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# Trace and boundary singularities of positive solutions of a class of quasilinear equations

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*A Juan-Luis por su 75 cumpleaños. Cuarenta y seis años de amistad, respeto y admiración*

## Abstract

We study positive functions satisfying (E)  $-\Delta u + m|\nabla u|^q - u^p = 0$  in a domain  $\Omega$  or in  $\mathbb{R}_+^N$  when  $p > 1$  and  $1 < q < 2$ . We give sufficient conditions for the existence of a solution to (E) with a nonnegative measure  $\mu$  as boundary data; these conditions are expressed in terms of Bessel capacities on the boundary. We also study removability of boundary singular sets, and solutions with an isolated singularity on  $\partial\Omega$ . The different results depend on two critical exponents for  $p = p_c := \frac{N+1}{N-1}$  and for  $q = q_c := \frac{N+1}{N}$ , and on the sign of  $q - \frac{2p}{p+1}$ .

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# 1 Introduction

In this article we study the boundary behaviour of positive solutions of the following class of quasilinear elliptic equations

$$-\Delta u + m|\nabla u|^q - |u|^{p-1}u = 0 \tag{1.1}$$

in a domain  $G$  of  $\mathbb{R}^N$  which can be either  $\mathbb{R}^N$ , or  $\mathbb{R}^N \setminus \{0\}$ , or  $\mathbb{R}_+^N$ , or a bounded domain  $\Omega$  with smooth boundary  $\partial\Omega$ , according the type of phenomenon we are interested in. We assume that  $p, q > 1$  and  $m \geq 0$ . We also consider the associated measure boundary data problem

$$\begin{aligned} -\Delta u + m|\nabla u|^q - |u|^{p-1}u &= 0 && \text{in } \Omega \\ u &= \mu && \text{in } \partial\Omega, \end{aligned} \tag{1.2}$$

in the case where  $G = \Omega$  and  $\mu$  is a positive Radon measure on  $\partial\Omega$ .

The wide variety of phenomena that exhibit the solutions of equation (1.1) comes from the opposition between the forcing term  $|u|^{p-1}u$  and the reaction term  $m|\nabla u|^q$ . Furthermore, in the specific case  $q = \frac{2p}{p+1}$ , the order of magnitude of the forcing and the reaction is the same, therefore the value of the coefficient  $m$  plays a fundamental role. This is due to the equivariance of the equation (1.1) under the transformation  $u \mapsto T_\ell[u]$  defined by

$$T_\ell[u](x) = \ell^{\frac{2}{p-1}}u(\ell x) \quad \text{where } \ell > 0. \tag{1.3}$$

This equation has been introduced by Chipot and Weissler in [11] in a parabolic setting. They also studied the one dimensional case of (1.1). Later on Serrin and Zou published two deep articles [26], [27] where they concentrate on the existence of radial ground states, introducing unexpected energy functions. In [25] they conduct a series of numerical experimentations showing the extreme complexity of this equation, even in the radial case, and many deep questions that they raised are still unanswered. More recently, Alarcón, García-Melián and Quaas proved several non-existence results of supersolutions in an exterior domain of a large class of equations containing in particular (1.1). Their results pointed out the role of some critical exponents,  $p = \frac{N}{N-2}$ ,  $p = \frac{N+2}{N-2}$  and  $q = \frac{N}{N-1}$  as well as  $q = \frac{2p}{p+1}$ . A priori estimates of solutions have been obtained in [23] in the case  $q < \frac{2p}{p+1}$  and  $p < \frac{N+2}{N-2}$ , and then extended in [4] to the case  $q = \frac{2p}{p+1}$  and  $p < \frac{N+2}{N-2}$  under a condition of smallness of  $m$  by a completely different method. The regular Dirichlet problem has been investigated in [24] in the subcritical case  $p < \frac{N}{N-2}$  and  $q < \frac{2p}{p+1}$ , and even extended to the  $m$ -Laplace equation, always in the corresponding subcritical case, but to our knowledge, nothing has already been published concerning the boundary behaviour of singular

solutions and the associated Dirichlet problem with measure data. The aim of this article is to fulfill some gaps in the knowledge of the properties of this equation, emphasizing the connection with an accurate description of the boundary behaviour.

We first prove an a priori estimate for positive solutions of

$$\begin{aligned} -\Delta u + m|\nabla u|^q - u^p &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \setminus \{0\}. \end{aligned} \quad (1.4)$$

when  $q \leq \frac{2p}{p+1}$ . We set

$$\alpha = \frac{2}{p-1}. \quad (1.5)$$

**Theorem 1.1** *Let  $\Omega$  be a bounded smooth domain such that  $0 \in \partial\Omega$ . Suppose  $1 < p < \frac{N+2}{N-2}$  and either  $1 < q < \frac{2p}{p+1}$  and  $m > 0$ , or  $q = \frac{2p}{p+1}$  and  $0 < m < \epsilon_0$  for some  $\epsilon_0 > 0$  depending on  $N$  and  $p$ . Then there exists a constant  $c = c(N, p, \Omega) > 0$  such that if  $u$  is a positive solution of (1.4), it satisfies*

$$u(x) \leq c|x|^{-\alpha} \quad \text{for all } x \in \overline{\Omega} \setminus \{0\}, \quad (1.6)$$

and

$$\frac{u(x)}{\rho(x)} + |\nabla u(x)| \leq c|x|^{-\alpha-1} \quad \text{for all } x \in \overline{\Omega} \setminus \{0\}, \quad (1.7)$$

where  $\rho(x) = \text{dist}(x, \partial\Omega)$ .

Thanks to this estimate we can describe the behaviour of positive functions satisfying (1.4). For this purpose we say that the bounded open set  $\Omega \subset \mathbb{R}^N$  is in *normal position with respect to*  $0 \in \partial\Omega$  if  $\partial\Omega$  is tangent to  $\partial\mathbb{R}_+^N$  at  $x = 0$  and if  $x_N > 0$  is the normal inward direction to  $\partial\Omega$ . We set  $\partial B_{1+} := \mathbb{R}_+^N \cap \partial B_1$ , identified with  $S_+^{N-1} := S^{N-1} \cap \mathbb{R}_+^N$  in spherical coordinates  $(r, s)$ . In the sequel we denote by  $\Delta'$  the Laplace-Beltrami operator on  $S^{N-1}$  and by  $\nabla'$  the covariant gradient identified with the tangential gradient to  $\partial B_1$ .

**Corollary 1.2** *Let  $\Omega$  be a bounded smooth domain in normal position with respect to  $0 \in \partial\Omega$ . Suppose  $1 < p < \frac{N+2}{N-2}$ ,  $1 < q < \frac{2p}{p+1}$  and  $m > 0$ , and  $u$  is a positive solution of (1.1) vanishing on  $\partial\Omega \setminus \{0\}$ , then either  $u$  can be extended as a continuous function in  $\overline{\Omega}$ , or one of the following situations occurs.*

1- If  $1 < p < \frac{N+1}{N-1}$ , there exists  $k > 0$  such that

$$\lim_{\substack{x \in \Omega \\ x \rightarrow 0}} \frac{u(x)}{\mathbb{P}_\Omega(x, 0)} = k, \quad (1.8)$$

where  $\mathbb{P}_\Omega$  is the Poisson kernel in  $\Omega$  with asymptotics given in (1.12).

2- If  $p = \frac{N+1}{N-1}$ ,

$$\lim_{\substack{x \in \Omega \\ x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow s}} |x|^{N-1} \left( \ln \frac{1}{|x|} \right)^{\frac{N-1}{2}} u(x) = \lambda_N \phi_1(s), \quad (1.9)$$

uniformly on any compact set of  $S_+^{N-1}$ , where  $\phi_1$  is the first eigenfunction of  $-\Delta'$  in  $W_0^{1,2}(S_+^{N-1})$  with maximum 1 (actually  $\phi_1(x/|x|) = \sin(x_N/|x|)$ ), and  $\lambda_N$  is a positive constant depending only on  $N$ .

3- If  $\frac{N+1}{N-1} < p < \frac{N+2}{N-2}$

$$\lim_{\substack{x \in \Omega \\ x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow s}} |x|^\alpha u(x) = \psi(s), \quad (1.10)$$

with  $\alpha = \frac{2}{p-1}$ , uniformly on any compact set of  $S_+^{N-1}$ , where  $\psi$  is the unique positive solution of

$$\begin{aligned} -\Delta' \psi + \alpha(N-2-\alpha)\psi - \psi^p &= 0 & \text{in } S_+^{N-1} \\ \psi &= 0 & \text{on } \partial S_+^{N-1}. \end{aligned} \quad (1.11)$$

A direct computation by matching asymptotic expansion shows that if  $\Omega$  is in normal position at  $0 \in \partial\Omega$  the Poisson kernel has the following asymptotic expression near  $x = 0$

$$\mathbb{P}_\Omega(x, 0) = c_N |x|^{1-N} \left( \sin\left(\frac{x_N}{|x|}\right) + O\left(\frac{x_N^2}{|x|^2}\right) \right) \quad \text{as } x \in \Omega, x \rightarrow 0, \quad (1.12)$$

for some explicit constant  $c_N$ .

In case 1, a solution which satisfies (1.8) is actually a weak solution of

$$\begin{aligned} -\Delta u + m|\nabla u|^q - u^p &= 0 & \text{in } \Omega \\ u &= k\delta_0 & \text{in } \mathcal{D}'(\partial\Omega). \end{aligned} \quad (1.13)$$

where  $\delta_0$  is the Dirac measure at 0. A solution which satisfies (1.9) or (1.10) is a weak solution of (1.1) in  $\Omega$  with zero boundary value in the sense of distributions in  $\partial\Omega$ , and this property still holds even when  $\frac{N+1}{N-1} \leq p < \frac{N+2}{N-2}$  and  $q = \frac{2p}{p+1}$ .

The proof of Corollary 1.2 is based upon the fact that if  $1 < q < \frac{2p}{p+1}$ , the a priori estimates of Theorem 1.1 imply that problem (1.4) is a perturbation of

$$\begin{aligned} -\Delta u - u^p &= 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \setminus \{0\} \end{aligned} \quad (1.14)$$

near  $x = 0$ , a problem which has been thoroughly studied in [3]. When  $q = \frac{2p}{p+1}$ , it is a consequence of the invariance of (1.1) under the transformations  $T_\ell$  that there could exist invariant solutions  $u$  which are the ones such that  $T_\ell[u] = u$  for any  $\ell > 0$ . We first consider self-similar solutions in whole  $\mathbb{R}^N$ . Using spherical coordinates  $(r, s) \in \mathbb{R}_+ \times S^{N-1}$ , these self-similar solutions have the form

$$u(x) = u(r, s) = r^{-\alpha} \omega(s), \quad (1.15)$$

where  $\alpha$  is defined in (1.5). Then  $\omega$  satisfies

$$-\Delta' \omega + \alpha(N-2-\alpha)\omega + m(\alpha^2 \omega^2 + |\nabla' \omega|^2)^{\frac{p}{p+1}} - |\omega|^{p-1} \omega = 0 \quad (1.16)$$

in  $S^{N-1}$ . Constant solutions are roots of the function

$$\mathcal{P}_m(X) = \alpha(N - 2 - \alpha)X + m\alpha^{\frac{2p}{p+1}}X^{\frac{2p}{p+1}} - |X|^{p-1}X. \quad (1.17)$$

In the study of the variations of  $\mathcal{P}_m$  on  $\mathbb{R}$  the following constant, defined if  $p < \frac{N}{N-2}$ , plays an important role

$$m^* = (p + 1) \left( \frac{N - p(N - 2)}{2p} \right)^{\frac{p}{p+1}}. \quad (1.18)$$

Concerning the self-similar solutions in  $\mathbb{R}^N$  we recall the result stated without proof in [4, Prop. 6.1],

**Proposition 1.3** *Assume  $N \geq 2$ .*

- (i) *If  $N \geq 3$ ,  $m > 0$  and  $p \geq \frac{N}{N-2}$  there exists a unique constant positive solution  $X_m$  to (1.16).*
- (ii) *If  $N \geq 2$ ,  $1 < p < \frac{N}{N-2}$  and  $m > m^*$  there exist two constant positive solutions  $0 < X_{1,m} < X_{2,m}$  to (1.16).*
- (iii) *If  $N \geq 2$   $1 < p < \frac{N}{N-2}$  and  $m = m^*$  there exists a unique constant positive solution  $X_{m^*}$  to (1.16).*
- (iv) *If  $N \geq 2$ ,  $1 < p < \frac{N}{N-2}$  and  $0 < m < m^*$  there exists no constant positive solution to (1.16).*

A more complete study of equation (1.16) and its role in the description of isolated singularities is developed in the forthcoming paper [7].

When the domain  $G$  in which we consider equation (1.1) has a non-empty boundary (in the sequel, either  $G = \mathbb{R}_+^N$  or  $G$  is a smooth bounded domain that we denote by  $\Omega$ ), it is natural to study solutions of (1.1) with an isolated singularity lying on the boundary. The understanding of boundary singularities is conditioned by the knowledge of positive self-similar solutions in  $\mathbb{R}_+^N$  vanishing on  $\partial\mathbb{R}_+^N$  except at  $x = 0$ . They are solutions of

$$\begin{aligned} -\Delta'\omega + \alpha(N - 2 - \alpha)\omega + m(\alpha^2\omega^2 + |\nabla'\omega|^2)^{\frac{p}{p+1}} - \omega^p &= 0 && \text{in } S_+^{N-1} \\ \omega &= 0 && \text{on } \partial S_+^{N-1}. \end{aligned} \quad (1.19)$$

There, the critical value for  $p$  is  $\frac{N+1}{N-1}$ . The main result concerning problem (1.19) states as follows,

**Theorem 1.4** *Let  $1 < p < \frac{N+1}{N-1}$ .*

- 1- *For any  $m \geq m^*$  there exists at least one positive solution  $\omega_m$  to (1.19).*
- 2- *There exists  $m_p \in (0, m^*)$  such that for any  $0 < m \leq m_p$  there exists no positive solution to (1.19).*

The value of  $m_p$  is explicit.

