. Moreover, there exists a constant $C > 0$, independent of $x \in D$ and $n \in \mathbb{N}$, such that

(i) under Assumption (A1), we have $\theta_{0,S} = \theta_{0,P}$, $j_S = j_P$, and, for all $i \in I_S = I_P$, $\eta_{S,i} \propto \eta_{P,i}$ and

$$\nu_{S,i} \propto \nu_{P,i} + \sum_{k \geq 0} \theta_{0,S}^{-k-1} \mathbb{P}_{\nu_{P,i}}(T_{D_2} = 1, X_{k+1} \in \cdot),$$

with inverse proportionality constants, and

$$\alpha_{S,n} = C \sum_{k=0}^n \left(\frac{\gamma}{\theta_{0,P}} \right)^k \left((1 + j_{0,P}) \alpha_{P,n-k} + j_{0,P} \frac{k}{n} \right);$$

(ii) under Assumption (A2), we have $\theta_{0,S} = \theta_{0,R}$ and, for all $x \in D$,

$$j_S(x) = \begin{cases} \max_{n \geq 0} j_R(\delta_x P^n Q) & \text{if } x \in D_1, \\ j_R(x) & \text{if } x \in D_2 \end{cases}$$

and for all $i \in I_S = I_R$,

$$\eta_{S,i}(x) = \mathbb{E}_x \left(\theta_{0,R}^{-T_{D_2}} \eta_{R,i}(X_{T_{D_2}}) \mathbb{1}_{j_R(X_{T_{D_2}}) = j_S(x)} \right)$$

$\nu_{S,i} = \nu_{R,i}$ and

$$\alpha_{S,n} = C \sum_{k=0}^n \left(\alpha_{R,n-k} + j_{0,R} \frac{k}{n} \right) \left(\frac{\gamma}{\theta_{0,R}} \right)^k;$$

(iii) under Assumption (A3), we have $\theta_{0,S} = \theta_{0,R} = \theta_{0,P}$,

$$j_S(x) = \begin{cases} 1 + \ell_* & \text{for all } x \in D_1, \\ j_R(x) & \text{for all } x \in D_2, \end{cases}$$

where ℓ_* is defined in (3.4)–(3.5), $I_S = I_R$, for all $i \in I_R$, $\nu_{S,i} = \nu_{R,i}$,

$$\eta_{S,i}(x) = \eta_{R,i}(x) + \frac{\theta_{0,P}^{-1}}{1 + \ell_*} \sum_{k \in I_P} \eta_{P,k}(x) \mathbb{E}_{\nu_{P,k}} \left(\eta_{R,i}(X_1) \mathbb{1}_{j_R(X_1) = \ell_*} \right), \quad \forall x \in D,$$

and

$$\alpha_{S,n} = \frac{C}{n} \left(\ell_* + \sum_{k=0}^n \left(\alpha_{P,k} + \alpha_{R,k} \frac{k^{\ell_*}}{n^{\ell_*}} \right) \right),$$

Remark 11. Note that, in conclusion (iii) of the last theorem, even if $\alpha_{P,n}$ and $\alpha_{S,n}$ converge geometrically to 0, $\alpha_{S,n}$ only converges to 0 in $O(1/n)$. Similarly, in conclusion (i) (resp. (ii)), if j_P (resp. j_R) is not identically equal to 0, then the convergence rate is $O(1/n)$. \triangle

Remark 12. Theorem 3.1 under Assumption (A1) and its proof adapt to the situation where j_P takes finite values but where $j_{0,P} = +\infty$, in which case $\alpha_{S,n}$ depends on the initial position x . Similarly, under Assumption (A2) or (A3), the result adapts to the situation where j_R takes finite values but where $j_{0,R} = +\infty$ (and with the additional assumption that $\max_{n \geq 0} j_R(\delta_x P^n Q) < +\infty$ for all $x \in D_1$ when working under Assumption (A2)), in which case $\alpha_{S,n}$ also depends on x . Thus, in these situations, one checks that S satisfies a version of Assumption (A) allowing dependence of $\alpha_{S,n}$ with respect to x .

In fact, it is possible to adapt the results of this paper to the situation where $\alpha_{S,n}$ (and hence $\alpha_{R,n}$ and/or $\alpha_{P,n}$) is allowed to depend on x . However, this leads to additional technical assumptions and cumbersome expressions. Since the case α independent of x is already rich enough and is sufficiently powerful to derive general results (see Sections 4 and 5), we decided to focus on this situation and leave the adaptation to the interested reader (see also Remark 23 in Appendix A.2 for the expressions obtained in this situation).

Finally, if j_S is not assumed integer valued, then again Theorem 3.1 holds true, but with $\alpha_{S,n}$ depending non-explicitly on the initial position of the process. \triangle

We will see in the proof that a weakened version of Assumption (A3), without the requirement $\eta_P > 0$ in D_1 and without (3.5) implies Assumption (A'') (as introduced in Remark 7). This entails the following corollary.

Corollary 3.2. *Assume that $j_{0,P} = 0$, $j_{0,R} < +\infty$, $\theta_{0,R} = \theta_{0,P}$, the process X restricted to D_1 satisfies Assumption (A) and the process X restricted to D_2 also satisfies Assumption (A). Assume also that (3.3) holds true and define $\ell_* \in \mathbb{Z}_+$ as the smallest ℓ such that, for all $x \in D_1$, $\mathbb{P}_x(j_R(X_{T_{D_2}}) \leq \ell) = 1$. Then for all $f \in L^\infty(W_S)$, all $n \geq 1$ and all $x \in D$,*

$$\left| \theta_{0,P}^{-n} n^{-j(x)} \mathbb{E}_x(f(X_n) \mathbb{1}_{n < \tau_\partial}) - \sum_{i \in I_R} \left(\eta_{R,i}(x) + \frac{\theta_{0,P}^{-1}}{1 + \ell_*} \sum_{k \in I_P} \eta_{P,k}(x) \mathbb{E}_{v_{P,k}}(\mathbb{1}_{j_R(X_1) = \ell_*} \eta_{R,i}(X_1)) \right) v_{R,i}(f) \right| \leq \alpha_{S,n} W_S(x) \|f\|_{W_S}, \quad (3.6)$$

where

$$j(x) = \begin{cases} 1 + \ell_* & \text{for all } x \in D_1, \\ j_R(x) & \text{for all } x \in D_2, \end{cases}$$

and

$$\alpha_{S,n} = \frac{C}{n} \left(\ell_* + \sum_{k=0}^n (\alpha_{P,k} + \alpha_{R,k}) \right),$$

for some constant $C > 0$, which does not depend on x and n . In addition, $j_R(x) = j(x)$ for all $x \in D_2$ and $j_R(x) \leq j(x)$ for all $x \in D_1$.

Remark 13. The difference between Theorem 3.1 (iii) and Corollary 3.2 is that we don't know if the polynomial exponent j involved in (3.6) is equal on D_1 to the polynomial convergence parameter j_S defined in (2.1). However, it follows from (3.6) that $j(x) = j_S(x)$ for all $x \in D_1$ such that

$$\sum_{i \in I_R} \sum_{k \in I_P} \eta_{P,k}(x) \mathbb{E}_{\nu_{P,k}} \left(\mathbb{1}_{j_R(X_1) = \ell_*} \eta_{R,i}(X_1) \right) > 0.$$

The introduction of an arbitrary polynomial exponent j instead j_S is the subject of Assumptions (A') and (A'') introduced in Remark 7. Using very similar proofs, processes with two successive sets can also be studied using Assumption (A'') in place of Assumption (A), then loosing the explicit expression of the polynomial convergence parameter. \triangle

Proof of Theorem 3.1 and Corollary 3.2. Under Assumptions (A1) or (A2) or (A3), we define the linear operators $\mathfrak{P} : \mathcal{M}(W_P) \rightarrow \mathcal{M}(W_P)$, $\mathfrak{Q} : \mathcal{M}(W_P) \rightarrow \mathcal{M}(W_R)$ and $\mathfrak{R} : \mathcal{M}(W_R) \rightarrow \mathcal{M}(W_R)$ (these notations implicitly assume that $\mathcal{M}(W_P) \subset \mathcal{M}(D_1)$ and $\mathcal{M}(W_R) \subset \mathcal{M}(D_2)$) by

$$\mathfrak{P}\mu = \theta_0^{-1} \mu P_1, \quad \mathfrak{Q}\mu = \theta_0^{-1} \mu Q, \quad \text{and} \quad \mathfrak{R}\mu = \theta_0^{-1} \mu R_1,$$

where $\theta_0 = \theta_{0,P}$ under Assumption (A1), $\theta_0 = \theta_{0,R}$ under Assumption (A2), and $\theta_0 = \theta_{0,P} = \theta_{0,R}$ under Assumption (A3). Under Assumption (A1), (A2) or (A3), all these operators are bounded. Our aim is to apply the results of Appendix A.2 to $\mathfrak{S} : \mathcal{M}(W_S) \rightarrow \mathcal{M}(W_S)$, where $\mathcal{M}(W_S) \equiv \mathcal{M}(W_P) \oplus \mathcal{M}(W_R)$, with $W_S = W_R + W_P$ and $\mathfrak{S} = \mathfrak{P} + \mathfrak{Q} + \mathfrak{R}$. Beware that $\mathfrak{P}, \mathfrak{R}, \mathfrak{Q}, \mathfrak{S}$ act on the left on μ while P_n, R_n, Q, S_n act on the right, so that, for instance, $\mathfrak{R}\mathfrak{P}\mu = \theta_0^{-2} \mu P_1 R_1$.

Under Assumption (A1). We make use of Proposition A.2. We define B_2 as the Banach space $\mathcal{M}(W_R)$ and observe that the operator $\mathfrak{R} : B_2 \rightarrow B_2$ is bounded

and

$$\sum_{n=0}^{\infty} \|\mathfrak{R}^n\| \leq \sum_{n=0}^{\infty} \theta_{0,P}^{-n} \sup_{\mu \in B_2, |\mu|(W_R)=1} |\mu| R_n W_R.$$

It follows from (3.1) that

$$\sum_{n=0}^{\infty} \|\mathfrak{R}^n\| \leq c_1 \sum_{n=0}^{\infty} \theta_{0,P}^{-n} \gamma^n \sup_{\mu \in B_2, |\mu|(W_R)=1} |\mu|(W_R) = \frac{c_1 \theta_0}{\theta_{0,P} - \gamma}.$$

Moreover, $\Gamma_n \leq c_1 \frac{\gamma^n / \theta_{0,P}^{n-1}}{\theta_{0,P} - \gamma}$ and $\gamma_n \leq c_1 \gamma^n / \theta_{0,P}^n$ (using the notations of Proposition A.2). In particular, if $x \in D_2$, then (2.3) holds true with $\eta_{S,i}(x) = 0$ for all $i \in I_S = I_P$, $j_S(x) = 0$, $W_S(x) = W_R(x)$ and $\alpha_{S,n} = c_1 \left(\frac{\gamma}{\theta_{0,P}}\right)^n$.

