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Abstract. The problem of reservation in a large distributed system is analyzed via a new mathematical model. A typical application is a station-based car-sharing system which can be described as a closed stochastic network where the nodes are the stations and the customers are the cars. The user can reserve the car and the parking space. In the paper, we study the evolution of the system when the reservation of parking spaces and cars is effective for all users. The asymptotic behavior of the underlying stochastic network is given when the number N of stations and the fleet size increase at the same rate. The analysis involves a Markov process on a state space with dimension of order N^2. It is quite remarkable that the state process describing the evolution of the stations, whose dimension is of order N, converges in distribution, although not Markov, to an non-homogeneous Markov process. We prove this mean-field convergence. We also prove, using combinatorial arguments, that the mean-field limit has a unique equilibrium measure when the time between reserving and picking up the car is sufficiently small. This result extends the case where only the parking space can be reserved.

Contents

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1. Introduction

The paper deals with a problem of reservation in a large distributed system. Our motivation is a car-sharing system in which a fleet of cars move around and is parked in a set of stations, mainly for electric issues. A crucial problem is the presence of empty and full stations. In an empty station, users cannot pick up a car, while in a full station, also called saturated, users cannot park the car. Reservation could help the user to find both a car at the departure and a parking space at the destination. In this paper, we focus on a reservation policy called *double reservation* which is to reserve both the car and the parking space at the same time, a moment before picking up the car. Note that if the time between the reservation and the pick-up is zero, the policy is to reserve the parking space when the car is picked up. This policy is called *simple reservation*. In free-floating systems, users cannot reserve the parking spaces as the cars are parked in the public space. These systems are outside the scope of the paper.

1.1. Simple reservation model.

The simple reservation model can be described as follows. It consists of N stations of finite capacity K and M_N cars. Users arrive at rate λ in each station. Such a user reserves a parking space at a randomly chosen destination station when he or she picks up a car. If it is not possible, the unhappy user leaves the system. Otherwise, after a trip with exponential distribution of parameter μ, the car is parked at its reserved parking space in the destination station.

The system is said to be large when N and M_N tend to infinity with $s_N = M_N/N$ tending to a constant s called *the average number of cars per station*. This sizing parameter s is a key parameter of the system. In [5], this policy is studied in a homogeneous framework by a mean-field approach, using a large-scale analysis similar to that of bike-sharing systems in [8].

Due to the the parking space reservation, the state of a station is described in [5] by two components: the number of cars and the number of reserved parking spaces, which is the main difference with [8]. This bi-dimensional process is Markovian. With standard arguments, mean-field convergence is established. Note that the analysis of the equilibrium point is much more delicate with parking space reservation.

The aim of the paper is to prove these results extended to the double reservation model introduced as follows. In particular, it gives the proof of [5, Theorem 2] omitted in [5] on the simple reservation model.

1.2. Double reservation model.

Station-based car-sharing systems offer the possibility of reserving a car and a parking space in the desired station online, a moment before actually picking up the car. In this paper, we present a model that meets this user demand. The user reserves a car at a given departure station and, at the same time, a parking space at a destination station. If there is no car or no parking space available, the user leaves the system. Otherwise, it takes a time called *reservation time* between the car reservation and the car pick-up. The reservation time has an exponential distribution with mean $1/\nu$. Then follows the trip whose duration is assumed to have an exponential distribution with mean $1/\mu$. Eventually the user parks the car at the reserved space in the destination station and leaves the system.

1.3. Mean-field approach.

A three-dimensional state space (reserved cars, reserved spaces and available cars) does not allow this process to be fully described as a Markov process, because the parking space reservation time is the sum of two exponentially distributed variables (car pick up time and trip time). This leads to the introduction of a fourth variable to distinguish between parking spaces reserved by *users not yet travelling* and *users travelling*. But even this state process is not Markov. As we will see on the transitions described in Section 2, the Markov process associated to this model, denoted by $(X^N(t))$, is more complicated and especially of dimension $(N+1)^2$, where N is the total number of stations. Therefore, for a given station, the state space is of dimension N, which is not suitable for an asymptotic analysis when N tends to infinity. To

solve this problem, we introduce the process, denoted by $(Z^N(t))$, which describes the state of each station i, $1 \leq i \leq N$, as a function of the Markov process $(X^N(t))$. Indeed $Z_i^N(t)$ has four components,

- $R_i^{r,N}(t)$ the number of parking places reserved by non-driving users at station i at time t,
- $R_i^N(t)$ the number of parking places reserved by users driving at station i at time t,
- $V_i^N(t)$ the number of cars available at station i at time t and
- $V_i^{r,N}(t)$ the number of reserved cars at station i at time t.

As previously mentioned, process $(Z^N(t)) = \left(R^{r,N}(t), R^N(t), V^N(t), V^{r,N}(t)\right)$ is of dimension $4N$ but is not a Markov process. The loss of the Markov property is the price to pay to this dimension reduction. Nevertheless, it is remarkable and we prove that, despite this non-Markovian description, the state $(Z_i^N(t))$ of a given station i $(1 \leq i \leq N)$ converges in distribution, as N goes to infinity, to a non-homogeneous Markov process $(\overline{Z}(t)) = (\overline{R^r}(t), \overline{R}(t), \overline{V}(t), \overline{V^r}(t))$ satisfying the following Fokker–Planck equation

