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# STABILITY PROPERTIES FOR A PROBLEM OF LIGHT SCATTERING IN A DISPERSIVE METALLIC DOMAIN

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**ABSTRACT.** In this work, we study the well-posedness and some stability properties of a PDE system that models the propagation of light in a metallic domain with a hole. This model takes into account the dispersive properties of the metal. It consists of a linear coupling between Maxwell's equations and a wave type system. We prove that the problem is well posed for several types of boundary conditions. Furthermore, we show that it is polynomially stable and that the exponential stability is conditional on the exponential stability of the Maxwell system.

**KEYWORDS.** Maxwell's equation, Stability analysis, dispersive media

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**1. Introduction.** In order to accurately model the propagation of light in a nanostructured metal, the recourse to dispersive models is mandatory in order to describe the non-instantaneous response of the electrons of the metal to light. To be more complete, one has also to take into account for the spatial dispersion that occurs through the electric polarization of the electrons: the response of the electron depends on the electric field all around the position of the electron. The linearized hydrodynamic Drude dispersive model is one of the most popular model that is able to tackle these two effects using a semi-classical approach. The system consists in the linear coupling of the set of Maxwell's equations to a wave type system driving the polarization source current and charge. This system has been studied numerically on several occasions and proves its efficiency on real physical scenario (see *e.g.* [24], [26]). Theoretical works on well-posedness do exist for this system on a bounded domain with standard boundary conditions. In this latter case, metallic boundary conditions for the Maxwell electromagnetic part and either no flux current or no charges boundary conditions for the polarization unknowns are considered (see [16], [10] and [21]). In [21], the authors furthermore investigated the stability properties but also numerical stability in a discontinuous Galerkin framework. However, even if these results provide essential steps for a complete understanding of the system, this precise setting still plays the role of a toy model. To the best knowledge of the two authors, well-posedness and theoretical stability studies have not

yet been addressed for more realistic settings. It requires especially to use more general boundary conditions. The consideration of an exterior problem is a first significant step toward future realistic settings. Therefore, in this work, we propose an extension of the theory developed in [21] and we focus on a particular setting that consists in an open metallic domain with a hole. We artificially truncate the domain and impose absorbing boundary conditions on the artificial boundary. For the Maxwell electromagnetic field, classical absorbing boundary conditions, that is to say first order Silver Müller boundary conditions, are considered. For the polarization current and charge (solving also a PDE), the question of the correct boundary conditions has not been investigated so far in the literature. We propose to keep it as general as possible and study a continuum of possible boundary conditions: from a "no charges" type boundary condition to a "no flux" boundary condition passing through a vanishing linear combination of the two. We show that the system is well-posed using the classical theory of unbounded operators and semigroup theory. We furthermore investigate the polynomial and exponential decay using a frequency domain approach. Since the kernel of the associated unbounded operator is non-zero, we base our strategy on characterizing the orthogonal of its kernel and prove that the restriction of the unbounded operator to this orthogonal generates a  $C_0$ -semigroup of contractions that is polynomially stable. Finally, we show that exponential stability is conditioned to exponential stability of the Maxwell system alone.

The paper is organized as follows. In section 2, we introduce the notations and state some useful preliminary results. In section 3, we concentrate on the proof of well posedness using semigroup theory. In section 4, we study the polynomial decay and finally section 5 focus on the study of the exponential stability.

## 2. Notations and preliminary results.

**2.1. Notations.** As mentioned in the introduction, we consider an exterior scattering problem by an obstacle. The obstacle will be denoted by  $O$  and is a bounded domain of  $\mathbb{R}^3$  with a boundary that we denote  $\Gamma_S$ . Then the computational domain  $\Omega$  of  $\mathbb{R}^3$  is obtained by truncating  $\mathbb{R}^3 \setminus \bar{O}$  with an artificial boundary  $\Gamma_A$ . The obtained domain is a bounded and simply connected open domain with a Lipschitz boundary. Its boundary  $\Gamma$  consists in the union of the structure boundary  $\Gamma_S$  and the absorbing boundary  $\Gamma_A$ , i.e.  $\Gamma = \Gamma_A \cup \Gamma_S$ , with  $\bar{\Gamma}_A \cap \bar{\Gamma}_S = \emptyset$ , see Figure 1 for a typical domain.

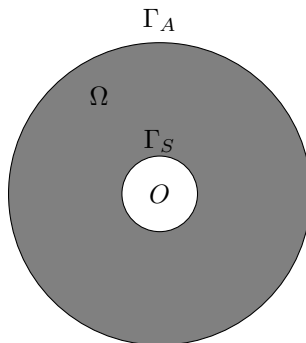


FIGURE 1. Type of considered domain

**Remark 1.** The artificial truncation of the domain is done first based on mathematical and numerical motivations. However, since real media are always of finite extension, through the

truncation of the domain, we also intend to investigate and give some preliminary answer to the difficult question of the adequate physical boundary conditions one could impose in real scenario.

As usual, the unit outward normal vector along  $\Gamma$  will be denoted by  $\mathbf{n}$ . We assume that  $\Gamma_A$  is  $C^2$  regular. Furthermore, we set  $D = \overline{\Omega} \cup \Omega$ . We will use the usual spaces  $L^2(\Omega)$  and  $H^s(\Omega)$  for  $s \in \mathbb{R}^+$ . The  $L^2(\Omega)$ -inner product (resp. norm) will be denoted by  $\langle \cdot, \cdot \rangle$  (resp.  $\| \cdot \|$ ). The usual norm and semi-norm of  $H^s(\Omega)$  ( $s \geq 0$ ) are denoted by  $\| \cdot \|_{s,\Omega}$  and  $| \cdot |_{s,\Omega}$ , respectively. For  $s = 0$  we drop the index  $s$ . Furthermore, for shortness, if  $f \in L^1(\Omega)$ , then we shall write  $\int_{\Omega} f$  instead of  $\int_{\Omega} f(x) dx$ . Similarly if  $\Gamma_0$  is a subset of  $\Gamma$ , and if  $g \in L^1(\Gamma_0)$ , then we write  $\int_{\Gamma_0} g$  for  $\int_{\Gamma_0} g(x) d\sigma(x)$ . If  $V$  is a Hilbert space and  $V'$  its dual, the duality pairing between  $V'$  and  $V$  will be denoted by  $\langle \cdot, \cdot \rangle_{V'-V}$ . Finally by  $a \lesssim b$ , we mean that there exists a constant  $C > 0$  independent of  $a, b$ , and the time variable such that  $a \leq Cb$ .

For further uses, let us introduce the following spaces:

$$H_0^1(\Omega) := \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma\}, \text{ and } H_{0,\Gamma_S}^1(\Omega) := \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma_S\},$$

that is a Hilbert space for the inner product  $\int_{\Omega} \nabla u \cdot \nabla v dx, \forall u, v \in H_{0,\Gamma_S}^1(\Omega)$ .

We also define usual spaces for fields with  $L^2$ -divergence and curl, see *e.g.* [14, 18]. For fields with square integrable divergence, we set

$$\begin{aligned} H_0(\text{div}; \Omega) &= \{\chi \in L^2(\Omega)^3 | \text{div } \chi \in L^2(\Omega) \text{ and } \chi \cdot \mathbf{n} = 0 \text{ on } \Gamma\}, \\ \mathcal{K}(\Omega) &= \{\chi \in L^2(\Omega)^3 | \text{div } \chi = 0\}, \\ H_{0,\Gamma_S}(\text{div}, \Omega) &= \{\chi \in L^2(\Omega)^3 | \text{div } \chi \in L^2(\Omega) \text{ and } \chi \cdot \mathbf{n} = 0 \text{ on } \Gamma_S\}, \\ \hat{\mathcal{K}}(\Omega) &= \mathcal{K}(\Omega) \cap H_{0,\Gamma_S}(\text{div}, \Omega) = \{\chi \in \mathcal{K}(\Omega) | \chi \cdot \mathbf{n} = 0 \text{ on } \Gamma_S\}. \end{aligned}$$

Similarly, for fields with square integrable curl, we set

$$\begin{aligned} H(\text{curl}, \Omega) &= \{\chi \in L^2(\Omega)^3 | \text{curl } \chi \in L^2(\Omega)^3\}, \\ H_0(\text{curl}, \Omega) &= \{\chi \in L^2(\Omega)^3 | \text{curl } \chi \in L^2(\Omega)^3 \text{ and } \chi \times \mathbf{n} = 0 \text{ on } \Gamma\}, \\ H_{0,\Gamma_S}(\text{curl}, \Omega) &= \{v \in H(\text{curl}, \Omega), v \times \mathbf{n} = 0, \text{ on } \Gamma_S\}. \end{aligned}$$

We furthermore introduce the following subspaces of  $H(\text{curl}; \Omega)$

$$\mathcal{V}_{\Gamma_A}(\Omega) = \{\chi \in H(\text{curl}; \Omega) | \chi \times \mathbf{n} \in L^2(\Gamma_A)^3\} \text{ and } H_{\text{mix}}(\text{curl}, \Omega) = \{\chi \in \mathcal{V}_{\Gamma_A}(\Omega) | \chi \times \mathbf{n} = 0 \text{ on } \Gamma_S\}.$$

Similarly the space  $\mathcal{V}(\Omega) = \{\chi \in H(\text{curl}; \Omega) | \chi \times \mathbf{n} \in L^2(\Gamma)^3\}$ , is a Hilbert space with the norm

$$\|\chi\|_{\mathcal{V}(\Omega)}^2 := \int_{\Omega} (|\chi|^2 + |\text{curl } \chi|^2) dx + \int_{\Gamma} |\chi \times \mathbf{n}|^2 d\sigma, \forall \chi \in \mathcal{V}(\Omega).$$

Recall that the spaces

$$X_T(\Omega) := H(\text{curl}; \Omega) \cap H_0(\text{div}, \Omega) \text{ and } X_N(\Omega) := H(\text{div}, \Omega) \cap H_0(\text{curl}; \Omega)$$

are Hilbert space with the norm  $\|\chi\|_{X(\Omega)}^2 = \int_{\Omega} (|\text{curl } \chi|^2 + |\text{div } \chi|^2) dx$ .

If  $\Gamma_0$  is an open subset of  $\Gamma$ , we introduce the spaces

$$V_{\pi}(\Gamma_0) = \{\mathbf{n} \times (G \times \mathbf{n}) : G \in H^{\frac{1}{2}}(\Gamma_0)^3\}, \quad V_{\gamma}(\Gamma_0) = \{G \times \mathbf{n} : G \in H^{\frac{1}{2}}(\Gamma_0)^3\},$$

and the trace mappings

$$\pi_{\tau} : H^1(\Omega)^3 \rightarrow V_{\pi}(\Gamma) : G \rightarrow G_{\parallel}, \quad \gamma_{\tau} : H^1(\Omega)^3 \rightarrow V_{\gamma}(\Gamma) : G \rightarrow G \times \mathbf{n},$$

where we recall that  $G_{\parallel} = \mathbf{n} \times (G \times \mathbf{n})$  is the tangential component of  $G$  on  $\Gamma$ . We recall from [9, p. 855] that  $\pi_{\tau}$  and  $\gamma_{\tau}$  are surjective and as  $\Gamma_A$  is  $C^2$  (and is disjoint to  $\Gamma_S$ ), from [9, p. 850] that

$$V_{\gamma}(\Gamma_A) = V_{\pi}(\Gamma_A) = TH^{\frac{1}{2}}(\Gamma_A)^3 := \{H \in H^{\frac{1}{2}}(\Gamma_A)^3 : H \cdot \mathbf{n} = 0 \text{ on } \Gamma_A\}, \quad (2.1)$$

and that  $V_{\gamma}(\Gamma_A)' = V_{\pi}(\Gamma_A)'$ .

