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Deviation and concentration inequalities for dynamical systems with subexponential decay of correlation.

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Abstract

We obtain large and moderate deviation estimates, as well as concentration inequalities, for a class of nonuniformly expanding maps with stretched exponential decay of correlations. In the large deviation regime, we also exhibit examples showing that the obtained upper bounds are essentially optimal.

MSC2020 subject classifications: 37A25; 37A50; 60F10.

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1 Introduction

Let X be a bounded metric space and $T: X \rightarrow X$ be a transformation, preserving a Borel probability measure ν . Suppose that $\varphi: X \rightarrow \mathbb{R}$ is a $\mathbb{L}^1(\nu)$ observable with $\int \varphi d\nu = 0$. We consider the Birkhoff sums

$$S_n(\varphi) = \sum_{k=0}^{n-1} \varphi \circ T^k \quad (1.1)$$

as a discrete time random process, defined on the probability space (X, ν) . The Birkhoff ergodic theorem asserts that if T is ergodic then $\lim_{n \rightarrow \infty} n^{-1} S_n(\varphi) = 0$ ν -a.s. Deviation inequalities aim to quantify the rate at which the Birkhoff sum goes to zero. One of the purposes of this paper is then to derive estimates of the quantity

$$\nu\left(\sqrt{\frac{a_n}{n}} |S_n(\varphi)| \geq x\right), \quad x > 0, \quad (1.2)$$

when $a_n \rightarrow 0$, in either the large deviation regime (so when $a_n = 1/n$) or in the moderate deviation regime (so when $na_n \rightarrow \infty$ with $a_n \rightarrow 0$).

In this paper, we shall consider nonuniformly expanding maps in the sense recalled in Section 2. It is well-known that the study of the deviation probability (1.2) is widely linked with the decay of correlations of some observables of the iterates of T (see for instance [17, 3]), which is actually linked with the moments of the return time to the basis of the induced map (see again Section 2 for the definition of the induced map and its associated return times). When the return time to the basis has an exponential moment (this is the case for instance of the dispersing billiards, see [6]) and φ is

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an Hölder observable, it has been proved in [22] (see their Theorems B and C) that the following large/moderate deviation principles hold: there exists a strictly convex function $I(x)$ vanishing only at $x = 0$ such that

$$\lim_{n \rightarrow \infty} a_n \log \nu \left(\sqrt{\frac{a_n}{n}} S_n(\varphi) \geq x \right) = -I(x).$$

In the large deviation regime (i.e. $a_n = 1/n$) we refer also to Theorem 2.1 in [18].

Next, when the return time to the basis has a moment of polynomial order and φ is either an Hölder observable or a bounded variation function, the deviation probability is still well understood both in the large deviation regime and in the moderate one. Indeed, it follows from [9] that if the return time R to the basis has a weak moment of order $p > 1$ (so when $m(R > n) \leq Cn^{-p}$ where m is the reference probability measure defined in Section 2), one has for any $x > 0$,

$$\limsup_{n \rightarrow \infty} n^{\alpha p - 1} \nu \left(|S_n(\varphi)| \geq xn^\alpha \right) \leq Cx^{-p},$$

for any $\alpha > 1/2$ such that $1/p \leq \alpha \leq 1$. Moreover it has been proved in [9] that this upper bound cannot be essentially improved.

However, when the return time R to the basis has subexponential (also called stretched exponential) moment of order $\gamma \in]0, 1[$ (so when $m(R > n) \leq Ce^{-\delta n^\gamma}$ for some $\delta > 0$), the situation is not so well understood. The recent paper [2] considers this case, but when the dynamical system can be modeled by a Young tower, the obtained bounds turn out to be suboptimal (see our Remark 3.2). In Section 3, we improve the estimates of the quantity (1.2) both in the large deviation regime and in the moderate one, when the return time to the basis has stretched moment of order $\gamma \in]0, 1[$ and when the observable φ is either Hölder continuous or with bounded variations. For instance, in case when T is the Viana map as introduced in [23], using that its associated return time to the basis exhibits stretched exponential moment of order at least $1/2$ (so that $\gamma = 1/2$) as proved in [13], it follows from our Corollary 3.1 that, for any Hölder observable φ there exist positive constants c_1 and c_2 such that for any $x > 0$,

$$\limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \nu \left(\frac{S_n(\varphi)}{n} \geq x \right) \leq -c_1 x^{1/2} \quad \text{and} \quad \limsup_{n \rightarrow \infty} a_n \log \nu \left(\sqrt{\frac{a_n}{n}} S_n(\varphi) \geq x \right) \leq -c_2 x^2,$$

where $a_n \rightarrow 0$ and $a_n n^{1/3} \rightarrow \infty$.

Our strategy of proof will be first to estimate the τ -mixing coefficients associated with the observable of the nonuniformly expanding map (see our Section 2 for the definition of these coefficients) and then to apply previous known results for τ -mixing sequences. Moreover, considering the family of interval maps T_γ , $\gamma \in]0, 1[$, introduced in [8, Appendix A] and which are such that the return time to the basis has stretched exponential moment (see also our Section 5) we will show that the large deviation upper bounds given in this paper are essentially optimal. Concerning the moderate deviation regime, we will also give in the Corollary 3.2 a moderate deviation principle, which implies (??) with a restriction (depending on γ) on the possible range of the sequence a_n .

We shall also be interested in proving concentration inequalities in the spirit of those obtained in [5, 14] but when the return time to the basis has stretched exponential moment of order $\gamma \in]0, 1[$ (the case of exponential moment and strong polynomial moments is handled in [5]; the case of weak polynomial moments is considered in [14]). Section 4 is devoted to the statement of these concentration inequalities in case of stretched exponential moment of the return time. All the proofs are postponed to Section 6.