$$
\begin{aligned}
\frac{d}{dt}\mathbb{E}\left(f(\overline{Z}(t))\right) &= \lambda\mathbb{P}(\overline{V}(t)>0)\mathbb{E}\left((f(\overline{Z}(t)+e_1)-f(\overline{Z}(t))1_{\{\bar{S}(t)<K\}}\right)\\
&+\nu\mathbb{E}\left((f(\overline{Z}(t)+e_2-e_1)-f(\overline{Z}(t)))1_{\{\overline{R^r}(t)>0\}}\right)\\
&+\mu\mathbb{E}\left((f(\overline{Z}(t)+e_3-e_2)-f(\overline{Z}(t)))1_{\{\overline{R}(t)>0\}}\right)\\
&+\lambda\mathbb{P}(\overline{S}(t)<K)\mathbb{E}\left((f(\overline{Z}(t)+e_4-e_3)-f(\overline{Z}(t)))1_{\{\overline{V}(t)>0\}}\right)\\
&+\nu\mathbb{E}\left((f(\overline{Z}(t)-e_4)-f(\overline{Z}(t)))1_{\{\overline{V^r}(t)>0\}}\right)
\end{aligned}
\tag{1}
$$

with $e_1=(1,0,0,0)$, $e_2=(0,1,0,0)$, $e_3=(0,0,1,0)$, $e_4=(0,0,0,1)$, f a function with finite support on $\mathbb{N}^4$ and $\overline{S}(t)=\overline{R^r}(t)+\overline{R}(t)+\overline{V}(t))+\overline{V^r}(t)$ the limiting number of unavailable parking places at station i at time t. The asymptotic process $(\overline{Z}(t)) = ((\overline{R^r}(t), \overline{R}(t), \overline{V}(t), \overline{V^r}(t))$ is a jump process with time-dependent rates, the so-called McKean–Vlasov process.

1.4. **The equilibrium.** For non-homogeneous Markov processes, recall that there can be several invariant measures. We prove that there is a unique invariant measure in a restricted framework. The proof is based on three main arguments. First, by applying queueing theory, the McKean–Vlasov process on the basic state space is identified with a tandem of four queues with an invariant measure of explicit product form. Thus, by simplifications, the problem of existence and uniqueness of the invariant measure of the non-homogeneous Markov process amounts to the same problem for a fixed point equation in dimension 2. Finally, the global inversion theorem and a monotonicity property allow us to conclude. The last two arguments are based on combinatorial calculations. Monotonicity is just proved when the mean reservation time is sufficiently small, but it seems to be true without this technical assumption.

1.5. **Related works.** For car-sharing systems, a first part of the literature investigates the location problem. In Boyaci et al. [6], OR optimization is used to plan an efficient car-sharing system in terms of the number, location and capacity of stations and fleet size, applied to the case of Nice, France.

Vehicle redistribution and staff rebalancing is a big issue in one-way vehicle sharing systems (see for example [11]). For simple reservation, called complete parking reservation (CPR), Kaspi et al. show by simulation in [9] and then with OR techniques in [10] that CPR outperforms NR (no reservation) for a specific user-oriented metric. This metric is the excess time users spend in the system due to lack of cars or parking spaces, i.e. the difference between actual time and ideal trip time, including walking times to stations. Some papers do not investigate the reservation impact,

but the charging problem. In [7], a queueing analysis for dimensioning the fleet size is proposed for a closed network taking account car charging.

Despite the potential of car sharing, even data analysis remains largely unexplored. This is also due to the lack of data provided by the operators. In [3], Boldrini et al. exploit one month (April 2015) of publicly available data from a large car-sharing system in Paris, France, with 960 stations and 2700 electric cars. We get an idea of the average car pickup rate and the availability of cars at a station. Furthermore, a dichotomy between Paris and the suburbs is highlighted.

Mean-field is an efficient tool for studying the behavior of large distributed systems or interacting systems in many different application domains. These systems have both a large number of nodes and customers (particles, cars, etc). To the best of our knowledge, our large-scale stochastic analysis is the first for a stochastic model of car sharing systems with double reservation. Our main result is that the non-Markovian state process for a given station converges to an non-homogeneous Markov process. Furthermore, we note that this limiting Markov process can be described using a simple queueing system. This result was first obtained by simulation in [4, Section 5]. In [4], an artificial Markov model on the state of the stations, called the approximate model process, is introduced for the double reservation. Its mean-field limit at equilibrium fits with the real dynamics at equilibrium, which allows the authors to guess such an outcome. The same phenomenon is proved in an entirely different framework, for a model of a network with failures by Aghajani, Robert and Sun [1]. Note that the framework in [1] induces simultaneous jumps, which make the proofs more technical. Our paper discusses a simpler framework which focuses on the main arguments and it provides the proof of the result expected in [4]. We believe that this methodology can be useful in many other contexts. A main contribution of the paper relies on the result of uniqueness of the equilibrium point in high dimension (dimension 4), thanks to a nice interpretation in terms of queues. As far as we know, there is no such result in the literature.

Outline of the paper. In Section 2, the Markov process $(X^N(t))$ associated to our model is defined and the stochastic evolution equations are given. In Section 3, the second process $(Z^N(t))$ describing the state of the stations is introduced. A heuristic computation of its McKean–Vlasov asymptotic process is derived. Theorem 1 which establishes the existence and uniqueness of this stochastic process is proved. Section 4 is devoted to Theorem 2 giving the mean-field convergence for $(Z^N(t))$ and highlights the probabilistic interpretation of the Mckean–Vlasov process. Section 5 analyses of its invariant distribution.

2. The model

In this section, we describe the dynamics of our stochastic model. We recall that the system has N stations of capacity K. The Markov process that gives the dynamics of the system is the following,

$$X(t) = (X^N_{i,j}(t), 0 \leq i, j \leq N)$$

where, for $1 \leq i, j \leq N$, at time t,

- $X^N_{i,j}(t)$ is the number of cars reserved at i with parking space reserved at j,
- $X^N_{0,j}(t)$ is the number of parking spaces reserved at j by users driving
- and $X^N_{i,0}(t)$ is the number of cars available at station i.

The total number of cars is M_N and the fleet size parameter, defined as the mean number of cars per station if they are all parked, is

$$s \stackrel{def.}{=} \lim_{N\to+\infty} \frac{M_N}{N}.$$

2.1. **Transitions of the Markov process.** The transitions of the Markov process $(X^N(t))$ are reservations, car picks up and car returns.