**2.2. Useful Green's formulas.** For further uses, let us now recall some useful Green's formulas. First we recall that the normal trace mapping

$$H(\operatorname{div}, \Omega) \rightarrow H^{-\frac{1}{2}}(\Gamma_S) : R' \rightarrow R'_{|\partial\Omega} \cdot \mathbf{n}$$

is continuous and surjective (see [14, Theorem I.2.5 and Corollary I.2.8]) and that the next Green's formula holds (see [14, (I.2.17)])

$$\int_{\Omega} (J \cdot \nabla \phi + \phi \operatorname{div} J) dx = \langle J \cdot \mathbf{n}, \phi \rangle_{H^{-\frac{1}{2}}(\Gamma) - H^{\frac{1}{2}}(\Gamma)}, \forall \phi \in H^1(\Omega), \forall J \in H(\operatorname{div}, \Omega). \quad (2.2)$$

We go on with Green's formula related to the curl operator. Recall from [9, Theorem 4.1] that for all  $E \in H(\operatorname{curl}, \Omega)$ , we have

$$E_{\parallel} \in H^{-\frac{1}{2}}(\operatorname{curl}_{\Gamma}, \Gamma) := \{\lambda \in V_{\gamma}(\Gamma)' : \operatorname{curl}_{\Gamma} \lambda \in H^{-\frac{1}{2}}(\Gamma)\}, \quad (2.3)$$

$$E \times \mathbf{n} \in H^{-\frac{1}{2}}(\operatorname{div}_{\Gamma}, \Gamma) := \{\lambda \in V_{\pi}(\Gamma)' : \operatorname{div}_{\Gamma} \lambda \in H^{-\frac{1}{2}}(\Gamma)\}, \quad (2.4)$$

from [9, (27)]<sup>1</sup>, for all  $E \in H(\operatorname{curl}, \Omega)$ , and all  $F \in H^1(\Omega)^3$ , we have

$$\int_{\Omega} (E \cdot \operatorname{curl} \bar{F} - \operatorname{curl} E \cdot \bar{F}) dx = \langle E \times \mathbf{n}, F_{\parallel} \rangle_{V_{\pi}(\Gamma)' - V_{\pi}(\Gamma)} = -\langle E_{\parallel}, F \times \mathbf{n} \rangle_{V_{\gamma}(\Gamma)' - V_{\gamma}(\Gamma)}. \quad (2.5)$$

As  $\mathcal{D}(\Omega)^3$  is dense in  $H_0(\operatorname{curl}, \Omega)$ , this identity directly implies that

$$\int_{\Omega} (E \cdot \operatorname{curl} \bar{F} - \operatorname{curl} E \cdot \bar{F}) dx = 0, \forall E \in H(\operatorname{curl}, \Omega), F \in H_0(\operatorname{curl}, \Omega). \quad (2.6)$$

We continue with a similar Green formula but for elements in  $\mathcal{V}(\Omega)$  (see Lemma 2.2 of [20]):

**Lemma 2.1.** For all  $E, F \in \mathcal{V}(\Omega)$ , we have

$$\int_{\Omega} (E \cdot \operatorname{curl} \bar{F} - \operatorname{curl} E \cdot \bar{F}) dx = \int_{\Gamma} (E \times \mathbf{n}) \cdot \bar{F}_{\parallel} d\sigma. \quad (2.7)$$

**Corollary 2.2.** For all  $E \in H_{\operatorname{mix}}(\operatorname{curl}; \Omega)$  and  $F \in \mathcal{V}_{\Gamma_A}(\Omega)$ , we have

$$\int_{\Omega} (E \cdot \operatorname{curl} \bar{F} - \operatorname{curl} E \cdot \bar{F}) dx = \int_{\Gamma_A} (E \times \mathbf{n}) \cdot \bar{F}_{\parallel} d\sigma. \quad (2.8)$$

*Proof.* Fix a cut-off function  $\eta \in \mathcal{D}(\mathbb{R}^3)$  such that  $\eta = 1$  in a neighborhood of  $\Gamma_S$  and  $\eta = 0$  in a neighborhood of  $\Gamma_A$ , then  $\eta E$  belongs to  $H_0(\operatorname{curl}, \Omega)$  and  $(1 - \eta)E$  belongs to  $\mathcal{V}(\Omega)$ . Using (2.6) for the pair  $(\eta E, F)$ , we get

$$\int_{\Omega} (\eta E \cdot \operatorname{curl} \bar{F} - \operatorname{curl}(\eta E) \cdot \bar{F}) dx = 0. \quad (2.9)$$

Now we fix a cut-off function  $\psi \in \mathcal{D}(\mathbb{R}^3)$  such that  $\psi = 1$  on a neighbourhood of the set  $\{x \in \mathbb{R}^3 : \eta(x) \neq 1\}$  and  $\psi = 0$  in a neighborhood of  $\Gamma_S$ . In that way  $\psi F$  belongs to  $\mathcal{V}(\Omega)$ . The property of  $\psi$  implying

$$\int_{\Omega} ((1 - \eta)E \cdot \operatorname{curl} \bar{F} - \operatorname{curl}((1 - \eta)E) \cdot \bar{F}) dx = \int_{\Omega} ((1 - \eta)E \cdot \operatorname{curl} \overline{\psi F} - \operatorname{curl}((1 - \eta)E) \cdot \overline{\psi F}) dx,$$

<sup>1</sup>the reader is referred to this reference for the definition of surfacic divergence and curl operators.

using (2.7) for the pair  $((1 - \eta)E, \psi F)$ . Since  $1 - \eta = \psi = 1$  on  $\Gamma_A$ , we get

$$\int_{\Omega} ((1 - \eta)E \cdot \operatorname{curl} \bar{F} - \operatorname{curl}((1 - \eta)E) \cdot \bar{F}) dx = \int_{\Gamma_A} (E \times \mathbf{n}) \cdot \bar{F}_{\parallel} d\sigma.$$

Taking the sum of this identity with (2.9), we arrive at (2.8).  $\square$

**3. The system and its well-posedness.** In this section we describe the PDE model on which we focus and investigate its well-posedness.

**3.1. The system of equations.** The PDE model is obtained using a hydrodynamic description of the metal. In this approximation, one models the collective behavior of the cloud of the free electrons of the metal as a fluid. Focusing on the linear response of the system (we refer the reader to [6] for the details), the resulting system of equations is a linear hyperbolic system of PDE's. These equations write formally as:

$$\left\{ \begin{array}{l} \varepsilon \partial_t E - \operatorname{curl} H = -J \text{ in } \mathcal{O} := \Omega \times ]0, +\infty[, \\ \mu \partial_t H + \operatorname{curl} E = 0 \text{ in } \mathcal{O}, \\ \partial_t J - \beta^2 \nabla Q = \varepsilon_0 \omega_p^2 E - \gamma J \text{ in } \mathcal{O}, \\ \partial_t Q - \operatorname{div} J = 0 \text{ in } \mathcal{O}. \end{array} \right. \quad (3.1)$$

The system of PDE thus consists in a linear coupling between Maxwell's equations (with  $(E, H)$  denoting the electromagnetic field) and a PDE system that describes the evolution of the polarization current  $J$  and the charge  $Q$ . As usual, we consider that  $\varepsilon_0$ , the vacuum permeability,  $\varepsilon$ , the permeability of the media and  $\mu$ , its permittivity, are physical electromagnetic constants. In addition,  $\omega_p$  denotes the plasma frequency and  $\beta$  is the so-called "nonlocal" parameter taking into account the fact that the reaction of the electron depends on the electromagnetic field existing in a vicinity of its position. One should notice that if  $\beta = 0$ , the system reduces to a linear coupling between Maxwell's equation and a simple ODE describing the evolution of the polarization current  $J$ : this is the so-called well-known Drude dispersive model. If  $\beta = 0$ , only time dispersion is taken into account (the electron reaction is influenced by all the history of the electric field).

The full system (3.1) (with  $\beta \neq 0$ ) has been studied theoretically in a bounded domain with standard boundary conditions: metallic boundary conditions for the Maxwell electromagnetic part ( $E \times \mathbf{n} = 0$ ) and either no flux ( $J \cdot \mathbf{n} = 0$ ) or no charges ( $Q = 0$ ). Several works (already mentioned in the introduction) exist: [16], [10] and [21]. In [21], a complete theoretical stability study is also provided. Here, we propose a first extension of [21]. We consider a metallic domain with a hole with physical properties described by the model (3.1).

**3.2. Boundary conditions and spaces.** On the structure boundary  $\Gamma_S$ , we impose the electric boundary conditions

$$E \times \mathbf{n} = 0, H \cdot \mathbf{n} = 0, \operatorname{div} J = 0, Q = 0, \quad (3.2)$$

On the absorbing boundary  $\Gamma_A$ , we impose the Silver-Müller condition

$$E \times \mathbf{n} - z(H \times \mathbf{n}) \times \mathbf{n} = 0 \text{ on } \Gamma_A, \text{ and} \quad (3.3)$$

$$\beta_1 J \cdot \mathbf{n} + \beta_2 Q = 0 \text{ on } \Gamma_A, \quad (3.4)$$

with two non negative real numbers  $\beta_1$  and  $\beta_2$  such that  $\beta_1 + \beta_2 > 0$  and  $z = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$

This allows to treat the boundary condition  $J \cdot \mathbf{n} = 0$  and  $Q = 0$  altogether corresponding respectively to the case  $\beta_2 = 0$  and  $\beta_1 = 0$ . In the last case,  $\beta_1 > 0$  and  $\beta_2 > 0$ , hence if we define  $r = \frac{\beta_1}{\beta_2} > 0$ , we can eliminate  $J \cdot \mathbf{n}$  (or  $Q$ ) since in this case (3.4) is equivalent to

$$Q = -rJ \cdot \mathbf{n}. \quad (3.5)$$

Alltogether we find the problem

$$\left\{ \begin{array}{l} \varepsilon \partial_t E - \operatorname{curl} H = -J \text{ in } \mathcal{O} := \Omega \times ]0, +\infty[, \\ \mu \partial_t H + \operatorname{curl} E = 0 \text{ in } \mathcal{O}, \\ \partial_t J - \beta^2 \nabla Q = \varepsilon_0 \omega_p^2 E - \gamma J \text{ in } \mathcal{O}, \\ \partial_t Q - \operatorname{div} J = 0 \text{ in } \mathcal{O}, \\ E \times \mathbf{n} = 0, H \cdot \mathbf{n} = 0, \operatorname{div} J = 0, Q = 0 \text{ on } \Sigma_S := \Gamma_S \times ]0, +\infty[, \\ E \times \mathbf{n} - z(H \times \mathbf{n}) \times \mathbf{n} = 0, \beta_1 J \cdot \mathbf{n} + \beta_2 Q = 0 \text{ on } \Sigma_A := \Gamma_A \times ]0, +\infty[, \\ E(\cdot, 0) = E_0, H(\cdot, 0) = H_0, J(\cdot, 0) = J_0, Q(\cdot, 0) = Q_0, \text{ in } \Omega. \end{array} \right. \quad (3.6)$$

Note that these equations imply some hidden constraints on  $H$  and  $\varepsilon E$ . Indeed the second equation in (3.6) formally yields  $(\operatorname{div} H)_t = 0$  in  $\mathcal{O}$ , while the first and the fourth equations in (3.6) formally yield  $(\operatorname{div}(\varepsilon E) + Q)_t = 0$  in  $\mathcal{O}$ . Therefore

$$(\operatorname{div}(\varepsilon E) + Q)(x, t) = (\operatorname{div}(\varepsilon E) + Q)(x, 0) \text{ and } \operatorname{div} H(x, t) = \operatorname{div} H(x, 0), \forall x \in \Omega, t > 0, \quad (3.7)$$

and if we assume the divergence free properties at  $t = 0$ , they will remain valid for  $t > 0$ .

Further in the case  $\beta_2 = 0$ , the boundary condition  $\beta_1 J \cdot \mathbf{n} + \beta_2 Q = 0$  on  $\Sigma_A$  reduces to

$$J \cdot \mathbf{n} = 0 \text{ on } \Sigma_A. \quad (3.8)$$

Hence if  $E$  and  $H$  are smooth enough, by multiplying the first equation by  $\nabla \varphi$ , with  $\varphi \in H_{0, \Gamma_S}^1(\Omega)$  such that  $\varphi = 1$  on  $\Gamma_A$ , and integrating in  $\Omega$ , one has

$$\int_{\Omega} (\varepsilon \partial_t E - \operatorname{curl} H + J) \cdot \nabla \varphi = 0.$$

Integrating by parts, and taking into account (3.8) we then find  $\int_{\Omega} (\varepsilon \partial_t E + J) \cdot \nabla \varphi = \int_{\Gamma_A} \varepsilon \partial_t E \cdot \mathbf{n}$ , as well as  $\int_{\Omega} \operatorname{curl} H \cdot \nabla \varphi = \int_{\Gamma} \nabla \varphi \times \mathbf{n} \cdot \operatorname{curl} H = 0$ . These last properties then lead to  $\partial_t \left( \int_{\Gamma_A} E \cdot \mathbf{n} \right) = 0$ .

and therefore if we assume that  $\int_{\Gamma_A} E(x, t) \cdot \mathbf{n} = 0$  at time  $t = 0$ , one has for  $t > 0$

$$\int_{\Gamma_A} E(x, t) \cdot \mathbf{n} = \int_{\Gamma_A} E(x, 0) \cdot \mathbf{n}. \quad (3.9)$$

We now define the energy space. Contrary to [21], we do not directly integrate these constraints in the energy space, they will be taken into account later on. We thus consider the Hilbert space

$$\mathcal{H} = \{(F, G, R, S)^\top \in L^2(\Omega)^3 \times L^2(\Omega)^3 \times L^2(\Omega)^3 \times L^2(\Omega)\} \equiv L^2(\Omega)^{10},$$

with the inner product

$$((F, G, R, S)^\top, (F', G', R', S')^\top)_{\mathcal{H}} := \int_{\Omega} (\varepsilon F \cdot \bar{F}' + \mu G \cdot \bar{G}' + \frac{1}{\varepsilon_0 \omega_p^2} R \cdot \bar{R}' + \frac{\beta^2}{\varepsilon_0 \omega_p^2} S \cdot \bar{S}') dx. \quad (3.10)$$

We define the unbounded operator  $\mathcal{A}$  by

$$\mathcal{A}(E, H, J, Q)^\top = (\varepsilon^{-1}(\operatorname{curl} H - J), -\mu^{-1} \operatorname{curl} E, \beta^2 \nabla Q + \varepsilon_0 \omega_p^2 E - \gamma J, \operatorname{div} J)^\top, \quad (3.11)$$

with domain

$$D(\mathcal{A}) := \left\{ (E, H, J, Q)^\top \in H_{0,\Gamma_S}(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega) \times H(\operatorname{div}, \Omega) \times H_{0,\Gamma_S}^1(\Omega) \mid \right. \\ \left. E \times \mathbf{n} + zH_\parallel = 0, \beta_1 J \cdot \mathbf{n} + \beta_2 Q = 0, \text{ on } \Gamma_A \right\}. \quad (3.12)$$

With this definition, introducing the vectorial unknown  $U = (E, H, J, Q)^\top$ , problem (3.6) can then be formally rewritten as a first order evolution equation

$$\partial_t U = \mathcal{A}U, \quad U(0) = U_0. \quad (3.13)$$

**3.3. The well-posedness result.** The well-posedness of our problem is based on semigroup theory, by showing that  $\mathcal{A}$  generates a strongly continuous semigroup.