In the next section of this article we study problem (1.2). We denote by  $L_\rho^1(\Omega)$  the space of measurable functions  $u$  in  $\Omega$  such that  $u\rho \in L^1(\Omega)$  where  $\rho(x) = \text{dist}(x, \partial\Omega)$ .

**Definition 1.5** Let  $p, q > 0$ ,  $m \in \mathbb{R}$  and  $\mu$  be a bounded measure on  $\partial\Omega$ . We say that a nonnegative Borel function  $u$  defined in  $\Omega$  is a weak solution of (1.2) if  $u \in L^1(\Omega)$ ,  $u^p \in L^1_\rho(\Omega)$ ,  $|\nabla u|^q \in L^1_\rho(\Omega)$  and

$$\int_{\Omega} (-u\Delta\zeta + (m|\nabla u|^q - u^p)\zeta) dx = - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\mu, \quad (1.20)$$

for all  $\zeta \in \mathbb{X}(\Omega) := \{\zeta \in C_c^1(\overline{\Omega}) : \Delta\zeta \in L^\infty(\Omega)\}$ .

For  $a > 0$  and  $1 < b < \infty$ , we denote by  $Cap_{a,b}^{\partial\Omega}$  the Bessel capacity on  $\partial\Omega$ . It is defined by local charts (see e.g. [18]). Our main existence result is the following.

**Theorem 1.6** Let  $p > 1$ ,  $1 < q < 2$  and  $m > 0$ . Assume  $\mu$  is a nonnegative measure on  $\partial\Omega$ . If  $\mu$  satisfies

$$\mu(K) \leq C_3 \min \left\{ Cap_{\frac{2-q}{q}, q'}^{\partial\Omega}(K), Cap_{\frac{2}{p}, p'}^{\partial\Omega}(K) \right\} \quad \text{for any compact set } K \subset \partial\Omega, \quad (1.21)$$

then one can find  $\epsilon_3 > 0$  such that for any  $0 < \epsilon \leq \epsilon_3$  there exists a weak solution  $u$  to problem (1.2) with  $\mu$  replaced by  $\epsilon\mu$ .

The main idea for proving this result is to associate to (1.1) the two problems

$$\begin{aligned} -\Delta v + m|\nabla v|^q &= 0 & \text{in } \Omega \\ v &= \mu & \text{in } \partial\Omega, \end{aligned} \quad (1.22)$$

and

$$\begin{aligned} -\Delta w - w^p &= 0 & \text{in } \Omega \\ w &= \mu & \text{in } \partial\Omega. \end{aligned} \quad (1.23)$$

We show that when (4.6) holds these two problems admit positive solutions respectively  $v_\mu$  and  $w_\mu$ , such that  $0 < v_\mu < w_\mu$  (with  $\mu$  replaced by  $\epsilon\mu$ ) which both satisfy the boundary trace relation as it is introduced in [17],

$$\lim_{\delta \rightarrow 0} \int_{\{\rho(x)=\delta\}} v_\mu Z dS = \lim_{\delta \rightarrow 0} \int_{\{\rho(x)=\delta\}} w_\mu Z dS = \int_{\partial\Omega} Z d\mu, \quad (1.24)$$

for any  $Z \in C(\overline{\Omega})$ . Since  $v_\mu$  and  $w_\mu$  are respectively a subsolution and a supersolution of (1.1), we derive the existence of a solution  $u$  of (1.1) in  $\Omega$  which satisfies also the boundary trace relation (1.24). This approach is linked to the dynamical construction of the boundary trace developed in [20]. As an easy consequence of Theorem 1.6 we have the following result.

**Corollary 1.7** Let  $p > 1$ ,  $1 < q < 2$  and  $m > 0$ . If  $\mu$  is a nonnegative measure on  $\partial\Omega$  there exists a positive weak solution to (1.2) with  $\mu$  replaced by  $\epsilon\mu$  under the following conditions.

- 1- If  $1 < p < \frac{N+1}{N-1}$  and  $1 < q < \frac{N+1}{N}$ .
- 2- If  $p \geq \frac{N+1}{N-1}$ ,  $1 < q < \frac{N+1}{N}$  and  $\mu$  satisfies

$$\mu(K) \leq C_2 Cap_{\frac{2}{p}, p'}^{\partial\Omega}(K) \quad \text{for any compact set } K \subset \partial\Omega. \quad (1.25)$$

3- If  $1 < p < \frac{N+1}{N-1}$ ,  $\frac{N+1}{N} \leq q < 2$  and  $\mu$  satisfies

$$\mu(K) \leq C_1 \text{Cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(K) \quad \text{for any compact set } K \subset \partial\Omega. \quad (1.26)$$

A more delicate corollary is based upon relations between Bessel capacities.

**Corollary 1.8** *Let  $p > 1$ ,  $1 < q < 2$  and  $m > 0$ . If  $\mu$  is a nonnegative measure on  $\partial\Omega$ , there exists a positive weak solution to (1.2) with  $\mu$  replaced by  $\epsilon\mu$  under the following conditions.*

1- If  $\frac{N+1}{N} \leq q < \frac{2p}{p+1}$ , when

$$\mu(K) \leq C_3 \text{Cap}_{\frac{2}{p}, p'}^{\partial\Omega}(K) \quad \text{for any compact set } K \subset \partial\Omega. \quad (1.27)$$

2- If  $p \geq \frac{N+1}{N-1}$  and  $q \geq \frac{2p}{p+1}$ , when

$$\mu(K) \leq C_4 \text{Cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(K) \quad \text{for any compact set } K \subset \partial\Omega. \quad (1.28)$$

It is noticeable that the results of Corollary 1.7 and Corollary 1.8 cover the full range of exponents  $(p, q) \in (1, \infty) \times (1, 2)$ . The sufficient conditions of Theorem 1.6 are stronger than the necessary conditions which are obtained below.

**Theorem 1.9** *Let  $p > 1$ ,  $1 < q < 2$  and  $m > 0$ . Assume there exists a nonnegative solution  $u$  of problem (1.4) for some  $\mu \in \mathfrak{M}_+(\partial\Omega)$ . Then  $\mu$  satisfies*

$$\text{Cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(K) = 0 \implies \mu(K) = 0 \quad \text{if } K \subset \partial\Omega \text{ is a compact set,} \quad (1.29)$$

and

$$\text{Cap}_{\frac{2}{p}, p'}^{\partial\Omega}(K) = 0 \implies \mu(K) = 0 \quad \text{if } K \subset \partial\Omega \text{ is a compact set.} \quad (1.30)$$

In the last section we study the boundary trace of positive solutions of (1.1). The notion of boundary trace is classical in harmonic analysis in the framework of bounded Borel measures. It has been extended to semilinear elliptic equations by Marcus and Véron in [17], [18], [20] with general Borel measures as a natural framework for the boundary trace.

**Definition 1.10** *Let  $\Omega \subset \mathbb{R}^N$  be a smooth bounded domain,  $p > 1$ ,  $1 < q < p$ ,  $m > 0$  and  $\mathcal{O}$  a relatively open subset of  $\partial\Omega$ . We say that a positive solution  $u$  of (1.1) in  $\Omega$  admits a boundary trace on  $\mathcal{O}$ , denoted by  $\text{Tr}_{\mathcal{O}}(u)$ , if there exist a relatively open subset  $\mathcal{R}(u)$  of  $\mathcal{O}$  and a nonnegative Radon measure  $\mu$  on  $\mathcal{R}(u)$  such that*

$$\lim_{\delta \rightarrow 0} \int_{\{\rho(x)=\delta\}} u Z dS = \int_{\partial\Omega} Z d\mu \quad (1.31)$$

for every  $Z \in C(\bar{\Omega})$  such that  $\text{supp}(Z|_{\mathcal{O}}) \subset \mathcal{R}(u)$ , and if for every  $z \in \mathcal{S}(u) := \mathcal{O} \setminus \mathcal{R}(u)$  and any  $\epsilon > 0$ , there holds

$$\lim_{\delta \rightarrow 0} \int_{\{\rho(x)=\delta\} \cap B_\epsilon(z)} u dS = \infty. \quad (1.32)$$



The boundary trace  $Tr_{\mathcal{O}}(u)$  is represented by the couple  $(\mathcal{S}(u), \mu)$  or equivalently by the outer Borel measure  $\mu_{\mathcal{O}}^*$  on  $\mathcal{O}$  defined as follows:

$$\mu_{\mathcal{O}}^*(\zeta) = \begin{cases} \int_{\mathcal{R}(u)} \zeta d\mu & \forall \zeta \in C^\infty(\partial\Omega) \text{ s.t. } \text{supp}(\zeta) \subset \mathcal{R}(u) \\ \infty & \forall \zeta \in C^\infty(\partial\Omega) \text{ s.t. } \text{supp}(\zeta) \cap \mathcal{S}(u) \neq \emptyset, \zeta \geq 0. \end{cases} \quad (1.33)$$

It is easy to prove that if a compact set  $K \subset \bar{\Omega}$  is such that  $u^p + |\nabla u|^q \in L^1_\rho(K)$ , then  $K \cap \partial\Omega \subset \mathcal{R}(u)$ . We first give a result where the trace is always a nonnegative Radon measure. The meaning of this result is that the absorption term is dominated by the reaction term and the solution behaves like a superharmonic function.

**Theorem 1.11** *Let  $\Omega$  be a bounded smooth domain and  $p > 1$ . Assume either  $1 < q < \frac{2p}{p+1}$  and  $m > 0$ , or  $q = \frac{2p}{p+1}$  and  $0 < m \leq m_1$  for some  $m_1 > 0$  depending on  $N$  and  $p$ . If  $u$  is a positive solution of (1.1) in  $\Omega$ , then  $u \in L^1(\Omega)$ ,  $u^p + |\nabla u|^q \in L^1_\rho(\Omega)$  and there exists a nonnegative Radon measure  $\mu$  on  $\partial\Omega$  such that  $u$  is a solution of (1.2).*

The boundary trace of a positive solution of (1.1) may not be a Radon measure, for example, if  $\frac{N+1}{N-1} < p < \frac{N}{N-2}$ ,  $q = \frac{2p}{p+1}$ ,  $m \geq m^*$  and  $u$  is the restriction to  $\Omega$  of a radial singular solution obtained in Proposition 1.3. In that case

$$\lim_{\delta \rightarrow 0} \int_{\{\rho(x)=\delta\}} u Z dS = \infty, \quad (1.34)$$

for any  $Z \in C_+(\bar{\Omega})$ , such that  $Z(0) > 0$ . We have the following result.

**Theorem 1.12** *Assume  $p > 1$ ,  $1 < q < p$  and  $m > 0$ . If  $u$  is a positive solution of (1.1) in  $\Omega$  and  $z \in \partial\Omega$ , we have that:*

- 1- *If there exists  $\epsilon > 0$  such that  $|\nabla u|^q \in L^1_\rho(B_\epsilon(z) \cap \Omega)$ , then  $u^p \in L^1_\rho(B_\epsilon(z) \cap \Omega)$  and  $u$  admits a boundary trace on  $\partial\Omega \cap B_\epsilon(z)$  which is a nonnegative Radon measure.*
- 2- *If there exists  $\epsilon > 0$  such that  $u^p \in L^1_\rho(B_\epsilon(z) \cap \Omega)$ , then  $u$  admits a boundary trace on  $\partial\Omega \cap B_\epsilon(z)$  which is a nonnegative outer regular Borel measure, not necessarily bounded.*

The last assertion shows how delicate is the construction of solutions with unbounded boundary trace. We give a few examples with one point blow-up on the boundary. In particular we prove that when  $0 \in \partial\Omega$ ,  $p > 1$  and  $\frac{2p}{p+1} < q < \frac{N+1}{N}$  there exist positive solutions  $u$  of (1.1) in  $\Omega$  (or  $\Omega \setminus K$  where  $K$  is compact), vanishing on  $\partial\Omega \setminus \{0\}$  satisfying

$$u(x) = |x|^{-\frac{2-q}{q-1}} \chi\left(\frac{x}{|x|}\right) (1 + o(1)) \quad \text{as } x \rightarrow 0, \quad (1.35)$$

for some positive function  $\chi$  defined on the  $S_+^{N-1}$ . Such solutions have boundary trace  $Tr_{\partial\Omega}(u) = (\{0\}, 0)$ .

The existence of a boundary trace in the case  $q > \frac{2p}{p+1}$  for any positive solution of (1.1) remains an open problem.

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## 2 Solutions with a boundary isolated singularity

### 2.1 A priori estimates

In this section  $\Omega$  denotes a bounded smooth domain of  $\mathbb{R}^N$  such that  $0 \in \partial\Omega$ . We prove an a priori estimate for positive solutions of (1.1) vanishing on  $\partial\Omega \setminus \{0\}$ .

**Proposition 2.1** *Suppose  $1 < p < \frac{N+2}{N-2}$  and either  $1 < q < \frac{2p}{p+1}$  and  $m > 0$ , or  $q = \frac{2p}{p+1}$  and  $0 < m < \epsilon_0$  for some  $\epsilon_0 > 0$  depending on  $N$  and  $p$ . Then there exists a constant  $c = c(N, p, \Omega) > 0$  such that if  $u$  is a positive solution of (1.4), it satisfies*

$$u(x) \leq c|x|^{-\alpha} \quad \text{for all } x \in \bar{\Omega} \setminus \{0\}. \quad (2.1)$$

The proof needs a series of intermediate results involving the Polacik et al. method [23], a result of Montoro [21] and a previous Liouville theorem proved in [4]. We first recall the doubling lemma.

**Lemma 2.2** *Let  $(X, d)$  be a complete metric space,  $\Gamma \subsetneq X$  and  $\gamma : X \setminus \Gamma \mapsto (0, \infty)$ . Assume that  $\gamma$  is bounded on all compact subsets of  $X \setminus \Gamma$ . Given  $k > 0$ , let  $y \in X \setminus \Gamma$  such that*

$$\gamma(y)\text{dist}(y, \Gamma) > 2k.$$

*Then there exists  $x \in X \setminus \Gamma$  such that*

- 1-  $\gamma(x)\text{dist}(x, \Gamma) > 2k$ ,
- 2-  $\gamma(x) \geq \gamma(y)$ ,
- 3-  $2\gamma(x) \geq \gamma(z)$ , for all  $z \in B_{\frac{k}{\gamma(x)}}(x)$ .