From now on we assume that $x \in D_1$ and consider the vector space $B_1 = \{\mu \in \mathcal{M}(W_P), j_P(|\mu|) \leq j_P(x)\}$. By Lemma 2.6, the operator $\mathfrak{P} : B_1 \rightarrow B_1$ satisfies Assumption (H) from the Appendix with $\mathfrak{J}_{\mathfrak{P}} = j_P(x)$,

$$E_{\mathfrak{P}} \mu = \sum_{i \in I_P} \mu(\mathbb{1}_{j_P(\cdot) = j_P(x)} \eta_{P,i}) \nu_{P,i} \quad (3.7)$$

and

$$\alpha_{\mathfrak{P},n} = \alpha_{P,n} + \mathbb{1}_{j_P(x) \geq 1} \frac{\|\sum_{i \in I_P} \eta_{P,i} \nu_{P,i}(W_P)\|_{W_P}}{n}$$

Note also that $\mathfrak{Q} : B_1 \rightarrow B_2$ is a bounded operator by (3.1). As a consequence, according to Proposition A.2, \mathfrak{S} restricted to $B = B_1 \oplus B_2$ also satisfies Assumption (H) with $\mathfrak{J}_{\mathfrak{S}} = j_P(x)$ and for all $\mu \in B_1$,

$$\begin{aligned} E_{\mathfrak{S}} \mu &= E_{\mathfrak{P}} \mu + \sum_{k \geq 0} \mathfrak{R}^k \mathfrak{Q} E_{\mathfrak{P}} \mu = \sum_{i \in I_P} \mu(\eta_{P,i}) \nu_{P,i} + \sum_{k \geq 0} \theta_{0,P}^{-k-1} \sum_{i \in I_P} \mu(\eta_{P,i}) \mathbb{P}_{\nu_{P,i}}(T_{D_2} = 1, X_{k+1} \in \cdot) \\ &= \sum_{i \in I_P} \mu(\eta_{P,i}) \left(\nu_{P,i} + \sum_{k \geq 0} \theta_{0,P}^{-k-1} \mathbb{P}_{\nu_{P,i}}(T_{D_2} = 1, X_{k+1} \in \cdot) \right). \end{aligned}$$

and

$$\begin{aligned} \alpha_{\mathfrak{S},n} &= \alpha_{\mathfrak{P},n} + C \Gamma_n + C \sum_{k=0}^{n-1} \gamma_k \left(\frac{(\alpha_{\mathfrak{P},n-k-1} + 1) \mathfrak{J}_{\mathfrak{P}}(k+1)}{n} + \alpha_{\mathfrak{P},n-k-1} \right) \\ &\leq C \sum_{k=0}^n \left(\frac{\gamma}{\theta_{0,P}} \right)^k \left((1 + j_P(x)) \alpha_{P,n-k} + \mathbb{1}_{j_P(x) \geq 1} \frac{1}{n+1-k} + j_P(x) \frac{k}{n} \right) \quad (3.8) \\ &\leq \alpha_{S,n} := C \sum_{k=0}^n \left(\frac{\gamma}{\theta_{0,P}} \right)^k \left((1 + j_{0,P}) \alpha_{P,n-k} + j_{0,P} \frac{k}{n} \right) \end{aligned}$$

for some constant $C > 0$ that may change from line to line, where we used $\mathbb{1}_{j_{0,P} \geq 1} \leq j_{0,P}$ and $\frac{1}{n-k+1} \leq \frac{k}{n}$. Using the fact that $\mathfrak{S}^n \mu = \theta_{0,P}^{-n} \mu S_n$ and taking $\mu = \delta_x$, we deduce that, for all $x \in D$ and all $f \in L^\infty(W_S)$,

$$\left| n^{-j_P(x)} \theta_{0,P}^{-n} S_n f(x) - \sum_{i \in I_P} \eta_{P,i}(x) \left(v_{P,i}(f) + \sum_{k \geq 0} \theta_{0,P}^{-k-1} \mathbb{E}_{v_{P,i}} \left(\mathbb{1}_{T_{D_2}=1} f(X_{k+1}) \right) \right) \right| \leq \alpha_{S,n} W_S(x) |f|. \quad (3.9)$$

It only remains to prove that $j_S(x) = j_P(x)$ for all $x \in D$ (recall that under our convention j_P is extended to D_2 by the value 0). On the one hand, the definitions of j_S , j_P and S clearly imply that $j_S(x) \geq j_P(x)$ for all $x \in D$. On the other hand, inequality (3.9) implies that, for all $\varepsilon > 0$,

$$\liminf_{n \rightarrow +\infty} n^{-(j_P(x)+\varepsilon)} \theta_{0,P}^{-n} S_n \mathbb{1}_D(x) = 0,$$

so that $j_S(x) \leq j_P(x) + \varepsilon$ for all $\varepsilon > 0$, and hence $j_S(x) \leq j_P(x)$. This concludes the proof of (i).

Under Assumption (A2). We make use of Proposition A.3. For all $x \in D_2$, we have $\delta_x S_n = \delta_x R_n$, so that (2.3) holds true with $I_S = I_R$, $\eta_{S,i}(x) = \eta_{R,i}(x)$, $j_S(x) = j_R(x)$ and $\alpha_{S,n} = \alpha_{R,n}$. This also implies that $\theta_S(x) = \theta_R(x)$ for all $x \in D_2$.

We fix now $x \in D_1$. We set

$$\tilde{\mathfrak{J}}(x) := \max_{n \geq 0} j_R(\delta_x P^n Q)$$

and consider the operators \mathfrak{P} , \mathfrak{R} and \mathfrak{S} restricted to the Banach space

$$B = B_1 \oplus B_2 \subset \mathcal{M}(W_S),$$

where

$$B_1 = \left\{ \mu \in \mathcal{M}(W_P), \max_{n \geq 0} j_R(|\mu| P^n Q) \leq \tilde{\mathfrak{J}}(x) \right\} \text{ and } B_2 = \{ \mu \in \mathcal{M}(W_R), j_R(|\mu|) \leq \tilde{\mathfrak{J}}(x) \}.$$

Note that B is indeed stable by \mathfrak{P} , \mathfrak{R} and \mathfrak{S} . In addition, Proposition 2.1 entails that B_1 is a Banach subspace of $\mathcal{M}(W_P)$.

We first observe that

$$\sum_{n=0}^{\infty} \|\mathfrak{P}^n\| \leq \sum_{n=0}^{\infty} \theta_{0,R}^{-n} \sup_{\mu \in B_1, |\mu|(W_P)=1} \mu P^n W_P \leq \frac{\theta_{0,R}}{\theta_{0,R} - \gamma}$$

and $\Theta_n \leq \frac{\gamma^n / \theta_{0,R}^{n-1}}{\theta_{0,R} - \gamma}$ and $\theta_n \leq \gamma^n / \theta_{0,R}^n$ (using the notations of Proposition A.3).

By Lemma 2.6, the operator $\mathfrak{R} : B_2 \rightarrow B_2$ satisfies Assumption (H) from the Appendix with $\mathfrak{J}_{\mathfrak{R}} = \mathfrak{J}(x)$,

$$E_{\mathfrak{R}}\mu = \sum_{i \in I_R} \mu(\mathbb{1}_{j_R(\cdot) = \mathfrak{J}_{\mathfrak{R}} \eta_{R,i}}) \nu_{R,i}$$

and

$$\alpha_{\mathfrak{R},n} = C \left(\alpha_{R,n} + \frac{\mathbb{1}_{\mathfrak{J}_{\mathfrak{R}} \geq 1}}{n} \right).$$

We thus deduce from Proposition A.3 that \mathfrak{S} restricted to B satisfies Assumption (H) with $\mathfrak{J}_{\mathfrak{S}} = \mathfrak{J}(x)$, for all $\mu \in B$,

$$\begin{aligned} E_{\mathfrak{S}}\mu &= E_{\mathfrak{R}}\mu + \sum_{k=0}^{\infty} E_{\mathfrak{R}}\mathfrak{Q}\mathfrak{P}^k\mu \\ &= \sum_{i \in I_R} \left(\mu(\mathbb{1}_{j_R(\cdot) = \mathfrak{J}_{\mathfrak{R}} \eta_{R,i}}) + \sum_{k=0}^{\infty} \theta_{0,R}^{-k-1} \mu P^k Q(\mathbb{1}_{j_R(\cdot) = \mathfrak{J}_{\mathfrak{R}} \eta_{R,i}}) \right) \nu_{R,i} \\ &= \sum_{i \in I_R} \mathbb{E}_{\mu} \left(\theta_{0,R}^{-T_{D_2}} \eta_{R,i}(X_{T_{D_2}}) \mathbb{1}_{j_R(X_{T_{D_2}}) = \mathfrak{J}(x)} \right) \nu_{R,i} \end{aligned} \quad (3.10)$$

and there exists a constant C independent of $x \in D_1$ such that

$$\begin{aligned} \alpha_{\mathfrak{S},n} &= \alpha_{\mathfrak{R},n} + C \sum_{k=0}^{n-1} \alpha_{\mathfrak{R},n-k-1} \theta_k + C \Theta_n + C \sum_{k=0}^{n-1} \frac{\mathfrak{J}_{\mathfrak{S}} k}{n} \theta_k \\ &\leq C \sum_{k=0}^n \left(\alpha_{R,n-k} + \frac{\mathbb{1}_{\mathfrak{J}(x) \geq 1}}{n-k+1} + \mathfrak{J}(x) \frac{k}{n} \right) \left(\frac{\gamma}{\theta_{0,R}} \right)^k \\ &\leq \alpha_{S,n} := C \sum_{k=0}^n \left(\alpha_{R,n-k} + j_{0,R} \frac{k}{n} \right) \left(\frac{\gamma}{\theta_{0,R}} \right)^k, \end{aligned} \quad (3.11)$$

where $\alpha_{R,0} := 1$. Since $\mathfrak{S}^n = \theta_{0,R}^{-n} S_n$, taking $\mu = \delta_x$ in (3.10), we finally deduce that, for all $x \in D$ and all $f \in L^\infty(W_S)$,

$$\begin{aligned} \left| \theta_{0,R}^{-n} n^{-\mathfrak{J}(x)} S_n f(x) - \sum_{i \in I_R} \mathbb{E}_x \left(\theta_{0,R}^{-T_{D_2}} \eta_{R,i}(X_{T_{D_2}}) \mathbb{1}_{j_R(X_{T_{D_2}}) = \mathfrak{J}(x)} \right) \nu_{R,i}(f) \right| \\ \leq \alpha_{S,n} W_S(x) \|f\|_{W_S}, \end{aligned} \quad (3.12)$$

where we extended \mathfrak{J} to D_2 by setting $\mathfrak{J}(x) := j_R(x)$ if $x \in D_2$.