Nevertheless, let us start with an hidden regularity for elements of  $D(\mathcal{A})$ .

**Lemma 3.1.** Let  $(F, G, R, S)^\top \in D(\mathcal{A})$ , then  $F \times \mathbf{n}, G \times \mathbf{n} \in H^{\frac{1}{2}}(\Gamma_A)^3$ , in particular  $F$  belongs to  $H_{\text{mix}}(\operatorname{curl}, \Omega)$  and  $G$  belongs to  $\mathcal{V}_{\Gamma_A}(\Omega)$

*Proof.* By its definition, for  $(F, G, R, S)^\top \in D(\mathcal{A})$ ,  $F$  and  $G$  belong to  $H(\operatorname{curl}, \Omega)$  but they are not in  $H(\operatorname{div}, \Omega)$ , so for any  $H \in L^2(\Omega)^3$ , we consider  $\varphi_H \in H_0^1(\Omega)$  the unique solution of

$$\int_{\Omega} \nabla \varphi_H \cdot \nabla v \, dx = \int_{\Omega} H \cdot \nabla v \, dx, \forall v \in H_0^1(\Omega),$$

whose existence follows from Lax-Milgram Lemma. This implies that  $\tilde{F} = F - \nabla \varphi_F$  and  $\tilde{G} = G - \nabla \varphi_G$  are divergence free, belong to  $H(\operatorname{curl}, \Omega)$  and satisfy  $\tilde{F} \times \mathbf{n} + z\tilde{G}_\parallel = 0$  on  $\Gamma_A$ . These facts imply that for any cut-off function  $\chi \in \mathcal{D}(\mathbb{R}^3)$  such that  $\chi = 0$  in a neighborhood of  $\Gamma_S$  and  $\chi = 1$  on  $\Gamma_A$ , the pair  $(\chi\tilde{F}, \chi\tilde{G})$  belongs to  $\mathcal{V}_{\Gamma_A}^2$  (as  $\Gamma_A = \partial D$ )

$$\mathbf{V}(D) = \{(E, H) \in H(\operatorname{curl}, D) \cap H(\operatorname{div}, D) : E \times \mathbf{n} + zH_\parallel = 0 \text{ on } \partial D\}. \quad (3.14)$$

As Lemma 4.5.5 of [12] (see [3] for a smooth boundary) shows that  $\mathbf{V}(D)$  is continuously embedded into  $H^1(D)^6$ , we deduce that  $\tilde{F} \times \mathbf{n}, \tilde{G} \times \mathbf{n} \in H^{\frac{1}{2}}(\Gamma_A)^3$ . This yields the result because  $F \times \mathbf{n} = \tilde{F} \times \mathbf{n}$  and  $G \times \mathbf{n} = \tilde{G} \times \mathbf{n}$ .  $\square$

We now continue with the characterization of the adjoint operator of  $\mathcal{A}$ . First for any  $(F, G, R, S)^\top \in \mathcal{H}$ , we set  $\mathcal{O}(F, G, R, S)^\top = (F, -G, -R, S)^\top$ . Note that  $\mathcal{O}$  is an isomorphism from  $\mathcal{H}$  into itself with  $\mathcal{O}^{-1} = \mathcal{O}$ .

**Lemma 3.2.** The domain of the adjoint  $\mathcal{A}^*$  of  $\mathcal{A}$  is  $D(\mathcal{A}^*) = \{U \in \mathcal{H} \mid \mathcal{O}U \in D(\mathcal{A})\}$ , and

$$\mathcal{A}^*U = \mathcal{O}\mathcal{A}\mathcal{O}U, \forall U \in D(\mathcal{A}^*). \quad (3.15)$$

*Proof.* By definition,  $U = (F, G, R, S)^\top$  in  $\mathcal{H}$  belongs to  $D(\mathcal{A}^*)$  if and only if there exists  $X \in \mathcal{H}$  such that

$$(U, \mathcal{A}U')_{\mathcal{H}} = (X, U')_{\mathcal{H}}, \forall U' \in D(\mathcal{A}), \quad (3.16)$$

<sup>2</sup>as  $\chi\tilde{F}$  and  $\chi\tilde{G}$  are zero in a neighborhood of  $\Gamma_A$ , we can extend them by zero in  $\mathcal{O}$



and in that case  $\mathcal{A}^*U = X$ . Using (3.11), (3.16) is clearly equivalent to

$$\int_{\Omega} \left( (\operatorname{curl} \bar{G}' - \bar{R}') \cdot F - \operatorname{curl} \bar{F}' \cdot G + \frac{1}{\varepsilon_0 \omega_p^2} (\beta^2 \nabla \bar{S}' + \varepsilon_0 \omega_p^2 \bar{F}' - \gamma \bar{R}') \cdot R + \frac{\beta^2}{\varepsilon_0 \omega_p^2} \operatorname{div} \bar{R}' \cdot S \right) dx = (X, U')_{\mathcal{H}}, \forall U' = (F', G', R', S')^{\top} \in D(\mathcal{A}). \quad (3.17)$$

First taking  $U' \in \mathcal{D}(\Omega)^{10}$ , we find

$$\langle R - \operatorname{curl} G, \bar{F}' \rangle + \langle \operatorname{curl} F, \bar{G}' \rangle - \langle F + \frac{1}{\varepsilon_0 \omega_p^2} \gamma R + \frac{\beta^2}{\varepsilon_0 \omega_p^2} \nabla S, \bar{R}' \rangle - \frac{\beta^2}{\varepsilon_0 \omega_p^2} \langle \operatorname{div} R, \bar{S}' \rangle = (X, U')_{\mathcal{H}}.$$

Consequently  $U$  belongs to  $H(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega) \times H(\operatorname{div}, \Omega) \times H^1(\Omega)$  and  $X = \mathcal{O}\mathcal{A}\mathcal{O}U$  (which is meaningful due to the previous regularity of  $U$ ). Using this identity and (3.17), we find

$$(U, \mathcal{A}U')_{\mathcal{H}} = (\mathcal{A}^*U, U')_{\mathcal{H}}, \forall U' \in D(\mathcal{A}). \quad (3.18)$$

It remains to find the boundary conditions satisfied by  $U$ . Let us start with the ones at  $\Gamma_S$ .

Let us now choose  $U' = (0, G', 0, 0)$  with  $G' = \eta G''$ , with  $G'' \in H^1(\Omega)^3$  and  $\eta$  is cut-off function (in  $\mathcal{D}(\mathbb{R}^3)$ ) such that  $\eta = 1$  on  $\Gamma_S$  and  $\eta = 0$  on  $\Gamma_A$ . In this way,  $U'$  belongs to  $D(\mathcal{A})$  and using (3.18), we find

$$(F, \operatorname{curl} G') = (\operatorname{curl} F, G').$$

As  $G' \in H^1(\Omega)^3$  and  $F \in H(\operatorname{curl}, \Omega)$ , using Green's formula (2.5), we find (see [9] for the details)

$$\langle F \times \mathbf{n}, G'_{\parallel} \rangle_{V_{\pi}(\Gamma_S)' - V_{\pi}(\Gamma_S)} = 0.$$

As  $\pi_{\tau} : H^1(\Omega)^3 \rightarrow V_{\pi} : G \rightarrow G_{\parallel}$  is surjective (see section 2.2), we deduce that  $F \times \mathbf{n} = 0$  on  $\Gamma_S$ .

Similarly we take  $U' = (0, 0, R', 0)^{\top}$  with  $R' = \eta R''$  with  $R'' \in H(\operatorname{div}, \Omega)$  and  $\eta$  as before so that  $R' = 0$  on  $\Gamma_A$ . This yields an element of  $D(\mathcal{A})$ , hence using this test function in (3.18) and Green's formula (2.2), we obtain

$$\langle R' \cdot \mathbf{n}, S \rangle_{H^{-\frac{1}{2}}(\Gamma_S) - H^{\frac{1}{2}}(\Gamma_S)} = 0.$$

As the trace mapping  $H(\operatorname{div}, \Omega) \rightarrow H^{-\frac{1}{2}}(\Gamma_S) : R' \rightarrow R'_{\parallel \partial \Omega} \cdot \mathbf{n}$  is surjective (see section 2.2), we conclude that  $S = 0$  on  $\Gamma_S$ .

Let us go on with the boundary conditions satisfied by  $U$  at  $\Gamma_A$ . First by the surjectivity of the trace operators  $\pi_{\tau}$  and  $\gamma_{\tau}$  for all  $H \in TH^{\frac{1}{2}}(\Gamma_A)^3$ , we can find  $F'', G'' \in H^1(\Omega)^3$  such that

$$F'' \times \mathbf{n} = H \text{ and } G''_{\parallel} = -zH \text{ on } \Gamma_A. \quad (3.19)$$

Multiplying  $F''$  and  $G''$  by  $1 - \eta$ , with  $\eta$  fixed below we find  $F' = (1 - \eta)F''$  and  $G' = (1 - \eta)G''$  such that  $U' = (F', G', 0, 0)$  belongs to  $D(\mathcal{A})$ . Choosing this element  $U'$  in (3.18) and using Green's formula (2.5), and recalling (2.1) we deduce that

$$\langle F \times \mathbf{n}, G'_{\parallel} \rangle_{TH^{\frac{1}{2}}(\Gamma_A)' - TH^{\frac{1}{2}}(\Gamma_A)} + \langle G'_{\parallel}, F' \times \mathbf{n} \rangle_{TH^{\frac{1}{2}}(\Gamma_A)' - TH^{\frac{1}{2}}(\Gamma_A)} = 0.$$

With the help of (3.19), we arrive at

$$\langle -zF \times \mathbf{n} + G'_{\parallel}, H \rangle_{TH^{\frac{1}{2}}(\Gamma_A)' - TH^{\frac{1}{2}}(\Gamma_A)} = 0, \forall H \in TH^{\frac{1}{2}}(\Gamma_A)^3.$$

Therefore we have find that  $F \times \mathbf{n} - zG'_{\parallel} = 0$  on  $\Gamma_A$ .

For the last boundary condition, we take an arbitrary  $\lambda \in H^{\frac{1}{2}}(\Gamma_A)$ . First we fix a lifting  $R\lambda \in H^1(\Omega)$  of  $\lambda$ , namely an element in  $H^1(\Omega)$  such that

$$R\lambda = \lambda \text{ on } \Gamma_A, \text{ and } R\lambda = 0 \text{ on } \Gamma_S.$$

Second recalling that the boundary  $\Gamma_A$  is  $C^2$ , there exists a vector field  $N \in C^1(\bar{\Omega})$  such that  $N(x) = \mathbf{n}(x), \forall x \in \Gamma_A$ . We then readily check that  $(R\lambda)N$  belongs to  $H^1(\Omega)^3$  and satisfies

$$(R\lambda)N \cdot \mathbf{n} = \lambda|\mathbf{n}|^2 = \lambda \text{ on } \Gamma_A.$$

Then we take  $S' = \beta_1 R\lambda$  and  $R' = -\beta_2(R\lambda)N$ , which yield  $U' = (0, 0, R', S') \in D(\mathcal{A})$ . Choosing this element  $U'$  in (3.18) and using Green's formula (2.2), we deduce that

$$\langle R \cdot \mathbf{n}, S' \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)} + \langle S, R' \cdot \mathbf{n} \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)} = 0.$$

By their definition, we find

$$\langle \beta_1 R \cdot \mathbf{n} - \beta_2 S, \lambda \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)} = 0, \forall \lambda \in H^{\frac{1}{2}}(\Gamma_A),$$

and shows that  $\beta_1 R \cdot \mathbf{n} - \beta_2 S = 0$  on  $\Gamma_A$ .

Alltogether we have shown the inclusion  $D(\mathcal{A}^*) \subset D^* := \{U \in \mathcal{H} \mid \mathcal{O}U \in D(\mathcal{A})\}$ . But for  $U \in D^*$ , by the different Green's formula mentioned before, one readily checks that

$$(U, \mathcal{A}U')_{\mathcal{H}} = (\mathcal{A}^*U, U')_{\mathcal{H}}, \forall U' \in D(\mathcal{A}),$$

with  $\mathcal{A}^*U$  defined by (3.15). This means that  $D^* \subset D(\mathcal{A}^*)$  and concludes the proof.  $\square$

**Theorem 3.1.** *The operator  $\mathcal{A}$  defined by (3.11) with domain (3.12) generates a  $C_0$ -semigroup of contractions  $(T(t))_{t \geq 0}$  on  $\mathcal{H}$ . Therefore for all  $U_0 \in \mathcal{H}$ , the problem (3.13) has a mild solution  $U \in C([0, \infty), \mathcal{H})$  given by  $U(t) = T(t)U_0$ . If moreover  $U_0 \in D(\mathcal{A})$ , the problem (3.13) has a strong solution  $U \in C([0, \infty), D(\mathcal{A})) \cap C^1([0, \infty), \mathcal{H})$ .*

*Proof.* We apply Corollary I.4.4 of [22] that guarantees that  $\mathcal{A}$  generates a  $C_0$ -semigroup of contractions  $(T(t))_{t \geq 0}$  on  $\mathcal{H}$  if  $\mathcal{A}$  is a densely defined closed operator and if  $\mathcal{A}$  and its adjoint  $\mathcal{A}^*$  are dissipative.