2 Mixing properties of nonuniformly expanding maps

Let (X, d) be a complete bounded separable metric space with the Borel σ -algebra. Let us introduce the class of dynamical systems that we consider in this paper. Suppose that $T: X \rightarrow X$ is a measurable transformation which admits an inducing scheme consisting of:

- a closed subset Y of X with a *reference* probability measure m on Y ;
- a finite or countable partition $\Gamma = \cup_{\alpha \in E} \Gamma_\alpha$ of Y (up to a zero measure set) with $m(a) > 0$ for all $a \in \Gamma$;
- an integrable *return time* function $R: Y \rightarrow \{1, 2, \dots\}$ which is constant on each $a \in \Gamma$ with value $R(a)$ and $T^{R(a)}(y) \in Y$ for all $y \in a$, $a \in \Gamma$. We require in addition that $\gcd\{R(y), y \in Y\} = 1$.

Let $F: Y \rightarrow Y$, $F(y) = T^{R(y)}(y)$ be the induced map. We assume that there are constants $\lambda > 1$ and $K > 0$ such that for each $a \in \Gamma$ and all $x, y \in a$:

- F restricts to a (measure-theoretic) bijection from a to Y ;
- $d(F(x), F(y)) \geq \lambda d(x, y)$;
- $d(T^k(x), T^k(y)) \leq K d(F(x), F(y))$ for all $0 \leq k \leq R(a)$;
- the inverse Jacobian $\zeta_a = \frac{dm}{dm \circ F}$ of the restriction $F: a \rightarrow Y$ satisfies

$$|\log |\zeta_a(x)| - \log |\zeta_a(y)|| \leq K d(F(x), F(y)).$$

In addition to the standard assumptions above, we rely on non-pathological coding of orbits under F allowed by the elements of Γ . Let \mathcal{A} be the set of all finite words in the alphabet Γ and $Y_w = \cap_{k=0}^n F^{-k}(a_k)$ for $w = a_0 \cdots a_n \in \mathcal{A}$. We require that

$$m(Y_w) = m(\bar{Y}_w) \quad \text{for every } w \in \mathcal{A}. \quad (2.1)$$

We say that the map T as above is nonuniformly expanding. Recalling that $F: Y \rightarrow Y$ denotes the induced map, it is standard [1, Cor. p. 199], [24, Proof of Thm. 1] that there is a unique absolutely continuous F -invariant probability measure ν_Y on Y with $\frac{1}{c} \leq d\nu_Y/dm \leq c$ for some $c > 0$. Let ν be the corresponding T -invariant probability measure on X .

We shall say that the return times of T have a subexponential (or stretched exponential) moment of order $\gamma \in]0, 1]$, if $\int e^{cR^\gamma} dm < \infty$ for some $c > 0$.

The aim of this section is to provide estimates of the τ -mixing coefficients associated with $(\varphi(T^i))_{i \geq 0}$ for a measurable function $\varphi: (X, \nu) \rightarrow \mathbb{R}$. More precisely, these coefficients are defined by

$$\tau_\varphi(n) = \sup_{\ell \geq 1} \frac{1}{\ell} \sup_{1 \leq i_1 < \dots < i_\ell} \tau(\sigma(\varphi(T^j), j \geq i_\ell + n), \varphi(T^{i_1}), \dots, \varphi(T^{i_\ell})) \quad (2.2)$$

where, for any Z taking values in \mathbb{R}^k and any σ -algebra \mathcal{M} ,

$$\tau(\mathcal{M}, Z) = \left\| \sup \left\{ \left| \int f(x) \mathbb{P}_{Z|\mathcal{M}}(dy) - \int f(x) \mathbb{P}_Z(dy) \right|, f \in \Lambda_1(\mathbb{R}^k) \right\} \right\|_1.$$

Above $\Lambda_1(\mathbb{R}^k)$ is the space of 1-Lipschitz functions from \mathbb{R}^k to \mathbb{R} with respect to the ℓ^1 distance on \mathbb{R}^k . Note that compared to the τ -mixing coefficients as defined in [10, Section 7], there is a time inversion in the definition of the coefficients defined in (2.2).

For an Hölder function φ from X to \mathbb{R} , with Hölder's index η , let

$$|\varphi|_\eta = \sup_{x,y \in X} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)^\eta} \quad \text{and} \quad \|\varphi\|_\eta = \|\varphi\|_\infty + |\varphi|_\eta$$

In case where $X = [0, 1]$, and φ is a bounded variation (BV) function, let also

$$\|\varphi\|_v = \|\varphi\|_\infty + \|d\varphi\|,$$

where $\|d\varphi\|$ is the variation norm of the measure $d\varphi$.

Proposition 2.1. *Assume that T is nonuniformly expanding with return time to the basis having a subexponential moment of order $\gamma \in]0, 1[$. Then, on (X, ν) , one has:*

1. *Let φ be an Hölder observable with Hölder's index η , then there exist two positive constants c_1, c_2 such that*

$$\tau_\varphi(n) \leq c_1 \|\varphi\|_\eta e^{-c_2 n^\gamma}. \quad (2.3)$$

2. *Let $X = [0, 1]$ and φ be a BV observable from $[0, 1]$ to $[0, 1]$. Assume in addition that there exist $\eta \in]0, 1[$ and $C > 0$ such that, for any $x, y \in [0, 1]$ with $x \leq y$,*

$$\nu([x, y]) \leq C(y - x)^\eta. \quad (2.4)$$

Then there exist two positive constants c_1, c_2 such that

$$\tau_\varphi(n) \leq c_1 \|\varphi\|_v e^{-c_2 n^\gamma}. \quad (2.5)$$

Remark 2.1. *Following the lines of the proof of Proposition 2.1 and taking into account Lemma 5.7, Proposition 3.2 and inequality (3.6) in [7], we infer that if T is nonuniformly expanding with return time to the basis having a moment of order $\beta > 1$ (meaning that $\int R^\beta dm < \infty$), then, for φ an Hölder observable, one has $\sum_{k \geq 1} k^{\beta-2} \tau_\varphi(k) < \infty$.*

Remark 2.2. *Assumption (2.4) in Item 2 of the proposition above is satisfied for the class of transformations described in [15], with the additional restriction that $\lim_{x \rightarrow 0} x^\varepsilon \phi(x) = \infty$, for some $\varepsilon > 0$. In particular it will be true for the example of Section 5.*

3 Deviations inequalities and Moderate Deviation Principle for Birkhoff sums

Starting from Proposition 2.1 and using the deviations inequality stated in [20, 19], the following corollary holds.