The next result is an extension of [3, Proposition 5.1].

**Lemma 2.3** *Suppose  $1 < p < \frac{N+2}{N-2}$  and either  $1 < q < \frac{2p}{p+1}$  and  $m > 0$ , or  $q = \frac{2p}{p+1}$  and  $0 < m < \epsilon_0$  for some  $\epsilon_0 > 0$  depending on  $N$  and  $p$ . Let  $0 < r < \frac{1}{2}\text{diam } \Omega$ . There exists a constant  $c > 0$  depending on  $p, m, q$  and  $\Omega$  such that any function  $u$  verifying*

$$\begin{aligned} -\Delta u + m|\nabla u|^q &= u^p && \text{in } \Omega \cap (B_{2r} \setminus \bar{B}_r) \\ u &\geq 0 && \text{in } \Omega \cap (B_{2r} \setminus \bar{B}_r) \\ u &= 0 && \text{in } \partial\Omega \cap (B_{2r} \setminus \bar{B}_r), \end{aligned} \quad (2.2)$$

*satisfies*

$$u(x) \leq c(\text{dist}(x, \Gamma_r))^{-\alpha} \quad \text{for all } x \text{ in } \Omega \cap (B_{2r} \setminus \bar{B}_r), \quad (2.3)$$

*where  $\Gamma_r = \bar{\Omega} \cap (\partial B_{2r} \cup \partial B_r)$ .*

*Proof.* We proceed by contradiction. For every  $k \geq 1$  there exist  $0 < r_k < \frac{1}{2}\text{diam } \Omega$ , a solution  $u_k$  of (2.2) with  $r = r_k$  and  $y_k \in \Omega \cap (B_{2r_k} \setminus \bar{B}_{r_k})$  such that

$$u_k(y_k) \geq (2k)^\alpha (\text{dist}(x, \Gamma_{r_k}))^{-\alpha}.$$

It follows from Lemma 2.2 applied with

$$X = \overline{\Omega} \cap (\overline{B}_{2r_k} \setminus B_{r_k}) \quad \text{and} \quad \gamma = u_k^{\frac{1}{\alpha}},$$

that there exists  $x_k \in X \setminus \Gamma_k$  such that

$$\begin{aligned} (i) \quad & u_k(x_k) \geq (2k)^\alpha (\text{dist}(x_k, \Gamma_{r_k}))^{-\alpha}, \\ (ii) \quad & u_k(x_k) \geq u_k(y_k), \\ (iii) \quad & 2^\alpha u_k(x_k) \geq u_k(z), \text{ for all } z \in B_{R_k}(x_k) \cap \Omega \text{ with } R_k = k(u_k(x_k))^{-\frac{1}{\alpha}}. \end{aligned} \tag{2.4}$$

Since (i) holds,  $R_k < \frac{1}{2} \text{dist}(x_k, \Gamma_{r_k})$ , hence

$$B_{R_k}(x_k) \cap \Gamma_{r_k} = \emptyset.$$

Since  $\text{dist}(x_k, \Gamma_{r_k}) \leq \frac{1}{2} r_k < \frac{1}{4} \text{diam } \Omega$ , we also have from (i),

$$u_k(x_k) \geq \left( \frac{8k}{\text{diam } \Omega} \right)^\alpha \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Next we set

$$t_k = (u_k(x_k))^{-\frac{1}{\alpha}}, \quad D_k = \{ \xi \in \mathbb{R}^N : |\xi| \leq k \text{ and } x_k + t_k \xi \in \Omega \},$$

and

$$v_k(\xi) = t_k^\alpha u_k(x_k + t_k \xi) \quad \text{for all } \xi \in D_k.$$

Then  $v_k$  is positive in  $D_k$  and satisfies

$$\begin{aligned} -\Delta v_k + m t_k^{\frac{2p-(p+1)q}{p-1}} |\nabla v_k|^q &= v_k^p && \text{in } D_k \\ 0 \leq v_k &\leq 2^\alpha && \text{in } D_k \\ v_k(0) &= 1. \end{aligned} \tag{2.5}$$

We encounter the following dichotomy:

(A) Either for every  $a > 0$  there exists  $k_a \geq 1$  such that for  $k \geq k_a$   $B_{at_k}(x_k) \cap \partial\Omega = \emptyset$ . The sequence  $\{v_k\}$  is locally uniformly bounded in  $\mathbb{R}^N$ . Since  $q \leq 2$ , standard a priori estimates in elliptic equations imply that  $\{v_k\}$  is eventually uniformly bounded in the  $C^{2,\tau}$  local topology in  $\mathbb{R}^N$  with  $0 < \tau < 1$ . Up to a subsequence still denoted by  $\{v_k\}$  it converges locally in  $C^2(\mathbb{R}^N)$  to a positive function  $v$  which satisfies  $v(0) = 1$ ,  $0 \leq v \leq 2^\alpha$  and either

$$-\Delta v = v^p \quad \text{in } \mathbb{R}^N \tag{2.6}$$

if  $1 < q < \frac{2p}{p+1}$ , or

$$-\Delta v + m |\nabla v|^q = v^p \quad \text{in } \mathbb{R}^N \tag{2.7}$$

if  $q = \frac{2p}{p+1}$ . In the first case it is proved in [13] that such a solution cannot exist. If  $q = \frac{2p}{p+1}$  it is proved in [4, Theorem E] that there exists  $\epsilon_0 > 0$  depending on  $N, p$  such that if  $|m| \leq \epsilon_0$  no such solution exists. Therefore if situation (A) occurs we obtain a contradiction.

(B) Or there exists some  $a_0 > 0$  such that  $B_{a_0 t_k}(x_k) \cap \partial\Omega \neq \emptyset$  for all  $k \in \mathbb{N}^*$ . Let  $x'_k \in \partial\Omega$  minimizing the distance from  $x_k$  and  $\partial\Omega$ , then  $|x_k - x'_k| \leq a_0 t_k$ . Since the function  $v_k$  is bounded in  $D_k$  and vanishes on  $\partial\Omega$  which is smooth, it remains locally bounded in  $W^{2,s}(\bar{\Omega} \cap B_{a'_0 t_k}(x_k))$  for all  $a'_0 < a_0$  and all  $s < \infty$ , thus  $\nabla v_k$  remains locally bounded therein. Then either  $|x_k - x'_k| \geq a'_0 t_k$  or  $|x_k - x'_k| < a'_0 t_k$ . In this case we set  $\xi'_k = t_k^{-1}(x'_k - x_k)$  and use the fact that  $v_k(\xi'_k) = 0$  and  $v_k(0) = 1$  combined with the uniform bound on  $\nabla v_k$  to infer that  $|\xi'_k| \geq a_1$  for some  $0 < a_1 < a_0$  independent of  $k$  which implies that  $|x_k - x'_k| \geq a_1 t_k$ . Up to a subsequence, we can assume that  $t_k^{-1} x_k \rightarrow x_0$ ,  $t_k^{-1} x'_k \rightarrow x'_0$  and that  $D_k \rightarrow H$  where  $H \sim \mathbb{R}_+^N$  is the half-space passing through  $x'_0$  with normal inward unit vector  $\mathbf{e}_N$ , and  $x_0 - x'_0 = a \mathbf{e}_N$  with  $a_1 \leq a \leq a_0$ . Let  $\tilde{H} \sim \mathbb{R}^N$  be the union of  $H$  and its reflection through  $\partial H$ . Performing the reflection of  $v_k$  through  $\partial(t_k^{-1}\Omega)$  (see [29, Lemma 3.3.2]) we deduce that the function  $\tilde{v}_k$  which coincides with  $v_k$  in  $\partial(t_k^{-1}\Omega)$  and with its odd reflection in the image by reflection of the set  $\partial(t_k^{-1}\Omega)$  vanishes on  $(t_k^{-1}\Omega)$  and converges locally in  $C^2(\mathbb{R}^N)$  to a positive function  $\tilde{v}$  defined in  $\tilde{H}$ , bounded therein, vanishing on  $\partial H$  and positive in  $H$  and the function  $v = \tilde{v}|_H$  is nonnegative and  $v(x_0) = 1$ . If  $q < \frac{2p}{p+1}$ ,  $v$  satisfies

$$\begin{aligned} -\Delta v &= v^p & \text{in } H \\ v &= 0 & \text{in } \partial H. \end{aligned} \tag{2.8}$$

By [13] such a function cannot exist. If  $q = \frac{2p}{p+1}$ , the function  $v$  satisfies

$$\begin{aligned} -\Delta v + m|\nabla v|^{\frac{2p}{p+1}} &= v^p & \text{in } H \\ v &= 0 & \text{in } \partial H. \end{aligned} \tag{2.9}$$

Since it is positive, bounded and  $\nabla v$  is also bounded, it follows from [21] that  $v$  is nondecreasing in the variable  $x_N$ . But by [4, Theorem E], the function  $v$  satisfies

$$v(x) \leq 2^{\frac{2}{p-1}} x_N^{-\frac{2}{p-1}}, \tag{2.10}$$

which is impossible because  $x_N \mapsto v(\cdot, x_N)$  is nondecreasing. This ends the proof.  $\square$

*Proof of Proposition 2.1.* We use [3, Lemma 4.4] with  $r = \frac{2|x|}{3}$  in  $\Omega \cap (B_{\frac{4|x|}{3}} \setminus B_{\frac{2|x|}{3}})$ .  $\square$

*Proof of Theorem 1.1.* It follows from standard regularity results and scaling techniques (see e.g. [29, Lemma 3.3.2]).  $\square$

## 2.2 Removability

**Theorem 2.4** *Let  $\Omega$  be a bounded smooth domain such that  $0 \in \partial\Omega$ ,  $\frac{N+1}{N-1} < p < \frac{N+2}{N-2}$  and either  $1 < q < \frac{2p}{p+1}$  and  $m > 0$ , or  $q = \frac{2p}{p+1}$  and  $0 < m < \epsilon^*$ . If  $u \in C^1(\bar{\Omega} \setminus \{0\}) \cap C^2(\Omega)$  is a positive solution of (1.1) in*

$\Omega$  vanishing on  $\partial\Omega \setminus \{0\}$ , then  $u \in L^p_\rho(\Omega)$ ,  $\nabla u \in L^q_\rho(\Omega)$  and the equation (1.1) holds in the sense that

$$\int_{\Omega} (-u\Delta\zeta + (u^p - m|\nabla u|^q)\zeta) dx = 0 \quad \text{for all } \zeta \in \mathbb{X}(\Omega). \quad (2.11)$$

*Proof.* Under the assumptions on  $p$  and  $q$ , there holds

$$u(x) \leq c|x|^{-\alpha} \quad \text{for all } 0 < |x| \leq R. \quad (2.12)$$

Since  $p > \frac{N+1}{N-1} > \frac{N+2}{N}$ , the function  $u$  belongs to  $L^1(\Omega) \cap L^p_\rho(\Omega)$ . It follows by Theorem 1.1 that,

$$|\nabla u(x)| \leq c|x|^{-\alpha-1} \quad \text{for all } 0 < |x| \leq \frac{R}{2}. \quad (2.13)$$

Since  $q \leq \frac{2p}{p+1}$  we have that  $|\nabla u(x)|^q \leq c^q|x|^{-\alpha-2}$ . Hence  $\nabla u$  belongs to  $L^q_\rho(\Omega)$ . Finally, let  $\{\zeta_n\}$  be a sequence of smooth functions such that  $0 \leq \zeta_n \leq 1$ ,  $\zeta_n(x) = 0$  if  $|x| \leq n^{-1}$ ,  $\zeta_n(x) = 1$  if  $|x| \geq 2n^{-1}$  with  $|\nabla\zeta_n(x)| \leq cn$  and  $|\Delta\zeta_n(x)| \leq cn^2$ . Let  $\phi \in \mathbb{X}(\Omega)$ . We have that

$$\begin{aligned} \int_{\Omega} u\Delta(\zeta_n\phi)dx &= \int_{\Omega} \zeta_n u\Delta\phi dx + \int_{\Omega} \phi u\Delta\zeta_n dx + 2 \int_{\Omega} u\nabla\phi \cdot \nabla\zeta_n dx \\ &= I(n) + II(n) + III(n). \end{aligned} \quad (2.14)$$

Clearly

$$I(n) \rightarrow \int_{\Omega} u\Delta\phi dx \quad \text{as } n \rightarrow \infty.$$

If  $\phi \in \mathbb{X}(\Omega)$ ,  $\rho^{-1}\phi$  is bounded in  $\Omega$ , hence

$$\begin{aligned} |II(n)| &\leq c_1 n^2 \|\rho^{-1}\phi\|_{L^\infty} \left( \int_{n^{-1} \leq |x| \leq 2n^{-1}} u^p \rho dx \right)^{\frac{1}{p}} \left( \int_{n^{-1} \leq |x| \leq 2n^{-1}} \rho dx \right)^{\frac{1}{p'}} \\ &\leq c'_1 n^{2 - \frac{N+1}{p'}} \|\rho^{-1}\phi\|_{L^\infty} \left( \int_{n^{-1} \leq |x| \leq 2n^{-1}} u^p \rho dx \right)^{\frac{1}{p}}. \end{aligned}$$

Since  $p > \frac{N+1}{N-1}$ ,  $p' < \frac{N+1}{2}$ , hence  $|II(n)| \rightarrow 0$  when  $n \rightarrow \infty$ . For the last term, we have from Theorem 1.1,

$$\begin{aligned} |III(n)| &\leq c_2 n \|\nabla\phi\|_{L^\infty} \int_{n^{-1} \leq |x| \leq 2n^{-1}} u dx \\ &\leq c'_2 n \|\nabla\phi\|_{L^\infty} \int_{n^{-1} \leq |x| \leq 2n^{-1}} |x|^{-\alpha-1} \rho dx \\ &\leq c''_2 n^{\alpha+1-N} \|\nabla\phi\|_{L^\infty}. \end{aligned}$$