Since  $\mathcal{D}(\Omega)^{10}$  is dense in  $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$  and since  $\mathcal{D}(\Omega)^{10}$  is clearly included into  $D(\mathcal{A})$ , the domain of  $\mathcal{A}$  is therefore dense in  $\mathcal{H}$ .

Concerning the closedness of  $\mathcal{A}$ , since  $D(\mathcal{A})$  is dense in  $\mathcal{H}$ , its adjoint  $\mathcal{A}^*$  is closed. Furthermore Lemma 3.2 yields that

$$G_{\mathcal{A}} = \tilde{\mathcal{O}}G_{\mathcal{A}^*}$$

where  $G_{\mathcal{A}}$  is the graph of  $\mathcal{A}$  and

$$\tilde{\mathcal{O}}(X, Y) = (\mathcal{O}X, \mathcal{O}Y), \forall X, Y \in \mathcal{H}.$$

As  $\tilde{\mathcal{O}}$  is an isomorphism from  $\mathcal{H} \times \mathcal{H}$  into itself, we directly deduce that  $\mathcal{A}$  is closed.

It remains to show that  $\mathcal{A}$  and  $\mathcal{A}^*$  are dissipative operators. Let us start with the dissipativeness of  $\mathcal{A}$ . For  $U = (E, H, J, Q)^{\top} \in D(\mathcal{A})$ , by Green's formulas (2.2) and (2.8), we find that

$$\begin{aligned} (\mathcal{A}U, U)_{\mathcal{H}} &= \int_{\Omega} (H \cdot \text{curl } \bar{E} - \text{curl } E \cdot \bar{H} + \frac{\beta^2}{\varepsilon_0 \omega_p^2} (\text{div } J \cdot \bar{Q} - \text{div } \bar{J} \cdot Q) + E \cdot \bar{J} - J \cdot \bar{E} - \frac{\gamma}{\varepsilon_0 \omega_p^2} |J|^2) dx \\ &\quad + \int_{\Gamma_A} H \cdot \bar{E} \times \mathbf{n} + \frac{\beta^2}{\varepsilon_0 \omega_p^2} \int_{\Gamma_A} Q \bar{J} \cdot \mathbf{n}. \end{aligned}$$

Now using the boundary conditions, we deduce that

$$\begin{aligned} (\mathcal{A}U, U)_{\mathcal{H}} &= \int_{\Omega} (H \cdot \text{curl } \bar{E} - \text{curl } E \cdot \bar{H} + \frac{\beta^2}{\varepsilon_0 \omega_p^2} (\text{div } J \cdot \bar{Q} - \text{div } \bar{J} \cdot Q) + E \cdot \bar{J} - J \cdot \bar{E} - \frac{\gamma}{\varepsilon_0 \omega_p^2} |J|^2) dx \\ &\quad - z \int_{\Gamma_A} H \times \mathbf{n} \cdot \bar{H} \times \mathbf{n} - \Sigma_A(J, Q). \end{aligned}$$

where  $\Sigma_A(J, Q) = 0$  if  $\beta_1 = 0$  or  $\beta_2 = 0$ , and in the contrary  $\Sigma_A(J, Q) := \frac{r\beta^2}{\varepsilon_0\omega_p^2} \int_{\Gamma_A} |J \cdot \mathbf{n}|^2$ . In both cases, we may notice that  $\Sigma_A(J, Q) \geq 0$ . Hence taking the real part of this identity, we obtain

$$\Re(\mathcal{A}U, U)_{\mathcal{H}} = -\frac{\gamma}{\varepsilon_0\omega_p^2} \int_{\Omega} |J|^2 dx - z \int_{\Gamma_A} |H \times \mathbf{n}|^2 - \Sigma_A(J, Q). \quad (3.20)$$

This shows that  $\mathcal{A}$  is dissipative.

As for any  $U \in D(\mathcal{A}^*)$ ,  $(\mathcal{A}^*U, U)_{\mathcal{H}} = (\mathcal{A}OU, OU)_{\mathcal{H}}$ , one directly gets  $\Re(\mathcal{A}^*U, U)_{\mathcal{H}} \leq 0$ .  $\square$

**4. Polynomial stability results.** Our stability results are based on a frequency domain approach, namely for the exponential decay of the energy we use the following result, see [15, 25]:

**Lemma 4.1.** A  $C_0$ -semigroup  $(e^{t\mathcal{L}})_{t \geq 0}$  of contractions on a Hilbert space  $H$  is exponentially stable, i.e., satisfies

$$\|e^{t\mathcal{L}}U_0\|_H \lesssim e^{-\omega t} \|U_0\|_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,$$

for some positive constant  $\omega$  if and only if

$$\rho(\mathcal{L}) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R}, \quad \text{and} \quad (4.1)$$

$$\sup_{\beta \in \mathbb{R}} \|(i\beta - \mathcal{L})^{-1}\| < \infty, \quad (4.2)$$

where  $\rho(\mathcal{L})$  denotes the resolvent set of the operator  $\mathcal{L}$ .

On the contrary the polynomial decay of the energy is based on the following result stated in Theorem 2.4 of [7] (see also [4, 5, 17] for weaker variants).

**Lemma 4.2.** A  $C_0$ -semigroup  $e^{t\mathcal{L}}$  of contractions on a Hilbert space satisfies

$$\|e^{t\mathcal{L}}U_0\| \lesssim t^{-\frac{1}{l}} \|U_0\|_{\mathcal{D}(\mathcal{L})}, \quad \forall U_0 \in D(\mathcal{L}), \quad \forall t > 1,$$

for some positive real number  $l$  if (4.1) holds as well as

$$\limsup_{|\xi| \rightarrow \infty} \xi^{-l} \|(i\xi - \mathcal{L})^{-1}\| < \infty. \quad (4.3)$$

We then first investigate the property (4.1). We start with the study of the kernel of  $\mathcal{A}$ . First we introduce the space

$$H^1(\Omega, \beta_2) = \begin{cases} \{v \in H_{0,\Gamma_S}^1(\Omega) : v \text{ is constant on } \Gamma_A\} & \text{if } \beta_2 = 0, \\ H_0^1(\Omega) & \text{if } \beta_2 > 0. \end{cases}$$

**Lemma 4.3.** By setting  $\delta = \frac{\beta^2}{\varepsilon_0\omega_p^2}$ , one has

$$\ker \mathcal{A} = \{(-\delta \nabla Q, \nabla \varphi, 0, Q)^\top \mid \varphi \in H_{0,\Gamma_A}^1(\Omega) \text{ and } Q \in H^1(\Omega, \beta_2)\}.$$

*Proof.*  $U = (E, H, J, Q)^\top \in D(\mathcal{A})$  belongs to  $\ker \mathcal{A}$  if and only if

$$\operatorname{curl} H - J = 0, \quad (4.4)$$

$$\operatorname{curl} E = 0, \quad (4.5)$$

$$\beta^2 \nabla Q + \varepsilon_0\omega_p^2 E - \gamma J = 0, \quad (4.6)$$

$$\operatorname{div} J = 0. \quad (4.7)$$

Recalling (3.20), we get

$$0 = \Re(\mathcal{A}U, U)_{\mathcal{H}} = -\frac{\gamma}{\varepsilon_0\omega_p^2} \int_{\Omega} |J|^2 dx - z \int_{\Gamma_A} |H \times \mathbf{n}|^2 - \Sigma_A(J, Q). \quad (4.8)$$

and since the three terms of this right-hand side are non negative, we obtain

$$J = 0 \text{ in } \Omega, \quad (4.9)$$

$$H \times \mathbf{n} = 0 \text{ on } \Gamma_A. \quad (4.10)$$

and if  $\beta_2 > 0$

$$Q = 0 \text{ on } \Gamma_A.$$

Now taking into account (4.9) in (4.6), we find that  $E = -\delta\nabla Q$ . If  $\beta_2 > 0$  as  $Q$  is already in  $H_{0,\Gamma_S}^1(\Omega)$ , we deduce that  $Q$  indeed belongs to  $H_0^1(\Omega)$ . On the contrary if  $\beta_2 = 0$ , since (4.10) and the absorbing boundary condition (3.3) imply that  $E \times \mathbf{n} = 0$  on  $\Gamma_A$ , we deduce that  $\nabla Q \times \mathbf{n} = 0$  on  $\Gamma_A$ , hence  $Q$  is constant on  $\Gamma_A$ . As before, we then deduce that  $Q$  indeed belongs to  $H^1(\Omega, 0)$ .

For  $H$ , we notice that (4.4) and (4.9) imply that  $H$  is curl free. As  $\Omega$  is simply connected there exists  $\psi \in H^1(\Omega)$  such that  $H = \nabla\psi$ . With (4.10), we deduce that  $\psi = c$  on  $\Gamma_A$  for some constant  $c$ . We then conclude by setting  $\varphi = \psi - c$ .  $\square$

**Remark 2.** Since  $\mathcal{A}$  is a closed operator,  $\ker \mathcal{A}$  is a closed subspace of  $\mathcal{H}$ . Alternatively this can be directly checked by using the following Poincaré's type inequalities

$$\|\varphi\|_\Omega \lesssim |\varphi|_{1,\Omega}, \forall \varphi \in H_{0,\Gamma_A}^1(\Omega) \text{ and } \|\psi\|_\Omega \lesssim |\psi|_{1,\Omega}, \forall \psi \in H_{0,\Gamma_S}^1(\Omega),$$

and by noticing that  $H^1(\Omega, \beta_2)$  is continuously embedded into  $H_{0,\Gamma_S}^1(\Omega)$ .

The same arguments allow to characterize  $\ker \mathcal{A}^*$  that in fact coincides with  $\ker \mathcal{A}$ , namely

**Lemma 4.4.** One has

$$\ker \mathcal{A}^* = \ker \mathcal{A}.$$

Since  $\ker \mathcal{A}$  is far to be reduced to  $\{0\}$ , we introduce its orthogonal space (in  $\mathcal{H}$ )

$$\tilde{\mathcal{H}} = (\ker \mathcal{A})^\perp,$$

that is an Hilbert space equipped with the same inner product than  $\mathcal{H}$ . Note that

$$R(\mathcal{A}) \subset \tilde{\mathcal{H}} \text{ and } R(\mathcal{A}^*) \subset \tilde{\mathcal{H}}, \quad (4.11)$$

because we always have [8, Corollary 2.18]

$$R(\mathcal{A}) \subset \overline{R(\mathcal{A})} = (\ker \mathcal{A})^\perp = \tilde{\mathcal{H}} \text{ and } R(\mathcal{A}^*) \subset \overline{R(\mathcal{A}^*)} \subset (\ker \mathcal{A}^*)^\perp = \tilde{\mathcal{H}}.$$

Owing to the first inclusion from (4.11), we can introduce the restriction  $\tilde{\mathcal{A}}$  of  $\mathcal{A}$  to  $\tilde{\mathcal{H}}$ , that is defined by

$$\tilde{\mathcal{A}}U = \mathcal{A}U, \forall U \in D(\tilde{\mathcal{A}}) = D(\mathcal{A}) \cap \tilde{\mathcal{H}}.$$

Let us characterize the adjoint of the operator  $\tilde{\mathcal{A}}$ .

**Lemma 4.5.**  $\tilde{\mathcal{A}}^*$  is the restriction of  $\mathcal{A}^*$  to  $\tilde{\mathcal{H}}$ , namely

$$\tilde{\mathcal{A}}^*U = \mathcal{A}^*U, \forall U \in D(\tilde{\mathcal{A}}^*) = D(\mathcal{A}^*) \cap \tilde{\mathcal{H}}. \quad (4.12)$$

*Proof.* Let us start with the inclusion

$$D(\tilde{\mathcal{A}}^*) \subset D(\mathcal{A}^*) \cap \tilde{\mathcal{H}}. \quad (4.13)$$

By definition  $U \in D(\tilde{\mathcal{A}}^*)$  if and only if there exists  $X \in \tilde{\mathcal{H}}$  such that

$$(U, \tilde{\mathcal{A}}U')_{\mathcal{H}} = (X, U')_{\mathcal{H}}, \forall U' \in D(\tilde{\mathcal{A}}). \quad (4.14)$$

Now fix  $U'' \in D(\mathcal{A})$ , then if we denote by  $P$  the orthogonal projection on  $\tilde{\mathcal{H}}$ , owing to Remark 2 we have

$$U'' = PU'' + (\mathbb{I} - P)U'',$$

with  $PU'' \in D(\tilde{\mathcal{A}})$  and  $(\mathbb{I} - P)U'' \in \ker \mathcal{A}$ . Therefore one has  $\mathcal{A}U'' = \mathcal{A}(PU'') = \tilde{\mathcal{A}}(PU'')$  and

$$(X, U'')_{\mathcal{H}} = (X, PU'')_{\mathcal{H}},$$

since  $X$  belongs to  $\tilde{\mathcal{H}}$ . Using (4.14) with  $U' = PU''$ , we then find

$$(U, \mathcal{A}U'')_{\mathcal{H}} = (X, U'')_{\mathcal{H}}, \forall U'' \in D(\mathcal{A}).$$

This means that  $U$  belongs to  $D(\mathcal{A}^*)$  with  $\mathcal{A}^*U = X = \tilde{\mathcal{A}}^*U$ , which yields (4.13) and (4.12).