Corollary 3.1. *Assume that T is nonuniformly expanding with return time to the basis having a subexponential moment of order $\gamma \in]0, 1[$. Let φ be an Hölder observable. Let $S_n(\varphi) = \sum_{i=0}^{n-1} \varphi(T^i) - n\nu(\varphi)$ and $V = \text{Var}_\nu(\varphi) + 2 \sum_{i \geq 1} |\text{cov}_\nu(\varphi, \varphi \circ T^i)|$. Then, on (X, ν) ,*

1. *if $\gamma \in]0, 1[$ then for any $n \geq 4$, there exist positive constants C_1, C_2, C_3 and C_4 depending on (γ, C) such that, for any positive x ,*

$$\nu\left(\sup_{j \leq n} |S_j(\varphi)| \geq x\right) \leq n \exp\left(-\frac{x^\gamma}{C_1}\right) + \exp\left(-\frac{x^2}{C_2(1+nV)}\right) + \exp\left(-\frac{x^2}{C_3 n} \exp\left(\frac{x^{\gamma(1-\gamma)}}{C_4 (\log x)^\gamma}\right)\right).$$

2. if $\gamma = 1$, then there exist positive constants C_1 , and C_2 such that, for any positive x ,

$$\nu\left(\sup_{j \leq n} |S_j(\varphi)| \geq x\right) \leq C_1 \exp\left(-\frac{x^2}{C_2(1 + nV + x(\log n)^2)}\right).$$

In addition, if $X = [0, 1]$, φ is a BV observable and ν satisfies (2.4), then the upper bounds given in Items 1 and 2 are satisfied.

Remark 3.1. Item 2 of Corollary 3.1 without the supremum is a consequence of Inequality (2.3) of [19, Theorem 2]. The fact that it also holds for the maximum $\sup_{j \leq n} |S_j(\varphi)|$ follows from [16, Example 2].

If $\gamma \in]0, 1[$, we deduce from Corollary 3.1 that the following large deviation estimate holds: for any $x > 0$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n^\gamma} \log \nu\left(\frac{S_n(\varphi)}{n} \geq x\right) \leq \frac{-x^\gamma}{C_1}. \quad (3.1)$$

It follows from Section 5 that the exponent of n in (3.1) is optimal under our assumption on the return times.

Remark 3.2. Note that an application of Theorem 2 in [2] gives the following large deviation estimate: for any $x > 0$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{\gamma/(\gamma+1)}} \log \nu\left(\frac{S_n(\varphi)}{n} \geq x\right) \leq \frac{-x^{2\gamma/(\gamma+1)}}{C_1},$$

giving then a suboptimal power of n .

We also deduce that the following moderate deviation estimate holds: if $a_n \rightarrow 0$ and $a_n n^{\gamma/(2-\gamma)} \rightarrow \infty$, then for any positive x ,

$$\limsup_{n \rightarrow \infty} a_n \log \nu\left(\sqrt{\frac{a_n}{n}} S_n(\varphi) \geq x\right) \leq \frac{-x^2}{C_2 V}. \quad (3.2)$$

In what follows, we shall give a more precise result than (3.2), and show that $(S_n(\varphi)/\sqrt{n})$ satisfies a Moderate Deviation Principle (MDP). Let us recall the definition. Let $(Z_n)_n$ be a sequence of random variables defined on (X, ν) . We say that the MDP holds for $(Z_n)_n$ with speed $a_n \rightarrow 0$ and good rate function $I(\cdot)$, if the level sets $\{x, I(x) \leq \lambda\}$ are compact for all $\lambda < \infty$, and for each Borel set A ,

$$-\inf_{t \in A^\circ} I(t) \leq \liminf_n a_n \log \nu(\sqrt{a_n} Z_n \in A) \leq \limsup_n a_n \log \nu(\sqrt{a_n} Z_n \in A) \leq -\inf_{t \in \bar{A}} I(t), \quad (3.3)$$

where \bar{A} denotes the closure of A and A° the interior of A .

Proposition 2.1 together with Corollary 1 in [20] give the following result.

Corollary 3.2. Assume that T is nonuniformly expanding with return time to the basis having a subexponential moment of order $\gamma \in]0, 1[$. Let φ be an Hölder observable. Let $S_n(\varphi) = \sum_{i=0}^{n-1} \varphi(T^i) - n\nu(\varphi)$ and $\sigma_n^2 = \text{Var} S_n(\varphi)$. Assume that $\sigma_n^2 \rightarrow \infty$. Then $\lim_{n \rightarrow \infty} \sigma_n^2/n = \sigma^2 > 0$. Moreover, for all positive sequences a_n with $a_n \rightarrow 0$ and $a_n n^{\gamma/(2-\gamma)} \rightarrow \infty$, $\{n^{-1/2} S_n(\varphi)\}$ satisfies (3.3) with the good rate function $I(t) = t^2/(2\sigma^2)$. In addition, this Moderate Deviation Principle still holds if $X = [0, 1]$, φ is a BV observable and ν satisfies (2.4).

Remark 3.3. If $\gamma = 1$ and $(a_n)_{n \geq 1}$ satisfies $a_n \rightarrow 0$ and $na_n \rightarrow \infty$, as $n \rightarrow \infty$, the moderate deviations principle holds for Hölder observables, as shown in [22, Theorem 4.6]. However, as far as we know, this result has not yet been proved when $X = [0, 1]$ and φ is a BV observable. But, applying Corollary 5 in [19] together with Proposition 2.1, one gets (assuming also (2.4)) that, for all positive sequences a_n with $a_n \rightarrow 0$ and $na_n/((\log n)^2(\log \log n)^2) \rightarrow \infty$, $\{n^{-1/2} S_n(\varphi)\}$ satisfies (3.3) with the good rate function $I(t) = t^2/(2\sigma^2)$.

4 Concentration inequalities

In this section, we follow the approach of Chazottes-Gouëzel [5] to derive concentration inequalities in case of nonuniformly expanding transformations with return time to the basis having a subexponential moment.