Since  $p > \frac{N+1}{N-1}$ ,  $\alpha + 1 - N < 0$ , we deduce that  $|III(n)| \rightarrow 0$  when  $n \rightarrow \infty$ . Therefore there holds

$$\int_{\Omega} (u^p - m|\nabla u|^q) \phi \zeta_n dx \rightarrow \int_{\Omega} (u^p - m|\nabla u|^q) \phi dx,$$

we obtain the claim.  $\square$

### 2.3 Proof of Corollary 1.2

The proof is an easy but technical adaptation of the computations in [3, Theorems 1.1, 1.2] and [22, Theorem 3.25], but for the sake of completeness, we briefly recall its technique. Since  $\Omega$  is in normal position with respect to  $0$  there exist a bounded open neighborhood  $G$  of  $0$  and a smooth function  $\phi : G \cap \partial\mathbb{R}_+^N \mapsto \mathbb{R}$  such that

$$G \cap \partial\Omega = \{x = (x', x_N) : x' \in G \cap \partial\mathbb{R}_+^N \text{ and } x_N = \phi(x')\}.$$

Furthermore  $\phi(x') = 0(|x'|^2)$ ,  $\nabla\phi(x') = 0(|x'|)$  and  $|D^2\phi(x')| \leq c$  if  $x' \in G \cap \partial\mathbb{R}_+^N$ . If  $u$  satisfies (1.1), we denote

$$u(x) = \tilde{u}(y) \text{ with } y_i = x_i \text{ when } 1 \leq i \leq N-1 \text{ and } y_N = x_N - \phi(x').$$

If we set  $r = |y|$ ,  $s = y/r$ ,  $t = \ln r$  and  $v(t, s) = r^\alpha \tilde{u}(r, s)$ , then  $v$  is bounded in  $C^2((-\infty, T_0] \times S_+^{N-1})$  and vanishes on  $(-\infty, T_0] \times \partial S_+^{N-1}$ . Using the computations in [22, Theorem 3.25] and [3, Lemma 6.1], it satisfies, with  $\mathbf{n} = \frac{y}{|y|}$ ,

$$\begin{aligned} (1 + \epsilon_1)v_{tt} + \Delta'v - (N - 2 + 2\alpha + \epsilon_2)v_t + (\alpha(N - 2 - \alpha) + \epsilon_3)v \\ + \Delta'v + \nabla'v \cdot \vec{\epsilon}_4 + \nabla'v_t \cdot \vec{\epsilon}_5 + \nabla'(\nabla'v \cdot \mathbf{e}_N) \cdot \vec{\epsilon}_6 + v^p \\ - me^{\frac{2p-q(p+1)}{p-1}t} [(v_t - \alpha v)\mathbf{n} + \nabla'v + ((v_t - \alpha v)\mathbf{n} + \nabla'v \cdot \mathbf{e}_N) \cdot \vec{\epsilon}_7]^q = 0, \end{aligned} \quad (2.15)$$

where  $\mathcal{B} := \{\mathbf{e}_1, \dots, \mathbf{e}_N\}$  denotes the canonical orthogonal basis in  $\mathbb{R}^N$ . The functions  $\epsilon_j$  (or  $\vec{\epsilon}_j$ ) are uniformly continuous and bounded for  $j = 1, \dots, 7$  and there holds

$$\begin{aligned} |\epsilon_j(t, \cdot)| &\leq ce^t && \text{for } j = 1, \dots, 7 \\ |\epsilon_{jt}(t, \cdot)| + |\nabla'\epsilon_j(t, \cdot)| &\leq ce^t && \text{for } j = 1, 5, 6, 7. \end{aligned} \quad (2.16)$$

By since  $v, v_t$  and  $\nabla'v$  are uniformly bounded, we infer by standard regularity results (see e.g. [14]) the following uniform estimate,

$$\|v(t, \cdot)\|_{C^{2,\tau}(\overline{S_+^{N-1}})} + \|v(t, \cdot)_t\|_{C^{1,\tau}(\overline{S_+^{N-1}})} + \|v_{tt}(t, \cdot)\|_{C^{0,\tau}(\overline{S_+^{N-1}})} \leq c \quad (2.17)$$

for any  $t \leq T_0$ , for some  $c > 0$  and  $\tau \in (0, 1)$ . Hence the limit set at  $-\infty$  of the trajectory  $\{v(t, \cdot)\}_{t \leq T_0}$  in  $C^2(\overline{S_+^{N-1}})$  is a connected non-empty compact subset of  $\{\omega \in C^2(\overline{S_+^{N-1}}) : \omega|_{\partial S_+^{N-1}} = 0\}$ . Next we write (2.15) under the form

$$v_{tt} + \Delta'v - (N - 2 + 2\alpha)v_t + \alpha(N - 2 - \alpha)v + v^p = e^{\theta t}\Theta, \quad (2.18)$$

where  $\Theta$  is bounded and  $\theta = \min\left\{1, \frac{2p-q(p+1)}{p-1}\right\}$ . Since  $N - 2 + 2\alpha \neq 0$ , the standard energy method (multiplication by  $v_t$ ) yields

$$\int_{-\infty}^{T_0} \int_{S_+^{N-1}} (v_t^2 + v_{tt}^2) dS dt < \infty.$$

Since  $v_t$  and  $v_{tt}$  are uniformly continuous, the above integrability condition yields

$$\lim_{t \rightarrow -\infty} \left( \|v_t(t, \cdot)\|_{L^2(S_+^{N-1})} + \|v_{tt}(t, \cdot)\|_{L^2(S_+^{N-1})} \right) = 0. \quad (2.19)$$

Therefore the limit set of the trajectory at  $-\infty$  is a compact connected subset of nonnegative solutions of (1.11). This implies that either  $v(t, \cdot)$  converges to the unique positive solution  $\psi$  of (1.11) in  $C^2(\overline{S_+^{N-1}})$  or it converges to 0. Note that the set of nonnegative solutions of (1.11) is reduced to 0 when  $1 < p \leq \frac{N+1}{N-1}$ .

If  $\frac{N+1}{N-1} < p < \frac{N+2}{N-2}$  and  $v(t, \cdot)$  does not converge to 0, then we have proved (1.10). If  $v(t, \cdot)$  converges to 0, then the proof of [3, Theorem 7.1] applies, the only difference being in the value of the term  $H$  therein [3, (7.3)] which is replaced by  $e^{\theta t} \Theta$  defined above. The remaining of the argument can be easily adapted.

If  $p = \frac{N+1}{N-1}$  then  $v(t, \cdot)$  converges to 0. The adaptation of [3, Theorem 9.1] is easy. We obtain that  $u$  satisfies

$$u(x) \leq c|x|^{1-N} \left( \ln \frac{1}{|x|} \right)^{-\frac{N-1}{2}} \quad \text{for all } x \in \Omega. \quad (2.20)$$

The completion of the proof follows by the same perturbation method as in [3, Lemma 9.1], by decomposing the function  $v(t, \cdot)$  into  $v(t, \cdot) = v_1 + v_2(t, \cdot)$  where  $v_1 \in \ker(\Delta' + (N-1)I)$  and  $v_2 \in (\ker(\Delta' + (N-1)I))^\perp$ . This yields

$$\|v_1(t, \cdot)\|_{L^2(S_+^{N-1})} \leq c(-t)^{-\frac{N-1}{2}} \quad \text{and} \quad \|v_2(t, \cdot)\|_{L^2(S_+^{N-1})} \leq ce^{\frac{\theta}{2}t} \quad \text{for } t \leq T_0. \quad (2.21)$$

The function  $w(t, s) = (-t)^{\frac{N-1}{2}} v(t, s)$  satisfies

$$\begin{aligned} w_{tt} - \left(N + \frac{N-1}{t}\right) w_t + \left(N - 1 + \frac{N^2-1}{4t^2}\right) w + \Delta' w \\ - \frac{1}{t} \left( w^{\frac{N+1}{N-1}} - \frac{N(N-1)}{2} w \right) = (-t)^{\frac{N-1}{2}} \Theta, \end{aligned} \quad (2.22)$$

where  $\Theta$  is bounded. The proof given in [3, Theorem 9.1] applies with almost no change, but for some straightforward ones. The main step is to introduce

$$z(t) = \int_{S_+^{N-1}} w(t, s) \phi_1(s) dS,$$

and to prove that  $z(t)$  admits a nonnegative limit  $\lambda \geq 0$  when  $t \rightarrow -\infty$ . If this limit is positive its value  $\lambda$  is given in the proof of [3, Theorem 1.3]. If this limit is zero, then

$$\lim_{y \rightarrow 0} |y|^{N-1} \left( \ln \frac{1}{|y|} \right)^{\frac{N-1}{2}} \tilde{u}(y) = 0,$$

and the conclusion follows easily from the proof [3, Theorem 7.2] (only the exponent in the perturbation term  $H$  therein is changed).

If  $1 < p < \frac{N+1}{N-1}$ , then  $v(t, \cdot)$  converges to 0 and (2.18) can be written under the form

$$v_{tt} + \Delta' v - (N - 2 + 2\alpha)v_t + \alpha(N - 2 - \alpha + \epsilon(t))v + \Delta' v = 0, \quad (2.23)$$

where  $\epsilon(t) \rightarrow 0$  when  $t \rightarrow -\infty$ . It is therefore a very standard but technical method of linearization [15, Theorem 5.1] to obtain, first an exponential decay of  $w(t, \cdot)$  at  $-\infty$ , and then the convergence of  $t \mapsto e^{(N-1-\alpha)t}v(t, \cdot)$  to  $k\phi_1$  for some  $k \geq 0$ , and then to deduce the regularity of  $u$  if  $k = 0$ .  $\square$

### 3 Separable solutions

#### 3.1 Separable solutions in $\mathbb{R}^N$

*Proof of Proposition 1.3.* Constant positive solutions of (1.16) are any positive roots of

$$\Phi(X) := X^{p-1} - m\alpha^{\frac{2p}{p+1}}X^{\frac{p-1}{p+1}} - \alpha(N - 2 - \alpha) = 0. \quad (3.1)$$

Set

$$\Phi(X) = \tilde{\Phi}(X^{\frac{p-1}{p+1}}), \quad (3.2)$$

where

$$\tilde{\Phi}(Y) = Y^{p+1} - m\alpha^{\frac{2p}{p+1}}Y - \alpha(N - 2 - \alpha). \quad (3.3)$$

Then  $\tilde{\Phi}'(Y) = (p+1)Y^p - m\alpha^{\frac{2p}{p+1}}$ , hence if  $m \leq 0$ ,  $\tilde{\Phi}$  is increasing and if  $m > 0$ ,  $\tilde{\Phi}$  is decreasing on  $[0, Y_0]$  and increasing on  $(Y_0, \infty)$  with

$$Y_0 = \left(\frac{m}{p+1}\right)^{\frac{1}{p}} \alpha^{\frac{2}{p+1}}. \quad (3.4)$$

From now we always assume  $m > 0$ . Then

$$\tilde{\Phi}(Y_0) = \left[ N - p(N-2) - 2p \left(\frac{m}{p+1}\right)^{\frac{p+1}{p}} \right] \frac{2}{(p-1)^2},$$

and

$$\tilde{\Phi}(0) = -\alpha(N-2-\alpha) = \frac{2(N-2)}{(p-1)^2} \left( \frac{N}{N-2} - p \right).$$

Therefore,  $\tilde{\Phi}(0) \leq 0$  if and only if  $p \geq \frac{N}{N-2}$ . In that case there exists a unique  $X_m > 0$  such that  $\Phi(X_m) = 0$ .

When

$$0 < \frac{N}{N-2} - p < \frac{2p}{N-2} \left(\frac{m}{p+1}\right)^{\frac{p+1}{p}}, \quad (3.5)$$

then  $\tilde{\Phi}(0) > 0$  and  $\tilde{\Phi}(Y_0) < 0$ , thus  $\tilde{\Phi}$  admits two positive roots. The same property is shared by  $\Phi$ , hence there exist  $X_{j,m}$ , for  $j = 1, 2$  such that  $\Phi(X_{j,m}) = 0$  and  $0 < X_{1,m} < Y_0^{\frac{p+1}{p-1}} < X_{2,m}$ .

When

$$0 = \frac{2p}{N-2} \left(\frac{m}{p+1}\right)^{\frac{p+1}{p}} \iff \left(\frac{m}{p+1}\right)^{\frac{p+1}{p}} = \frac{N-p(N-2)}{2p}, \quad (3.6)$$



then  $\tilde{\Phi}$  admits a unique positive root. Hence  $\Phi > 0$  on  $\mathbb{R}_+ \setminus \{X_{m^*}\}$  and vanishes at  $X_{m^*}$ , where

$$X_{m^*} = \left( \frac{m^*}{p+1} \right)^{\frac{p+1}{p'}} \alpha^{\frac{2}{p-1}} \quad \text{with } m^* = (p+1) \left( \frac{N-p(N-2)}{2p} \right)^{\frac{p}{p+1}}. \quad (3.7)$$

If  $0 < m < m^*$ ,  $\tilde{\Phi}$  and thus  $\Phi$  are positive on  $\mathbb{R}_+$ , hence there exists no root to  $\Phi$ . The proof of Proposition 1.3 is complete.  $\square$

### 3.2 Separable solutions in $\mathbb{R}_+^N$

If  $u$  is a nonnegative separable solution of (1.1) in  $\mathbb{R}_+^N$  which vanishes on  $\partial\mathbb{R}_+^N \setminus \{0\}$ , the function  $\omega$  is a nonnegative solution of (1.19).