Let us prove the converse inclusion

$$D(\mathcal{A}^*) \cap \tilde{\mathcal{H}} \subset D(\tilde{\mathcal{A}}^*). \quad (4.15)$$

Let  $U \in D(\mathcal{A}^*) \cap \tilde{\mathcal{H}}$  then  $(U, \mathcal{A}U')_{\mathcal{H}} = (\mathcal{A}^*U, U')_{\mathcal{H}}$ , for all  $U' \in D(\mathcal{A})$ . Note that  $\mathcal{A}^*U$  belongs to  $\tilde{\mathcal{H}}$  due to (4.11). As  $D(\tilde{\mathcal{A}}) \subset D(\mathcal{A})$ , we directly get

$$(U, \tilde{\mathcal{A}}U')_{\mathcal{H}} = (\mathcal{A}^*U, U')_{\mathcal{H}}, \forall U' \in D(\tilde{\mathcal{A}}).$$

As  $U$  belongs to  $\tilde{\mathcal{H}}$ , this implies that  $U$  belongs to  $D(\tilde{\mathcal{A}}^*)$  with  $\tilde{\mathcal{A}}^*U = \mathcal{A}^*U$ . This proves (4.15) and concludes the proof.  $\square$

Before going on, we want to give an explicit characterization of the space  $\tilde{\mathcal{H}}$ . Denote for simplicity

$$\begin{aligned} \mathcal{H}_0 &:= \{(F, G, R, S)^\top \in H(\operatorname{div}, \Omega) \times \hat{\mathcal{K}}(\Omega) \times L^2(\Omega)^3 \times L^2(\Omega) \mid \operatorname{div}(\varepsilon F) = -S \text{ in } \Omega\}, \\ \mathcal{H}_{00} &:= \{(F, G, R, S)^\top \in \mathcal{H}_0 \mid \langle F \cdot \mathbf{n}, 1 \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)} = 0\}. \end{aligned}$$

**Lemma 4.6.** If  $\beta_2 > 0$ , one has  $\tilde{\mathcal{H}} = \mathcal{H}_0$ , while if  $\beta_2 = 0$ , one has  $\tilde{\mathcal{H}} = \mathcal{H}_{00}$ .

*Proof.* Let  $(F, G, R, S)^\top \in \tilde{\mathcal{H}}$  or equivalently satisfying

$$\int_{\Omega} (-\varepsilon \delta F \cdot \nabla \bar{Q} + \mu G \cdot \nabla \bar{\varphi} + \delta S \bar{Q}) \, dx = 0, \forall \varphi \in H_{0, \Gamma_A}^1(\Omega), Q \in H^1(\Omega, \beta_2). \quad (4.16)$$

Taking first  $Q = 0$  and  $\varphi \in \mathcal{D}(\Omega)^3$ , we find  $\operatorname{div} G = 0$  in  $\Omega$ , and hence  $G$  belongs to  $H(\operatorname{div}, \Omega)$ . Taking again  $Q = 0$  and  $\varphi \in H_{0, \Gamma_A}^1(\Omega)$  and using Green's formula (2.2), we deduce as before that  $G \cdot \mathbf{n} = 0$  on  $\Gamma_S$ , which means that  $G \in \hat{\mathcal{K}}(\Omega)$ .

Conversely by taking  $\varphi = 0$  and  $Q \in \mathcal{D}(\Omega)^3$ , we find  $\operatorname{div}(\varepsilon F) + S = 0$  in  $\Omega$ . If  $\beta_2 > 0$ , we get no more conditions since  $\mathcal{D}(\Omega)$  is dense in  $H_0^1(\Omega)$ , on the contrary if  $\beta_2 = 0$  by taking  $\varphi = 0$ ,  $Q \in H^1(\Omega, 0)$  with  $Q = 1$  on  $\Gamma_A$  and again using Green's formula (2.2), we deduce that

$$\langle F \cdot \mathbf{n}, 1 \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)} = 0.$$

This proves the inclusion  $\tilde{\mathcal{H}} \subset \mathcal{H}_0$  if  $\beta_2 > 0$  and  $\tilde{\mathcal{H}} \subset \mathcal{H}_{00}$  else.

As the converse inclusions directly follow from (2.2) the proof is complete.  $\square$

**Remark 3.** By Theorem 3.1, the restriction  $\tilde{\mathcal{A}}$  of  $\mathcal{A}$  to  $\tilde{\mathcal{H}}$  generates a  $C_0$ -semigroup of contractions  $(\tilde{T}(t))_{t \geq 0}$  on  $\tilde{\mathcal{H}}$  (given by  $\tilde{T}(t)U_0 = T(t)U_0$ , for all  $U_0 \in \tilde{\mathcal{H}}$ ). Hence for  $U_0 \in \tilde{\mathcal{H}}$ , the problem (3.13) has a weak solution  $U \in C([0, \infty), \tilde{\mathcal{H}})$ , in particular  $U(t)$  belongs to  $\tilde{\mathcal{H}}$  for all  $t > 0$ . This fact is in accordance with the hidden properties (3.7) (and (3.9) if  $\beta_2 = 0$ ) mentioned before.

For further purposes, we need another hidden regularity for elements of  $D(\tilde{\mathcal{A}})$ .

**Lemma 4.7.** Fix one cut-off function  $\chi \in \mathcal{D}(\mathbb{R}^3)$  such that  $\chi = 0$  in a neighborhood of  $\Gamma_S$  and  $\chi = 1$  on  $\Gamma_A$ . Then for any  $(F, G, R, S)^\top \in D(\tilde{\mathcal{A}})$ ,  $\chi F$  and  $\chi G$  belongs to  $H^1(\Omega)^3$  with the estimate

$$\|\chi F\|_{1, \Omega} + \|\chi G\|_{1, \Omega} \lesssim \|F\|_{H(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)} + \|G\|_{H(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)}. \quad (4.17)$$

*Proof.* The proof is similar to the one of Lemma 3.1, except that here by Lemma 4.6 for  $(F, G, R, S)^\top \in D(\tilde{\mathcal{A}})$ ,  $F$  and  $G$  are in  $H(\operatorname{div}, \Omega)$ . Hence the pair  $(\chi F, \chi G)$  belongs to  $\mathbf{V}(D)$  defined by (3.14). As we have seen in the proof of Lemma 3.1 that  $\mathbf{V}(D)$  is continuously embedded into  $H^1(D)^6$ , the conclusion follows.  $\square$

We are ready to start the proof of the property (4.1). First we show that  $i\xi\mathbb{I} - \tilde{\mathcal{A}}$  (and its adjoint) is injective, for any  $\xi \in \mathbb{R}$ . By the definition of  $\tilde{\mathcal{A}}$  and Lemma 4.5 for  $\tilde{\mathcal{A}}^*$ , the case  $\xi = 0$  requires no investigation, so we distinguish these two cases.

**Lemma 4.8.**  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{A}}^*$  are injective, in other words

$$\ker \tilde{\mathcal{A}} = (\ker \mathcal{A}) \cap \tilde{\mathcal{H}} = \{0\}, \text{ and } \ker \tilde{\mathcal{A}}^* = (\ker \mathcal{A}^*) \cap \tilde{\mathcal{H}} = \{0\}.$$

Let us then concentrate on the injectivity of  $i\xi\mathbb{I} - \tilde{\mathcal{A}}$ , and  $i\xi\mathbb{I} - \tilde{\mathcal{A}}^*$ , for  $\xi \in \mathbb{R} \setminus \{0\}$ .

**Lemma 4.9.** For all  $\xi \in \mathbb{R} \setminus \{0\}$ , we have

$$\ker(i\xi\mathbb{I} - \tilde{\mathcal{A}}) = \ker(i\xi\mathbb{I} - \tilde{\mathcal{A}}^*) = \{0\}.$$

*Proof.* We have that  $U = (E, H, J, Q)^\top \in D(\tilde{\mathcal{A}})$  belongs to  $\ker(i\xi\mathbb{I} - \tilde{\mathcal{A}})$  if and only if

$$-i\xi E + \varepsilon^{-1} \operatorname{curl} H - \varepsilon^{-1} J = 0, \quad (4.18)$$

$$-i\xi H - \mu^{-1} \operatorname{curl} E = 0, \quad (4.19)$$

$$-i\xi J + \beta^2 \nabla Q + \varepsilon_0 \omega_p^2 E - \gamma J = 0, \quad (4.20)$$

$$-i\xi Q + \operatorname{div} J = 0. \quad (4.21)$$

As in the proof of Lemma 4.3 the dissipativeness of  $\mathcal{A}$  yields (4.8) and then  $J = 0$  (see (4.9)). Using this property in (4.21) yields  $Q = 0$  (since  $\xi \neq 0$ ). Then by (4.20) we directly get  $E = 0$  and finally (4.19) reduces to  $H = 0$ . This proves that  $\ker(i\xi\mathbb{I} - \tilde{\mathcal{A}}) = \{0\}$ . The same and simple proof yields the second assertion.  $\square$

Let us go on with the surjectivity of  $i\xi\mathbb{I} - \tilde{\mathcal{A}}$ .

**Lemma 4.10.** For  $\xi \in \mathbb{R}$ ,  $i\xi\mathbb{I} - \tilde{\mathcal{A}}$  has a closed range and

$$R(i\xi\mathbb{I} - \tilde{\mathcal{A}}) = \tilde{\mathcal{H}}. \quad (4.22)$$

*Proof.* For the first assertion, we prove that <sup>3</sup>

$$\|U\|_{\mathcal{H}} \lesssim \|(i\xi\mathbb{I} - \tilde{\mathcal{A}})U\|_{\mathcal{H}}, \forall U \in D(\tilde{\mathcal{A}}), \quad (4.23)$$

which directly implies that the range of  $\tilde{\mathcal{A}}$  is closed because the operator  $\mathcal{A}$  is closed. We now prove (4.23) by contradiction. Assume that (4.23) is false, this means that there exists a sequence  $(U_n)_{n \in \mathbb{N}}$  of elements of  $D(\tilde{\mathcal{A}})$  such that

$$\|U_n\|_{\mathcal{H}} = 1, \forall n \in \mathbb{N}, \text{ and} \quad (4.24)$$

$$\|(i\xi\mathbb{I} - \tilde{\mathcal{A}})U_n\|_{\mathcal{H}} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (4.25)$$

If we set  $U_n = (E_n, H_n, J_n, Q_n)^\top$ , we remark that (4.24) implies that the sequences  $(E_n)_n$ ,  $(H_n)_n$ ,  $(J_n)_n$  are bounded in  $L^2(\Omega)^3$  and that the sequence  $(Q_n)_n$  is bounded in  $L^2(\Omega)$ . By these properties and (4.25), we deduce that the sequences  $(E_n)_n$ ,  $(H_n)_n$  are bounded in  $H(\operatorname{curl}, \Omega)$ , while the sequence  $(Q_n)_n$  is bounded in  $H^1(\Omega)$ .

Furthermore as  $U_n$  belongs to  $\mathcal{H}_0$ , we have

$$\operatorname{div}(\varepsilon E_n) = -J_n \text{ and } \operatorname{div} H_n = 0 \text{ in } \Omega. \quad (4.26)$$

<sup>3</sup>within this proof the constants hidden in  $\lesssim$  may depend on  $\xi$ .