Recall that a function $K : X^n \rightarrow \mathbb{R}$ is said to be a separately Hölder function on X^n of order $\eta \in [0, 1]$ if for all i there exists a constant L_i with

$$|K(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) - K(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)| \leq L_i d^\eta(x_i, x'_i),$$

for all points x_1, \dots, x_n, x'_i in X .

The following deviation bound holds.

Theorem 4.1. *Assume that T is nonuniformly expanding with return time to the basis having a subexponential moment of order $\gamma \in]0, 1[$. Let K be a separately Hölder function on X^n . Then, there exists a positive constant κ such that for any positive integer n and any $t > 0$,*

$$\nu\left(x : K(x, T(x), \dots, T^{n-1}(x)) - \mathbb{E}(K) \geq t\right) \leq 2 \exp\left(-\frac{t^2}{\kappa(\sum_{i=0}^{n-1} L_i^2 + 1 + t^{2-\gamma})}\right) \quad (4.1)$$

where $\mathbb{E}(K) = \int K(x, T(x), \dots, T^{n-1}(x)) d\nu(x)$.

Remark 4.1. *To prove (4.1), the assumption (2.1) on (T, m) is not needed.*

Remark 4.2. *When $\gamma = 1$, Chazottes and Gouëzel [5] have proved the following concentration inequality: there exists a positive constant κ such that for any positive integer n and any $t > 0$,*

$$\nu\left(x : K(x, T(x), \dots, T^{n-1}(x)) - \mathbb{E}(K) \geq t\right) \leq \exp\left(-\frac{t^2}{\kappa \sum_{i=0}^{n-1} L_i^2}\right). \quad (4.2)$$

5 An example of nonuniformly expanding system with stretched exponential return times

Let us consider the following examples of interval maps, as defined in [8, Annex A], whose return times to the basis satisfy $m(R > n) \sim e^{-\kappa n^\gamma}$, where $\gamma \in]0, 1[$ is a parameter and $\kappa = \kappa(\gamma) > 0$.

Let $T : [0, 1] \rightarrow [0, 1]$,

$$T(x) = \begin{cases} x(1 + \frac{c}{|\log x|^\beta}), & x \leq 1/2 \\ 2x - 1, & x > 1/2 \end{cases} \quad (5.1)$$

with $\beta = \gamma^{-1} - 1$ and $c = (\log 2)^\beta$ so that $T(1/2) = 1$. Note that this map belongs to the class of transformations considered in Holland [15].

Let $Y =]1/2, 1[$ be a base, $R : Y \rightarrow \mathbb{N}$, $R(x) = \inf\{k \geq 1 : T^k(x) \in Y\}$ be the first return time and $F : Y \rightarrow Y$, $F(x) = T^{R(x)}(x)$ be the induced map. Let Γ denote the partition of Y into the intervals where R is constant. Let m denote the Lebesgue measure.

Items a)-d) of the next result have been established in [8, Theorem A.A.]. To prove its last part concerning the invariant density, it suffices to follow the lines of the proof of [15, Lemma 3].

Theorem 5.1. *T is a nonuniformly expanding map with basis Y , return time τ and reference measure m . That is, there exists $C > 0$ such that for every $a \in \Gamma$ and all $x, y \in a$,*

- a) $F: a \rightarrow Y$ is a nonsingular bijection;
- b) F is expanding: $|F(y) - F(x)| \geq 2|y - x|$;
- c) F has bounded distortion: $|\log F'(y) - \log F'(x)| \leq C|F(y) - F(x)|$.

Further, there exist $\eta_1, \eta_2 > 0$ such that for all $n \geq 1$,

$$d) e^{-\eta_2 n^\gamma} \leq m(R \geq n) \leq e^{-\eta_1 n^\gamma}.$$

In addition, there exists a unique T -invariant measure ν absolutely continuous with respect to the Lebesgue measure, whose density φ is such that: φ is lower bounded by a strictly positive constant on $[0, 1]$, φ is upper bounded on any interval $[\varepsilon, 1]$ with $\varepsilon > 0$, and there exist positive reals δ, c_1 and c_2 such that, $c_1 \leq |\log(x)|^{-\beta} \varphi(x) \leq c_2$ on $]0, \delta]$.

Theorem 5.2. Let T be defined by (5.1). For any measurable function $f: [0, 1] \rightarrow \mathbb{R}$, let $S_n(f) = \sum_{k=0}^{n-1} f(T^k) - n\nu(f)$.

- a) There exist a Lipschitz function f and positive constants c_1, c_2, κ_1 and κ_2 such that, for $n > 2/\nu(f)$,

$$c_1 e^{-\kappa_1 n^\gamma} \leq \nu(|S_n(f)| > n\nu(f)/2) \leq c_2 e^{-\kappa_2 n^\gamma}.$$

- b) There exist an unbounded measurable function f satisfying $A_1 e^{-B_1 t^{1/\delta}} \leq \nu(|f| > t) \leq A_2 e^{-B_2 t^{1/\delta}}$ for some $A_1, A_2, B_1, B_2 > 0$ and $\delta > 0$, and positive constants c_1, c_2, κ_1 and κ_2 such that

$$c_1 e^{-\kappa_1 n^{\gamma/(1+\gamma\delta)}} \leq \nu(|S_n(f)| > n) \leq c_2 e^{-\kappa_2 n^{\gamma/(1+\gamma\delta)}}. \quad (5.2)$$

Remark 5.1. As it will be clear from the proof, for the function of Item b), one can take $f(x) = |\log(x)|^\delta$. For this function, following the proof of Item b) and applying Theorem 1 in [20], we derive that the following deviation Inequality holds: for any $n \geq 4$, there exist positive constants C_1, C_2, C_3 and C_4 depending on (γ, C) such that, for any positive x ,

$$\mathbb{P}\left(\sup_{j \leq n} |S_j(f)| \geq x\right) \leq n \exp\left(-\frac{x^{\gamma_0}}{C_1}\right) + \exp\left(-\frac{x^2}{C_2(1+nV)}\right) + \exp\left(-\frac{x^2}{C_3 n} \exp\left(\frac{x^{\gamma_0(1-\gamma_0)}}{C_4(\log x)^{\gamma_0}}\right)\right),$$

where $\gamma_0 = \gamma(1 + \gamma\delta)^{-1}$ and $V = \text{Var}_\nu(f) + 2 \sum_{i \geq 1} |\text{cov}_\nu(f, f \circ T^i)|$. From [20], one can also get a moderate deviations principle for $S_n(f)$.