*Proof of Theorem 1.4.* If  $1 < \frac{2p}{p+1} < \frac{N+1}{N}$ , equivalently  $1 < p < \frac{N+1}{N-1}$ , it is proved in [22, Theorem 3.21] that there exists a unique positive function  $\eta := \eta_m \in C^2(\overline{S_+^{N-1}})$  satisfying

$$\begin{aligned} -\Delta' \eta + \alpha(N-2-\alpha)\eta + m(\alpha^2 \eta^2 + |\nabla' \eta|^2)^{\frac{p}{p+1}} &= 0 & \text{in } S_+^{N-1} \\ \eta &= 0 & \text{on } \partial S_+^{N-1}. \end{aligned} \quad (3.8)$$

By uniqueness,  $\eta_m = m^{-\frac{p+1}{p-1}} \eta_1$ , and by the maximum principle

$$m^{-\frac{p+1}{p-1}} \sup_{S_+^{N-1}} \eta_1 = \sup_{S_+^{N-1}} \eta_m \leq \frac{1}{\alpha} \left( \frac{\alpha+2-N}{m} \right)^{\frac{p+1}{p-1}}. \quad (3.9)$$

If  $\bar{\eta}_m = \sup_{S_+^{N-1}} \eta_m$ , then

$$-m\alpha^{\frac{2p}{p+1}} \bar{\eta}_m^{\frac{2p}{p+1}} - \alpha(N-2-\alpha)\bar{\eta}_m \geq 0.$$

Hence  $\Phi(\bar{\eta}_m) > 0$ , where  $\Phi$  has been defined in (3.1). Therefore

$$\begin{aligned} (i) \quad & \text{either} \quad \bar{\eta}_m > X_{2,m} \quad (\text{resp. } \bar{\eta}_{m^*} > X_{m^*}), \\ (ii) \quad & \text{or} \quad \bar{\eta}_m < X_{1,m} \quad (\text{resp. } \bar{\eta}_{m^*} < X_{m^*}). \end{aligned} \quad (3.10)$$

For  $\epsilon \in (0, 1)$ ,  $\epsilon\eta_m$  is a subsolution of (3.8), hence it is a subsolution of (1.19) too. For  $\epsilon > 0$  small enough it is smaller than  $X_{2,m}$  (resp.  $X_{m^*}$ ) and it belongs to  $W_0^{1,\infty}(S_+^{N-1})$ . By the result of Boccardo, Murat and Puel [10] there exists a solution  $\omega \in W_0^{1,2}(S_+^{N-1})$  of (1.19), and it satisfies

$$\epsilon\eta_m < \omega \leq X_{2,m} \quad (\text{resp. } \epsilon\eta_m < \omega \leq X_{m^*}). \quad (3.11)$$

For proving the second assertion, we set  $\omega = \phi^b$  for some  $b > 1$ , then

$$\begin{aligned} -\Delta' \phi - (b-1) \frac{|\nabla \phi|^2}{\phi} - \frac{\alpha(\alpha+2-N)}{b} \phi - \frac{1}{b} \phi^{1+b(p-1)} \\ + \frac{m}{b} \phi^{\frac{(p-1)(b-1)}{p+1}} (\alpha^2 \phi^2 + b^2 |\nabla \phi|^2)^{\frac{p}{p+1}} = 0. \end{aligned} \quad (3.12)$$

Since

$$(\alpha^2\phi^2 + b^2|\nabla\phi|^2)^{\frac{p}{p+1}} \leq \alpha^{\frac{2p}{p+1}}\phi^{\frac{2p}{p+1}} + b^{\frac{2p}{p+1}}|\nabla\phi|^{\frac{2p}{p+1}}, \quad (3.13)$$

(3.12) implies

$$\begin{aligned} -\Delta'\phi + \frac{m\alpha^{\frac{2p}{p+1}}}{b}\phi^{1+b\frac{p-1}{p+1}} + mb^{\frac{p-1}{p+1}}\phi^{\frac{(b-1)(p-1)}{p+1}}|\nabla\phi|^{\frac{2p}{p+1}} \\ \geq (b-1)\frac{|\nabla\phi|^2}{\phi} + \frac{1}{b}\phi^{1+b(p-1)} + \frac{\alpha(\alpha+2-N)}{b}\phi. \end{aligned} \quad (3.14)$$

For any  $\theta > 0$  we have by Hölder's inequality,

$$mb^{\frac{p-1}{p+1}}\phi^{\frac{(b-1)(p-1)}{p+1}}|\nabla\phi|^{\frac{2p}{p+1}} \leq \frac{mpb^{\frac{p-1}{p+1}}}{(p+1)\theta^{\frac{p+1}{p}}}\frac{|\nabla\phi|^2}{\phi} + \frac{mb^{\frac{p-1}{p+1}}\theta^{p+1}}{p+1}\phi^{1+b(p-1)},$$

we deduce the inequality

$$\begin{aligned} -\Delta'\phi \geq \left(b-1 - \frac{mpb^{\frac{p-1}{p+1}}}{(p+1)\theta^{\frac{p+1}{p}}}\right)\frac{|\nabla\phi|^2}{\phi} + \frac{1}{b}\left(1 - \frac{mb^{\frac{p-1}{p+1}}\theta^{p+1}}{p+1}\right)\phi^{1+b(p-1)} \\ + \frac{\alpha(\alpha+2-N)}{b}\phi. \end{aligned} \quad (3.15)$$

If the following two conditions are satisfied

$$\begin{aligned} (i) \quad b-1 - \frac{mpb^{\frac{p-1}{p+1}}}{(p+1)\theta^{\frac{p+1}{p}}} &\geq 0, \\ (ii) \quad 1 - \frac{mb^{\frac{p-1}{p+1}}\theta^{p+1}}{p+1} &\geq 0, \end{aligned} \quad (3.16)$$

we infer that there holds

$$(N-1)\int_{S_+^{N-1}}\phi\phi_1dS > \frac{\alpha(\alpha+2-N)}{b}\int_{S_+^{N-1}}\phi\phi_1dS, \quad (3.17)$$

where  $\phi_1$  denotes the first normalized and positive eigenfunction of  $-\Delta'$  in  $W_0^{1,2}(S_+^{N-1})$ , with corresponding eigenfunction  $\lambda_1 = N-1$ . Hence, if (3.16) is verified and there holds

$$N-1 \leq \frac{\alpha(\alpha+2-N)}{b}, \quad (3.18)$$

there exists no positive solution. We proceed as follows for solving (3.16)-(3.18). If  $1 < p < \frac{N+1}{N-1}$ , then  $\alpha(\alpha+2-N) > N-1$ . We define  $b_p > 1$  by

$$b_p = \frac{\alpha(\alpha+2-N)}{N-1}. \quad (3.19)$$

For such  $b = b_p$ , the optimality is achieved in (3.16) when  $b_p - 1 = \frac{mpb_p^{\frac{p-1}{p+1}}}{(p+1)\theta^{\frac{p+1}{p}}}$  and  $1 = \frac{mb_p^{\frac{p-1}{p+1}}\theta^{p+1}}{p+1}$ . This gives an implicit maximal value of

$m_p$  through the relation

$$m_p = \frac{(p+1)(b_p-1)\theta^{\frac{p+1}{p}}}{pb_p^{\frac{p-1}{p+1}}} = \frac{p+1}{b_p^{\frac{2p}{p+1}}\theta^{p+1}}. \quad (3.20)$$

Then the value of the corresponding  $\theta := \theta_p$  is expressed by

$$\theta_p = \frac{p}{b_p(b_p-1)},$$

and we infer

$$m_p = \frac{p+1}{b_p^{\frac{2p}{p+1}}\theta_p^{p+1}} = \frac{(p+1)}{p^{p+1}}(b_p-1)^{p+1}b_p^{\frac{p^2+1}{p+1}}. \quad (3.21)$$

Hence if  $m \leq m_p$  problem (3.13) admits no positive solution.  $\square$

*Remark.* The case  $p \geq \frac{N+1}{N-1}$  is open. It can be noticed that the constant solution  $X_m$  obtained in Proposition 1.3-(i) cannot be used as a super-solution for solving problem (1.19) as it is done in Theorem 1.4. If  $\omega$  is a positive solution of (1.19) and  $\bar{\omega}$  is its maximal value, then

$$-\Delta\bar{\omega} = \bar{\omega}\Phi(\bar{\omega}).$$

Hence  $\Phi(\bar{\omega}) \geq 0$  which implies that  $\bar{\omega} > X_m$ .

## 4 Boundary data measures

### 4.1 Sufficient conditions

We associate to (1.2) the following two problems

$$\begin{aligned} -\Delta v + m|\nabla v|^q &= 0 && \text{in } \Omega \\ v &= \mu && \text{in } \partial\Omega, \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} -\Delta w - w^p &= 0 && \text{in } \Omega \\ w &= \mu && \text{in } \partial\Omega. \end{aligned} \quad (4.2)$$

Problem (4.1) has been solved in the case  $1 < q < \frac{N+1}{N}$  in [22]. There, it is proved that for any nonnegative bounded measure  $\mu$  on  $\partial\Omega$  there exists a weak solution  $v_\mu$  to (4.1). Furthermore the correspondance  $\mu \mapsto v_\mu$  is sequentially stable. When  $\frac{N+1}{N} \leq q < 2$  it is proved in [8, Theorem 1.6] that if a measure  $\mu$  satisfies

$$|\mu|(K) \leq C_1 \text{Cap}_{\frac{2-q}{2}, q}^{\partial\Omega}(K) \quad \text{for any compact set } K \subset \partial\Omega, \quad (4.3)$$

then there exists  $\epsilon_0 > 0$  such that for any  $0 < \epsilon \leq \epsilon_0$  there exists a solution  $v_{\epsilon\mu}$  to (4.1) (i.e. with  $\mu$  replaced by  $\epsilon\mu$ ).

Problem (4.2) has been solved in the case  $1 < p < \frac{N+1}{N-1}$  in [9] where it is proved that for any nonnegative measure  $\mu$  there exists  $\epsilon_1 > 0$  such that for any  $0 < \epsilon \leq \epsilon_1$  there exists a positive solution  $w := w_{\epsilon\mu}$  to (4.2)

provided  $\mu$  is replaced by  $\epsilon\mu$ . In the supercritical case  $p \geq \frac{N+1}{N-1}$  it is shown in [8, Theorem 1.6] that if a positive measure  $\mu$  satisfies

$$\mu(K) \leq C_2 \text{Cap}_{\frac{2}{p}, p'}^{\partial\Omega}(K) \quad \text{for any compact set } K \subset \partial\Omega, \quad (4.4)$$

then existence of a positive solution  $w_{\epsilon\mu}$  to problem (4.2) holds with  $\mu$  replaced by  $\epsilon\mu$ , under the condition  $0 < \epsilon \leq \epsilon_2$ , for some  $\epsilon_2 > 0$  depending on  $\mu$ .

*Proof of Theorem 1.6.* We assume that (1.21) holds and we set  $\epsilon_3 = \min\{\epsilon_0, \epsilon_1, \epsilon_2\}$ , take  $\epsilon \leq \epsilon_3$  and for the sake of clarity, replace  $\epsilon\mu$  by  $\mu$ . We denote by  $v_\mu$  and  $w_\mu$  the solutions of (4.1) and (4.2) respectively with boundary data  $\mu$ . Since there holds

$$v_\mu \leq \mathbb{P}_\Omega[\mu] \leq w_\mu,$$

and  $v_\mu$  is a subsolution of (1.1) and  $w_\mu$  a supersolution in  $\Omega$ , it follows from [29, Theorem 1.4.6] that there exists a solution  $u$  to (1.1) such that  $v_\mu \leq u \leq w_\mu$ . This implies that  $u \in L^1(\Omega)$  and  $u^p \in L^1_\rho(\Omega)$ . Because  $v$  and  $w$  satisfy

$$\lim_{\delta \rightarrow 0} \int_{\{\rho(x)=\delta\}} v Z dS = \lim_{\delta \rightarrow 0} \int_{\{\rho(x)=\delta\}} w Z dS = \int_{\partial\Omega} Z d\mu \quad (4.5)$$

for any  $Z \in C(\overline{\Omega})$ , it follows that

$$\lim_{\delta \rightarrow 0} \int_{\{\rho(x)=\delta\}} u Z dS = \int_{\partial\Omega} Z d\mu. \quad (4.6)$$

Let  $\phi_\delta$  be the first eigenfunction of  $-\Delta$  in  $W_0^{1,2}(\Omega'_\delta)$  ( $\Omega'_\delta$  is defined in (5.26)), normalized by  $0 \leq \phi_\delta \leq 1 = \max\{\phi_\delta(x) : x \in \Omega'_\delta\}$  and  $\lambda_\delta$  the eigenvalue. Then

$$m \int_{\Omega'_\delta} |\nabla u|^q \phi_\delta dx = \int_{\Omega'_\delta} (u^p - \lambda_\delta u) \phi_\delta dx - \int_{\Sigma_\delta} \frac{\partial \phi_\delta}{\partial \mathbf{n}} u(x) dS. \quad (4.7)$$

Because  $\phi_\delta \rightarrow \phi := \phi_0$  and  $\lambda_\delta \rightarrow \lambda := \lambda_0$ , and the left-hand side of (4.7) is convergent, it follows by Fatou's lemma that

$$m \int_{\Omega} |\nabla u|^q \phi dx \leq \int_{\Omega} (u^p - \lambda u) \phi dx - \int_{\partial\Omega} \frac{\partial \phi}{\partial \mathbf{n}} d\mu.$$

Hence  $\nabla u \in L^q_\rho(\Omega)$ , thus (1.20) holds and this ends the proof.  $\square$

In several cases the sufficient condition can be weakened either by comparison between capacities or because one at least of the two exponents  $p$  or  $q$  is subcritical.