Therefore the sequences  $(E_n)_n, (H_n)_n$  are bounded in  $H(\operatorname{div}, \Omega)$ . By Lemma 4.7 (with  $\chi$  defined there), we deduce that (see (4.17))

$$\|\chi E_n\|_{1,\Omega} + \|\chi H_n\|_{1,\Omega} \lesssim 1, \quad (4.27)$$

By the compact embedding of  $H^1(\Omega)$  into  $L^2(\Omega)$ , up to a subsequence still denoted by  $(Q_n), (E_n)$  and  $(H_n)$ , there exists  $Q \in L^2(\Omega), X_E, X_H \in L^2(\Omega)^3$  such that

$$Q_n \rightarrow Q \text{ in } L^2(\Omega) \text{ as } n \rightarrow \infty, \quad (4.28)$$

$$\chi E_n \rightarrow X_E \text{ in } L^2(\Omega)^3 \text{ as } n \rightarrow \infty, \quad (4.29)$$

$$\chi H_n \rightarrow X_H \text{ in } L^2(\Omega)^3 \text{ as } n \rightarrow \infty. \quad (4.30)$$

For the sequence  $((1 - \chi)E_n)_n$  (resp.  $((1 - \chi)H_n)_n$ ), we may notice that it is bounded in  $X_N(\Omega)$  (resp.  $X_T(\Omega)$ ) and since these spaces are compactly embedded into  $L^2(\Omega)^3$  [27], up to a subsequence still denoted by  $(E_n)$  and  $(H_n)$ , there exists  $Y_E, Y_H \in L^2(\Omega)^3$  such that

$$(1 - \chi)E_n \rightarrow Y_E \text{ in } L^2(\Omega)^3 \text{ as } n \rightarrow \infty \text{ and } (1 - \chi)H_n \rightarrow Y_H \text{ in } L^2(\Omega)^3 \text{ as } n \rightarrow \infty.$$

Altogether setting  $E = X_E + Y_E$  and  $H = X_H + Y_H$  we have shown that

$$E_n \rightarrow E \text{ in } L^2(\Omega)^3 \text{ as } n \rightarrow \infty, \quad (4.31)$$

$$H_n \rightarrow H \text{ in } L^2(\Omega)^3 \text{ as } n \rightarrow \infty. \quad (4.32)$$

Finally for the sequence  $(J_n)_n$ , the properties (3.20), (4.24) and (4.25) directly imply that

$$J_n \rightarrow 0 \text{ in } L^2(\Omega)^3 \text{ as } n \rightarrow \infty. \quad (4.33)$$

At this stage owing to (4.28) and (4.31)-(4.33), we have shown

$$U_n \rightarrow (E, H, 0, Q)^\top \text{ in } \tilde{\mathcal{H}} \text{ as } n \rightarrow \infty. \quad (4.34)$$

This convergence property combined with the definition of  $\mathcal{A}U_n$ , (4.25) and (4.26) allows to conclude that

$$\begin{aligned} E_n &\rightarrow E \text{ in } H_{0,\Gamma_S}(\operatorname{curl}, \Omega) \text{ as } n \rightarrow \infty, \\ H_n &\rightarrow H \text{ in } H(\operatorname{curl}, \Omega) \text{ as } n \rightarrow \infty, \\ J_n &\rightarrow 0 \text{ in } H(\operatorname{div}, \Omega) \text{ as } n \rightarrow \infty, \\ Q_n &\rightarrow Q \text{ in } H_{0,\Gamma_S}^1(\Omega) \text{ as } n \rightarrow \infty. \end{aligned}$$

This implies that  $(E, H, 0, Q)$  belongs to  $D(\tilde{\mathcal{A}})$  and again owing to (4.25) that

$$(i\xi\mathbb{I} - \tilde{\mathcal{A}})(E, H, 0, Q)^\top = (0, 0, 0, 0)^\top.$$

By Lemmas 4.8 and 4.9, we obtain  $(E, H, 0, Q)^\top = (0, 0, 0, 0)^\top$ , which is a contradiction because (4.34) and (4.24) yield  $\|(E, H, 0, Q)^\top\|_{\mathcal{H}} = 1$ . In conclusion (4.23) is valid.

For the second assertion, the first assertion is equivalent to (see [8, Theorem 2.19] for instance)

$$(\ker(i\xi\mathbb{I} - \tilde{\mathcal{A}})^*)^\perp = \tilde{\mathcal{H}},$$

where here the orthogonal is taken in  $\tilde{\mathcal{H}}$ . As  $(i\xi\mathbb{I} - \tilde{\mathcal{A}})^* = -i\xi\mathbb{I} - \tilde{\mathcal{A}}^*$ , with the help of Lemmas 4.8 and 4.9, we conclude that (4.22) holds.  $\square$

As a direct consequence of the previous Lemmas we get the

**Corollary 4.11.** The resolvent set  $\rho(\tilde{\mathcal{A}})$  of  $\tilde{\mathcal{A}}$  contains the imaginary axis, namely (4.1) holds.

We go on with the asymptotic behavior of the resolvent of  $\tilde{\mathcal{A}}$  on the imaginary axis. Without further assumption, we will show a polynomial stability. An exponential stability holds if the Maxwell system with the electric boundary condition on  $\Gamma_S$  and the Silver-Müller condition on  $\Gamma_A$  is exponentially stable but we postpone this result to the next section.

**Lemma 4.12.** If  $\beta_2 > 0$ , then the resolvent of the operator  $\tilde{\mathcal{A}}$  satisfies (4.3) with  $l = 2$ , i.e.,

$$\limsup_{|\xi| \rightarrow \infty} \xi^{-2} \|(i\xi\mathbb{I} - \tilde{\mathcal{A}})^{-1}\| < \infty. \quad (4.35)$$

On the contrary if  $\beta_2 = 0$ , then the resolvent of the operator  $\tilde{\mathcal{A}}$  satisfies (4.3) with  $l = 6$ , i.e.,

$$\limsup_{|\xi| \rightarrow \infty} \xi^{-6} \|(i\xi\mathbb{I} - \tilde{\mathcal{A}})^{-1}\| < \infty. \quad (4.36)$$

*Proof.* We use a contradiction argument, i.e., we suppose that (4.3) is false with some  $l \geq 2$ . Then there exist a sequence of real numbers  $\xi_n \rightarrow +\infty$  and a sequence of vectors  $Z_n = (E_n, H_n, J_n, Q_n)^\top$  in  $D(\tilde{\mathcal{A}})$  with

$$\|Z_n\|_{\mathcal{H}} = 1, \forall n \in \mathbb{N}, \quad (4.37)$$

such that

$$\xi_n^l \|(i\xi_n - \mathcal{A})Z_n\|_{\mathcal{H}} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (4.38)$$

By (3.11), this is equivalent to

$$\xi_n^l \|i\varepsilon\xi_n E_n - \text{curl} H_n + J_n\|_{\Omega} \rightarrow 0, \text{ as } n \rightarrow +\infty, \quad (4.39)$$

$$\xi_n^l \|i\mu\xi_n H_n + \text{curl} E_n\|_{\Omega} \rightarrow 0, \text{ as } n \rightarrow +\infty, \quad (4.40)$$

$$\xi_n^l \|i\xi_n J_n - \beta^2 \nabla Q_n - \varepsilon_0 \omega_p^2 E_n + \gamma J_n\|_{\Omega} \rightarrow 0, \text{ as } n \rightarrow +\infty, \quad (4.41)$$

$$\xi_n^l \|i\xi_n Q_n - \text{div} J_n\|_{\Omega} \rightarrow 0, \text{ as } n \rightarrow +\infty. \quad (4.42)$$

We now notice that

$$\Re((i\xi_n - \mathcal{A})Z_n, Z_n)_{\mathcal{H}} \leq \|(i\xi_n - \mathcal{A})Z_n\|_{\mathcal{H}} \|Z_n\|_{\mathcal{H}} = \|(i\xi_n - \mathcal{A})Z_n\|_{\mathcal{H}}, \quad (4.43)$$

and the dissipativeness of  $\mathcal{A}$  yields

$$\Re((i\xi_n - \mathcal{A})Z_n, Z_n)_{\mathcal{H}} = -\Re(\mathcal{A}Z_n, \bar{Z}_n)_{\mathcal{H}} = \frac{\gamma}{\varepsilon_0 \omega_p^2} \int_{\Omega} |J_n|^2 dx + z \int_{\Gamma_A} |H_n \times \mathbf{n}|^2 + \Sigma_A(J_n, Q_n).$$

We then obtain

$$\frac{\gamma}{\varepsilon_0 \omega_p^2} \int_{\Omega} |J_n|^2 dx + z \int_{\Gamma_A} |H_n \times \mathbf{n}|^2 + \Sigma_A(J_n, Q_n) \leq \|(i\xi_n - \mathcal{A})Z_n\|_{\mathcal{H}}.$$

>From (4.38) we deduce that

$$\xi_n^l \int_{\Omega} |J_n|^2 dx \rightarrow 0, \text{ and } \xi_n^l \int_{\Gamma_A} |H_n \times \mathbf{n}|^2 \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Additionally if  $\beta_2 > 0$ , then as  $\Sigma_A(J_n, Q_n) = \frac{\tau\beta^2}{\varepsilon_0 \omega_p^2} \int_{\Gamma_A} |J_n \cdot \mathbf{n}|^2$ , by (4.38), we get

$$\xi_n^l \int_{\Gamma_A} |J_n \cdot \mathbf{n}|^2 \rightarrow 0, \text{ as } n \rightarrow +\infty. \quad (4.44)$$

Using (3.5), we further deduce that

$$\xi_n^l \int_{\Gamma_A} |Q_n|^2 \rightarrow 0, \text{ as } n \rightarrow +\infty.$$



This means that

$$\xi_n^{\frac{1}{2}} J_n \rightarrow 0, \text{ in } L^2(\Omega)^3, \text{ as } n \rightarrow +\infty, \quad (4.45)$$

$$\xi_n^{\frac{1}{2}} (H_n \times \mathbf{n}) \rightarrow 0, \text{ in } L^2(\Gamma_A)^3, \text{ as } n \rightarrow +\infty, \quad (4.46)$$

and if  $\beta_2 > 0$ , that

$$\xi_n^{\frac{1}{2}} Q_n \rightarrow 0, \text{ in } L^2(\Gamma_A), \text{ as } n \rightarrow +\infty. \quad (4.47)$$

The property (4.45) and (4.41) (recalling that  $l \geq 2$ ) imply that

$$\|\beta^2 \nabla Q_n + \varepsilon_0 \omega_p^2 E_n\| \rightarrow 0, \text{ as } n \rightarrow +\infty. \quad (4.48)$$

One has that  $(E_n)$  is bounded in  $(L^2(\Omega))^3$ , so (4.48) implies that

$$\|\nabla Q_n\|_{L^2(\Omega)^3} \lesssim 1. \quad (4.49)$$

Hence one has

$$\left| \int_{\Omega} (\beta^2 \nabla Q_n + \varepsilon_0 \omega_p^2 E_n) \cdot \nabla \bar{Q}_n \right| \lesssim \|\beta^2 \nabla Q_n + \varepsilon_0 \omega_p^2 E_n\|, \quad (4.50)$$

which implies that

$$\int_{\Omega} (\beta^2 \nabla Q_n + \varepsilon_0 \omega_p^2 E_n) \cdot \nabla \bar{Q}_n \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

For the second term of this left-hand side, we apply Green's formula (2.2) to get (recalling Lemma 4.7)

$$\beta^2 \|\nabla Q_n\|^2 + \varepsilon^{-1} \varepsilon_0 \omega_p^2 \|Q_n\|^2 + \varepsilon_0 \omega_p^2 \int_{\Gamma_A} Q_n (E_n \cdot \mathbf{n}) \rightarrow 0, \text{ as } n \rightarrow +\infty, \quad (4.51)$$

where we used that  $\varepsilon \operatorname{div} E_n = -Q_n$ . Now we want to show that the boundary term in this left-hand side tends to zero as  $n$  goes to infinity. For that purpose, we first use Cauchy-Schwarz's inequality and a trace formula to get

$$\left| \int_{\Gamma_A} Q_n (E_n \cdot \mathbf{n}) \right| \lesssim \|\chi E_n\|_{1,\Omega} \|Q_n\|_{\Gamma_A},$$

where the function  $\chi$  is the cut-off function from Lemma 4.7. By the estimate (4.17) from this lemma, we therefore have

$$\left| \int_{\Gamma_A} Q_n (E_n \cdot \mathbf{n}) \right| \lesssim (\|E_n\|_{H(\operatorname{curl},\Omega) \cap H(\operatorname{div},\Omega)} + \|H_n\|_{H(\operatorname{curl},\Omega) \cap H(\operatorname{div},\Omega)}) \|Q_n\|_{\Gamma_A}. \quad (4.52)$$

Now we show that

$$\|E_n\|_{H(\operatorname{curl},\Omega) \cap H(\operatorname{div},\Omega)} + \|H_n\|_{H(\operatorname{curl},\Omega) \cap H(\operatorname{div},\Omega)} \lesssim \xi_n. \quad (4.53)$$

Indeed by  $\|Z_n\|_{\mathcal{H}} = 1$ , and (4.39)-(4.40), we have

$$\|E_n\|_{H(\operatorname{curl},\Omega)} + \|H_n\|_{H(\operatorname{curl},\Omega)} \lesssim \xi_n,$$

while by the conditions  $\varepsilon \operatorname{div} E_n + Q_n = \operatorname{div} H_n = 0$  and  $\|Z_n\|_{\mathcal{H}} = 1$ , we surely have

$$\|E_n\|_{H(\operatorname{div},\Omega)} + \|H_n\|_{H(\operatorname{div},\Omega)} \lesssim 1.$$

These two estimates then yield (4.53). Using this estimate in (4.52), we get

$$\left| \int_{\Gamma_A} Q_n (E_n \cdot \mathbf{n}) \right| \lesssim \xi_n \|Q_n\|_{\Gamma_A}. \quad (4.54)$$

In the case  $\beta_2 > 0$ , by (4.47) we conclude that, with the choice  $l = 2$ ,

$$\int_{\Gamma_A} Q_n(E_n \cdot \mathbf{n}) \rightarrow 0, \text{ as } n \rightarrow +\infty. \quad (4.55)$$

For  $\beta_2 = 0$ , we have to use another method. Namely by (4.42) and since  $\|Q_n\| \lesssim 1$  we have

$$\xi_n^{\frac{l}{2}} \int_{\Omega} (i\xi_n Q_n - \operatorname{div} J_n) \bar{Q}_n \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Integrating by parts in the second term and recalling that  $J_n \cdot \mathbf{n} = 0$  on  $\Gamma_A$  as  $\beta_2 = 0$ , we find

$$\xi_n^{\frac{l}{2}} \int_{\Omega} (i\xi_n |Q_n|^2 + J_n \cdot \nabla \bar{Q}_n) \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

By (4.49) and (4.45), we deduce that

$$\xi_n^{\frac{l}{4} + \frac{1}{2}} \|Q_n\| \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Hence with  $l$  chosen such that  $\frac{l}{4} + \frac{1}{2} = 2$ , or equivalently  $l = 6$ , we find

$$\xi_n^2 \|Q_n\| \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Finally using the second trace formula

$$\|Q_n\|_{\Gamma_A}^2 \lesssim \|Q_n\| \|Q_n\|_{1,\Omega},$$

by (4.49) and  $\|Q_n\| \lesssim 1$  we find that

$$\xi_n^2 \|Q_n\|_{\Gamma_A}^2 \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Using this property in (4.54), we again conclude that (4.55) holds but with the choice  $l = 6$ .