Some observables as $x \mapsto |\log(x)|^\delta$ are considered by Nicol and Török [21]. In particular, they obtained the lower bound of Item b) for the doubling map which corresponds to $\gamma = 1$ in (5.1).

6 Proofs

6.1 Proof of Proposition 2.1

From Corollary 2.5 in [7], recall that

$$(\varphi(T^n))_{n \geq 0} = (\psi(g_n, g_{n+1}, \dots))_{n \geq 0} \quad \text{in distribution,}$$

where $(g_i)_{i \geq 0}$ is a strictly stationary Markov chain. This chain is generated by a random variable g_0 and a sequence of iid innovations $(\varepsilon_i)_{i \geq 1}$ independent of g_0 . Let $\mathcal{F}_k = \sigma(g_i, i \leq k)$ and $\mathcal{G}_m = \sigma(g_i, i \geq m)$. The chain is also β mixing in the sense that

$$\sup_{k \geq 0} \beta(\mathcal{F}_k, \mathcal{G}_{k+n}) = O(e^{-Bn^\gamma}). \quad (6.1)$$

(See [4, Chapter 3] for a definition of the β -mixing coefficients, Relation (3.6) in [7] and Lemma 2.1 in [8] for the upper bound (6.1)). Let $X_n = \psi(g_n, g_{n+1}, \dots)$. We want to prove that

$$\tau(n) = \sup_{\ell \geq 1} \frac{1}{\ell} \sup_{1 \leq i_1 < \dots < i_\ell} \tau(\mathcal{G}_{i_\ell+n}, X_{i_1}, \dots, X_{i_\ell}) = O(e^{-Cn^\gamma}), \quad (6.2)$$

which will imply (2.3). To do so, we need an independent copy $(\varepsilon'_i)_{i \geq 1}$ of $(\varepsilon_i)_{i \geq 1}$, this copy being also independent of g_0 . Let now

$$X'_{i_k} = \psi(g_{i_k}, g_{i_k+1}, \dots, g_{i_k+[n/2]}, g'_{i_k+[n/2]+1}, g'_{i_k+[n/2]+2}, \dots),$$

where the ' means that we have used the innovations $(\varepsilon'_i)_{i \geq 1}$ to continue the trajectory of the chain. From [8, Proposition 2.3], we know that

$$\|X'_{i_k} - X_{i_k}\|_1 \leq K|\varphi|_\eta e^{-\delta n^\gamma},$$

for some constant δ depending on η . Hence, from the definition of $\tau(n)$, we infer that

$$\tau(n) \leq K|\varphi|_\eta e^{-\delta n^\gamma} + \sup_{\ell \geq 1} \frac{1}{\ell} \sup_{1 \leq i_1 < \dots < i_\ell} \tau(\mathcal{G}_{i_\ell+n}, X'_{i_1}, \dots, X'_{i_\ell}). \quad (6.3)$$

Now, since $\|\psi\|_\infty = \|\varphi\|_\infty$,

$$\sup_{\ell \geq 1} \frac{1}{\ell} \sup_{1 \leq i_1 < \dots < i_\ell} \tau(\mathcal{G}_{i_\ell+n}, X'_{i_1}, \dots, X'_{i_\ell}) \leq \|\varphi\|_\infty \sup_{k \geq 0} \beta(\mathcal{F}'_k, \mathcal{G}_{k+[n/2]}) \quad (6.4)$$

where $\mathcal{F}'_k = \mathcal{F}_k \vee \sigma(\varepsilon'_i, i \geq 1)$, and we have used that $\beta(\mathcal{F}'_k, \mathcal{G}_{k+[n/2]}) = \beta(\mathcal{G}_{k+[n/2]}, \mathcal{F}'_k)$. Since $(\varepsilon'_i)_{i \geq 1}$ is independent of $(g_i)_{i \geq 0}$, standard arguments (see [4, Theorem 6.2]) show that

$$\beta(\mathcal{F}'_k, \mathcal{G}_{k+[n/2]}) \leq \beta(\mathcal{F}_k, \mathcal{G}_{k+[n/2]}). \quad (6.5)$$

The estimate (6.2) follows from (6.1), (6.3), (6.4) and (6.5).

We turn now to Item 2; namely, the case of a BV function φ on $[0, 1]$, when $T : [0, 1] \rightarrow [0, 1]$ is a nonuniformly expanding map with return time to the basis satisfying $m(R > n) = O(e^{-\delta n^\gamma})$ for $\gamma \in]0, 1]$ and some $\delta > 0$. We want to control $\tau_\varphi(n)$, and we recall that

$$\varphi(x) = \varphi(0) + \int_0^x d\varphi(t).$$

Let now

$$f_\epsilon(t, x) = \mathbf{1}_{t \leq x} + \frac{1}{\epsilon}(x + \epsilon - t)\mathbf{1}_{x < t \leq x + \epsilon} \quad \text{and} \quad \varphi_\epsilon(x) = \varphi(0) + \int_0^1 f_\epsilon(t, x) d\varphi(t).$$

Since ν satisfies (2.4), note that

$$\|\varphi - \varphi_\epsilon\|_{1, \nu} \leq C_\eta \|d\varphi\| \epsilon^\eta.$$

On the other hand, using that $u \mapsto f_\epsilon(t, u)$ is $\frac{1}{\epsilon}$ -Lipschitz, we have

$$|\varphi_\epsilon(x) - \varphi_\epsilon(y)| \leq \int_0^1 |f_\epsilon(t, x) - f_\epsilon(t, y)| |d\varphi|(t) \leq \frac{\|d\varphi\|}{\epsilon} |x - y|.$$

From the definition of $\tau_\varphi(n)$ and Item 1, we infer that

$$\tau_\varphi(n) \leq C_\eta \|\varphi\|_\nu \epsilon^\eta + \frac{K \|\varphi\|_\nu (1 + \epsilon)}{\epsilon} e^{-Cn^\gamma}.$$

The upper bound (2.5) easily follows, with $c_2 = (C_\eta)/(\eta + 1)$.