- *Reservations.* At station i, at rate λ, an available car is replaced by a reserved car and an available parking space is reserved at the same time at a random station, say j. If there is either no available car at i or no available parking space at j, the reservation fails. If a reservation is made at time t,
$$\begin{cases} X^N_{i,0}(t) = X^N_{i,0}(t^-) - 1 & \text{for } 1 \le i \le N, \\ X^N_{i,j}(t) = X^N_{i,j}(t^-) + 1 & \text{for } 1 \le i, j \le N. \end{cases}$$
where the limit from the left of function f at t is denoted by $f(t^-)$.
- *Car picks up.* After a reservation, the user takes a time with an exponential distribution with parameter ν to come and pick the car. So, at rate ν, each car reserved at station i disappears and the associated parking space reserved at station j moves to a parking space reserved by a driving user. When taking such a car at time t,
$$\begin{cases} X^N_{i,j}(t) = X^N_{i,j}(t^-) - 1, & \text{for } 1 \le i, j \le N, \\ X^N_{0,j}(t) = X^N_{0,j}(t^-) + 1 & \text{for } 1 \le j \le N, \end{cases}$$
- *Car returns.* After a trip with exponential distribution with parameter μ, a user driving returns his car. Thus, at rate μ, each parking space at station j reserved by a user driving his car becomes an available car. When a car is returned at station j at time t,
$$\begin{cases} X^N_{0,j}(t) = X^N_{0,j}(t^-) - 1, \\ X^N_{j,0}(t) = X^N_{j,0}(t^-) + 1. \end{cases}$$

Note that $(X^N(t))$ is an irreducible Markov process on the finite state space

$$\{x = (x_{i,j}) \in \mathbb{N}^{(N+1)^2}, \sum_{j=0}^{N} x_{i,j} + \sum_{j=0}^{N} x_{j,i} \le K, \sum_{0 \le i,j \le N} x_{i,j} = M_N\}, \tag{2}$$

where K is the finite capacity of each station, thus $(X^N(t))$ is ergodic.

2.2. **Stochastic evolution equations.** The dynamics of $(X^N(t))$ can be given in terms of stochastic integrals with respect to Poisson processes. Let us introduce the following notations.

- A Poisson process on $\mathbb{R}_+$ with parameter ξ is denoted by $\mathcal{N}_\xi$. A sequence of such i.i.d. processes is denoted by $(\mathcal{N}_{\xi,i}, i \in \mathbb{N})$.

- A Poisson process on $\mathbb{R}^2_+$ with intensity $\xi dtdh$ is denoted by $\overline{\mathcal{N}}_\xi$. A sequence of such i.i.d. processes is denoted by $(\overline{\mathcal{N}}_{\xi,i}, i \in \mathbb{N})$.

- A marked Poisson process (t_n, U_n), where $(t_n, n \in \mathbb{N})$ is a Poisson process on $\mathbb{R}_+$ with parameter ξ and $(U_n, n \in \mathbb{N})$ is a sequence of i.i.d random variables with uniform distribution on $\{1, \dots, N\}$, is denoted by $\mathcal{N}^{U,N}_\xi$. Note that, for $1 \le i \le N$, $\mathcal{N}^{U,N}_\xi(., \{i\})$ is a Poisson process on $\mathbb{R}_+$ with parameter ξ/N and $\mathcal{N}^{U,N}_\xi(., \mathbb{N})$ is a Poisson process on $\mathbb{R}_+$ with parameter ξ.

Let us introduce the following independent point processes. A reservation of a car in station i is a point of a Poisson process $\mathcal{N}^{U,N}_{\lambda,i}$ on $\mathbb{R}^2_+$. For reservations of both a car at station i and a parking space at station j, the times from the moment the user makes a reservation to the moment the car is picked up form a Poisson process $\mathcal{N}_{\nu,i,j}$ on $\mathbb{R}_+$. We need a sequence of such i.i.d processes $(\mathcal{N}_{\nu,i,j,l}, l \in \mathbb{N})$ as cars are picked up independently. And the same for the trip times of cars returned at station j, associated to a sequence $(\mathcal{N}_{\mu,j,l}, l \in \mathbb{N})$ of independent Poisson processes.

Using the previous notations, process $(X^N(t))$ is given by the following stochastic differential equations. For $1 \leq i, j \leq N$ and $t \geq 0$,

$$dX^N_{i,j}(t) = -\sum_{l=1}^{\infty} \mathbf{1}_{\{l \leq X^N_{i,j}(t^-)\}} \mathcal{N}_{\nu,i,j,l}(dt) + \mathbf{1}_{\{X^N_{i,0}(t^-)>0,\, \sum_{k=0}^{N}\left(X^N_{k,j}(t^-)+X^N_{j,k}(t^-)\right)<K\}}\, \mathcal{N}^{U,N}_{\lambda,i}(dt,\{j\}), \tag{3}$$

$$dX^N_{0,j}(t) = \sum_{l=1}^{\infty}\sum_{i=1}^{N} \mathbf{1}_{\{l \leq X^N_{i,j}(t^-)\}} \mathcal{N}_{\nu,i,j,l}(dt) - \sum_{l=1}^{\infty} \mathbf{1}_{\{l \leq X^N_{0,j}(t^-)\}} \mathcal{N}_{\mu,j,l}(dt) \tag{4}$$

and

$$dX^N_{i,0}(t) = -\sum_{j=1}^{N} \mathbf{1}_{\{X^N_{i,0}(t^-)>0,\, \sum_{k=0}^{N}\left(X^N_{k,j}(t^-)+X^N_{j,k}(t^-)\right)<K\}}\, \mathcal{N}^{U,N}_{\lambda,i}(dt,\{j\}) + \sum_{l=1}^{\infty} \mathbf{1}_{\{l \leq X^N_{0,i}(t^-)\}} \mathcal{N}_{\mu,i,l}(dt). \tag{5}$$

3. The asymptotic process

3.1. **Introduction to the asymptotic process.** Some other notations are needed for that. The state of each station i is given by the quadruplet

$$Z^N_i(t) = (R^{r,N}_i(t), R^N_i(t), V^N_i(t), V^{r,N}_i(t))$$

where, at node i at time t,

- $R^{r,N}_i(t)$ is the number of parking spaces reserved by non-driving users,
- $R^N_i(t)$ is the number of reserved parking spaces by users driving,
- $V^N_i(t)$ is the number of available cars,
- $V^{r,N}_i(t)$ is the number of reserved cars.