In order to conclude, it remains to prove that $\theta_{0,S} = \theta_{0,R}$ and that $j_S(x) = \tilde{\mathfrak{J}}(x)$ for all $x \in D$. Inequality (3.12) with $f = \mathbb{1}_D$ implies that $\theta_S(x) \leq \theta_{0,R}$ for all $x \in D$, so that $\theta_{0,S} \leq \theta_{0,R}$. Moreover, for all $x \in D_2$, $\theta_S(x) = \theta_R(x)$, and thus $\theta_{0,S} \geq \theta_{0,R}$. We deduce that $\theta_{0,S} = \theta_{0,R}$ and hence, using again (3.12) with $f = \mathbb{1}_D$, we deduce that $j_S(x) \leq \tilde{\mathfrak{J}}(x)$ for all $x \in D$. On the one hand, for all $x \in D_2$, we have $j_S(x) = j_R(x) = \tilde{\mathfrak{J}}(x)$. On the other hand, for $x \in D_1$, we observe that, for any $n \geq 0$ such that $\tilde{\mathfrak{J}}(x) = j_S(\delta_x P^n Q)$, we have the inequality $\delta_x S^{n+1} \mathbb{1}_D \geq \delta_x P^n Q \mathbb{1}_D$, and hence

$$j_S(x) = j_S(\delta_x S^{n+1}) \geq j_S(\delta_x P^n Q) = \tilde{\mathfrak{J}}(x).$$

We thus proved that $j_S(x) \geq \tilde{\mathfrak{J}}(x)$ for all $x \in D$, which concludes the proof of (ii).

Under Assumption (A3). If $x \in D_2$, then (2.3) holds true with $I_S = I_R$ and $\eta_{S,i}(x) = \eta_{R,i}(x)$, $j_S(x) = j_R(x)$ and $\alpha_{S,n} = \alpha_{R,n}$.

Fix $x \in D_1$. We consider \mathfrak{P} , \mathfrak{R} and \mathfrak{S} restricted to the Banach space

$$B = B_1 \oplus B_2 \subset \mathcal{M}(W_S),$$

where

$$B_1 = \mathcal{M}(W_P) \text{ and } B_2 = \{\mu \in \mathcal{M}(W_R), j_R(|\mu|) \leq \ell_*\}$$

Note that it follows Proposition 2.1 and the assumption that $\mathbb{P}_X(j_R(X_{T_{D_2}}) \leq \ell_*) = 1$ that $\mathfrak{Q}B_1 \subset B_2$, and from the rest of Assumption (A3) that $\mathfrak{P} : B_1 \rightarrow B_1$, $\mathfrak{R} : B_2 \rightarrow B_2$ and $\mathfrak{Q} : B_1 \rightarrow B_2$ are bounded operators.

As in the previous step, the operator \mathfrak{R} satisfies Assumption (H) in the Appendix with $\tilde{\mathfrak{J}}_{\mathfrak{R}} = \ell_*$,

$$E_{\mathfrak{R}}\mu = \sum_{i \in I_R} \mu(\mathbb{1}_{j_R(\cdot) = \ell_*} \eta_{R,i}) \nu_{R,i},$$

and

$$\alpha_{\mathfrak{R},n} = C \left(\alpha_{R,n} + \frac{\mathbb{1}_{\ell_* \geq 1}}{n} \right),$$

for some constant $C > 0$. Moreover, Assumption (A) for P_n implies that \mathfrak{P} satisfies Assumption (H) with $\tilde{\mathfrak{J}}_{\mathfrak{P}} = 0$, $E_{\mathfrak{P}}\mu = \sum_{j \in I_P} \mu(\eta_{P,j}) \nu_{P,j}$ and $\alpha_{\mathfrak{P},n} = \alpha_{P,n}$. We conclude from Proposition A.4 that \mathfrak{S} satisfies Assumption (H1) with $\tilde{\mathfrak{J}}_{\mathfrak{S}} = 1 + \ell_*$,

$$E_{\mathfrak{S}}\mu = \frac{1}{\tilde{\mathfrak{J}}_{\mathfrak{S}}} E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}} \mu = \frac{\theta_{0,P}^{-1}}{1 + \ell_*} \sum_{i \in I_R} \sum_{k \in I_P} \mu(\eta_{P,k}) \mathbb{E}_{\nu_{P,k}} \left(\mathbb{1}_{j_R(X_1) = \ell_*} \eta_{R,i}(X_1) \right) \nu_{R,i},$$

and

$$\begin{aligned}\alpha_{\mathfrak{S},n} &= \frac{C}{n} \left(\mathfrak{J}_{\mathfrak{R}} + \sum_{k=0}^n \alpha_{\mathfrak{P},k} + \left(\max_{k \geq 0} \alpha_{\mathfrak{P},k} + 1 \right) \sum_{k=0}^n \alpha_{\mathfrak{R},n-k} \left(\frac{n-k}{n} \right)^{\mathfrak{J}_{\mathfrak{R}}} \right) \\ &\leq \alpha_{S,n} := \frac{C}{n} \left(\ell_* + \sum_{k=0}^n \left(\alpha_{P,k} + \alpha_{R,k} \frac{k^{\ell_*}}{n^{\ell_*}} \right) \right),\end{aligned}$$

with $\alpha_{P,0} = \alpha_{R,0} = 1$. Since $\mathfrak{S}^n \mu = \theta_{0,P}^{-n} \mu S_n$, we deduce that, for all $x \in D_1$ and all $f \in L^\infty(W_S)$,

$$\left| \theta_{0,P}^{-n} n^{-(1+\ell_*)} S_n f(x) - \frac{\theta_{0,P}^{-1}}{1+\ell_*} \sum_{i \in I_R} \sum_{k \in I_P} \eta_{P,k}(x) \mathbb{E}_{\nu_{P,k}} \left(\mathbb{1}_{j_R(X_1) = \ell_*} \eta_{R,i}(X_1) \right) \nu_{R,i}(f) \right| \leq \alpha_{S,n} W_S(x) \|f\|_{W_S}. \quad (3.13)$$

This implies that $\theta_S(x) \leq \theta_{0,P}$ for all $x \in D_1$, so $\theta_{0,S} \leq \theta_{0,P} \vee \theta_{0,R} = \theta_{0,P} = \theta_{0,R}$. Conversely, since $Sf \geq Rf$ for all positive f , we have $\theta_{0,S} \geq \theta_{0,R}$. We thus deduce that $\theta_{0,S} = \theta_{0,P} = \theta_{0,R}$. We also have, by definition of S , $j_S(x) = j_R(x)$ for all $x \in D_2$. Moreover, for all $x \in D_1$, (3.13) implies that $j_S(x) \leq 1 + \ell_*$. Note that, up to now, we have neither used the assumption that $\eta_P > 0$, nor the second part of (3.4). Therefore, we have proved Corollary 3.2.

It remains to prove that $j_S(x) \geq 1 + \ell_*$ for all $x \in D_1$. Fix $x \in D_1$ until the end of the proof. Since $\eta_P(x) > 0$, we deduce from Proposition 2.5 (i) that

$$\nu := \frac{1}{\eta_P(x)} \sum_{k \in I_P} \eta_{P,k}(x) \nu_{P,k}$$

is a quasi-stationary distribution for the semigroup $(P_n)_{n \geq 0}$ with exponential convergence parameter $\theta_{0,P}$. Let us first prove that $j_R(\nu Q) = \ell_*$. Since $(R_n)_{n \geq 0}$ satisfies (A), we have for all $y \in D_2$ such that $j_R(y) \leq \ell_*$

$$\theta_{0,R}^{-n} n^{-\ell_*} \delta_y R_n \mathbb{1}_{D_2} \xrightarrow{n \rightarrow +\infty} \begin{cases} \sum_{i \in I_R} \eta_{R,i}(y) & \text{if } j_R(y) = \ell_*, \\ 0 & \text{if } j_R(y) < \ell_*, \end{cases}$$

where the convergence holds in $L^\infty(W_R)$. Therefore,

$$\theta_{0,R}^{-n} n^{-\ell_*} \nu Q R_n \mathbb{1}_{D_2} \xrightarrow{n \rightarrow +\infty} \sum_{i \in I_R} \nu Q (\eta_{R,i} \mathbb{1}_{j_R(\cdot) = \ell_*}). \quad (3.14)$$

Now, using that ν is a quasi-stationary distribution,

$$\begin{aligned}\mathbb{E}_\nu\left(\eta_R(X_{T_{D_2}})\mathbb{1}_{j_R(X_{T_{D_2}})=\ell_*}\right) &= \sum_{m=0}^{+\infty} \nu P_m Q(\eta_R(\cdot)\mathbb{1}_{j_R(\cdot)=\ell_*}) \\ &= \frac{1}{1-\theta_{0,P}} \nu Q(\eta_R(\cdot)\mathbb{1}_{j_R(\cdot)=\ell_*}).\end{aligned}$$

By the second part of Assumption (3.4), the left-hand side is positive, so we have proved that $\theta_{0,R}^{-n} n^{-\ell_*} \nu Q R_n \mathbb{1}_{D_2}$ converges to a positive limit. This shows that $j_R(\nu Q) = \ell_*$.