*Proof of Corollary 1.7.* It follows easily from [9], [22] and the previous theorem.  $\square$

*Proof of Corollary 1.8 1-* As in the proof of [6, Corollary 1.5], we have from [1, Theorem 5.5.1]

$$\text{Cap}_{\frac{2}{p}, p'}^{\partial\Omega}(K) \leq c^* \text{Cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(K).$$

It implies the following inequality

$$\begin{aligned}\mu(K) &\leq C_3 \text{Cap}_{\frac{2}{p}, p'}^{\partial\Omega}(K) = C_3 \min \left\{ \text{Cap}_{\frac{2}{p}, p'}^{\partial\Omega}(K), c^* \text{Cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(K) \right\} \\ &\leq C_3(1 + c^*) \min \left\{ \text{Cap}_{\frac{2}{p}, p'}^{\partial\Omega}(K), \text{Cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(K) \right\}.\end{aligned}$$

2- Similarly, as in the proof of [6, Corollary 1.4], we have from [1, Theorem 5.5.1]

$$\text{Cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(K) \leq c^{**} \text{Cap}_{\frac{2}{p}, p'}^{\partial\Omega}(K),$$

therefore

$$\begin{aligned}\mu(K) &\leq C_4 \text{Cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(K) = C_4 \min \left\{ c^{**} \text{Cap}_{\frac{2}{p}, p'}^{\partial\Omega}(K), \text{Cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(K) \right\} \\ &\leq C_4(1 + c^{**}) \min \left\{ \text{Cap}_{\frac{2}{p}, p'}^{\partial\Omega}(K), \text{Cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(K) \right\}.\end{aligned}$$

This completes the proof.  $\square$

## 4.2 Necessary conditions

*Proof of Theorem 1.9.*

*Step 1: proof of (1.29).* We follow the notations of the proof of [22, Theorem 4-5]. Let  $\eta \in C^2(\partial\Omega)$  be a nonnegative function with value 1 in a neighborhood  $\mathcal{U}$  of the compact set  $K$ , and  $\zeta = (\mathbb{P}_\Omega[\eta])^{2q'} \phi$ . Then we have

$$\int_{\Omega} (|\nabla u|^q \zeta - u \Delta \zeta) dx = \int_{\Omega} u^p \zeta dx - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} dS \geq - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} dS.$$

Since  $\eta = 1$  on  $K$ , there holds by Hopf boundary lemma

$$- \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} dS \geq c_1 \mu(K).$$

The same computation as in [22, Theorem 4-5] yields, with  $\lambda = \lambda_1(\Omega)$ ,

$$c_1 \mu(K) \leq \int_{\Omega} (|\nabla u|^q + \lambda u) \zeta dx + c_2 \left(1 + \|\nabla u\|_{L^q_\rho}^q\right)^{\frac{1}{q}} \|\eta\|_{W^{\frac{2-q}{q}, q'}}. \quad (4.8)$$

Since  $\text{Cap}_{\frac{2-q}{q}, q'}^{\partial\Omega}(K) = 0$ , there exists a sequence  $\{\eta_n\} \subset C^2(\partial\Omega)$  satisfying  $0 \leq \eta_n \leq 1$  and  $\eta_n = 1$  in a neighborhood of  $K$ , such that  $\|\eta_n\|_{W^{\frac{2-q}{q}, q'}} \rightarrow 0$  as  $n \rightarrow \infty$ ; thus  $\eta_n \rightarrow 0$  in  $L^1(\partial\Omega)$  and  $\zeta_n := (\mathbb{P}_\Omega[\eta_n])^{2q'} \phi \rightarrow 0$  a.e. in  $\Omega$ . This implies that the right-hand side of (4.8) with  $\eta$  replaced by  $\eta_n$  tends to 0 as  $n \rightarrow \infty$  and thus  $\mu(K) = 0$ .

*Step 2: proof of (1.30).* We recall that a positive lifting is a mapping  $\eta \mapsto R[\eta]$  from  $C^2(\partial\Omega)$  to  $C^2(\bar{\Omega})$  satisfying

$$R[\eta]|_{\partial\Omega} = \eta \text{ and } \eta \geq 0 \implies R[\eta] \geq 0.$$

If  $\eta \in C^2(\partial\Omega)$  satisfies  $0 \leq \eta \leq 1$ ,  $\eta = 1$  in a neighborhood of  $K$  we take for test function  $\zeta = (R[\eta])^{p'}\phi$ . There holds

$$\begin{aligned} \Delta\zeta &= -\lambda\zeta + p'\phi(R[\eta])^{p'-1}\Delta R[\eta] + p'(p'-1)\phi(R[\eta])^{p'-2}|\nabla R[\eta]|^2 \\ &\quad + 2(p'-1)(R[\eta])^{p'-1}\nabla\phi \cdot \nabla R[\eta]. \end{aligned}$$

As in [18, Lemma 1.1] we have

$$-\int_{\Omega} u\Delta\zeta dx \leq \left(\int_{\Omega} u^p\zeta dx\right)^{\frac{1}{p}} \left(\lambda\left(\int_{\Omega} \zeta dx\right)^{\frac{1}{p'}} + p'\left(\int_{\Omega} |L(\eta)|^{p'} dx\right)^{\frac{1}{p'}}\right),$$

where

$$L(\eta) = |\phi^{\frac{1}{p'}}\Delta R[\eta]| + 2|\phi^{-\frac{1}{p}}\nabla\phi \cdot \nabla R[\eta]|.$$

From (1.20) we have (see [18, formula (1.2)])

$$\begin{aligned} \left(\int_{\partial\Omega} \eta d\mu\right)^{p'} + C_{\mu} \int_{\Omega} u^p\zeta dx &\leq mC_{\mu} \int_{\Omega} |\nabla u|^q \zeta dx \\ &\quad + C_{\mu} \left(\int_{\Omega} u^p\zeta dx\right)^{\frac{1}{p}} \left(\lambda\left(\int_{\Omega} \zeta dx\right)^{\frac{1}{p'}} + p'\left(\int_{\Omega} |L(\eta)|^{p'} dx\right)^{\frac{1}{p'}}\right), \end{aligned} \tag{4.9}$$

where

$$C_{\mu} = \left(\int_{\partial\Omega} \left|\frac{\partial\phi}{\partial\mathbf{n}}\right|^{-\frac{p'}{p}} d\mu\right)^{\frac{p'}{p}}.$$

The "optimal lifting" introduced in [18] has the property that the mapping  $\eta \mapsto L(\eta)$  is continuous from  $W^{\frac{2}{p}, p'}(\partial\Omega)$  into  $L^{p'}(\Omega)$ . Note that with  $R[\eta] = \mathbb{P}_{\Omega}[\eta]$ , which is a positive lifting, the continuity of the mapping  $L$  holds only when  $1 < p' < 2$ . This is why the construction in [18] is much more elaborate. We conclude as in Step 1 by considering a sequence  $\{\eta_n\} \subset C^2(\partial\Omega)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  in a neighborhood of  $K$ , such that  $\|\eta_n\|_{W^{\frac{2}{p}, p'}} \rightarrow 0$ . Then  $\eta_n \rightarrow 0$  in  $L^1(\partial\Omega)$ ,  $\zeta_n \rightarrow 0$  a.e. and  $L(\eta_n) \rightarrow 0$  in  $L^{p'}(\Omega)$ . Thus the right-hand side of (4.9) tends to 0. This ends the proof.  $\square$

*Remark.* We conjecture that (1.30) could be strengthened and replaced by: There exists a constant  $c > 0$  such that

$$\mu(K) \leq cCap_{\frac{2}{p}, p'}^{\partial\Omega}(K) \quad \text{for any compact set } K \subset \partial\Omega. \tag{4.10}$$

This is a necessary condition when  $m = 0$  (see [8]).

## 5 The boundary trace

### 5.1 The regular boundary trace

*Proof of Theorem 1.11.* Set  $u = v^b$  for some  $b > 1$ , then we have that

$$-\Delta v = (b-1)\frac{|\nabla v|^2}{v} + \frac{1}{b}v^{1+b(p-1)} - mb^{q-1}v^{(q-1)(b-1)}|\nabla v|^q := F. \tag{5.1}$$

By Hölder's inequality,

$$mb^{q-1}v^{(q-1)(b-1)}|\nabla v|^q \leq \frac{b-1}{2} \frac{|\nabla v|^2}{v} + mb^{q-1} \left( \frac{2mb^{q-1}}{b-1} \right)^{\frac{q}{2-q}} v^{\frac{2b(q-1)}{2-q}+1}. \quad (5.2)$$

*Case 1:*  $q < \frac{2p}{p+1}$ . There holds  $\frac{2b(q-1)}{2-q} + 1 < 1 + b(p-1)$  independently of  $b$ . Hence for any  $\delta > 0$  there exists  $C = C(\delta, b, m, p, q) > 0$  such that

$$mb^{q-1} \left( \frac{2mb^{q-1}}{b-1} \right)^{\frac{q}{2-q}} v^{\frac{2b(q-1)}{2-q}+1} \leq \frac{\delta}{b} v^{1+b(p-1)} + C. \quad (5.3)$$

Therefore

$$F \geq \frac{b-1}{2} \frac{|\nabla v|^2}{v} + \frac{1-\delta}{b} v^{1+b(p-1)} - C. \quad (5.4)$$

If  $\psi = \mathbb{G}_{B_R}[1]$  (ie. the solution of  $-\Delta\psi = 1$  in  $B_R$  vanishing on  $\partial B_R$ ), we have

$$-\Delta(v + C\psi) \geq \frac{b-1}{2} \frac{|\nabla v|^2}{v} + \frac{1-\delta}{b} v^{1+b(p-1)} \geq 0.$$

By Doob's theorem on positive superharmonic functions (see [12]) we have that  $\frac{|\nabla v|^2}{v} + v^{1+b(p-1)} \in L^1_\rho(\Omega)$ . We put  $a = b^{-1} - 1$ , then  $a < 0$  and  $v = u^{\frac{1}{b}} = u^{1+a}$ . Therefore

$$\nabla v = (1+a)u^a \nabla u, \quad \frac{|\nabla v|^2}{v} = (1+a)^2 u^{a-1} |\nabla u|^2 \text{ and } v^{1+b(p-1)} = u^{p+a},$$

consequently

$$u^{a-1} |\nabla u|^2 + u^{p+a} \in L^1_\rho(\Omega).$$

Let  $1 < \ell < \frac{2p}{p+1} < 2$ , then

$$\begin{aligned} \int_\Omega |\nabla u|^\ell \rho dx &= \int_\Omega |u^{\frac{a-1}{2}} \nabla u|^\ell u^{\frac{(1-a)\ell}{2}} \rho dx \\ &\leq \epsilon \int_\Omega u^{a-1} |\nabla u|^2 \rho dx + C(\epsilon) \int_\Omega u^{\frac{(1-a)\ell}{2-\ell}} \rho dx. \end{aligned} \quad (5.5)$$

We fix  $a < 0$  such that  $\frac{(1-a)\ell}{2-\ell} = p + a$ , or equivalently

$$a = -\frac{p+1}{2} \left( \frac{2p}{p+1} - \ell \right).$$

Finally, we infer that for any  $\ell < \frac{2p}{p+1}$ ,  $|\nabla u|^\ell \in L^1_\rho(\Omega)$ . This implies in particular that  $|\nabla u|^q \in L^1_\rho(\Omega)$ .

Let  $\Psi = m\mathbb{G}_\Omega[|\nabla u|^q]$ , then  $\Psi > 0$  and

$$-\Delta(u + \Psi) = u^p.$$

Clearly the function  $u + \Psi$  is positive and superharmonic in  $\Omega$ . By using again Doob's theorem [12], it follows that  $[-\Delta(u + \Psi)] = u^p \in L^1_\rho(\Omega)$  and there exists a nonnegative Radon measure  $\mu$  on  $\partial\Omega$  such that

$$u = \mathbb{G}_\Omega[u^p] - \Psi + \mathbb{P}_\Omega[\mu] = \mathbb{G}_\Omega[u^p - m|\nabla u|^q] + \mathbb{P}_\Omega[\mu],$$

where  $\mathbb{P}_\Omega$  is the Poisson operator in  $\Omega$ . This implies that (1.24) holds.

*Case 2:*  $q = \frac{2p}{p+1}$ . We proceed as in the proof of Theorem 1.11, setting  $u = v^b$ ,  $b > 1$ . Since  $q = \frac{2p}{p+1}$ , inequality (5.2) becomes

$$mb^{\frac{p-1}{p+1}} v^{\frac{(p-1)(b-1)}{p+1}} |\nabla v|^{\frac{2p}{p+1}} \leq \frac{b-1}{2} \frac{|\nabla v|^2}{v} + mb^{\frac{p-1}{p+1}} \left( \frac{2mb^{\frac{p-1}{p+1}}}{b-1} \right)^p v^{1+b(p-1)}. \quad (5.6)$$

Defining  $m_1$  by the identity,

$$m_1 = \left( \frac{b-1}{2b} \right)^{\frac{p}{p+1}}, \quad (5.7)$$

we deduce that for  $0 < m < m_1$  and some  $\delta \in (0, 1)$ , there holds

$$-\Delta v \geq \frac{b-1}{2} \frac{|\nabla v|^2}{v} + \frac{b-1}{2} v^{1+b(p-1)}. \quad (5.8)$$

Again, by Doob's theorem,  $\frac{|\nabla v|^2}{v} + v^{1+b(p-1)} \in L_\rho^1(\Omega)$ , which implies that  $\sqrt{v} \in W_\rho^{1,2}(\Omega)$ . Using Sobolev type imbedding theorem for weighted Sobolev spaces (see e.g. [16, Section 19]),

$$\left( \int_\Omega (\sqrt{v})^{\frac{2(N+1)}{N-1}} \rho dx \right)^{\frac{N-1}{N}} \leq c \int_\Omega ((\sqrt{v})^2 + |\nabla \sqrt{v}|^2) \rho dx. \quad (5.9)$$

If we choose in particular  $b = \frac{N+1}{p(N-1)}$  we deduce that  $u^p \in L_\rho^1(\Omega)$ . Actually, for any  $1 \leq \tilde{p} < \frac{N+1}{N-1}$ ,  $u^{\tilde{p}} \in L_\rho^1(\Omega)$  and for any  $\epsilon > 0$ ,  $\frac{|\nabla u|^2}{u^{1+\epsilon}} \in L_\rho^1(\Omega)$ . We have from (5.5) with  $\ell = \frac{2p}{p+1}$ ,

$$\begin{aligned} \int_\Omega |\nabla u|^{\frac{2p}{p+1}} \rho dx &= \int_\Omega u^{-\frac{(1+\epsilon)p}{p+1}} |\nabla u|^{\frac{2p}{p+1}} u^{\frac{(1+\epsilon)p}{p+1}} \rho dx \\ &\leq s \int_\Omega \frac{|\nabla u|^2}{u^{1+\epsilon}} \rho dx + C(s) \int_\Omega u^{(1+\epsilon)p} \rho dx. \end{aligned} \quad (5.10)$$

If  $\epsilon$  is chosen such that  $(1+\epsilon)p = \tilde{p} < \frac{N+1}{N-1}$ , we infer that  $|\nabla u|^{\frac{2p}{p+1}} \in L_\rho^1(\Omega)$ . We end the proof as in Case 1.  $\square$

*Remark.* The same regularity and boundary trace results hold if it is assumed that  $u$  is a nonnegative supersolution of 1.1 in  $\Omega$ .