Process $(Z^N(t))$ takes values on

$$\mathcal{S}^N = \{(w_i,x_i,y_i,z_i)_{1\leq i\leq N}, (w_i,x_i,y_i,z_i) \in \chi, \sum_{i=1}^{N} w_i + x_i + y_i + z_i = M_N\}$$

where

$$\chi = \{(w,x,y,z) \in \mathbb{N}^4, w + x + y + z \leq K\}.$$

Moreover, let us denote by

$$S^N_i(t) = R^{r,N}_i(t) + R^N_i(t) + V^N_i(t) + V^{r,N}_i(t)$$

the number of unavailable parking spaces at station i at time t. Note that $S^N_i(t)$ is a function of $Z^N_i(t)$. This process $(Z^N(t))$ gives some refined state of the stations and can be obtained as a function of the Markov process $(X^N(t))$ by

$$R^{r,N}_i(t) = \sum_{j=1}^{N} X^N_{j,i}(t), \qquad R^N_i(t) = X^N_{0,i}(t),$$

$$V^N_i(t) = X^N_{i,0}(t), \qquad V^{r,N}_i(t) = \sum_{j=1}^{N} X^N_{i,j}(t).$$

So the evolution equations of $(Z^N(t))$ can be obtained from equations (3)–(5) as follows

$$dR_i^{r,N}(t) = \sum_{j=1}^{N} \mathbf{1}_{\{V_j^N(t^-)>0,\ S_i^N(t^-)<K\}}\, \mathcal{N}_{\lambda,j}^{U,N}(dt,\{i\}) - \sum_{j=1}^{N}\left(\sum_{l=1}^{\infty} \mathbf{1}_{\{l\le X_{j,i}^N(t^-)\}} \mathcal{N}_{\nu,j,i,l}(dt)\right), \tag{6}$$

$$dR_i^{N}(t) = \sum_{l=1}^{\infty}\sum_{j=1}^{N} \mathbf{1}_{\{l\le X_{j,i}^N(t^-)\}} \mathcal{N}_{\nu,j,i,l}(dt) - \sum_{l=1}^{\infty} \mathbf{1}_{\{l\le R_i^N(t^-)\}} \mathcal{N}_{\mu,i,l}(dt), \tag{7}$$

$$dV_i^{N}(t) = \sum_{l=1}^{\infty} \mathbf{1}_{\{l\le R_i^N(t^-)\}} \mathcal{N}_{\mu,i,l}(dt) - \sum_{j=1}^{N} \mathbf{1}_{\{V_i^N(t^-)>0,\ S_j^N(t^-)<K\}}\, \mathcal{N}_{\lambda,i}^{U,N}(dt,\{j\}), \tag{8}$$

$$dV_i^{r,N}(t) = \sum_{j=1}^{N} \mathbf{1}_{\{V_i^N(t^-)>0,\ S_j^N(t^-)<K\}}\, \mathcal{N}_{\lambda,i}^{U,N}(dt,\{j\}) - \sum_{j=1}^{N}\left(\sum_{l=1}^{\infty} \mathbf{1}_{\{l\le X_{i,j}^N(t^-)\}} \mathcal{N}_{\nu,i,j,l}(dt)\right). \tag{9}$$

This process $(Z^N(t))$, in dimension $4N$, is not a Markov process because the evolution equations for $(Z^N(t))$ are not autonomous. They depend on the Markov process $(X^N(t))$ which lives in dimension $(N+1)^2$. See equation (2). We introduce process $(Z^N(t))$ because it is sufficient to capture the performance of the system, such the fact that a station is empty or full. The quite remarkable property is that the limit as N gets large of $(Z^N(t))$ is a non-linear Markov process. We will present this process in this section and prove the convergence result in the next section.

3.2. **Heuristic computation of the asymptotic process.** The proof of the convergence will be given in the next section. Here we show how we can guess the asymptotic process. Suppose that, for $1 \le i \le N$, $(Z_i^N(t))$ converges in distribution to some process

$$(\overline{Z}(t)) = (\overline{R^r}(t), \overline{R}(t), \overline{V}(t), \overline{V^r}(t)).$$

Let $\mathcal{P}_i^N$ be the random measure on $\mathbb{R}_+$ defined by

$$\mathcal{P}_i^N([0,t]) = \int_0^t \sum_{j=1}^{N}\left(\sum_{l=1}^{\infty} \mathbf{1}_{\{l\le X_{j,i}^N(s^-)\}} \mathcal{N}_{\nu,j,i,l}(ds)\right). \tag{10}$$

It shows that $\mathcal{P}_i^N$ is a counting process (with jump size 1) on $\mathbb{R}_+$ and its compensator is

$$\nu \int_0^t \sum_{j=1}^{N} X_{j,i}^N(s)\, ds = \nu \int_0^t R_i^{N,r}(s)\, ds.$$

See Robert [12, Proposition A.9] for example. Thus, due to the convergence in distribution of $(Z_i^N(t))$ and the standard results on convergence of point processes, $\mathcal{P}_i^N$ converges to an non-homogeneous Poisson process $\mathcal{P}^\infty$ with intensity $(\nu\overline{R^r}(t))$. It can be written as follows

$$\mathcal{P}^\infty(dt) = \int_{\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \overline{R^r}(t^-)\}} \overline{\mathcal{N}}_{\nu,1}(dt,dh)$$

where, with our notations, $\overline{\mathcal{N}}_{\nu,1}$ is a Poisson process with intensity $\nu dhdt$, using the characterization of a Poisson process by the martingale of its stochastic integral.