For all $n \geq 1$, using the fact that $S_n = P_n + R_n + \sum_{k=0}^{n-1} P_{n-k-1} Q R_k$, we have

$$\begin{aligned}n^{-(\ell_*+1)} \theta_0^{-n} \nu S_n \mathbb{1}_D &\geq n^{-(\ell_*+1)} \theta_0^{-n} \sum_{k=0}^{n-1} \nu P_{n-k-1} Q R_k \mathbb{1}_{D_2} \\ &= n^{-(\ell_*+1)} \theta_0^{-1} \sum_{k=1}^{n-1} \theta_0^{-k} \nu Q R_k \mathbb{1}_{D_2} \\ &= n^{-(\ell_*+1)} \theta_0^{-1} \sum_{k=1}^{n-1} k^{\ell_*} \left[k^{-\ell_*} \theta_0^{-k} \nu Q R_k \mathbb{1}_{D_2} \right].\end{aligned}$$

Using that, by (3.14), $k^{-\ell_*} \theta_0^{-k} \nu Q R_k \mathbb{1}_{D_2}$ converges to a positive limit when $k \rightarrow +\infty$, we deduce that

$$\liminf_{n \rightarrow +\infty} n^{-(\ell_*+1)} \theta_0^{-n} \nu S_n \mathbb{1}_D > 0.$$

This shows that $j_S(\nu) \geq \ell_* + 1$. Finally, we have $\theta_0^{-n} \delta_x P_n \rightarrow \sum_{k \in I_P} \eta_{P,k}(x) \nu_{P,k} = \eta_P(x) \nu$ in $\mathcal{M}(W_S)$ when $n \rightarrow +\infty$. Since $\eta_P(x) > 0$, we deduce from the lower semi-continuity of j_S (see Proposition 2.1) that

$$\liminf_{n \rightarrow +\infty} j_S(\theta_0^{-n} \delta_x P_n) \geq j_S(\eta_P(x) \nu) = j_S(\nu) = \ell_* + 1.$$

Using again Proposition 2.1, we have $j_S(x) = j_S(\theta_0^{-n} \delta_x S_n) \geq j_S(\theta_0^{-n} \delta_x P_n)$ for all $n \geq 0$. So we finally deduce that

$$j_S(x) = \liminf_{n \rightarrow +\infty} j_S(\theta_0^{-n} \delta_x S_n) \geq \liminf_{n \rightarrow +\infty} j_S(\theta_0^{-n} \delta_x P_n) \geq j_S(\nu_P) = \ell_* + 1.$$

This concludes the proof of Theorem 3.1. □

4 Quasi-stationary distributions in reducible state spaces

Our goal is to study quasi-stationary distributions on general reducible state spaces, a situation which naturally leads to non-zero polynomial convergence parameters. In particular, we extend the results of [10, Section 6.2], which are stated under conditions ensuring that the polynomial convergence parameter of the process vanishes. We refer the reader to [27, 28] where an in-depth study of the quasi-stationary distributions on finite reducible state spaces has been conducted (see also the survey [29], an earlier work [22] resumed in [13, Section 9], and the more recent works [7, 5]). The quasi-stationary distribution of particular processes on reducible state spaces with finitely many communication classes have also previously been studied in [26] (for multi-type Galton-Watson processes), [17, Section 3] (for discrete state space processes, under conditions ensuring that the polynomial convergence parameter vanishes), [8] (for multitype Dawson-Watanabe processes).

We consider a Markov process X with semigroup S on a general state space D that can be decomposed into finitely many disjoint sets $E_0, E_1, E_2, \dots, E_k$, where $k \geq 2$. We denote, for all $i \in \{0, 1, \dots, k\}$, by $Y^{(i)}$ the process

$$Y_n^{(i)} = \begin{cases} X_n & \text{if } n < T_{\cup_{j \neq i} E_j \cup \{\partial\}}, \\ \partial & \text{otherwise} \end{cases}$$

and define by $\theta_{0,i}$ its exponential convergence parameter. We assume that, for all $i \in \{1, \dots, k\}$,

$$\theta_{0,i} = \bar{\theta} := \max_{j \in \{1, \dots, k\}} \theta_{0,j}.$$

and we assume that there exists $\gamma < \bar{\theta}$ and a function $W_0 \geq 1$ such that, for all $x \in E_0$ and for some constant $C > 0$,

$$\mathbb{E}_x(W_0(Y_n^{(0)})) \leq C\gamma^n W_0(x), \quad \forall n \geq 0. \quad (4.1)$$

Assume also that, for all $i \in \{1, \dots, k\}$, the process $Y^{(i)}$ satisfies Assumption (A) with the objects $\theta_{0,i}$, j_i , $\alpha_{i,n}$, W_i , ν_i and η_i such that $j_i \equiv 0$, $I_i = \{1\}$ and $\eta_i > 0$ on E_i (note that we omit the second index for $\eta_{i,1}$ and $\nu_{i,1}$). Many references provide practical criteria to check Assumption (A) with $I_S = \{1\}$, $j_S \equiv 0$, $\eta_S > 0$ and with $\alpha_{S,n}$ converging exponentially fast to 0, which corresponds to the classical irreducible situation, see for example [16, 9, 10, 11, 15, 20, 1, 18, 19, 21, 4].

The process $Y^{(i)}$ is called the process X restricted to E_i . More generally, for all $M \subset D$, we call process X restricted to M the process X killed after its first exit time from M . We set $W = \sum_{i=1}^k W_i$ and assume that there exists a constant $C_W > 0$ such that

$$\mathbb{E}_x(W(X_1)) \leq C_W W(x), \quad \forall x \in D. \quad (4.2)$$

We assume that the set $\{1, \dots, k\}$ can be equipped with a partial strict order $<$ such that $i < j$ if and only if E_i is accessible from E_j in the sense that: for all $i, j \in \{1, \dots, k\}$, if $i < j$, then

$$\forall x \in E_j, \quad \mathbb{P}_x(T_{E_i} < +\infty) > 0 \quad (4.3)$$

and, if $i \not< j$, then

$$\forall x \in E_j, \quad \mathbb{P}_x(T_{E_i} < +\infty) = 0.$$

and

$$\forall x \in E_j, \quad \mathbb{P}_x(\exists n \geq T_{E_j^c} \text{ such that } X_n \in E_j) = 0,$$

where $E_j^c = D \cup \{\partial\} \setminus E_j$.

To state the main result of this section, we introduce the following notations. We define the set

$$\bar{F}_0 := \{\text{minimal elements in } \{1, \dots, k\} \text{ for the order } <\}.$$

and, for all $\ell \geq 0$, we define by induction

$$\bar{F}_{\ell+1} := \left\{ \text{minimal elements in } \bar{F} \setminus \left(\bar{F}_0 \cup \dots \cup \bar{F}_\ell \right) \text{ for the order } < \right\}.$$

We denote by $\bar{\ell}$ the non-negative integer such that $\bar{F}_\ell = \emptyset$ iff $\ell > \bar{\ell}$. For all $x \in \bigcup_{i=1}^k E_i$, we define

$$\text{index}(x) = \ell \in \{0, \dots, \bar{\ell}\} \text{ such that } \exists i \in \bar{F}_\ell \text{ with } x \in E_i.$$

For all $x \in E_0$, we define

$$\text{index}(x) = \max\{\ell \geq 0 \text{ such that } \exists i \in \bar{F}_\ell, \mathbb{P}_x(T_i < \infty) > 0\},$$

with $\max \emptyset = -1$.

Theorem 4.1. *Under the above assumptions, the process X satisfies Condition (A) with*

$$\theta_{0,S} = \bar{\theta}, \quad I_S = \bar{F}_0, \quad W_S = \sum_{i=0}^k W_i,$$

$$j_S(x) = \text{index}(x) \vee 0, \text{ for all } x \in D,$$

and, for all $i \in I_S$,

$$v_{S,i} \propto v_i + \sum_{\ell \geq 0} \theta_{0,S}^{-\ell-1} \mathbb{P}_{v_i}(X_1 \notin E_i, X_{\ell+1} \in \cdot) \quad (4.4)$$

and

$$\eta_{S,i}(x) > 0 \text{ for all } x \in D \text{ with } \mathbb{P}_x(T_{E_i} < \infty) > 0 \quad (4.5)$$

and

$$\eta_{S,i}(x) = 0 \text{ for all } x \in D \text{ with } \mathbb{P}_x(T_{E_i} < \infty) = 0. \quad (4.6)$$

Remark 14. In this theorem, the functions $\eta_{S,i}$ can also be expressed in terms of the parameters of the problem, since they are constructed in the proof below with an inductive argument, with explicit expressions at each step. \triangle

Remark 15. Similarly, the speed of convergence $\alpha_{S,n}$ is also constructed explicitly with an inductive argument in the proof below. In particular, if it is assumed that $\alpha_{i,n}$ converges exponentially fast to 0 for all i such that $\theta_{0,i} = \bar{\theta}$, one can easily check that $\alpha_{S,n}$ also converges to 0 exponentially fast if $j_S \equiv 0$, and converges to 0 polynomially in $O(1/n)$ otherwise. \triangle

Remark 16. It follows from the last theorem and Corollary 2.4 that the set of quasi-stationary distributions ν for X such that $\nu(W_S) < +\infty$ and $\nu(\eta_S) > 0$ has dimension $\#\bar{F}_0$ and is spanned (in the sense of convex hulls) by $\nu_{S,i}$, $i \in \bar{F}_0$. Our result also allows to characterize all quasi-stationary distributions ν of X such that $\nu(W_S) < \infty$: one can obtain the other quasi-stationary distributions by applying Theorem 4.1 to the process X restricted to E_0 and by proceeding recursively. All the new quasi-stationary distributions obtained this way have an exponential convergence parameter (strictly) smaller than $\bar{\theta}$.

In the particular case where the state space D is finite, our result are thus reminiscent of [28] (see in particular Theorems 4.3 and 5.1 therein). These results are already quite complete, and one of our main contributions to the problem in the finite state space situation is to determine explicitly the polynomial convergence parameter associated to each communication classes, and to emphasize the support of the functions $\eta_{S,i}$. \triangle

Remark 17. In cases where some of the η_i , $i \in \bar{F}$, is not positive on its domain E_i , one could obtain a similar result using Corollary 3.2 instead of Theorem 3.1 (iii) in the proof below. For this, Assumption (4.3) needs to be replaced by

$$\forall i, j \in \{1, \dots, k\} \text{ such that } i < j, \forall x \in E_j, \sup_{n \geq 0} \mathbb{E}_x(\eta_i(X_n)) > 0$$

to make the induction argument used in the proof below work. Then, one obtains the same expressions for $\eta_{S,i}$ and $v_{S,i}$ for the process X in (2.3), but with a non-optimal exponent j instead of j_S . However, as noticed in Remark 13, we have $j(x) = j_S(x)$ for all x such that $\eta_S(x) = \sum_{i \in I_S} \eta_{S,i} > 0$. It is easy to check from the proof below that $\eta_S(x) > 0$ for all $x \in D$ such that there exists $i \in \bar{F}_0$ satisfying $\mathbb{P}_x(T_{E_i} < \infty, \eta_i(X_{T_{E_i}}) > 0) > 0$. \triangle

Remark 18. The above result also allows to study reducible processes with denumerably many communication classes. In particular, E_0 may contain infinitely many communication classes. \triangle

Proof of Theorem 4.1. In what follows, we set, for all index value $\ell \in \{-1, 0, 1, 2, \dots\}$,

$$E_0^{(\ell)} := \{x \in E_0, \text{ such that } \text{index}(x) = \ell\}.$$

We prove the result by induction on $\bar{\ell}$. In Step 1, we prove that the result holds true for the process restricted to the set $\cup_{i \in \bar{F}_0} E_i \cup E_0^{(-1)}$. In Step 2, we prove that, if the result holds true for some value of $\bar{\ell} \geq 0$ when the set $E_0^{\bar{\ell}}$ is empty, then it holds true when $E_0^{(\bar{\ell})}$ is non-empty. In Step 3, we assume that the result holds true for some value $\bar{\ell} = n \geq 0$, and show that the result holds true when $\bar{\ell} = n + 1$ with E_0^{n+1} empty.