## 5.2 The singular boundary trace

*Proof of Theorem 1.12. Assertion 1.* We assume that  $F := |\nabla u|^q \in L_\rho^1(B_\epsilon(z) \cap \Omega)$ . We set  $F_\epsilon = F \mathbf{1}_{B_\epsilon(z) \cap \Omega}$  and  $\Psi_\epsilon = \mathbb{G}_{B_\epsilon(z) \cap \Omega}[F_\epsilon]$ . Then  $\Psi_\epsilon$  has boundary trace zero on  $B_\epsilon \cap \partial\Omega$  and

$$-\Delta(u + m\Psi_\epsilon) = u^p \quad \text{in } B_\epsilon(z) \cap \Omega.$$

Thus  $u + m\Psi_\epsilon$  is a positive super-harmonic function in  $B_\epsilon \cap \Omega$ . Hence  $u^p \in L_\rho^1(B_\epsilon(z) \cap \Omega)$  and there exists a Radon measure  $\mu_\epsilon$  such that



$u + m\Psi_\epsilon$  admits for boundary trace  $\mu_\epsilon$  on  $B_\epsilon(z) \cap \partial\Omega$ . This implies that  $u$  admits the same boundary trace on  $B_\epsilon(z) \cap \partial\Omega$ . **1 Assertion 2.** We assume that  $H := u^p \mathbf{1}_{B_\epsilon(z) \cap \Omega} \in L_\rho^1(B_\epsilon(z) \cap \Omega)$ . If  $F_\epsilon = |\nabla u|^q \mathbf{1}_{B_\epsilon(z) \cap \Omega} \in L_\rho^1(B_\epsilon(z) \cap \Omega)$ , we deduce from Assertion 1 that  $u$  admits the boundary trace  $\mu_\epsilon \in \mathfrak{M}_+(B_\epsilon(z) \cap \partial\Omega)$  on  $B_\epsilon(z) \cap \partial\Omega$ . If for any  $\epsilon' \in (0, \epsilon]$

$$\int_{B_{\epsilon'}(z) \cap \Omega} |\nabla u|^q \rho dx = \infty,$$

there holds

$$\int_{B_{\epsilon'}(z) \cap \Omega} (m|\nabla u|^q - u^p) \rho dx = \infty.$$

For  $0 < \delta < \frac{\epsilon'}{2}$ , set  $\Theta_{\delta, \epsilon'} = B_{\epsilon'}(z) \cap \Omega \cap \{x \in \Omega : \rho(x) > \delta\}$  and denote by  $\phi_{\delta, \epsilon'}$  the first eigenfunction of  $-\Delta$  in  $H_0^1(\Theta_{\delta, \epsilon'})$  normalized by  $\sup \phi_{\delta, \epsilon'} = 1$  and let  $\lambda_{\delta, \epsilon'}$  be the corresponding eigenvalue. Then  $\phi_{\delta, \epsilon'} \rightarrow \phi_{0, \epsilon'}$ , uniformly,  $\lambda_{\delta, \epsilon'} \downarrow \lambda_{0, \epsilon'}$  and  $\frac{\partial \phi_{\delta, \epsilon'}}{\partial \mathbf{n}} \rightarrow \frac{\partial \phi_{0, \epsilon'}}{\partial \mathbf{n}}$  in the sense that

$$\frac{\partial \phi_{\delta, \epsilon'}}{\partial \mathbf{n}}(x + \delta \mathbf{n}) \rightarrow \frac{\partial \phi_{0, \epsilon'}}{\partial \mathbf{n}}(x) \quad \text{uniformly for } x \in \partial\Omega \cap B_{\epsilon'}(z).$$

Let  $v_{\epsilon', \delta}$  be the solution of

$$\begin{aligned} -\Delta v + m|\nabla u|^q - u^p &= 0 && \text{in } \Theta_{\delta, \epsilon'} \\ v &= u && \text{on } \partial\Theta_{\delta, \epsilon'}^{up} := \overline{\Theta}_{\delta, \epsilon'} \cap \{x : \rho(x) = \delta\} \\ v &= 0 && \text{on } \partial\Theta_{\delta, \epsilon'}^{lat} := \partial\Theta_{\delta, \epsilon'} \cap \{x : \rho(x) > \delta\}. \end{aligned}$$

Then  $u \geq v_{\epsilon', \delta}$  in  $\overline{\Theta}_{\delta, \epsilon'}$  and

$$\int_{\Theta_{\delta, \epsilon'}} (\lambda_{\delta, \epsilon'} v + m|\nabla u|^q - u^p) \phi_{\delta, \epsilon'} dx = - \int_{\partial\Theta_{\delta, \epsilon'}^{up}} \frac{\partial \phi_{\delta, \epsilon'}}{\partial \mathbf{n}} u dS. \quad (5.11)$$

Since the left-hand side of (5.11) tends to  $\infty$  when  $\delta \rightarrow 0$ , we deduce that

$$\lim_{\delta \rightarrow 0} \int_{\Omega \cap B_{\epsilon'}(z)} u dS = \infty. \quad (5.12)$$

Thus  $z \in \mathcal{S}(u)$ .  $\square$

*Remark.* Note also that if  $p > 2$ , then  $u^p \in L_\rho^1(\Omega)$  implies  $u \in L^1(\Omega)$  and the assertion 2 follows from [19, Lemma 2.8]. If  $p > \frac{N+1}{N-1}$  and if we assume that  $u$  satisfies

$$u(x) \leq c(\rho(x))^{-\frac{2}{p-1}}, \quad (5.13)$$

then  $u^p \in L_\rho^1(\Omega)$ .

In order to describe the boundary singularities of solutions we introduce the following equation studied in [22]

$$\begin{aligned} -\Delta' \chi - \beta(\beta + 2 - N)\chi + m(\beta^2 \chi^2 + |\nabla' \chi|^2)^{\frac{\beta}{2}} &= 0 && \text{in } S_+^{N-1} \\ \chi &= 0 && \text{in } \partial S_+^{N-1}, \end{aligned} \quad (5.14)$$

where  $m > 0$  and

$$\beta = \frac{2-q}{q-1} \quad (5.15)$$

It is proved in [22] that if  $1 < q < \frac{N+1}{N}$ , (5.14) admits a unique solution  $\chi_m$ . The function  $V_{\chi_m}(x) = V_{\chi_m}(r, s) = r^{-\beta} \chi_m(s)$  where  $(r, s) \in \mathbb{R}_+ \times S_+^{N-1}$  is the only positive solution of

$$-\Delta v + m|\nabla v|^q = 0 \quad \text{in } \mathbb{R}_+^N \quad (5.16)$$

which vanishes on  $\partial\mathbb{R}_+^N \setminus \{0\}$  and satisfies

$$\limsup_{x \rightarrow 0} |x|^{N-1} v(x) = \infty. \quad (5.17)$$

It is a consequence of uniqueness that

$$\chi_m = m^{-\frac{1}{q-1}} \chi_1 := m^{-\frac{1}{q-1}} \chi. \quad (5.18)$$

Furthermore, if  $v_{k\delta_0}$  is the unique positive solution of

$$\begin{aligned} -\Delta v + m|\nabla v|^q &= 0 & \text{in } \mathbb{R}_+^N \\ v &= k\delta_0 & \text{on } \partial\mathbb{R}_+^N, \end{aligned} \quad (5.19)$$

then  $v_{k\delta_0} \uparrow v_{\chi_m}$  when  $k \rightarrow \infty$ . If  $\mathbb{R}_+^N$  is replaced by a bounded smooth subset  $\Omega$ , the previous statements still hold provided some adaptations are performed. We assume that  $\Omega$  is in normal position with respect to  $0 \in \partial\Omega$ . The next result is proved in [22].

**Theorem 5.1** *Let  $\Omega$  be as described above,  $m > 0$  and  $1 < q < \frac{N+1}{N}$ .*

1- *Then for any  $k > 0$  there exists a unique positive weak solution  $v_{k\delta_0}$  of*

$$\begin{aligned} -\Delta v + m|\nabla v|^q &= 0 & \text{in } \Omega \\ v &= k\delta_0 & \text{on } \partial\Omega. \end{aligned} \quad (5.20)$$

Furthermore

$$\lim_{\substack{x \in \Omega \\ x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow s}} |x|^{N-1} v_k(x) = c_N k \phi_1(s) \quad \text{locally uniformly in } s \in S_+^{N-1}. \quad (5.21)$$

2- *The function  $v_k$  is stable in the sense that if  $\{\mu_n\}$  is a sequence of positive Radon measures on  $\partial\Omega$  which converges weakly to  $k\delta_0$ , then the corresponding sequence of solutions  $\{v_{\mu_n}\}$  of*

$$\begin{aligned} -\Delta v + m|\nabla v|^q &= 0 & \text{in } \Omega \\ v &= \mu_n & \text{on } \partial\Omega, \end{aligned} \quad (5.22)$$

*converges locally uniformly in  $\Omega$  to  $v_{k\delta_0}$ .*

3- *Finally, when  $k \uparrow \infty$ ,  $v_{k\delta_0} \uparrow v_{\chi_m}$  where  $v_{\chi_m}$  is the unique positive solution of*

$$-\Delta v + m|\nabla v|^q = 0 \quad \text{in } \Omega, \quad (5.23)$$

*which vanishes on  $\partial\Omega \setminus \{0\}$  and satisfies (5.17). Furthermore  $v_{\chi_m}$  verifies the following limits, locally uniformly on  $S_+^{N-1}$ ,*

$$\lim_{\substack{x \in \Omega \\ x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow s}} |x|^\beta v_{\chi_m}(x) = \chi_m(s), \quad (5.24)$$

and

$$\begin{aligned} \lim_{\substack{x \in \Omega \\ x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow s}} |x|^{\beta+1} \frac{x}{|x|} \cdot \nabla v_{\chi_m}(x) &= -\beta \chi_m(s), \\ \lim_{\substack{x \in \Omega \\ x \rightarrow 0 \\ \frac{x}{|x|} \rightarrow s}} |x|^{\beta+1} \nabla_{\text{tang}} v_{\chi_m}(x) &= \nabla_{\text{tang}} \chi_m(s), \end{aligned} \quad (5.25)$$

where  $\nabla_{\text{tang}} = r^{-1} \nabla'$  denotes the tangential gradient.

We set

$$\Omega'_\delta = \{x \in \Omega : \rho(x) > \delta\}, \quad \Omega_\delta = \{x \in \Omega : 0 < \rho(x) < \delta\} \quad \text{and} \quad \Sigma_\delta = \partial\Omega'_\delta. \quad (5.26)$$

It is known that  $\Sigma_\delta$  is smooth for  $\delta$  small enough. The following variant is proved in [22, Corollary 2.4].

**Corollary 5.2** *Under the assumptions on  $N$ ,  $q$  and  $m$  of Theorem 5.1, assume that  $\{\delta_n\}$  is a sequence decreasing to 0,  $\{\mu_n\}$  is a sequence of positive bounded Radon measures on  $\Sigma_{\delta_n}$  which converges in the sense of measures in  $\bar{\Omega}$  to a measure  $\mu$  on  $\partial\Omega$ . Then the sequence  $\{v_{\mu_n}\}$  of solutions of*

$$\begin{aligned} -\Delta v + m|\nabla v|^q &= 0 && \text{in } \Omega'_{\delta_n} \\ v &= \mu_n && \text{on } \Sigma_{\delta_n}, \end{aligned} \quad (5.27)$$

converges up to a subsequence locally uniformly in  $\Omega$  to a positive solution  $v_\mu$  of

$$\begin{aligned} -\Delta v + m|\nabla v|^q &= 0 && \text{in } \Omega \\ v &= \mu && \text{on } \partial\Omega. \end{aligned} \quad (5.28)$$

**Proposition 5.3** *Let  $p > 1$ ,  $1 < q < \frac{N+1}{N}$  and  $m > 0$ . Let  $u$  be a positive solution of (1.1) in  $\Omega$  such that there exist a sequence  $\{z_n\} \subset \partial\Omega$  converging to  $z$  and two decreasing sequences  $\{\epsilon_n\}$  and  $\{\delta_n\}$  converging to 0 such that*

$$\lim_{n \rightarrow \infty} \int_{B_{\epsilon_n}(z_n) \cap \Sigma_{\delta_n}} u dx = \infty, \quad (5.29)$$

then there holds

$$\liminf_{\substack{x \in \Omega \\ x \rightarrow z \\ \frac{x-z}{|x-z|} \rightarrow s}} |x-z|^{\frac{2-q}{q-1}} u(x) \geq \chi(s) \quad \text{locally uniformly in } s \in S_+^{N-1}. \quad (5.30)$$

*Proof.* For  $k > 0$ , there exists  $n_0$  such that for  $n \geq n_0$ ,

$$\int_{B_{\epsilon_n}(z_n) \cap \Sigma_{\delta_n}} u dx > k.$$

Hence there exists  $\ell := \ell_n > 0$  such that

$$\int_{B_{\epsilon_n}(z_n) \cap \Sigma_{\delta_n}} \min\{u, \ell\} dx = k.$$

We set  $\mu_{n,k} = \min\{u, \ell\} \lfloor_{\Sigma_{\delta_n}} \mathbf{1}_{\Sigma_{\delta_n} \cap B_{\epsilon_n}(z_n)}$  and denote by  $v_{\mu_{n,k}}$  the corresponding solution of (5.27) in  $\Omega'_{\delta_n}$ . Then  $u \geq v_{\mu_{n,k}}$  in  $\Omega'_{\delta_n}$ . Up to a

rotation we can assume that  $\partial\mathbb{R}_+^N$  is tangent to  $\partial\Omega$  at  $z$ . Using Corollary 5.2 we obtain  $u \geq v_{k\delta_z}$ . Letting  $k \rightarrow \infty$  and using Theorem 5.1-3 we deduce that

$$\liminf_{\substack{x \in \Omega \\ x \rightarrow z \\ \frac{x-z}{|x-z|} \rightarrow s}} |x-z|^\beta u(x) \geq \chi(s) \quad \text{locally uniformly in } s \in S_+^{N-1}. \quad (5.31)$$

□

In the sequel we denote  $\chi_1 = \chi$  and  $v_{\chi_1} = v_\chi$ . In the next theorem we show the existence of positive singular solution of (1.4) with a strong blow-up in  $|x|^{-\beta}$  provided the function  $v_{\chi_1}$  has no critical point in  $\Omega$  and  $\frac{2p}{p+1}q < \frac{N+1}{N}$ . If it is the case the constant  $M_{v_\chi}$  defined below is positive because of (5.25) and Hopf boundary lemma,

$$M_{v_\chi} = \min_{x \in \Omega} \frac{|\nabla v_\chi(x)|^q}{v_\chi^p(x)}. \quad (5.32)$$