In both cases, using (4.55) in (4.51) we thus deduce that

$$Q_n \rightarrow 0 \text{ in } H_{0,\Gamma_S}^1(\Omega), \text{ as } n \rightarrow +\infty. \quad (4.56)$$

As a consequence, by (4.48), we arrive at

$$E_n \rightarrow 0 \text{ in } L^2(\Omega)^3, \text{ as } n \rightarrow +\infty. \quad (4.57)$$

>From (4.39) and the above results:

$$\xi_n^{-1} \operatorname{curl} H_n \rightarrow 0 \text{ in } L^2(\Omega)^3, \text{ as } n \rightarrow +\infty. \quad (4.58)$$

By Green's formula (2.8) and the Silver-Müller condition (3.3), one has

$$\int_{\Omega} \operatorname{curl} E_n \cdot \bar{H}_n \, dx = \int_{\Omega} E_n \cdot \operatorname{curl} \bar{H}_n \, dx + z \int_{\Gamma_A} |H_n \times \mathbf{n}|^2$$

and by (4.46) we get

$$\xi_n^{-1} \int_{\Omega} \operatorname{curl} E_n \cdot \bar{H}_n \, dx = o(1). \quad (4.59)$$

Now by (4.40) and the fact that  $\|H_n\| \lesssim 1$ , we have

$$\xi_n^{-1} \int_{\Omega} (i\mu\xi_n H_n + \operatorname{curl} E_n) \cdot \bar{H}_n \, dx = o(1),$$

and by (4.59) we get  $H_n \rightarrow 0$ , in  $L^2(\Omega)^3$ , as  $n \rightarrow +\infty$ .

In conclusion, we have shown that  $Z_n \rightarrow 0$ , in  $\mathcal{H}$ , as  $n \rightarrow +\infty$ , which contradicts  $\|Z_n\|_{\mathcal{H}} = 1$ .  $\square$

The previous Lemmas allow to check the hypotheses of Lemma 4.2 and then lead to the next stability results.

**Theorem 4.1.** *Problem (3.13) is polynomially stable in  $\tilde{\mathcal{H}}$ , more precisely*

$$\|T(t)U_0\|_{\mathcal{H}} \lesssim t^{-\frac{1}{l}} \|U_0\|_{\mathcal{D}(\mathcal{A})}, \forall t > 0, U_0 \in \mathcal{D}(\tilde{\mathcal{A}}),$$

with  $l = 2$  if  $\beta_2 > 0$ , and  $l = 6$  else.

**5. An exponential stability result.** As mentioned before this section is devoted to an exponentially decay result based on a similar result for the Maxwell system.

**Lemma 5.1.** Suppose that the Maxwell system with Silver-Müller boundary condition on  $\Gamma_A$  and the electric boundary condition on  $\Gamma_S$

$$\left\{ \begin{array}{l} \varepsilon \partial_t E = \text{curl } H, \partial_t H = -\text{curl } E \text{ in } \Omega \times (0, +\infty), \\ \text{div}(\varepsilon E) = \text{div } H = 0 \text{ in } \Omega \times (0, +\infty), \\ E \times \mathbf{n} = 0, H \cdot \mathbf{n} = 0, \text{ on } \Sigma_S, \\ E \times \mathbf{n} - z(H \times \mathbf{n}) \times \mathbf{n} = 0, \text{ on } \Sigma_A, \\ E(\cdot, 0) = E_0, H(\cdot, 0) = H_0, \text{ in } \Omega, \end{array} \right. \quad (5.1)$$

is exponentially stable in its energy space  $\mathcal{K}(\Omega) \times \hat{\mathcal{K}}(\Omega)$  (namely with initial data  $(E_0, H_0)$  in  $\mathcal{K}(\Omega) \times \hat{\mathcal{K}}(\Omega)$ ). Then the resolvent of the operator  $\tilde{\mathcal{A}}$  satisfies condition (4.2) in  $\tilde{\mathcal{H}}$ .

*Proof.* We again use a contradiction argument, i.e., we suppose that (4.2) is false. Then there exist a sequence of real numbers  $\xi_n \rightarrow +\infty$  and a sequence of vectors  $Z_n = (E_n, H_n, J_n, Q_n)^\top$  in  $D(\tilde{\mathcal{A}})$  satisfying (4.37) and (4.38) (or equivalently (4.39)-(4.42)) with  $l = 0$ . Using the dissipativeness of  $\mathcal{A}$ , we directly find that (4.45) and (4.46) hold with  $l = 0$ , and if  $\beta_2 > 0$ , that (4.47) holds with  $l = 0$ . Coming back to (4.39)-(4.40), and using (4.45), we find

$$i\varepsilon \xi_n E_n - \text{curl } H_n = F_n - J_n \rightarrow 0 \text{ in } L^2(\Omega)^3, \text{ as } n \rightarrow +\infty, \quad (5.2)$$

$$i\mu \xi_n H_n + \text{curl } E_n := G_n \rightarrow 0 \text{ in } L^2(\Omega)^3, \text{ as } n \rightarrow +\infty. \quad (5.3)$$

As  $Z_n \in D(\tilde{\mathcal{A}})$ , the pair  $(E_n, H_n)$  satisfies the boundary conditions from (5.1)

$$E_n \times \mathbf{n} = 0, H_n \cdot \mathbf{n} = 0 \text{ on } \Gamma_S, \quad E_n \times \mathbf{n} - z(H_n \times \mathbf{n}) \times \mathbf{n} = 0, \text{ on } \Gamma_A.$$

Further  $H_n$  is divergence free, but as  $\text{div}(\varepsilon E_n) + Q_n = 0$ ,  $E_n$  is not divergence free. Hence in order to use the exponential stability of (5.1), we then need to correct  $E_n$ . Therefore as in the proof of Lemma 5.3 of [19] we consider  $\varphi_n \in H_0^1(\Omega)$  such that

$$\int_{\Omega} \varepsilon \nabla \varphi_n \cdot \nabla \psi \, dx = \int_{\Omega} \varepsilon E_n \cdot \nabla \psi \, dx, \forall \psi \in H_0^1(\Omega), \quad (5.4)$$

and set  $\tilde{E}_n = E_n - \nabla \varphi_n$ , that belongs to  $H_{\text{mix}}(\text{curl}; \Omega)$  and satisfies  $\text{div}(\varepsilon \tilde{E}_n) = 0$  in  $\Omega$ . Now the identities (5.2) and (5.4) imply that

$$\int_{\Omega} \varepsilon |\nabla \varphi_n|^2 \, dx = \frac{1}{i\xi_n} \int_{\Omega} (F_n + \text{curl } H_n - J_n) \cdot \nabla \varphi_n \, dx = \frac{1}{i\xi_n} \int_{\Omega} (F_n - J_n) \cdot \nabla \varphi_n \, dx,$$

since by Green's formula (2.6), one has  $\int_{\Omega} \text{curl } H_n \cdot \nabla \varphi_n \, dx = 0$ , because  $\nabla \varphi_n \in H_0(\text{curl}, \Omega)$ . Hence by Cauchy-Schwarz's inequality we find that  $\|\nabla \varphi_n\| \lesssim \xi_n^{-1} \|F_n - J_n\|$ . By (5.2), we conclude that

$$\|i\xi_n \nabla \varphi_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (5.5)$$

At this stage we remark that the pair  $(\tilde{E}_n, H_n)$  satisfies the Silver-Müller boundary condition on  $\Gamma_A$ , the electric boundary condition on  $\Gamma_S$ ,  $\operatorname{div}(\varepsilon \tilde{E}_n) = \operatorname{div} H_n = 0$  in  $\Omega$  and Maxwell's equations

$$\begin{cases} i\xi_n \varepsilon \tilde{E}_n - \operatorname{curl} H_n &= \tilde{F}_n = F_n - J_n - i\xi_n \varepsilon \nabla \varphi_n, \\ i\xi_n H_n + \operatorname{curl} \tilde{E}_n &= G_n. \end{cases} \quad (5.6)$$

With the help of (5.2) and (5.5), we have

$$\tilde{F}_n, G_n \rightarrow 0 \text{ in } L^2(\Omega)^3, \text{ as } n \rightarrow \infty. \quad (5.7)$$

As by assumption, system (5.1) is exponentially stable, applying Lemma 4.1, its resolvent is uniformly bounded on the imaginary axis. In other words, the solution  $(\tilde{E}_n, H_n)$  of (5.6) satisfies

$$\|\tilde{E}_n\| + \|H_n\| \lesssim \|\tilde{F}_n\| + \|G_n\|.$$

The property (5.7) then yields  $\|\tilde{E}_n\| + \|H_n\| \rightarrow 0$ , as  $n \rightarrow \infty$ . By (5.5), we conclude that

$$\|E_n\| + \|H_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (5.8)$$

It remains to manage the  $L^2$ -norm of  $Q_n$ . As  $\|Q_n\| \lesssim 1$ , by (4.42) with  $l = 0$ , we have

$$i\xi_n \int_{\Omega} |Q_n|^2 = \int_{\Omega} \operatorname{div} J_n \bar{Q}_n + o(1).$$

Using Green's formula (2.2), we get

$$i\xi_n \int_{\Omega} |Q_n|^2 = \int_{\Omega} J_n \cdot \nabla \bar{Q}_n + \langle J_n \cdot \mathbf{n}, Q_n \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)} + o(1). \quad (5.9)$$

At this stage we again distinguish between the case  $\beta_2 = 0$  or not. In the first case, we have  $J_n \cdot \mathbf{n} = 0$  and hence

$$i\xi_n \int_{\Omega} |Q_n|^2 = \int_{\Omega} J_n \cdot \nabla \bar{Q}_n + o(1). \quad (5.10)$$

Conversely if  $\beta_2 > 0$ , then by (3.5), (5.9) reduces to

$$i\xi_n \int_{\Omega} |Q_n|^2 = \int_{\Omega} J_n \cdot \nabla \bar{Q}_n - r \|J_n \cdot \mathbf{n}\|_{\Gamma_A}^2 + o(1).$$

Due to (4.44) with  $l = 0$ , we conclude that (5.10) also holds in that case.

Now using (4.41) with  $l = 0$  in (5.10) we find

$$i\xi_n \int_{\Omega} |Q_n|^2 = \frac{-i\xi_n + \gamma}{\beta^2} \int_{\Omega} |J_n|^2 - \delta \int_{\Omega} J_n \cdot \bar{E}_n + o(1).$$

By dividing this expression by  $i\xi_n$ , we find

$$\int_{\Omega} |Q_n|^2 = \frac{-i\xi_n + \gamma}{i\xi_n \beta^2} \int_{\Omega} |J_n|^2 - \frac{\delta}{i\xi_n} \int_{\Omega} J_n \cdot \bar{E}_n + o(1).$$

As  $\|E_n\| \lesssim 1$ , using (4.45) with  $l = 0$ , we arrive at

$$\|Q_n\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Using this property, (4.45) with  $l = 0$  and (5.8), we arrive at a contradiction with (4.37).  $\square$

**Remark 4.** Due to [23, Theorem 4.1], system (5.1) is exponentially stable if in addition to our previous assumptions,  $\Gamma_A$  satisfies the Geometric Control Condition in the sense that there exists  $T > 0$  such that every ray of geometrical optics that propagates in  $\Omega$  and is reflected on its boundary intersects  $\Gamma_A$  in time less than  $T$ .

This last Lemma combined with Corollary 4.11 allows to apply Huang-Prüss Lemma 4.1 to get

**Corollary 5.2.** If the Maxwell system (5.1) is exponentially stable in  $\mathcal{K}(\Omega) \times \hat{\mathcal{K}}(\Omega)$ , then system (3.6) is exponentially stable in  $\tilde{\mathcal{H}}$ .

It turns out that the converse implication is also valid, namely we have the following result.

**Lemma 5.3.** Suppose that system (3.6) is exponentially stable in  $\tilde{\mathcal{H}}$ , then the Maxwell system (5.1) is exponentially stable in  $\mathcal{K}(\Omega) \times \hat{\mathcal{K}}(\Omega)$ .