Step 1. Our aim is to apply Theorem 3.1 under Assumption (A1) with

$$D_1 = \cup_{i \in \bar{F}_0} E_i \text{ and } D_2 = E_0^{(-1)}.$$

Note that $D_1 \cup D_2 \cup \{\partial\}$ is closed, and that $D_2 \cup \{\partial\}$ is closed too. In what follows, we set $W_R = W \mathbb{1}_{D_2}$ and $W_P = C_W W \mathbb{1}_{D_1}$. According to (4.2), we have, for all $x \in D_1$,

$$\mathbb{E}_x(W_R(X_1)) = \mathbb{E}_x(W(X_1)) \leq C_W W(x) = W_P(x),$$

so that the first part of (3.1) holds true. In addition, since D_2 is closed, we have, for all $x \in D_2$,

$$\mathbb{E}_x(W_R(X_n) \mathbb{1}_{n < T_\partial}) = \mathbb{E}_x(W(Y_n^{(0)} \mathbb{1}_{n < T_{\cup_{j \neq 0} E_j \cup \{\partial\}}})) \leq C \gamma^n W(x),$$

where we used (4.1) for the last inequality. Hence the second part of part of (3.1) holds true.

Finally, for all $i \in \bar{F}_0$ and $x \in E_i$, we have by assumption

$$\left| \bar{\theta}^{-n} \mathbb{E}_x(f(X_n) \mathbb{1}_{n < T_{\cup_{j \neq i} E_j \cup \{\partial\}}}) - \eta_i(x) \nu_i(f) \right| \leq \alpha_{i,n} W_i(x) \|f\|_W.$$

Since $D_1 \cup D_2 \cup \{\partial\}$ is closed, we deduce that

$$\left| \bar{\theta}^{-n} \mathbb{E}_x(f(X_n) \mathbb{1}_{n < T_{\cup_{j \in \bar{F}_0, j \neq i} E_j \cup D_2 \cup \{\partial\}}}) - \eta_i(x) \nu_i(f) \right| \leq \alpha_{i,n} W_i(x) \|f\|_W.$$

Since in addition, for all $i \neq j \in \bar{F}_0$, $j \not\prec i$, we deduce that

$$\left| \bar{\theta}^{-n} \mathbb{E}_x(f(X_n) \mathbb{1}_{n < T_{D_2 \cup \{\partial\}}}) - \eta_i(x) \nu_i(f) \right| \leq \alpha_{i,n} W_i(x) \|f\|_W.$$

Summing over $i \in \bar{F}_0$, we conclude that, for all $x \in D_1$,

$$\left| \bar{\theta}^{-n} \mathbb{E}_x(f(X_n) \mathbb{1}_{n < T_{D_2 \cup \{\partial\}}}) - \sum_{i \in \bar{F}_0} \eta_i(x) \nu_i(f) \right| \leq \sum_{i \in \bar{F}_0} \alpha_{i,n} W_i(x) \|f\|_W.$$

In particular, the process restricted to D_1 satisfies Assumption (A) with $j_{0,S} = 0$. We conclude that Assumption (A1) holds true and hence, according to Theorem 3.1, that the process restricted to D satisfies Assumption (A) with $W_S = W \mathbb{1}_{D_2} + C_W W \mathbb{1}_{D_1}$, $j_S \equiv 0$, $I_S = \bar{F}_0$,

$$\nu_{S,i} \propto \nu_i + \sum_{\ell \geq 0} \theta_{0,S}^{-\ell-1} \mathbb{P}_{\nu_i}(X_1 \notin E_i, X_{\ell+1} \in \cdot),$$

and $\eta_{S,i}(x) > 0$ if and only if $x \in E_i$ with $i \in \bar{F}_0$. This proves the result when $\bar{\ell} = 0$, with $E_0^0 = \emptyset$.

Step 2. Assume that the result holds true for some value of $\bar{\ell} \geq 0$ when the set $E_0^{\bar{\ell}}$ is empty, and let us show that it holds true when $E_0^{\bar{\ell}}$ is non-empty.

In order to do so, we aim to apply Theorem 3.1 under Assumption (A2) with

$$D_1 = E_0^{(\bar{\ell})} \text{ and } D_2 = \bigcup_{\ell=0}^{\bar{\ell}} \bigcup_{i \in \bar{F}_\ell} E_i \cup \bigcup_{\ell=-1}^{\bar{\ell}-1} E_0^{(\ell)}.$$

We observe that $D_2 \cup \{\partial\}$ is a closed set for the process X and that our induction assumption applies to D_2 , so that the semi-group R of the process X restricted

to D_2 satisfies Condition (A) with

$$\theta_{0,R} = \bar{\theta}, \quad I_R = \bar{F}_0, \quad W_R = \sum_{\ell=0}^{\bar{\ell}} \sum_{i \in \bar{F}_\ell}^k W_i,$$

$$j_R(x) = \text{index}(x) \vee 0 \text{ for all } x \in \bigcup_{\ell=0}^{\bar{\ell}} \bigcup_{i \in \bar{F}_\ell} E_i \cup \bigcup_{\ell=-1}^{\bar{\ell}-1} E_0^{(\ell)}$$

and, for all $i \in I_R$,

$$v_{R,i} \propto v_i + \sum_{k \geq 0} \bar{\theta}^{-k-1} \mathbb{P}_{v_i}(X_1 \notin E_i, X_{k+1} \in \cdot) \quad (4.7)$$

and

$$\eta_{R,i}(x) > 0 \text{ for all } x \in E_i \text{ with } \mathbb{P}_x(T_{E_i} < \infty) > 0, \quad (4.8)$$

and

$$\eta_{R,i}(x) = 0 \text{ for all } x \in E_i \text{ with } \mathbb{P}_x(T_{E_i} < \infty) = 0. \quad (4.9)$$

In addition, similarly to Step 1, one checks that the first and second part of (3.2) hold true. We can thus apply Theorem 3.1 under Assumption (A2). Since, for all $x \in D_1$ there exists $i \in \bar{F}_{\bar{\ell}}$ such that $\mathbb{P}_x(T_{E_i} < +\infty) > 0$, we deduce that, for all $x \in D_1$, $\max_{n \geq 0} j_R(\delta_x P^n Q) = \bar{\ell} = \text{index}(x)$. This and Theorem 3.1 proves the result when $E_0^{(\bar{\ell})}$ is not empty.

Step 3. Assume that the result holds true for some value $\bar{\ell} = n \geq 0$, and let us show that the result holds true when $\bar{\ell} = n + 1$. In this case, we apply Theorem 3.1 under Assumption (A3) with

$$D_1 = \bigcup_{i \in \bar{F}_{n+1}} E_i \text{ and } D_2 = \bigcup_{\ell=0}^n \bigcup_{i \in \bar{F}_\ell} E_i \cup \bigcup_{\ell=-1}^n E_0^{(\ell)}.$$

Using our induction assumption, we deduce that Assumption (A) holds true for the process restricted to D_2 , with $j_{0,R} = n$ and $\theta_{0,R} = \bar{\theta}$. As in Step 1, it is also clear that Assumption (A) holds true for the process restricted to D_1 , with $j_{0,P} = 0$ and $\theta_{0,P} = \bar{\theta}$. As in Step 1, we also observe that (3.3) holds true with $W_R = W \mathbb{1}_{D_2}$ and $W_P = C_W W \mathbb{1}_{D_1}$.

In order to check that Assumption (A3) holds true, it remains to prove (3.4) and (3.5), with $\ell^* = n$. Since $j_{0,R} = n = \max_{x \in D_2} j_R(x)$, the equality (3.4) is

immediate. For (3.5), we observe that, for all $x \in D_1$, $\text{index}(x) = n + 1$ and hence that the process starting from x can reach a point of index n (otherwise, its index would be smaller or equal to n), and hence that $\mathbb{P}_x(\text{index}(X_{T_{D_2}}) = n \text{ and } T_{D_2} < \infty) > 0$. Since $\text{index}(y) = n$ implies that $j_R(y) = n$ and $\eta_R(y) > 0$ (by induction assumption), we deduce that (3.5) also holds true.

Thus we can apply Theorem 3.1 under Assumption (A3), which concludes the proof. \square

5 Lyapunov type criterion for Markov processes in discrete state spaces

Let $X = (X_n, n \in \mathbb{Z}_+)$ be a Markov chain on a discrete state space $D \cup \{\partial\}$, with ∂ absorbing. It is well known that, when X is aperiodic and irreducible, that is when $\mathbb{P}_x(\exists n \geq 0, X_n = y) > 0$ for all $x, y \in D$, existence of a quasi-stationary distribution is implied by the existence of a Lyapunov type function (see for instance [14, 10], see also [29] for a general account on quasi-stationary distributions for discrete state space models). We show in this section that the irreducibility assumption can actually be removed entirely.

In the following result, we say that X is aperiodic if all states in D are aperiodic (with the usual convention that a state $x \in D$ such that $\mathbb{P}_x(\exists n \geq 0, X_n = x) = 0$ is said aperiodic).

Theorem 5.1. *Assume that X is aperiodic, that there exists $x_0 \in D$ such that $\mathbb{P}_{x_0}(\exists n \geq 0, X_n = x_0) > 0$ and that there exists a function $V : D \rightarrow [1, +\infty)$ such that $\{x \in D, V(x) \leq C\}$ is finite for all constants $C > 0$ and such that*

$$\frac{\mathbb{E}_x(V(X_1) \mathbb{1}_{1 < \tau_\partial})}{V(x)} \xrightarrow{V(x) \rightarrow +\infty} 0. \quad (5.1)$$

Then Assumption (A) holds true with $W_S = V$ and, in particular, X admits a quasi-stationary distribution. In addition, $\mathbb{P}_{x_0}(X_n \in \cdot \mid n < \tau_\partial)$ converges in $\mathcal{M}(V)$ when $n \rightarrow +\infty$ toward a quasi-stationary distribution of X .