Following the same lines, it gives also that $\tilde{\mathcal{P}}_i^N$, the random measure on $\mathbb{R}_+$ defined by

$$\tilde{\mathcal{P}}_i^N([0,t]) = \int_0^t \sum_{j=1}^N \left(\sum_{l=1}^\infty \mathbf{1}_{\{l \leq X_{i,j}^N(s^-)\}} \mathcal{N}_{\nu,i,j,l}(ds) \right) \tag{11}$$

is a counting process (with jump size 1) on $\mathbb{R}_+$ with compensator

$$\nu \int_0^t \sum_{j=1}^N X_{i,j}^N(s)\, ds = \nu \int_0^t V_i^{N,r}(s)\, ds.$$

It converges to an non-homogeneous Poisson process $\tilde{\mathcal{P}}^\infty$ with intensity $(\nu \overline{V^r}(t))$, i.e.

$$\tilde{\mathcal{P}}^\infty(dt) = \int_{\mathbb{R}_+} \mathbf{1}_{\{0 \leq h \leq \overline{V^r}(t^-)\}} \overline{\mathcal{N}}_{\nu,2}(dt, dh).$$

Remark 1. *Point processes $\overline{\mathcal{N}}_{\nu,1}$ and $\overline{\mathcal{N}}_{\nu,2}$ are independent because $\mathcal{P}_i^N$ (respectively $\tilde{\mathcal{P}}_i^N$) is a function of $(\mathcal{N}_{\nu,i,j,.}, j)$ (respectively $(\mathcal{N}_{\nu,j,i,.}, j)$), where $(\mathcal{N}_{\nu,i,j,.}, j \neq i)$ and $(\mathcal{N}_{\nu,j,i,.}, j \neq i)$ are independent.*

Then let us consider the random measure $\mathcal{Q}_i^N$ on $\mathbb{R}_+$ defined by

$$\mathcal{Q}_i^N([0,t]) = \int_0^t \sum_{j=1}^N \mathbf{1}_{\{V_j^N(s^-)>0,\, S_i^N(s^-)<K\}}\, \mathcal{N}_{\lambda,j}^{U,N}(ds, \{i\}) \tag{12}$$

with compensator

$$\lambda \int_0^t \mathbf{1}_{\{S_i^N(s)<K\}} \frac{1}{N} \sum_{j=1}^N \mathbf{1}_{\{V_j^N(s)>0\}} ds.$$

Heuristically, by asymptotic independence of stations and the law of large numbers, as N tends to infinity,

$$\frac{1}{N} \sum_{j=1}^N \mathbf{1}_{\{V_j^N(t)>0\}} \to \mathbb{P}(\overline{V}(t) > 0).$$

Thus formally, as N tends to $+\infty$, $\mathcal{Q}_i^N$ converges to an non-homogeneous Poisson process $\mathcal{Q}^\infty$ with intensity $(\lambda \mathbf{1}_{\{\overline{S}(t)<K\}} \mathbb{P}(\overline{V}(t) > 0))$. In other words,

$$\mathcal{Q}^\infty(dt) = \int_{\mathbb{R}_+} \mathbf{1}_{\{0 \leq h \leq \mathbf{1}_{\overline{S}(t^-)<K\}} \mathbb{P}(\overline{V}(t^-)>0)\}} \overline{\mathcal{N}}_{\lambda,1}(dt, dh).$$

With exactly the same work, $\tilde{\mathcal{Q}}_i^N$ on $\mathbb{R}_+$ defined by

$$\tilde{\mathcal{Q}}_i^N([0,t]) = \int_0^t \sum_{j=1}^N \mathbf{1}_{\{V_i^N(s^-)>0,\, S_j^N(s^-)<K\}}\, \mathcal{N}_{\lambda,i}^{U,N}(ds, \{j\}) \tag{13}$$

formally converges in distribution when N tends to $+\infty$ to the non-homogeneous Poisson process $\tilde{\mathcal{Q}}^\infty$ with intensity $(\lambda \mathbf{1}_{\{\overline{V}(t)>0\}} \mathbb{P}(\overline{S}(t) < K))$ also defined by

$$\tilde{\mathcal{Q}}^\infty(dt) = \int_{\mathbb{R}_+} \mathbf{1}_{\{0 \leq h \leq \mathbf{1}_{\{\overline{V}(t^-)>0\}} \mathbb{P}(\overline{S}(t^-)<K)\}} \overline{\mathcal{N}}_{\lambda,2}(dt, dh).$$

Note that for the same reason as previously, $\overline{\mathcal{N}}_{\lambda,2}$ is independent of $\overline{\mathcal{N}}_{\lambda,1}$. By formally taking the limit in equations (6)–(9), it leads to

$$
(14)\quad \begin{cases} d\overline{R^r}(t) = & \int_{\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \mathbf{1}_{\{\overline{S}(t^-)<K\}}\mathbb{P}(\overline{V}(t^-)>0)\}}\overline{\mathcal{N}}_{\lambda,1}(dt,dh) - \int_{\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \overline{R^r}(t^-)\}}\overline{\mathcal{N}}_{\nu,1}(dt,dh), \\ d\overline{R}(t) = & \int_{\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \overline{R^r}(t^-)\}}\overline{\mathcal{N}}_{\nu,1}(dt,dh) - \sum_{l=1}^{+\infty} \mathbf{1}_{\{l\le \overline{R}(t^-)\}}\mathcal{N}_{\mu,l}(dt), \\ d\overline{V}(t) = & \sum_{l=1}^{+\infty} \mathbf{1}_{\{l\le \overline{R}(t^-)\}}\mathcal{N}_{\mu,l}(dt) - \int_{\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \mathbf{1}_{\{\overline{V}(t^-)>0\}}\mathbb{P}(\overline{S}(t^-)<K)\}}\overline{\mathcal{N}}_{\lambda,2}(dt,dh), \\ d\overline{V^r}(t) = & \int_{\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \mathbf{1}_{\{\overline{V}(t^-)>0\}}\mathbb{P}(\overline{S}(t^-)<K)\}}\overline{\mathcal{N}}_{\lambda,2}(dt,dh) - \int_{\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \overline{V^r}(t^-)\}}\overline{\mathcal{N}}_{\nu,2}(dt,dh). \end{cases}
$$