6.2 Proof of Theorem 4.1

First, we recall that we can associate to the transformation T a Young tower \mathcal{T} and a transformation \bar{T} . More precisely, \mathcal{T} is the space

$$\mathcal{T} = \{(y, i) : y \in Y, i < R(y)\}$$

and the map \bar{T} on \mathcal{T} is defined by

$$\bar{T}(y, i) = \begin{cases} (y, i + 1) & \text{if } i < R(y) - 1 \\ (T^{R(y)}(y), 0) & \text{if } i = R(y) - 1. \end{cases}$$

For any $\alpha \in E$, define the height h_α by $h_\alpha = R(y)$ for $y \in \Gamma_\alpha$. One can then define the floors of the tower $\Delta_{\alpha, i}$ for $\alpha \in E$ and $i \in \{0, \dots, h_\alpha - 1\}$: $\Delta_{\alpha, i} = \{(y, i) : y \in \Gamma_\alpha\}$. These floors define a partition of \mathcal{T} :

$$\mathcal{T} = \bigcup_{\alpha \in E, i \in \{0, \dots, h_\alpha - 1\}} \Delta_{\alpha, i}.$$

On the tower, there exists a reference measure \bar{m} defined as follows: if \bar{B} is a set included in $\Delta_{\alpha, i}$, that can be written as $\bar{B} = B \times \{i\}$ with $B \subset \Gamma_\alpha$, then $\bar{m}(\bar{B}) = m(B)$. From Young [24] (see also [12], Proposition 1.3.18), it follows that on the tower, there exists a unique \bar{T} -invariant probability measure $\bar{\nu}$ which is absolutely continuous with respect to \bar{m} . On another hand, the distance on the tower is defined by $\delta(x, y) = \lambda^{-s(x, y)}$ where $s(x, y)$ is the separation time, i.e. the number of returns to the basis $\bar{Y} = \{(y, 0), y \in Y\}$ before the iterates of the points x and y are not in the same element of the partition. Let now π be the ‘‘projection’’ from \mathcal{T} to X defined by $\pi(y, i) = T^i(y)$. Then, one has $\pi \circ \bar{T} = T \circ \pi$ and for any x, y in \mathcal{T} , there exists $C > 0$, such that

$$d(\pi(x), \pi(y)) \leq C\delta(x, y).$$

Moreover it can be checked that the T -invariant measure ν defined in Section 2, is the image measure of $\bar{\nu}$ by π . Let \bar{R} be the function from \bar{Y} to $R(Y)$ such that $\bar{R}(y, 0) = R(y)$. The quantity $\bar{\nu}(\{(y, 0) \in \bar{Y} : \bar{R}((y, 0)) > k\})$ is exactly of the same order as $m(\{y \in Y : R(y) > k\})$ (see [12], Proposition 1.1.24). From all these considerations, we infer that if T is nonuniformly expanding then \bar{T} is also nonuniformly expanding with respect to the distance δ^η whatever $\eta \in]0, 1]$. On another hand, recalling that ν is the image measure of $\bar{\nu}$ by π , and that $\pi \circ \bar{T} = T \circ \pi$, we get, for any $t > 0$,

$$\begin{aligned} \nu(x : K(x, T(x), \dots, T^{n-1}(x)) - \mathbb{E}(K) \geq t) \\ = \bar{\nu}(z : K(\pi(z), T \circ \pi(z), \dots, T^{n-1} \circ \pi(z)) - \mathbb{E}(K) \geq t) \\ = \bar{\nu}(z : K(\pi(z), \pi \circ \bar{T}(z), \dots, \pi \circ \bar{T}^{n-1}(z)) - \mathbb{E}(K) \geq t). \end{aligned}$$

Next, defining \bar{K} on $\mathcal{T}^{\mathbb{N}}$ by

$$\bar{K}(z_1, \dots, z_n) = K(\pi(z_1), \dots, \pi(z_n)),$$

one sees that it satisfies

$$\left| \bar{K}(z_1, \dots, z_n) - \bar{K}(z'_1, \dots, z'_n) \right| \leq \sum_{i=1}^n L_i \delta^\eta(z_i, z'_i).$$

So \bar{K} is separately Lipschitz with respect to δ^η . Hence, it suffices to proof Theorem 4.1 with \bar{K} instead of K , \bar{T} instead of T and $\bar{\nu}$ instead of ν . Since the tower is also nonuniformly expanding with respect to δ^η , we shall only consider the case $\eta = 1$ as in [5].

From now on, we shall use the same notations as in [5, 14]. Hence, we shall now use T, K, ν instead of $\bar{T}, \bar{K}, \bar{\nu}$

It is convenient to consider K as a function defined on the space $\tilde{\mathcal{T}} = \mathcal{T}^{\mathbb{N}}$ endowed with the probability measure $\tilde{\nu} = \nu \otimes \delta_{Tx} \otimes \delta_{T^2x} \otimes \dots$. Let \mathcal{F}_k be the σ -algebra generated by indices starting with k . Let

$$K_k(x_k, \dots) = \mathbb{E}(K | \mathcal{F}_k)(x_k, \dots) = \sum_{T^k x = x_k} g^{(k)}(x) K$$

where $g^{(k)}(x) = g(x) \dots g(T^{k-1}(x))$ is the inverse of the jacobian of T^k . Next define $D_k = K_k - K_{k+1}$. Clearly D_k is \mathcal{F}_k -measurable and such that $\mathbb{E}(D_k | \mathcal{F}_{k+1}) = 0$. Hence $(D_k)_{k \geq 1}$ is a sequence of reversed martingale differences w.r.t. the decreasing filtration $(\mathcal{F}_k)_{k \geq 1}$.