Remark 19. Despite its generality, the assumption that there exists $x_0 \in D$ such that $\mathbb{P}_{x_0}(\exists n \geq 0, X_n = x_0) > 0$ is actually not necessary for the existence of a quasi-stationary distribution. Consider for instance the process with $D = \{1, 2, \dots\}$ and $\partial = 0$, with almost sure transition from i to $i - 1$ for all $i \geq 1$. Then,

choosing $\nu(i) = \frac{\theta}{1-\theta}\theta^i$ for all $i \geq 1$ and any $\theta \in (0, 1)$, we have

$$\mathbb{P}_\nu(X_1 = i) = \nu(i+1) = \frac{\theta}{1-\theta}\theta^{i+1} = \theta \nu(i),$$

so that ν is a quasi-stationary distribution. \triangle

Remark 20. In (5.1), we assumed for simplicity that $\frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1 < \tau_\partial})}{V(x)} \xrightarrow{V(x) \rightarrow +\infty} 0$. However, a straightforward adaptation of the proof leads to a finer result: denoting by C_i , $i \in I$ with $I = \mathbb{N} := \{1, 2, \dots\}$ or $I = \{1, \dots, n\}$ for some $n \geq 1$, the collection of communication classes of the process, and by θ_i the exponential convergence parameter associated to each C_i , it is sufficient to assume that

$$\limsup_{V(x) \rightarrow +\infty} \frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1 < \tau_\partial})}{V(x)} < \sup_{i \in I} \theta_i.$$

Another natural and straightforward adaptation of the result is to replace by V by any function $V : D \rightarrow [1, +\infty)$ without assuming that $\{x \in D, V(x) \leq C\}$ is finite for all $C \geq 0$, but such that, for a non-decreasing sequence of finite sets $(K_n)_{n \geq 0}$ such that $D = \cup_n K_n$, we have

$$\limsup_{n \rightarrow +\infty} \inf_{x \notin K_n} \frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1 < \tau_\partial})}{V(x)} < \sup_{i \in \mathbb{N}} \theta_i.$$

\triangle

Remark 21. The aperiodicity assumption is actually not needed for all $x \in D$: one easily checks that it is only required over communication classes whose exponential convergence parameter is maximal. More generally, adaptation of these results to periodic processes is common procedure, and we leave its details to the interested reader. \triangle

Proof of Theorem 5.1. For all $x \in D$, let C_x be the communication class of x , and let $(x_i)_{i \in I}$, where I is either \mathbb{N} or $\{1, \dots, n\}$ for some n , be such that D is the disjoint union of the sets C_{x_i} , $i \in I$. We take (without loss of generality) $x_1 = x_0$ and write C_i instead of C_{x_i} .

Let $i \in I$ be such that $\mathbb{P}_{x_i}(\exists n \geq 0, X_n = x_i) > 0$. By assumption, this is the case for $i = 1$. Then the process X restricted to C_i is irreducible and satisfies Assumption (E) in [9] (this is a direct adaptation to the discrete time setting of the proof of Theorem 5.1 in the last reference). In particular (see for instance [11]), the process X restricted to C_i satisfies Assumption (A) with $j_S \equiv 0$, η_S positive

and $W_S = V|_{C_i}$. We denote by θ_i the associated exponential convergence parameter.

Let $i \in I$ be such that $\mathbb{P}_{x_i}(\exists n \geq 0, X_n = x_i) = 0$. Then $C_i = \{x_i\}$ and

$$\frac{\mathbb{E}_{x_i}(V(X_1)\mathbb{1}_{X_1 \in C_i})}{V(x_i)} = 0.$$

Now, define

$$J := \left\{ i \in I, \frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1 < \tau_\partial})}{V(x)} < \theta_1 \ \forall x \in C_i \right\}.$$

By assumption, there exists only finitely many points $x \in D$ such that $\frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1 < \tau_\partial})}{V(x)} \geq \theta_1/2$, and hence there exists $\rho < \theta_1$ such that

$$J = \left\{ i \in I, \frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{1 < \tau_\partial})}{V(x)} \leq \rho \ \forall x \in C_i \right\}$$

In particular, for all $x \in \cup_{j \in J} C_j$,

$$\frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{X_1 \in \cup_{j \in J} C_j})}{V(x)} \leq \rho,$$

so, for all $n \geq 1$,

$$\mathbb{E}_x \left(V(X_n) \mathbb{1}_{T_{\{\partial\} \cup D \cup \cup_{j \in J} C_j} > n} \right) \leq \rho^n V(x).$$

Note that, by assumption, $I \setminus J$ is finite. In addition, all $i \in I \setminus J$ is either such that $\mathbb{P}_{x_i}(\exists n \geq 0, X_n = x_i) > 0$ or such that $\mathbb{P}_{x_i}(\exists n \geq 0, X_n = x_i) = 0$, hence the process restricted to C_i either satisfies Assumption (A) with $j_S \equiv 0$, $\eta_S > 0$ and $W_S = V|_{C_i}$, or satisfies

$$\sup_{x \in C_i} \frac{\mathbb{E}_x(V(X_1)\mathbb{1}_{X_1 \in C_i})}{V(x)} = 0.$$

We define

$$J' := \left\{ i \in I \text{ such that } \mathbb{E}_x \left(V(X_n) \mathbb{1}_{T_{\{\partial\} \cup D \cup C_i} > n} \right) = 0, \ \forall x \in C_i \right\}$$

and, setting $\bar{\theta} = \max_{i \in I \setminus (J \cup J')} \theta_i$,

$$J'' := \left\{ i \in I \setminus (J \cup J') \text{ such that } \theta_i < \bar{\theta} \right\}.$$

We now set

$$E_0 = \bigcup_{j \in J \cup J' \cup J''} C_j$$

and enumerate the C_i , $i \in I \setminus (J \cup J' \cup J'')$, as E_1, \dots, E_k .

Since J'' is finite, $\max_{i \in J''} \theta_i < \bar{\theta}$ and one checks that there exists $\gamma < \bar{\theta}$ such that

$$\mathbb{E}_x \left(V(X_n) \mathbb{1}_{n < T_{E_0^c}} \right) \leq C \gamma^n V(x),$$

and that Assumption (A) holds true on each of the E_i , $i \in \{1, \dots, k\}$ with $j_i \equiv 0$, $I_i = \{1\}$ and $\eta_i > 0$ on E_i . Therefore, the assumptions of Theorem 4.1 are satisfied by the process X , so we deduce that X satisfies Assumption (A).

In order to prove the last sentence of Theorem 5.1, we apply the above proof to the process X restricted to $D_{x_0} \cup \{\partial\}$, where

$$D_{x_0} = \{x \in D \text{ such that } \mathbb{P}_{x_0}(\exists n \geq 0, X_n = x) > 0\}.$$

We deduce from (4.5) in Theorem 4.1 that $\eta_S(x) > 0$ in Assumption (A) for this process, so that Proposition 2.5(iii) entails the claim for X restricted to $D_{x_0} \cup \{\partial\}$. But the definition of D_{x_0} clearly implies that $T_{\{\partial\} \cup D \setminus D_{x_0}} = \tau_\partial$ \mathbb{P}_{x_0} -a.s., which concludes the proof. \square

A Polynomial decay for linear operators on Banach spaces

A.1 Polynomial decay and first property

Let \mathfrak{B} be a bounded linear operator on a normed real vector space $(B, |\cdot|)$. We consider the following assumption.

Assumption (H). There exists a bounded linear operator $E_{\mathfrak{B}}$ on B and $\tilde{\mathfrak{J}}_{\mathfrak{B}} \in \mathbb{R}_+$ such that, for all $x \in B$ and all $n \geq 1$,

$$\left| n^{-\tilde{\mathfrak{J}}_{\mathfrak{B}}} \mathfrak{B}^n x - E_{\mathfrak{B}} x \right| \leq \alpha_{\mathfrak{B}, n} |x|, \quad (\text{A.1})$$

where $(\alpha_{\mathfrak{B}, n})_{n \in \mathbb{N}}$ is a numerical sequence which converges to 0 when $n \rightarrow +\infty$.

We also set $\alpha_{\mathfrak{B}, 0} = 1$.

Remark 22. One natural generalization of Assumption (H) is to consider the situation where $\alpha_{\mathfrak{P},n} = \alpha_{\mathfrak{P},n}(x)$ is allowed to depend on x . One easily checks that the following result and its proof remain valid in this setting (replacing $\alpha_{\mathfrak{P},n}$ by $\alpha_{\mathfrak{P},n}(x)$ in (i)). \triangle

Proposition A.1. *Assume that (H) holds. Then,*

(i) *for all $n \geq 1$ and all $\ell \geq 0$,*

$$\|\mathfrak{P}^n\| \leq (\alpha_{\mathfrak{P},n} + \|E_{\mathfrak{P}}\|) n^{\mathfrak{J}_{\mathfrak{P}}};$$

(ii) *we have $\mathfrak{P}E_{\mathfrak{P}} = E_{\mathfrak{P}}\mathfrak{P} = E_{\mathfrak{P}}$; in particular, $E_{\mathfrak{P}}$ takes its values in the vector space of eigenvectors of \mathfrak{P} associated to 1;*

(iii) *if x is an eigenvector of \mathfrak{P} associated to 1, then $E_{\mathfrak{P}}x = \mathbb{1}_{\mathfrak{J}_{\mathfrak{P}}=0}x$.*

Proof. The first assertion is immediate. For the second one, we observe that

$$\begin{aligned} \|\mathfrak{P}E_{\mathfrak{P}} - E_{\mathfrak{P}}\| &\leq \|n^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - \mathfrak{P}E_{\mathfrak{P}}\| \\ &\quad + \|(n+1)^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - n^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1}\| + \|(n+1)^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - E_{\mathfrak{P}}\| \end{aligned}$$

where

$$\|n^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - \mathfrak{P}E_{\mathfrak{P}}\| = \|\mathfrak{P}(n^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^n - E_{\mathfrak{P}})\| \leq \|\mathfrak{P}\| \alpha_{\mathfrak{P},n}$$

and

$$\|(n+1)^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - n^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1}\| \leq \left(\frac{(n+1)^{\mathfrak{J}_{\mathfrak{P}}}}{n^{\mathfrak{J}_{\mathfrak{P}}}} - 1 \right)$$

and

$$\|(n+1)^{\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - E_{\mathfrak{P}}\| \leq \alpha_{\mathfrak{P},n+1},$$

so that $\|\mathfrak{P}E_{\mathfrak{P}} - E_{\mathfrak{P}}\| \rightarrow 0$ when $n \rightarrow +\infty$, and hence

$$\mathfrak{P}E_{\mathfrak{P}} = E_{\mathfrak{P}}.$$

Similarly, for the third assertion, we have

$$\begin{aligned} \|E_{\mathfrak{P}}\mathfrak{P} - E_{\mathfrak{P}}\| &\leq \|n^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - E_{\mathfrak{P}}\mathfrak{P}\| \\ &\quad + \|(n+1)^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - n^{-\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1}\| + \|(n+1)^{\mathfrak{J}_{\mathfrak{P}}}\mathfrak{P}^{n+1} - E_{\mathfrak{P}}\|, \end{aligned}$$

where

$$\|n^{-\tilde{\gamma}_{\mathfrak{P}}} \mathfrak{P}^{n+1} - E_{\mathfrak{P}} \mathfrak{P}\| \leq \|\mathfrak{P}\| \|n^{-\tilde{\gamma}_{\mathfrak{P}}} \mathfrak{P}^n - E_{\mathfrak{P}}\| \leq \alpha_{\mathfrak{P},n},$$

and the other terms go to 0 as in the previous case. Hence, we deduce that

$$E_{\mathfrak{P}} \mathfrak{P} = E_{\mathfrak{P}}.$$

The last assertion is immediate. \square

A.2 Polynomial decay for upper triangular matrix of linear operators

Let B_1 and B_2 be two Banach spaces, and $\mathfrak{P} : B_1 \rightarrow B_1$, $\mathfrak{Q} : B_1 \rightarrow B_2$, $\mathfrak{R} : B_2 \rightarrow B_2$ three bounded operators. We define the Banach space B as the direct sum of B_1 and B_2 and consider the operator $\mathfrak{S} = \mathfrak{P} + \mathfrak{Q} + \mathfrak{R}$ on B , where $\mathfrak{P}|_{B_2} = \mathfrak{Q}|_{B_2} = \mathfrak{R}|_{B_1} = 0$. Formally, \mathfrak{S} can be represented as the following upper triangular matrix of linear operators:

$$\mathfrak{S} = \begin{bmatrix} \mathfrak{P} & \mathfrak{Q} \\ 0 & \mathfrak{R} \end{bmatrix},$$

so that

$$\mathfrak{S}^n = \begin{bmatrix} \mathfrak{P}^n & \sum_{k=1}^n \mathfrak{R}^{n-k} \mathfrak{Q} \mathfrak{P}^{k-1} \\ 0 & \mathfrak{R}^n \end{bmatrix}.$$

The study of the spectrum of such upper triangular matrices of linear operators over a Banach space has already been considered in the literature, see for instance [3, 6, 2, 30] and references therein. In the following propositions, we are interested in the polynomial decay of the operator \mathfrak{S} , which is related to the algebraic multiplicity of its leading eigenvalue.