3.3. **A first result.** Then the first result gives the existence and uniqueness of a stochastic process solution of the system of SDEs (14). Let $T>0$ be fixed. Let $\mathcal{D}_T = \mathcal{D}([0,T],\mathcal{P}(\chi))$ be the set of cad-lag functions from $[0,T]$ to $\mathcal{P}(\chi)$.

Theorem 1 (McKean–Vlasov process)**.** *For every $(w,x,y,z)\in\chi$, the system of equations*

$$
(15)\quad \begin{cases} \overline{R^r}(t) = & w + \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \mathbf{1}_{\{\overline{S}(s^-)<K\}}\mathbb{P}(\overline{V}(s^-)>0)\}}\overline{\mathcal{N}}_{\lambda,1}(ds,dh) \\ & - \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \overline{R^r}(s^-)\}}\overline{\mathcal{N}}_{\nu,1}(ds,dh), \\ \overline{R}(t) = & x + \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \overline{R^r}(s^-)\}}\overline{\mathcal{N}}_{\nu,1}(ds,dh) \\ & - \int_0^t \sum_{l=1}^{+\infty} \mathbf{1}_{\{l\le \overline{R}(s^-)\}}\mathcal{N}_{\mu,l}(ds), \\ \overline{V}(t) = & y + \int_0^t \sum_{l=1}^{+\infty} \mathbf{1}_{\{l\le \overline{R}(s^-)\}}\mathcal{N}_{\mu,l}(ds) \\ & - \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \mathbf{1}_{\{\overline{V}(s^-)>0\}}\mathbb{P}(\overline{S}(s^-)<K)\}}\overline{\mathcal{N}}_{\lambda,2}(ds,dh) \\ \overline{V^r}(t) = & z + \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \mathbf{1}_{\{\overline{V}(s^-)>0\}}\mathbb{P}(\overline{S}(s^-)<K)\}}\overline{\mathcal{N}}_{\lambda,2}(ds,dh) \\ & - \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \overline{V^r}(s^-)\}}\overline{\mathcal{N}}_{\nu,2}(ds,dh), \end{cases}
$$

has a unique solution $(\overline{R^r}(t),\overline{R}(t),\overline{V}(t),\overline{V^r}(t))$ in $\mathcal{D}_T$.

Note that the solution of equation (15) satisfies the Fokker–Planck equation (1) in the introduction.

Proof. Let us introduce the Wasserstein distances on $\mathcal{P}(\mathcal{D}_T)$. For π_1, $\pi_2\in\mathcal{P}(\mathcal{D}_T)$,

$$
W_T(\pi_1,\pi_2) = \inf_{\pi\in\mathcal{C}_T(\pi_1,\pi_2)} \int_{\omega=(\omega_1,\omega_2)\in\mathcal{D}_T^2} (d_T(\omega_1,\omega_2)\wedge 1)\, d\pi(\omega),
$$

$$
\rho_T(\pi_1,\pi_2) = \inf_{\pi\in\mathcal{C}_T(\pi_1,\pi_2)} \int_{\omega=(\omega_1,\omega_2)\in\mathcal{D}_T^2} (\|\omega_1-\omega_2\|_{\infty,T}\wedge 1)\, d\pi(\omega)
$$

where, for $f\in\mathcal{D}_T$, $\|f\|_{\infty,T} = \sup\{\|f\|, 0\le t\le T\} = \sup\{\sum_{i=1}^4 |f_i(t)|, 0\le t\le T\}$, $\mathcal{C}_T(\pi_1,\pi_2)$ is the set of couplings of π_1 and π_2, i.e. the subset of $\mathcal{P}(\mathcal{D}_T^2)$ with first marginal is π_1 and the second is π_2, $(\mathcal{D}_T,d_T)$ with d_T the distance associated to the Skorohod topology is complete and separable thus $(\mathcal{P}(\mathcal{D}_T),W_T)$ is complete and separable and $W_T(\pi_1,\pi_2)\le\rho_T(\pi_1,\pi_2)$.

Let us define

$$\Phi : (\mathcal{P}(\mathcal{D}_T), W_T) \to (\mathcal{P}(\mathcal{D}_T), W_T)$$
$$\pi \mapsto \Phi(\pi)$$

where $\Phi(\pi)$ is the distribution of $(Z_\pi(t)) = (R^r_\pi(t), R_\pi(t), V_\pi(t), V^r_\pi(t))$, unique solution of the SDEs