*Proof.* As our geometrical assumptions guarantee that the operator  $\mathcal{A}_M$  associated with (5.1) satisfies (4.1), we are reduced to show that the resolvent of  $\mathcal{A}_M$  satisfies (4.2). As before we use a contradiction argument, i.e., we suppose that (4.2) is false. Then there exist a sequence of real numbers  $\xi_n \rightarrow +\infty$  and a sequence of vectors  $(\tilde{E}_n, \tilde{H}_n)^\top$  in  $D(\mathcal{A}_M)$  satisfying

$$\|\tilde{E}_n\| + \|\tilde{H}_n\| = 1, \quad (5.11)$$

and such that

$$i\varepsilon\xi_n\tilde{E}_n - \operatorname{curl}\tilde{H}_n = \varepsilon\tilde{F}_n \rightarrow 0 \text{ in } L^2(\Omega)^3, \text{ as } n \rightarrow +\infty, \quad (5.12)$$

$$i\mu\xi_n\tilde{H}_n + \operatorname{curl}\tilde{E}_n = \tilde{G}_n \rightarrow 0 \text{ in } L^2(\Omega)^3, \text{ as } n \rightarrow +\infty. \quad (5.13)$$

Recall that  $(\tilde{F}_n, \tilde{G}_n)^\top$  belongs to  $\mathcal{H}_M := \mathcal{K}(\Omega) \times \hat{\mathcal{K}}(\Omega)$  and that

$$D(\mathcal{A}_M) = \left\{ (\tilde{E}_n, \tilde{H}_n) \in \mathcal{H}_M \cap (H_{0,\Gamma_S}(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega)) \mid E \times \mathbf{n} + zH_{\parallel} = 0, \text{ on } \Gamma_A \right\}.$$

As before the dissipativeness of  $\mathcal{A}_M$  implies that (compare with (4.46))

$$\tilde{H}_n \times \mathbf{n} \rightarrow 0, \text{ in } L^2(\Gamma_A)^3, \text{ as well as } \tilde{E}_n \times \mathbf{n} \rightarrow 0, \text{ in } L^2(\Gamma_A)^3, \text{ as } n \rightarrow +\infty. \quad (5.14)$$

Based on the sequence  $(\tilde{E}_n, \tilde{H}_n)^\top$ , we want to build another sequence of elements  $Z_n = (E_n, H_n, J_n, Q_n)^\top$  in  $D(\tilde{\mathcal{A}})$  satisfying  $\|(i\xi_n - \mathcal{A})Z_n\|_{\mathcal{H}} \rightarrow 0$  as  $n \rightarrow \infty$ .

For the case  $\beta_2 = 0$ , we consider the unique function  $\theta \in H_{0,\Gamma_S}^1(\Omega)$  such that (for its existence, see for instance [2, Proposition 3.18])

$$\begin{cases} \Delta\theta = 0 \text{ in } \Omega, \\ \theta = 1 \text{ on } \Gamma_A, \\ \langle \partial_{\mathbf{n}}\theta, 1 \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)} \neq 0. \end{cases} \quad (5.15)$$

Then we set

$$E_n = \tilde{E}_n - \alpha_n \nabla\theta, \quad (5.16)$$

where  $\alpha_n = 0$  if  $\beta_2 > 0$ , otherwise  $\alpha_n = (\langle \partial_{\mathbf{n}}\theta, 1 \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)})^{-1} \langle \tilde{E}_n \cdot \mathbf{n}, 1 \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)}$ , so that  $\langle E_n \cdot \mathbf{n}, 1 \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)} = 0$ .

Before going on, we may notice that  $E_n$  is divergence free, since  $\tilde{E}_n$  and  $\nabla\theta$  are, i.e.

$$\operatorname{div}(\varepsilon E_n) = 0. \quad (5.17)$$

Therefore the natural choice for  $Q_n$  is  $Q_n = 0$ . Note further that

$$\xi_n \alpha_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (5.18)$$

Indeed as  $\tilde{E}_n$  is divergence free, we have  $\int_{\Omega} \operatorname{div} \tilde{E}_n \theta = 0$ , and using Green's formula (2.2), we get

$$- \int_{\Omega} \tilde{E}_n \cdot \nabla\theta + \langle \tilde{E}_n \cdot \mathbf{n}, \theta \rangle_{H^{-\frac{1}{2}}(\Gamma) - H^{\frac{1}{2}}(\Gamma)} = 0.$$

Hence (5.15) yields

$$\langle \tilde{E}_n \cdot \mathbf{n}, 1 \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)} = \int_{\Omega} \tilde{E}_n \cdot \nabla \theta.$$

Now (5.12) implies that

$$\langle \tilde{E}_n \cdot \mathbf{n}, 1 \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)} = \frac{1}{i\varepsilon\xi_n} \int_{\Omega} (\operatorname{curl} \tilde{H}_n + \varepsilon \tilde{F}_n) \cdot \nabla \theta.$$

Since  $\nabla \theta$  belongs to  $H_0(\operatorname{curl}, \Omega)$ , Green's formula (2.6) yields  $\int_{\Omega} \operatorname{curl} \tilde{H}_n \cdot \nabla \theta = 0$ , so we obtain

$$\langle \tilde{E}_n \cdot \mathbf{n}, 1 \rangle_{H^{-\frac{1}{2}}(\Gamma_A) - H^{\frac{1}{2}}(\Gamma_A)} = \frac{1}{i\xi_n} \int_{\Omega} \tilde{F}_n \cdot \nabla \theta.$$

By Cauchy-Schwarz's inequality and (5.12) we arrive at (5.18).

Note that (5.11) and (5.18) imply that

$$\|E_n\| \lesssim 1. \quad (5.19)$$

We further introduce the unique solution  $\eta_n \in H_{0,\Gamma_S}^1(\Omega)$  of

$$\int_{\Omega} \nabla \eta_n \cdot \nabla \bar{v} = -\varepsilon_0 \omega_p^2 \int_{\Omega} E_n \cdot \nabla \bar{v}, \forall v \in H_{0,\Gamma_S}^1(\Omega). \quad (5.20)$$

Hence  $R_n = \nabla \eta_n$  satisfies

$$\begin{cases} \operatorname{div} R_n = -\varepsilon_0 \omega_p^2 \operatorname{div} E_n = 0 \text{ in } \Omega, \\ R_n \times \mathbf{n} = 0 \text{ on } \Gamma_S, \\ R_n \cdot \mathbf{n} = -\varepsilon_0 \omega_p^2 E_n \cdot \mathbf{n} \text{ on } \Gamma_A, \end{cases} \quad (5.21)$$

due to (5.17). By taking  $v = \eta_n$  in (5.20), we have

$$\int_{\Omega} |R_n|^2 = -\varepsilon_0 \omega_p^2 \int_{\Omega} E_n \cdot \bar{R}_n, \quad (5.22)$$

and using Cauchy-Schwarz's inequality and (5.19) we find

$$\|R_n\| \lesssim 1. \quad (5.23)$$

As  $\tilde{E}_n \times \mathbf{n}$  belongs to  $L^2(\Gamma_A)^3$ , Theorem 2 of [11] allows to conclude that  $\tilde{E}_n \cdot \mathbf{n}$  also belongs to  $L^2(\Gamma_A)^3$  with the estimate (see [11, Remark, p. 367])

$$\|\tilde{E}_n \cdot \mathbf{n}\|_{\Gamma_A} \lesssim \|\tilde{E}_n\| + \|\operatorname{curl} \tilde{E}_n\| + \|\tilde{E}_n \times \mathbf{n}\|_{\Gamma_A}.$$

Hence using (5.11), (5.13), and (5.14), we arrive at

$$\|E_n \cdot \mathbf{n}\|_{\Gamma_A} \lesssim \xi_n. \quad (5.24)$$

Using the definition of  $E_n$ , we get  $\|E_n \cdot \mathbf{n}\|_{\Gamma_A} \lesssim \|\tilde{E}_n \cdot \mathbf{n}\|_{\Gamma_A} + |\alpha_n|$ . Hence using (5.18) and (5.24) we arrive at

$$\|E_n \cdot \mathbf{n}\|_{\Gamma_A} \lesssim \xi_n. \quad (5.25)$$

By (5.21), again Theorem 2 of [11] yields that  $R_n \times \mathbf{n}$  belongs to  $L^2(\Gamma_A)^3$  with

$$\|R_n \times \mathbf{n}\|_{\Gamma_A} \lesssim \|R_n\| + \|R_n \cdot \mathbf{n}\|_{\Gamma_A}.$$

>From (5.21) we then have

$$\|R_n \times \mathbf{n}\|_{\Gamma_A} \lesssim \|R_n\| + \|E_n \cdot \mathbf{n}\|_{\Gamma_A}.$$

By (5.23) and (5.25), we obtain

$$\|R_n \times \mathbf{n}\|_{\Gamma_A} \lesssim \xi_n. \quad (5.26)$$

Coming back to (5.22), using the definition (5.16) of  $E_n$ , we have

$$\int_{\Omega} |R_n|^2 = -\varepsilon_0 \omega_p^2 \int_{\Omega} \left( \tilde{E}_n - \alpha_n \nabla \theta \right) \cdot \bar{R}_n.$$

Using (5.18), and (5.23), we deduce

$$\int_{\Omega} |R_n|^2 = o(1) - \varepsilon_0 \omega_p^2 \int_{\Omega} \tilde{E}_n \cdot \bar{R}_n.$$

Taking into account (5.12) and again (5.23), we obtain

$$\int_{\Omega} |R_n|^2 = o(1) - \frac{\varepsilon_0 \omega_p^2}{i \varepsilon \xi_n} \int_{\Omega} \operatorname{curl} \tilde{H}_n \cdot \bar{R}_n.$$

As  $\tilde{H}_n \times \mathbf{n}$  belongs to  $L^2(\Gamma_A)^3$  and  $\tilde{H}_n \cdot \mathbf{n} = 0$  on  $\Gamma_S$ , Theorem 2 of [11] guarantees that  $\tilde{H}_n$  belongs to  $\mathcal{V}(\Omega)$ , hence Green's formula (2.7) yields

$$\int_{\Omega} |R_n|^2 = o(1) - \frac{\varepsilon_0 \omega_p^2}{i \varepsilon \xi_n} \int_{\Gamma_A} (\bar{R}_n \times \mathbf{n}) \cdot \tilde{H}_n \cdot \mathbf{n}.$$

By Cauchy-Schwarz's inequality, we get

$$\int_{\Omega} |R_n|^2 \lesssim o(1) + \xi_n^{-1} \|R_n \times \mathbf{n}\|_{\Gamma_A} \|\tilde{H}_n \times \mathbf{n}\|_{\Gamma_A}.$$

The estimate (5.26) and the property (5.14) allow to conclude that

$$R_n \rightarrow 0 \text{ in } L^2(\Omega)^3, \text{ as } n \rightarrow +\infty. \quad (5.27)$$

Finally we set  $J_n = (i \xi_n + \gamma)^{-1} (\varepsilon_0 \omega_p^2 E_n + R_n)$  that satisfies

$$J_n \rightarrow 0 \text{ in } L^2(\Omega)^3, \text{ as } n \rightarrow +\infty, \quad (5.28)$$

owing to (5.19) and (5.27). Further due to (5.21), we see that  $J_n \cdot \mathbf{n} = 0$  on  $\Gamma_A$ , and that  $\operatorname{div} J_n = 0$ .

In conclusion we have found elements  $(E_n, \tilde{H}_n, J_n, Q_n)^\top \in D(\tilde{\mathcal{A}})$  that satisfy

$$i \varepsilon \xi_n E_n - \operatorname{curl} \tilde{H}_n + J_n = \varepsilon F_n, \quad (5.29)$$

$$i \mu \xi_n \tilde{H}_n + \operatorname{curl} E_n = \tilde{G}_n, \quad (5.30)$$

$$i \xi_n J_n - \beta^2 \nabla Q_n - \varepsilon_0 \omega_p^2 E_n + \gamma J_n = R_n, \quad (5.31)$$

$$i \xi_n Q_n - \operatorname{div} J_n = 0, \quad (5.32)$$

with  $Q_n = 0$ ,  $F_n = \tilde{F}_n - i \xi_n \alpha_n \nabla \theta + \varepsilon^{-1} J_n$ . This equivalently means that  $(i \xi_n \mathbb{I} - \tilde{\mathcal{A}})(E_n, \tilde{H}_n, J_n, Q_n)^\top = (F_n, \tilde{G}_n, R_n, 0)^\top$  and by our assumption, we deduce that

$$\|(E_n, \tilde{H}_n, J_n, Q_n)^\top\|_{\mathcal{H}} \leq \|(F_n, \tilde{G}_n, R_n, 0)^\top\|_{\mathcal{H}}. \quad (5.33)$$

But due to (5.12), (5.18), and (5.28), one has

$$F_n \rightarrow 0 \text{ in } L^2(\Omega), \text{ as } n \rightarrow +\infty.$$

This property combined with (5.13) and (5.27) yield

$$\|(F_n, \tilde{G}_n, R_n, 0)^\top\|_{\mathcal{H}} \rightarrow 0, \text{ as } n \rightarrow +\infty,$$

and by (5.33), one gets

$$\|(E_n, \tilde{H}_n, J_n, Q_n)^\top\|_{\mathcal{H}} \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Taking into account (5.16), and (5.18), this implies that

$$\|\tilde{E}_n\| + \|\tilde{H}_n\| \rightarrow 0, \text{ as } n \rightarrow +\infty,$$

which contradicts (5.11).  $\square$

**6. Conclusion.** We have investigated a problem of light propagation in metal with a hole. The light propagation properties inside the metal is described by a linearized hydrodynamical Drude model that express the non-local reaction both in time and space of the electron cloud of the metal through its polarization. We proved that for a whole range of possible boundary conditions on the polarization unknowns, the problem is well posed, polynomially stable and exponentially stable if the corresponding Maxwell system itself is. Furthermore, we obtain a smaller polynomial decay when a no flux type boundary condition is considered for the polarization current. The question of numerical stability has not been addressed here and will be part of a future work. Finally, this paper opens the route toward the study of more relevant physical settings such as a complete scattering problem by a metallic structure, that will also be part of our future investigations. For a complete scattering problem (that corresponds to an unbounded obstacle problem, i.e., if  $\Omega = \mathbb{R}^3 \setminus \bar{O}$ ), in analogy with [1, 13], we may expect to get a polynomial decay rate.

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