As quoted after the statement of [5, Theorem 3.1], we can assume without loss of generality that $\sup_{i \geq 0} L_i \leq \varepsilon_0$ for some $\varepsilon_0 > 0$ (the appropriate ε_0 will be chosen later and will appear in the constant κ of inequality (4.1)). Noticing that $K - \mathbb{E}(K) = \sum_{k=0}^n D_k$, inequality (4.1) will follow from the reversed martingale differences sequences version of Theorem 2.1 in Fan et al [11] (see also their inequality (1.5)). Note that [11, Theorem 2.1] is stated for sums of martingale differences but with the same proof it also holds for partial sums associated with reversed martingale differences sequences. Let us state it for reader convenience. Assume that $(D_p)_{p \geq 1}$ is a sequence of reversed martingale differences with respect to the filtration $(\mathcal{F}_p)_{p \geq 1}$ (so D_p is \mathcal{F}_p -adapted and such that $\mathbb{E}(D_p | \mathcal{F}_{p+1}) = 0$ a.s.), then setting

$$u_n = \max \left(\left\| \sum_{p=1}^n \mathbb{E}(D_p^2 e^{(D_p^+)^{\gamma}} | \mathcal{F}_{p+1}) \right\|_{\infty}, 1 \right),$$

one has, for any $x > 0$,

$$\mathbb{P} \left(\max_{1 \leq k \leq n} \sum_{i=1}^k D_i \geq x \right) \leq 2 \exp \left\{ - \frac{x^2}{2(u_n + x^{2-\gamma})} \right\}.$$

Hence we need to prove that

$$\left\| \sum_{p=1}^n \mathbb{E}(D_p^2 e^{(D_p^+)^{\gamma}} | \mathcal{F}_{p+1}) \right\|_{\infty} \leq C \sum_{i=1}^n L_i^2. \quad (6.6)$$

As noticed at the beginning of the proof of [5, Lemma 3.3], if $x_{p+1} \notin \Delta_0 = \cup_{\alpha \in E} \Delta_{\alpha,0}$ (hence when x_{p+1} is not in the basis of the tower), $\mathbb{E}[D_p^2 e^{(D_p^+)^{\gamma}} | \mathcal{F}_{p+1}](x_{p+1}, \dots) = 0$. Now when $x_{p+1} \in \Delta_0$, denoting by z_{α} its preimages with respective heights h_{α} , we have

$$\mathbb{E}[D_p^2 e^{(D_p^+)^{\gamma}} | \mathcal{F}_{p+1}](x_{p+1}, \dots) = \sum_{\alpha} g(z_{\alpha}) e^{(A_p^+(z_{\alpha}))^{\gamma}} A_p^2(z_{\alpha}),$$

where

$$A_p(z) := D_p(z, x_{p+1}, \dots) = K_p(z, x_{p+1}, \dots) - K_{p+1}(x_{p+1}, \dots).$$

According to Lemma 4.2 in [14] and taking into account that $m(R > n) = O(e^{-\delta n^{\gamma}})$, for some $\delta > 0$, it follows that if z is at height h and $x_{k+1} = Tz \in \Delta_0$, then there exist positive constants C and κ such that

$$|A_p(z)| \leq C \sum_{a=0}^{p-h} L_a e^{-\kappa(p-h-a)^{\gamma}} + \sum_{a=p-h+1}^p L_a. \quad (6.7)$$

Since, it is assumed that $\sup_{i \geq 0} L_i \leq \varepsilon_0$, (6.7) implies that $|A_p(z)| \leq c_0(h+1)\varepsilon_0$. Using again the upper bound (6.7), it follows that

$$\|\mathbb{E}[D_p^2 e^{(D_p^+)^{\gamma}} | \mathcal{F}_{p+1}]\|_{\infty} \ll \sum_{h \geq 0} \nu(R = h+1) e^{(c_0 \varepsilon_0 h)^{\gamma}} \left(\sum_{a=0}^{p-h} L_a^2 e^{-\kappa(p-h-a)^{\gamma}} + (h+1) \sum_{a=p-h+1}^p L_a^2 \right).$$

Hence, for ε_0 such that $(c_0 \varepsilon_0)^{\gamma} = A/2$, using that $m(R > n) = O(e^{-\delta n^{\gamma}})$, there exists a positive constant c_1 such that

$$\begin{aligned} \|\mathbb{E}[D_p^2 e^{(D_p^+)^{\gamma}} | \mathcal{F}_{p+1}]\|_{\infty} &\ll \sum_{h \geq 0} e^{-c_1 h^{\gamma}} \left(\sum_{a=0}^{p-h} L_a^2 e^{-\kappa(p-h-a)^{\gamma}} + (h+1) \sum_{a=p-h+1}^p L_a^2 \right) \\ &\ll \sum_{a=0}^p L_a^2 \left\{ \sum_{0 \leq h \leq p-a} e^{-c_1 h^{\gamma}} e^{-\kappa(p-h-a)^{\gamma}} + \sum_{h \geq p-a} (h+1) e^{-c_1 h^{\gamma}} \right\}. \end{aligned}$$

By splitting $\sum_{0 \leq h \leq p-a}$ into two sums: $\sum_{0 \leq h \leq [(p-a)/2]}$ and $\sum_{[(p-a)/2] < h \leq p-a}$, we derive that there exists a positive constant c_2 such that

$$\|\mathbb{E}[D_p^2 e^{(D_p^+)^{\gamma}} | \mathcal{F}_{p+1}]\|_{\infty} \ll \sum_{a=0}^p L_a^2 e^{-c_2(p-a)^{\gamma}}.$$

Hence

$$\sum_{p=1}^n \|\mathbb{E}[D_p^2 e^{(D_p^+)^{\gamma}} | \mathcal{F}_{p+1}]\|_{\infty} \ll \sum_{a=0}^n L_a^2 \sum_{p \geq a} e^{-c_2(p-a)^{\gamma}}$$

which leads to (6.6) and ends the proof of the theorem. \square

6.3 Proof of Theorem 5.2

We start with some preliminary considerations. Let $S := T_{[0,1/2]}^{-1}$ be the inverse left branch of T and $U := T_{[1/2,1]}^{-1}$.