$$\left\{\begin{array}{rl} R^r_\pi(t) = & w + \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \mathbf{1}_{\{\|Z_\pi(s^-)\|<K\}}\pi(v(s^-)>0)\}}\overline{\mathcal{N}}_{\lambda,1}(ds,dh) \\ & - \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0\le h\le R^r_\pi(s^-)\}}\overline{\mathcal{N}}_{\nu,1}(ds,dh), \\ R_\pi(t) = & x + \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0\le h\le R^r_\pi(s^-)\}}\overline{\mathcal{N}}_{\nu,1}(ds,dh) \\ & - \int_0^t \sum_{l=1}^{+\infty} \mathbf{1}_{\{l\le R_\pi(s^-)\}}\mathcal{N}_{\mu,l}(ds), \\ V_\pi(t) = & y + \int_0^t \sum_{l=1}^{+\infty} \mathbf{1}_{\{l\le R_\pi(s^-)\}}\mathcal{N}_{\mu,l}(ds) \\ & - \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \mathbf{1}_{\{V_\pi(s^-)>0\}}\pi(\|z(s^-)\|<K)\}}\overline{\mathcal{N}}_{\lambda,2}(ds,dh) \\ V^r_\pi(t) = & z + \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0\le h\le \mathbf{1}_{\{V_\pi(s^-)>0\}}\pi(\|z(s^-)\|<K)\}}\overline{\mathcal{N}}_{\lambda,2}(ds,dh) \\ & - \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0\le h\le V^r_\pi(s^-)\}}\overline{\mathcal{N}}_{\nu,2}(ds,dh). \end{array}\right. \tag{16}$$

Note that $\pi(v(t)>0) = \int_{z=(r^r,r,v,v^r)\in\mathcal{D}_T} 1_{\{v(t)>0\}} d\pi(z)$. The existence and uniqueness of a solution to (15) is equivalent to the existence and uniqueness of a fixed point $\pi = \Phi(\pi)$.

Let us prove that $\pi = \Phi(\pi)$ has a unique fixed point. For π_1, $\pi_2 \in \mathcal{P}(\mathcal{D}_T)$, let Z_{π_1} and Z_{π_2} be solutions of (16). Thus (Z_{π_1}, Z_{π_2}) is a coupling of $\Phi(\pi_1)$ and $\Phi(\pi_2)$ and, for $t \le T$, $\rho_t(\Phi(\pi_1), \Phi(\pi_2)) \le \mathbb{E}\left(\|Z_{\pi_1} - Z_{\pi_2}\|_{\infty,t}\right)$.

For $t \le T$, using the definition (16) of Z_{π_1} and Z_{π_2},

$$\begin{aligned} &\|Z_{\pi_1} - Z_{\pi_2}\|_{\infty,t} \\ &= \sup_{0\le s\le t} \left(|R^r_{\pi_1}(s) - R^r_{\pi_2}(s)| + |R_{\pi_1}(s) - R_{\pi_2}(s)| + |V_{\pi_1}(s) - V_{\pi_2}(s)| + |V^r_{\pi_1}(s) - V^r_{\pi_2}(s)|\right) \\ &\quad \le \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{A_{\pi_1}(s^-)\wedge A_{\pi_2}(s^-)\le h\le A_{\pi_1}(s^-)\vee A_{\pi_2}(s^-)\}}\overline{\mathcal{N}}_{\lambda,1}(ds,dh) \\ &\quad + 2\int_0^t \sum_{l=1}^{+\infty} |\mathbf{1}_{\{l\le R_{\pi_1}(s^-)\}} - \mathbf{1}_{\{l\le R_{\pi_2}(s^-)\}}|\mathcal{N}_{\mu,l}(ds) \\ &\quad + 2\iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{R^r_{\pi_1}(s^-)\wedge R^r_{\pi_2}(s^-)\le h\le R^r_{\pi_1}(s^-)\vee R^r_{\pi_2}(s^-)\}}\overline{\mathcal{N}}_{\nu,1}(ds,dh) \\ &\quad + 2\iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{B_{\pi_1}(s^-)\wedge B_{\pi_2}(s^-)\le h\le B_{\pi_1}(s^-)\vee B_{\pi_2}(s^-)\}}\overline{\mathcal{N}}_{\lambda,2}(ds,dh) \\ &\quad + \iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{V^r_{\pi_1}(s^-)\wedge V^r_{\pi_2}(s^-)\le h\le V^r_{\pi_1}(s^-)\vee V^r_{\pi_2}(s^-)\}}\overline{\mathcal{N}}_{\nu,2}(ds,dh). \end{aligned} \tag{17}$$

where

$$A_\pi(t) = \mathbf{1}_{\{\|Z_\pi(t)\|<K\}}\pi(v(t)>0) \text{ and} \qquad B_\pi(t) = \mathbf{1}_{\{V_\pi(t)>0\}}\pi(\|z(t)\|<K).$$

The mean of each term of the right-hand side of the previous equation is bounded as follows. For the first term,

$$\begin{aligned}
&\mathbb{E}\left(\iint_{[0,t]\times\mathbb{R}_+}\mathbf{1}_{\{A_{\pi_1}(s^-)\wedge A_{\pi_2}(s^-)\le h\le A_{\pi_1}(s^-)\vee A_{\pi_2}(s^-)\}}\overline{\mathcal{N}}_{\lambda,1}(ds,dh)\right)\\
&\le\lambda\int_0^t|\pi_1(v(s)>0)-\pi_2(v(s)>0)|ds\\
&=\lambda\int_0^t|\pi(z,v_1(s)>0)-\pi(z,v_2(s)>0)|ds\\
&=\lambda\int_0^t\int_{\omega=(z_1,z_2)\in\mathcal{D}_T^2}|\mathbf{1}_{\{v_1(s)>0)\}}-\mathbf{1}_{\{v_2(s)>0)\}}|\pi(d\omega)ds\\
&\le\lambda\int_0^t\int_{\omega=(z_1,z_2)\in\mathcal{D}_T^2}|v_1(s)-v_2(s)|\wedge 1\quad\pi(d\omega)ds\\
&\le\lambda\int_0^t\rho_s(\pi_1,\pi_2)ds.
\end{aligned}$$