Remark 23. Adapting the results of this section and their proofs to the situation where Assumption (H) allows $\alpha_{\mathfrak{P},n}$ to depend on x is straightforward, but leads in some places to additional technical assumptions and cumbersome expressions. Since the situation where $\alpha_{\mathfrak{P},n}$ does not depend on x is sufficiently rich to illustrate the implications of the results below, we decided to only expose this simplest situation. We leave the interested reader to check that if $\alpha_{\mathfrak{P},n} = \alpha_{\mathfrak{P},n}(x)$ is allowed to depend on x , then

- Proposition A.2 still holds true, replacing $\alpha_{\mathfrak{P},\ell}$ by $\alpha_{\mathfrak{P},\ell}(x)$ in the expression of $\alpha_{\mathfrak{S},n}$;

- Proposition A.3 still holds true, assuming in addition that

$$\beta_n(x) := \sum_{k=0}^{n-1} \alpha_{\mathfrak{R}, n-k-1}(\mathfrak{Q}\mathfrak{P}^k x) \theta_k \xrightarrow{n \rightarrow +\infty} 0$$

and replacing $\sum_{k=0}^{n-1} \alpha_{\mathfrak{R}, n-k-1} \theta_k$ (resp. $\alpha_{\mathfrak{R}, n}$) by $\beta_n(x)$ (resp. $\alpha_{\mathfrak{R}, n}(x)$) in the expression of $\alpha_{\mathfrak{S}, n}$;

- Proposition A.4 still holds true, assuming in addition that

$$\beta'_n(x) := \frac{1}{n} \sum_{k=1}^n \alpha_{\mathfrak{R}, n-k}(\mathfrak{Q}\mathfrak{P}_{k-1})(x) \left(\frac{n-k}{n}\right)^{\mathfrak{J}\mathfrak{R}} \xrightarrow{n \rightarrow +\infty} 0.$$

and replacing $\sum_{k=1}^n \alpha_{\mathfrak{R}, n-k} \left(\frac{n-k}{n}\right)^{\mathfrak{J}\mathfrak{R}}$ (resp. $\alpha_{\mathfrak{R}, k}$) by $\beta'_n(x)$ (resp. $\alpha_{\mathfrak{R}, k}(x)$) in the expression of $\alpha_{\mathfrak{S}, n}$.

△

For all $n \geq 0$, we set $\gamma_n = \|\mathfrak{R}^n\|$, $\Gamma_n = \sum_{k \geq n} \gamma_k$, $\theta_n = \|\mathfrak{P}^n\|$ and $\Theta_n = \sum_{k \geq n} \theta_k$. We consider the case $\Gamma_0 < +\infty$ in Proposition A.2, the case $\Theta_0 < +\infty$ in Proposition A.3, and the case $\Gamma_0 = \Theta_0 = +\infty$ in Proposition A.4.

Proposition A.2. *Assume that $\sum_{n=0}^{\infty} \|\mathfrak{R}^n\| < +\infty$ and that the operator \mathfrak{P} satisfies Assumption (H). Then \mathfrak{S} satisfies assumption (H) with*

$$\mathfrak{J}_{\mathfrak{S}} = \mathfrak{J}_{\mathfrak{P}} \text{ and } E_{\mathfrak{S}} = E_{\mathfrak{P}} + \sum_{\ell \geq 0} \mathfrak{R}^{\ell} \mathfrak{Q} E_{\mathfrak{P}},$$

and

$$\alpha_{\mathfrak{S}, n} = \alpha_{\mathfrak{P}, n} + C\Gamma_n + C \sum_{k=0}^{n-1} \gamma_k \left(\frac{(\alpha_{\mathfrak{P}, n-k-1} + 1)\mathfrak{J}_{\mathfrak{P}}(k+1)}{n} + \alpha_{\mathfrak{P}, n-k-1} \right),$$

for some positive constant $C > 0$ (which depends on $\|\mathfrak{Q}\|$ and $\|E_{\mathfrak{P}}\|$) and with the convention that $\alpha_{\mathfrak{P}, 0} = 1$.

Proof. Fix $n \geq 1$ and $x \in B_1$. Then

$$\begin{aligned} |n^{-\mathfrak{J}\mathfrak{P}} \mathfrak{S}^n x - E_{\mathfrak{S}} x| &\leq |n^{-\mathfrak{J}\mathfrak{P}} \mathfrak{P}^n x - E_{\mathfrak{P}} x| + \sum_{k=0}^{n-1} \left| n^{-\mathfrak{J}\mathfrak{P}} \mathfrak{R}^k \mathfrak{Q} \mathfrak{P}^{n-k-1} x - \mathfrak{R}^k \mathfrak{Q} E_{\mathfrak{P}} x \right| \\ &\quad + \sum_{k=n}^{\infty} |\mathfrak{R}^k \mathfrak{Q} E_{\mathfrak{P}} x|. \quad (\text{A.2}) \end{aligned}$$

By Assumption (H), the first term on the right hand side is bounded by $\alpha_{\mathfrak{P},n}(x)|x|$. Moreover, since \mathfrak{Q} and $E_{\mathfrak{P}}$ are bounded operators, $|\mathfrak{R}^k \mathfrak{Q} E_{\mathfrak{P}} x| \leq \|\mathfrak{Q} E_{\mathfrak{P}}\| \gamma_k |x|$ and hence

$$|n^{-\tilde{\mathfrak{J}}_{\mathfrak{P}}} \mathfrak{P}^n x - E_{\mathfrak{P}} x| + \left| \sum_{k=n}^{\infty} \mathfrak{R}^k \mathfrak{Q} E_{\mathfrak{P}} x \right| \leq \alpha_{\mathfrak{P},n} |x| + \|\mathfrak{Q} E_{\mathfrak{P}}\| \Gamma_n |x|.$$

For the second term in the r.h.s. of (A.2), we have for all $k \in \{0, \dots, n-2\}$,

$$\begin{aligned} \left| n^{-\tilde{\mathfrak{J}}_{\mathfrak{P}}} \mathfrak{R}^k \mathfrak{Q} \mathfrak{P}^{n-k-1} x - \mathfrak{R}^k \mathfrak{Q} E_{\mathfrak{P}} x \right| &\leq \left| 1 - \left(\frac{n-k-1}{n} \right)^{\tilde{\mathfrak{J}}_{\mathfrak{P}}} \right| \|\mathfrak{R}^k \mathfrak{Q}\| \left| (n-k-1)^{-\tilde{\mathfrak{J}}_{\mathfrak{P}}} \mathfrak{P}^{n-k-1} x \right| \\ &\quad + \|\mathfrak{R}^k \mathfrak{Q}\| \left| (n-k-1)^{-\tilde{\mathfrak{J}}_{\mathfrak{P}}} \mathfrak{P}^{n-k-1} x - E_{\mathfrak{P}} x \right| \\ &\leq \|\mathfrak{Q}\| \frac{\tilde{\mathfrak{J}}_{\mathfrak{P}}(k+1)}{n} \gamma_k (\alpha_{\mathfrak{P},n-k-1} + \|\mathfrak{E}_{\mathfrak{P}}\|) |x| \\ &\quad + \|\mathfrak{Q}\| \gamma_k \alpha_{\mathfrak{P},n-k-1} |x|, \end{aligned}$$

where we used Proposition A.1 (i) and Assumption (H). For $k = n-1$, we observe that

$$\left| n^{-\tilde{\mathfrak{J}}_{\mathfrak{P}}} \mathfrak{R}^k \mathfrak{Q} \mathfrak{P}^{n-k-1} x - \mathfrak{R}^k \mathfrak{Q} E_{\mathfrak{P}} x \right| \leq 2(\|\mathfrak{Q}\| + \|\mathfrak{Q} E_{\mathfrak{P}}\|) \gamma_{n-1} |x|.$$

If $x \in B_2$, then $\mathfrak{S}^n x = \mathfrak{R}^n x$ and $E_{\mathfrak{P}} x = 0$, so that $|\mathfrak{S}^n x - E_{\mathfrak{S}} x| \leq \gamma_n |x| \leq \Gamma_n |x|$.

The above bounds and (A.2) show that \mathfrak{S} satisfies Assumption (H) with

$$\alpha_{\mathfrak{S},n} = \alpha_{\mathfrak{P},n} + C \Gamma_n + C \sum_{k=0}^{n-1} \gamma_k \left(\frac{(\alpha_{\mathfrak{P},n-k-1} + 1) \tilde{\mathfrak{J}}_{\mathfrak{P}}(k+1)}{n} + \alpha_{\mathfrak{P},n-k-1} \right),$$

which converges to 0 when $n \rightarrow +\infty$. □

Proposition A.3. *Assume that $\sum_{n=0}^{\infty} \|\mathfrak{P}^n\| < +\infty$ and that the operator \mathfrak{R} satisfies Assumption (H). Then \mathfrak{S} satisfies assumption (H) with*

$$\tilde{\mathfrak{J}}_{\mathfrak{S}} = \tilde{\mathfrak{J}}_{\mathfrak{R}} \text{ and } E_{\mathfrak{S}} = E_{\mathfrak{R}} + \sum_{\ell \geq 0} E_{\mathfrak{R}} \mathfrak{Q} \mathfrak{P}^{\ell},$$

and

$$\alpha_{\mathfrak{S},n} = \alpha_{\mathfrak{R},n} + C \Theta_n + C \sum_{k=0}^{n-1} \theta_k \left(\alpha_{\mathfrak{R},n-k-1} + \frac{\tilde{\mathfrak{J}}_{\mathfrak{S}} k}{n} \right),$$

for some positive constant $C > 0$ (which depends on $\|\mathfrak{Q}\|$ and $\|\mathfrak{E}_{\mathfrak{R}} \mathfrak{Q}\|$) and with the convention that $\alpha_{\mathfrak{R},0} = 1$.