**Theorem 5.4** *Let  $\Omega$  be a bounded smooth domain with  $0 \in \partial\Omega$ ,  $p > 1$  and  $\frac{2p}{p+1} < q < \frac{N+1}{N}$ . If*

$$m > m_{v_\chi} =: \frac{p-1}{p-q} \left( \frac{p-q}{(q-1)M_{v_\chi}} \right)^{\frac{q-1}{p-1}}, \quad (5.33)$$

*then there exists a positive solution of (1.4) which satisfies*

$$\lim_{\substack{x \in \Omega \\ x \rightarrow 0 \\ \frac{x-z}{|x-z|} \rightarrow s}} |x|^\beta u(x) = \chi_m(s) \quad \text{locally uniformly in } s \in S_+^{N-1}, \quad (5.34)$$

*where  $\chi_m$  is the unique positive solution of (5.14).*

*Proof.* The function  $\chi$  is the unique positive solution of (5.14), and since it depends on  $m > 0$ , we denote it by  $\chi_m$ . Clearly  $\chi_m = m^{-\frac{1}{q-1}}\chi$ . Then  $v_{\chi_m} = m^{-\frac{1}{q-1}}v_\chi$  is the solution of (5.23) which is obtained in Theorem 5.1, since this solution is the unique positive solution of (5.23) which satisfies (5.24)-(5.25). We also set

$$L_{m,p,q}u = -\Delta u + m|\nabla u|^q - u^p.$$

The function  $v_{\chi_m}$  is a subsolution of (1.4). Let  $0 < \tilde{m} < m$ , then  $v_{\chi_m} < v_{\chi_{\tilde{m}}}$ . Furthermore

$$\begin{aligned} L_{m,p,q}v_{\chi_m} &= (m - \tilde{m})|\nabla v_{\chi_{\tilde{m}}}|^q - v_{\chi_{\tilde{m}}}^p \\ &= (m - \tilde{m})\tilde{m}^{-\frac{q}{q-1}}|\nabla v_\chi|^q - \tilde{m}^{-\frac{p}{q-1}}v_\chi^p \\ &\geq \left( (m - \tilde{m})\tilde{m}^{-\frac{q}{q-1}}M_{v_\chi} - \tilde{m}^{-\frac{p}{q-1}} \right) v_\chi^p \\ &\geq \left( m - \left( \tilde{m} + \frac{1}{\tilde{m}^{\frac{p-q}{q-1}}M_{v_\chi}} \right) \right) \tilde{m}^{-\frac{q}{q-1}}M_{v_\chi}v_\chi^p. \end{aligned}$$

Then

$$\min_{X>0} \left\{ X + \frac{1}{X^{\frac{p-q}{q-1}} M_{v_\chi}} \right\} = \frac{p-1}{p-q} \left( \frac{p-q}{(q-1)M_{v_\chi}} \right)^{\frac{q-1}{p-1}} := m_{v_\chi} \quad (5.35)$$

and the minimum is achieved for

$$X = X_0 = \left( \frac{p-q}{(q-1)M_{v_\chi}} \right)^{\frac{q-1}{p-1}}. \quad (5.36)$$

If we fix  $\tilde{m} = X_0$  it follows that for  $m > m_{v_\chi}$ , the function  $v_{\chi_{\tilde{m}}}$  satisfies  $L_{m,p,q}v_{\chi_{\tilde{m}}} \geq 0$  in  $\Omega$  and it is larger than the subsolution  $v_{\chi_m}$ . Hence there exists a solution  $u$  of (1.4) in  $\Omega$  and it satisfies

$$v_{\chi_m} \leq u \leq v_{\chi_{\tilde{m}}} \quad \text{in } \Omega. \quad (5.37)$$

The end of the proof is standard. For  $\ell > 0$  we set  $S_\ell[v](x) = \ell^\beta v(\ell x)$ . Then  $u_\ell := S_\ell[u]$  satisfies

$$-\Delta u_\ell + m|\nabla u_\ell|^q - \ell^{\frac{q(p+1)-2p}{q-1}} u_\ell^p = 0 \quad \text{in } \Omega_\ell := \frac{1}{\ell}\Omega, \quad (5.38)$$

and

$$S_\ell[v_{\chi_m}] \leq u_\ell \leq S_\ell[v_{\chi_{\tilde{m}}}] \quad \text{in } \Omega_\ell.$$

By Theorem 1.1,

$$|\nabla u_\ell(x)| + \frac{\ell u_\ell(x)}{\rho(\ell x)} \leq c|x|^{-\beta-1} \quad \text{in } \bar{\Omega}_\ell \setminus \{0\}. \quad (5.39)$$

Since  $\partial\Omega$  is smooth, there exists  $\epsilon_0 > 0$  such that  $c_2\ell\rho_\ell(x) \leq \rho(\ell x) \leq c_1\ell\rho_\ell(x)$  for  $|x| \leq \epsilon_0$ , in which formula we denote  $\rho_\ell(x) = \text{dist}(x, \Omega_\ell)$ . Since  $q(p+1) - 2p > 0$ ,  $\ell^{\frac{q(p+1)-2p}{q-1}} u_\ell^p \rightarrow 0$  when  $\ell \rightarrow 0$ , locally uniformly in  $\Omega_\ell \cap B_\delta^c$  for any  $\delta > 0$  and by standard elliptic equations regularity results [14],  $D^2 u_\ell$  is also locally bounded in  $\Omega_\ell \cap B_\delta^c$ . When  $\ell \rightarrow 0$ ,  $S_\ell[v_{\chi_m}]$  and  $S_\ell[v_{\chi_{\tilde{m}}}]$  converge respectively to  $x \mapsto |x|^{-\beta}\chi_m(\frac{x}{|x|})$  and  $x \mapsto |x|^{-\beta}\chi_{\tilde{m}}(\frac{x}{|x|})$ . Therefore, if  $u = \lim_{n \rightarrow \infty} u_{\ell_n}$  for some sequence  $\{\ell_n\}$  converging to 0, the function  $u$  is nonnegative and satisfies

$$-\Delta u + m|\nabla u|^q = 0 \quad (5.40)$$

in  $\mathbb{R}_+^N$  and there holds

$$|x|^{-\beta}\chi_m\left(\frac{x}{|x|}\right) \leq u(x) \leq |x|^{-\beta}\chi_{\tilde{m}}\left(\frac{x}{|x|}\right).$$

Since (5.40) admits a unique positive solution vanishing on  $\partial\mathbb{R}_+^N \setminus \{0\}$  such that  $\limsup_{x \rightarrow 0} |x|^\beta u(x) > 0$  (see [22, Proposition 3.24-Step 2]), it follows that  $u(x) = |x|^{-\beta}\chi_m(\frac{x}{|x|})$ . Uniqueness implies that  $u_\ell \rightarrow u$  and (5.34) holds.  $\square$

*Remark.* The assumption that  $v_\chi$  admits no critical point in  $\Omega$  is uneasy to verify. At least it is easy to see that  $v_\chi$  cannot have any non-degenerate

critical point in  $\Omega$ . Furthermore, because of Hopf boundary lemma and the behaviour of  $v_\chi$  near  $x = 0$  given by (5.24), the critical points of  $v_\chi$  are located in a compact subset  $N$  of  $\Omega$ , possibly empty. For  $\epsilon > 0$  we set

$$N_\epsilon = \{x \in \Omega : \text{dist}(x, N) < \epsilon\}.$$

If  $\epsilon$  is small enough  $\overline{N}_\epsilon \subset \Omega$ . Denote

$$M_{v_\chi}^\epsilon = \min_{x \in \Omega \setminus N_\epsilon} \frac{|\nabla v_\chi(x)|^q}{v_\chi^p(x)} \quad \text{and} \quad m_{v_\chi}^\epsilon =: \frac{p-1}{p-q} \left( \frac{p-q}{(q-1)M_{v_\chi}^\epsilon} \right)^{\frac{q-1}{p-1}}. \quad (5.41)$$

The proof of the next result is similar to the one of Theorem 5.4.

**Theorem 5.5** *Let  $\Omega$  be a bounded smooth domain with  $0 \in \partial\Omega$ ,  $p > 1$  and  $\frac{2p}{p+1} < q < \frac{N+1}{N}$ . If  $N$  denotes the set of critical points of  $v_\chi$ , then for any  $\epsilon > 0$  small enough and  $m > m_{v_\chi}^\epsilon$  there exists a positive solution of (1.1) in  $\Omega \setminus N_\epsilon$  which vanishes on  $\partial\Omega$  and satisfies (5.34).*

## References

- [1] D. Adams, L. Hedberg. *Function Spaces and Potential Theory*, Theory, Springer-Verlag, London-Berlin-Heidelberg-New York (1996).
- [2] S. Alarcón, J. García-Melián, A. Quaas. Nonexistence of positive supersolutions to some nonlinear elliptic problems. *J. Math. Pures Appl.* **90**, 618-634 (2013).
- [3] M.F. Bidaut-Véron, A. Ponce, L. Véron. Isolated boundary singularities of semilinear elliptic equations. *Calc. Var. Part. Diff. Equ.* **40**, 183-221 (2011).
- [4] M.F. Bidaut-Véron, M. Garcia-Huidobro, L. Véron. A priori estimates for elliptic equations with reaction terms involving the function and its gradient. *Math. Annalen* **378**, 13-58 (2020).
- [5] M.F. Bidaut-Véron, M. Garcia-Huidobro, L. Véron. Measure data problems for a class of elliptic equations with mixed absorption-reaction. *Adv. Nonlinear. Stud.* **21**, 261-280 (2020).
- [6] M.F. Bidaut-Véron, M. Garcia-Huidobro, L. Véron. Boundary singular solutions of a class of equations with mixed absorption-reaction. *Calc. Var. part. Diff. Equ.* **61** 113, pp.1-46, (2022),
- [7] M.F. Bidaut-Véron, M. Garcia-Huidobro, L. Véron. New results on the Chipot-Weissler quasilinear equation. In preparation.
- [8] M.F. Bidaut-Véron, G. Hoang, Q. H. Nguyen, L. Véron. An elliptic semilinear equation with source term and boundary measure data: the supercritical case, *J. Funct. Anal.* **269**, 1995-2017 (2015).
- [9] M.F. Bidaut-Véron, L. Vivier. An elliptic semilinear equation with source term involving boundary measures: the subcritical case, *Rev. Mat. Iberoamericana* **16**, 477-513 (2000).

- [10] L. Boccardo, F. Murat, J. P. Puel. Résultats d'existence pour certains problèmes elliptiques quasilineaires, *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze Ser. 4, T. 11*, 213-235 (1984).
- [11] M. Chipot, F. Weissler. Some blowup results for a nonlinear parabolic equation with a gradient term. *SIAM J on Num. Anal.* **20**, 886-907 (1989).
- [12] J. L. Doob. *Classical Potential Theory and Its Probabilistic Counterpart*, Springer-Verlag, London-Berlin-Heidelberg-New York (1984).
- [13] B. Gidas, J. Spruck. Global and local behaviour of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.* **34** (1981), 525-598.
- [14] D. Gilbarg, N. Trudinger. *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, London-Berlin-Heidelberg-New York (1983).
- [15] A. Gmira, L. Véron. Boundary singularities of solutions of some nonlinear elliptic equations. *Duke Math. J.* **64**, 271-324 (1991).
- [16] A. Kufner, B. Opic. *Hardy-type inequalities*. Pitman Research Notes in Mathematics Series, **219**, Longman Scientific Technical, Harlow, xii+333 (1990).
- [17] M. Marcus, L. Véron. The Boundary trace of positive solutions of semilinear elliptic equations: the Subcritical Case. *Arch. Rat. Mech. Anal.* **144**, 200-231 (1998).
- [18] M. Marcus, L. Véron. Removable singularities and boundary traces. *Jl. Math. Pures Appl.* **80**, 879-900 (2001).
- [19] M. Marcus, L. Véron. The boundary trace and generalized boundary value problem for semilinear elliptic equations with coercive absorption. *Comm. Pure Appl. Math.* **LVI**, 689-731 (2003).
- [20] M. Marcus, L. Véron. *Nonlinear Elliptic Equations involving Measures*. De Gruyter Series in Nonlinear Analysis and Applications. **21** xiii+pp. 1-262 (2013).
- [21] L. Montoro. Harnack inequalities and qualitative properties for some quasilinear elliptic equations. *Nonlinear Diff. Equ. Appl.* **26**, 33 pp (2019).
- [22] P. T. Nguyen, L. Véron. Boundary singularities of solutions to elliptic viscous Hamilton-Jacobi equations. *Jl. Funct. Anal.* **263**, 1487-1538 (2012).
- [23] P. Polacik, P. Quittner, Ph. Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. *Duke Math. J.* **139**, 555-579 (2007).
- [24] D. Ruiz. A priori estimates and existence of positive solutions for strongly nonlinear problems. *Jl. Diff. Equ.* **199**, 96-114 (2004).
- [25] J. Serrin, Y. Yan, H. Zou. A numerical study of existence and non-existence of the ground states and bifurcation for the equation of Chipot and Weissler. *AHPCRC Preprint 93-056*, Univesrity of Minnesota (1993).

- [26] J. Serrin, H. Zou. Existence and non-existence results for ground states of quasilinear elliptic equations. *Arch. Rational Mech. Anal.* **121**, 101-130 (1992).
- [27] J. Serrin, H. Zou. Classification of positive solutions of quasilinear elliptic equations. *Top. Meth. in Nonlinear Anal.* **3**, 1-26 (1994).
- [28] L. Véron. Singular solutions of some nonlinear elliptic equations. *Nonlinear Anal. Theory, Methods Appl.* **5**, 225-242 (1981).
- [29] L. Véron. *Local and Global Aspects of Quasilinear Degenerate Elliptic Equations*, World Scientific xvi+pp. 1-457 (2017).
- [30] F. X. Voirol. *Étude de Quelques Équations Elliptiques Fortement Non Linéaires*, PhD Thesis, University of Metz (1994).