For every $n \in \mathbb{N}_0$, set $y_n = S^n(1/2)$, $I_n := (0, y_n]$ and $J_n := U(I_n) = (1/2, y_n/2 + 1/2]$.

It follows from [8, Annex A] that there exist $v_1, v_2 > 0$ such that

$$e^{-v_1 n^{\gamma}} \leq y_n \leq e^{-v_2 n^{\gamma}}.$$

Then, using the control of the density φ , we infer that there exist $K, L > 0$ and $v_3, v_4 > 0$ such that

$$L e^{-v_3 n^{\gamma}} \leq \nu(I_n) \leq K e^{-v_4 n^{\gamma}}$$

Proof of Item a). The upper bound given in Item a) comes from an application of inequality (4.1) when $\gamma > 1$ and of inequality (4.2) when $\gamma = 1$. To construct a Lipschitz function for which the lower bound holds the idea is to take the function $\mathbf{1}_{(1/2,1]}$ and to extend it to a Lipschitz function.

For every $x \in (1/2, 1]$, set $f(x) = 1$ and for every $x \in I_1$ set $f(x) = 0$. For every $x \in (y_1, 1/2]$ set $f(x) = \frac{x-y_1}{1/2-y_1}$.

Let $n \in \mathbb{N}$ and $x \in J_n \subset (1/2, 1]$. Then, $f(x) = 1$ and for every $k \in \llbracket 1, n-1 \rrbracket$, $f \circ T^k(x) = 0$. So $S_n(f)(x) = 1 - n\nu(f)$ and for $n > 2/\nu(f)$, $|S_n(f)(x)| \geq n\nu(f)/2$. So, there exists a positive constant $c > 0$ such that

$$\nu(|S_n(f)| \geq n\nu(f)/2) \geq \nu(J_n) \geq c y_n \geq c e^{-v_1 n^{\gamma}},$$

proving the lower bound.

Proof of Item b). Let $\delta > 0$ and $f := (|\log x|)^\delta$. Note that

$$\nu(|f| > t) = \nu(]0, e^{-t^{1/\delta}}]) \leq C \int_0^{e^{-t^{1/\delta}}} (1 + |\log x|^\beta) dx \leq Ae^{-Bt^{1/\delta}}.$$

The lower bound may be proved similarly.

Let us prove the lower bound in (5.2). We start by noticing that

$$\nu(f) \leq (\log 2)^\delta \nu((1/2, 1]) + \sum_{n \in \mathbb{N}_0} (v_1(n+1)^\gamma)^\delta \nu(I_n \setminus I_{n+1}) \leq C + K \sum_{n \in \mathbb{N}} (v_1(n+1)^\gamma)^\delta e^{-v_4 n^\gamma} < \infty.$$

Let $1 \leq m \leq n$ be integers. Let $x \in I_m$ (then $T^k x \in I_{m-k}$ for every integer $k \in [0, m-1]$). We have

$$\begin{aligned} S_n(f) &:= \sum_{k=0}^{n-1} (f \circ T^k(x) - \nu(f)) = \left(\sum_{k=0}^{n-1} f \circ T^k(x) \right) - n\nu(f) \\ &\geq \sum_{k=0}^{m-1} (v_2(m-k)^\gamma)^\delta - n\nu(f) \geq \frac{v_2^\delta m^{1+\gamma\delta}}{1+\gamma\delta} - n\nu(f). \end{aligned}$$

Let $\varepsilon > 0$ and take

$$m = \left\lceil \left[\frac{1+\gamma\delta}{v_2^\delta} (\nu(f) + \varepsilon) n \right]^{1/(1+\gamma\delta)} \right\rceil,$$

which is smaller than n , for n large enough since $\gamma\delta > 0$. It follows that, for every $x \in I_m$, $S_n(f)(x) \geq \varepsilon n$. Moreover,

$$\nu(I_m) \geq Le^{-v_3 m^\gamma} \geq Ce^{-cn^{\gamma/(1+\gamma\delta)}},$$

which proves the lower bound.

The upper bound in (5.2) will follow by applying Theorem 1 in [20] with $\gamma_1 = \gamma$ and $\gamma_2 = 1/\delta$ provided one can prove that $\tau_f(n) = O(e^{-Cn^\gamma})$ where we recall that $f := |\log x|^\delta$. With this aim, let $\varepsilon \in [0, 1[$ and define $f_\varepsilon(x) = (\log(1/\varepsilon))^\delta \mathbf{1}_{[0, \varepsilon]}(x) + f(x) \mathbf{1}_{] \varepsilon, 1]}(x)$. Assume first that $\delta \geq 1$. In this case, note that f_ε is a Lipschitz function with Lipschitz constant $\varepsilon^{-1} \delta |\log \varepsilon|^{\delta-1}$. Hence, by Item 1 of Proposition 2.1, $\tau_{f_\varepsilon}(n) \leq K\varepsilon^{-1} |\log \varepsilon|^{\delta-1} e^{-Cn^\gamma}$. On another hand, for ε small enough,

$$\nu(|f - f_\varepsilon|) \leq C \int_0^\varepsilon ((\log(1/x))^\delta - (\log(1/\varepsilon))^\delta) dx \leq \tilde{C}\varepsilon.$$

Hence, for n large enough, choosing $\varepsilon = e^{-Cn^{\gamma/3}}$, and using the definition of τ_f , we get that, if $\delta \geq 1$,

$$\tau_f(n) \leq \tilde{C}\varepsilon + K\varepsilon^{-1} |\log \varepsilon|^{\delta-1} e^{-Cn^\gamma} = O(e^{-\kappa n^\gamma}).$$

Assume now that $\delta \in]0, 1[$. In this case, since the function $x \mapsto |x|^\delta$ is δ -Hölder, we derive that f_ε is a δ -Hölder function with Hölder constant equals to $\varepsilon^{-\delta}$, meaning that $|f_\varepsilon(x) - f_\varepsilon(y)| \leq \varepsilon^{-\delta} |x - y|^\delta$. Using again Item 1 of Proposition 2.1 and arguing as above, the desired upper bound for $\tau_f(n)$ follows.

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