Proof. We have, for all $n \geq 1$ and all $x \in B$,

$$\begin{aligned}
\left| n^{-\mathfrak{J}_{\mathfrak{S}}} \mathfrak{S}^n x - E_{\mathfrak{S}} x \right| &\leq \left| n^{-\mathfrak{J}_{\mathfrak{R}}} \mathfrak{R}^n x - E_{\mathfrak{R}} x \right| + n^{-\mathfrak{J}_{\mathfrak{R}}} |\mathfrak{P}^n x| \\
&\quad + n^{-\mathfrak{J}_{\mathfrak{R}}} \sum_{k=0}^{n-1} \left| \mathfrak{R}^{n-k-1} \mathfrak{Q} \mathfrak{P}^k x - (n-k-1)^{\mathfrak{J}_{\mathfrak{R}}} E_{\mathfrak{R}} \mathfrak{Q} \mathfrak{P}^k x \right| \\
&\quad + \sum_{k=0}^{n-1} \left(1 - \left(\frac{n-k-1}{n} \right)^{\mathfrak{J}_{\mathfrak{R}}} \right) |E_{\mathfrak{R}} \mathfrak{Q} \mathfrak{P}^k x| \\
&\quad + \sum_{k=n}^{\infty} |E_{\mathfrak{R}} \mathfrak{Q} \mathfrak{P}^k x|.
\end{aligned}$$

Using Assumption (H) for \mathfrak{R} and the fact that \mathfrak{Q} is a bounded operator, we deduce that the first three terms are bounded by

$$\begin{aligned}
\alpha_{\mathfrak{R},n}|x| + \theta_n|x| + \|\mathfrak{Q}\| \sum_{k=0}^{n-1} \alpha_{\mathfrak{R},n-k-1} \left(\frac{n-k-1}{n} \right)^{\mathfrak{J}_{\mathfrak{R}}} \theta_k|x| \\
\leq \left(\alpha_{\mathfrak{R},n} + \|\mathfrak{Q}\| \sum_{k=0}^{n-1} \alpha_{\mathfrak{R},n-k-1} \theta_k \right) |x|.
\end{aligned}$$

The fourth and fifth terms are bounded by

$$\begin{aligned}
\sum_{k=0}^{n-1} \|\mathfrak{E}_{\mathfrak{R}} \mathfrak{Q}\| \left(1 - \left(\frac{n-k-1}{n} \right)^{\mathfrak{J}_{\mathfrak{R}}} \right) \theta_k|x| + \sum_{k=n}^{\infty} \|\mathfrak{E}_{\mathfrak{R}} \mathfrak{Q}\| \theta_k|x| \\
\leq \|\mathfrak{E}_{\mathfrak{R}} \mathfrak{Q}\| \left(\sum_{k=0}^{n-1} \frac{\mathfrak{J}_{\mathfrak{R}} k}{n} \theta_k + \Theta_n \right) |x|.
\end{aligned}$$

We finally deduce that

$$\left| n^{-\mathfrak{J}_{\mathfrak{S}}} \mathfrak{S}^n x - E_{\mathfrak{S}} x \right| \leq \alpha_{\mathfrak{S},n}|x|,$$

where

$$\alpha_{\mathfrak{S},n} = \alpha_{\mathfrak{R},n} + \|\mathfrak{Q}\| \sum_{k=0}^{n-1} \alpha_{\mathfrak{R},n-k-1} \theta_k + (\|\mathfrak{E}_{\mathfrak{R}} \mathfrak{Q}\| + 1) \Theta_n + \|\mathfrak{E}_{\mathfrak{R}} \mathfrak{Q}\| \sum_{k=0}^{n-1} \frac{\mathfrak{J}_{\mathfrak{S}} k}{n} \theta_k. \quad \square$$

Proposition A.4. Assume \mathfrak{P} and \mathfrak{R} both satisfy Assumption (H), with $\mathfrak{J}_{\mathfrak{P}} = 0$. Then \mathfrak{S} satisfies Assumption (H) with

$$\mathfrak{J}_{\mathfrak{S}} = 1 + \mathfrak{J}_{\mathfrak{R}} \text{ and } E_{\mathfrak{S}} = \frac{1}{\mathfrak{J}_{\mathfrak{S}}} E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}}$$

and

$$\alpha_{\mathfrak{S},n} = \frac{C}{n} \left(\tilde{\mathfrak{J}}_{\mathfrak{R}} + \sum_{k=0}^n \alpha_{\mathfrak{P},k} + \left(\max_{k \geq 0} \alpha_{\mathfrak{P},k} + 1 \right) \sum_{k=0}^n \alpha_{\mathfrak{R},n-k} \left(\frac{n-k}{n} \right)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}} \right)$$

for some positive constant C (which depends on $\|E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}}\|$, $\|E_{\mathfrak{R}} \mathfrak{Q}\|$, $\|\mathfrak{Q}\|$ and $\|E_{\mathfrak{P}}\|$), and with the convention that $\alpha_{\mathfrak{P},0} = \alpha_{\mathfrak{R},0} = 1$.

Proof. Using the fact that $\mathfrak{S}^n x = \mathfrak{P}^n x + \sum_{k=1}^n \mathfrak{R}^{n-k} \mathfrak{Q} \mathfrak{P}^{k-1} x$, we deduce that

$$\begin{aligned} \left| n^{-\tilde{\mathfrak{J}}_{\mathfrak{S}}} \mathfrak{S}^n x - E_{\mathfrak{S}} x \right| &\leq n^{-\tilde{\mathfrak{J}}_{\mathfrak{S}}} \mathfrak{P}^n x + n^{-\tilde{\mathfrak{J}}_{\mathfrak{S}}} \mathfrak{R}^n x \\ &\quad + n^{-\tilde{\mathfrak{J}}_{\mathfrak{S}}} \sum_{k=1}^n \left| \mathfrak{R}^{n-k} \mathfrak{Q} \mathfrak{P}^{k-1} x - (n-k)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}} E_{\mathfrak{R}} \mathfrak{Q} \mathfrak{P}^{k-1} x \right| \\ &\quad + n^{-\tilde{\mathfrak{J}}_{\mathfrak{S}}} \sum_{k=1}^n (n-k)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}} \left| E_{\mathfrak{R}} \mathfrak{Q} \mathfrak{P}^{k-1} x - E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}} x \right| \\ &\quad + \left| n^{-\tilde{\mathfrak{J}}_{\mathfrak{S}}} \sum_{k=1}^n (n-k)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}} - \frac{1}{\tilde{\mathfrak{J}}_{\mathfrak{S}}} \right| \left| E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}} x \right| \end{aligned} \quad (\text{A.3})$$

For the first two terms on the right hand side, we deduce from Proposition A.1 (i) that

$$n^{-\tilde{\mathfrak{J}}_{\mathfrak{S}}} |\mathfrak{P}^n x| + n^{-\tilde{\mathfrak{J}}_{\mathfrak{S}}} \mathfrak{R}^n x \leq (\alpha_{\mathfrak{P},n} + \alpha_{\mathfrak{R},n} + \|E_{\mathfrak{P}}\| + \|E_{\mathfrak{R}}\|) n^{-1} |x|. \quad (\text{A.4})$$

For the third term, we use that, for all $n \geq 1$, Proposition A.1 (i) and the boundedness of \mathfrak{Q} imply

$$|\mathfrak{Q} \mathfrak{P}^{n-1} x| \leq \|\mathfrak{Q}\| (\alpha_{\mathfrak{P},n-1} + \|E_{\mathfrak{P}}\|) |x|.$$

Hence, using Assumption (H) for \mathfrak{R} , we obtain, for all $k \geq 1$,

$$\left| \mathfrak{R}^{n-k} \mathfrak{Q} \mathfrak{P}^{k-1} x - (n-k)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}} E_{\mathfrak{R}} \mathfrak{Q} \mathfrak{P}^{k-1} x \right| \leq \alpha_{\mathfrak{R},n-k} (n-k)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}} \|\mathfrak{Q}\| (\alpha_{\mathfrak{P},k-1} + \|E_{\mathfrak{P}}\|) |x|.$$

Thus

$$\begin{aligned} n^{-\tilde{\mathfrak{J}}_{\mathfrak{S}}} \sum_{k=1}^n \left| \mathfrak{R}^{n-k} \mathfrak{Q} \mathfrak{P}^{k-1} x - (n-k)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}} E_{\mathfrak{R}} \mathfrak{Q} \mathfrak{P}^{k-1} x \right| \\ \leq \|\mathfrak{Q}\| |x| \left(\max_{k \geq 0} \alpha_{\mathfrak{P},k} + \|E_{\mathfrak{P}}\| \right) \frac{1}{n} \sum_{k=1}^n \alpha_{\mathfrak{R},n-k} \left(\frac{n-k}{n} \right)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}}. \end{aligned} \quad (\text{A.5})$$

For the fourth term, we use Assumption (H) for \mathfrak{P} to derive

$$n^{-\tilde{\mathfrak{J}}_{\mathfrak{E}}} \sum_{k=1}^n (n-k)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}} \left| E_{\mathfrak{R}} \mathfrak{Q} \mathfrak{P}^{k-1} x - E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}} x \right| \leq \frac{\|E_{\mathfrak{R}} \mathfrak{Q}\| |x|}{n} \sum_{k=1}^n \alpha_{\mathfrak{P}, k-1} (1-k/n)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}}. \quad (\text{A.6})$$

Finally, the fifth term in (A.3) is bounded by

$$\|E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}}\| \left| n^{-\tilde{\mathfrak{J}}_{\mathfrak{E}}} \sum_{k=1}^n (n-k)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}} - \frac{1}{\tilde{\mathfrak{J}}_{\mathfrak{E}}} \right| |x| \quad (\text{A.7})$$

$$\begin{aligned} &\leq \|E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}}\| \sum_{k=1}^n \left| \frac{1}{n} (1-k/n)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}} - \int_{(k-1)/n}^{k/n} (1-u)^{\tilde{\mathfrak{J}}_{\mathfrak{R}}} du \right| |x| \\ &\leq \|E_{\mathfrak{R}} \mathfrak{Q} E_{\mathfrak{P}}\| \frac{\tilde{\mathfrak{J}}_{\mathfrak{R}}}{n} |x|. \end{aligned} \quad (\text{A.8})$$

Combining (A.3) and the bounds (A.4), (A.5), (A.6), (A.8) ends the proof of Proposition A.4. \square

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