For the second term of the right-hand side of equation (17),

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t\sum_{l=1}^{+\infty}|\mathbf{1}_{\{l\le R_{\pi_1}(s^-)\}}-\mathbf{1}_{\{l\le R_{\pi_2}(s^-)\}}|\mathcal{N}_{\mu,l}(ds)\right)\\
&\le\mu\int_0^t\mathbb{E}\left(\sum_{l=1}^{+\infty}|\mathbf{1}_{\{l\le R_{\pi_1}(s)\}}-\mathbf{1}_{\{l\le R_{\pi_2}(s)\}}|\right)ds\\
&\le\mu\int_0^t\mathbb{E}\left(|R_{\pi_1}(s)-R_{\pi_2}(s)|\right)ds\\
&\le\mu\int_0^t\mathbb{E}\left(\|Z_{\pi_1}-Z_{\pi_2}\|_{\infty,s}\right)ds.
\end{aligned}$$

For the third term of the right-hand side of (17),

$$\begin{aligned}
&\mathbb{E}\left(\iint_{[0,t]\times\mathbb{R}_+}\mathbf{1}_{\{R^r_{\pi_1}(s^-)\wedge R^r_{\pi_2}(s^-)\le h\le R^r_{\pi_1}(s^-)\vee R^r_{\pi_2}(s^-)\}}\overline{\mathcal{N}}_{\nu,1}(ds,dh)\right)\\
&\le\nu\int_0^t\mathbb{E}(|R^r_{\pi_1}(s)-R^r_{\pi_2}(s)|)ds\\
&\le\nu\int_0^t\mathbb{E}(\|Z_{\pi_1}-Z_{\pi_2}\|_{\infty,s})ds.
\end{aligned}$$

For the fourth term,

$$\begin{aligned}
&\iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{B_{\pi_1}(s^-)\wedge B_{\pi_2}(s^-)\le h\le B_{\pi_1}(s^-)\vee B_{\pi_2}(s^-)\}}\overline{\mathcal{N}}_{\lambda,2}(ds,dh)\\
&\le \lambda\int_0^t \mathbb{E}\left(|\mathbf{1}_{\{V_{\pi_1}(s)>0\}}\pi_1(\|z(s)\|<K)-\mathbf{1}_{\{V_{\pi_2}(s)>0\}}\pi_2(\|z(s)\|<K)|\right)ds\\
&\le \lambda\int_0^t \mathbb{E}\left(|\mathbf{1}_{\{V_{\pi_1}(s)>0\}}-\mathbf{1}_{\{V_{\pi_2}(s)>0\}}|\right)ds\\
&\le \lambda\int_0^t \mathbb{E}\left(|V_{\pi_1}(s)-V_{\pi_2}(s)|\right)ds\\
&\le \lambda\int_0^t \mathbb{E}\left(\|Z_{\pi_1}-V_{\pi_2}\|_{\infty,s}\right)ds.
\end{aligned}$$

As for the third term, for the fifth term,

$$\begin{aligned}
&\mathbb{E}\left(\iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{V^r_{\pi_1}(s^-)\wedge V^r_{\pi_2}(s^-)\le h\le V^r_{\pi_1}(s^-)\vee V^r_{\pi_2}(s^-)\}}\overline{\mathcal{N}}_{\nu,2}(ds,dh)\right)\\
&\le \nu\int_0^t \mathbb{E}(\|Z_{\pi_1}-Z_{\pi_2}\|_{\infty,s})ds.
\end{aligned}$$

Thus

$$\mathbb{E}(\|Z_{\pi_1}-Z_{\pi_2}\|_{\infty,t}) \le (2\mu+3\nu+2\lambda)\int_0^t \mathbb{E}(\|Z_{\pi_1}-Z_{\pi_2}\|_{\infty,s})ds + \lambda\int_0^t \rho_s(\pi_1,\pi_2)ds.$$

By Grönwall's inequality,

$$\mathbb{E}(\|Z_{\pi_1}-Z_{\pi_2}\|_{\infty,t}) \le C_t\int_0^t \rho_s(\pi_1,\pi_2)ds$$

with $C_t = \lambda\exp((2\mu+3\nu+2\lambda)t)$. For $t\le T$,

$$\rho_t(\Phi(\pi_1),\Phi(\pi_2)) \le C_T\int_0^t \rho_s(\pi_1,\pi_2)ds. \tag{18}$$

It leads to the uniqueness of the solution of $\Phi(\pi)=\pi$. Indeed, if π_1 and π_2 are fixed points of Φ, then equation (18) is rewritten

$$\rho_t(\pi_1,\pi_2) \le C_T\int_0^t \rho_s(\pi_1,\pi_2)ds.$$

By Grönwall's inequality again, for each $t\le T$, $\rho_t(\pi_1,\pi_2)=0$ thus $\pi_1=\pi_2$. The existence is proved by an iteration argument. Let $\pi_0\in\mathcal{P}(\mathcal{D}_T)$ and $\pi_{n+1}=\Phi(\pi_n)$. By equation (18),

$$\begin{aligned}
W_T(\pi_{n+1},\pi_n) &\le \rho_T(\pi_{n+1},\pi_n)\\
&\le C_T^n\rho_T(\pi_1,\pi_0)\int_{0\le s_1\le s_2\ldots s_n\le T} ds_1\ldots ds_n \le \frac{(C_TT)^n}{n!}\rho_T(\pi_1,\pi_0).
\end{aligned}$$

Thus (π_n) converges since $(\mathcal{P}(\mathcal{D}_T),W_T)$ is complete. Because of equation (18), Φ is continuous for the Skorohod topology and its limit is a fixed point of Φ. □

4. Mean-field limit

Recall that, by Theorem 1, the so-called McKean–Vlasov process $(\overline{Z}(t))$ is the unique solution of the system of equations (15). The empirical distribution $\Lambda^N(t)$ of $(Z_i^N(t), 1 \le i \le N)$ is defined, for f on χ, by

$$\Lambda^N(t)(f) = \frac{1}{N}\sum_{i=1}^{N} f(Z_i^N(t)) = \frac{1}{N}\sum_{i=1}^{N} f(R_i^{r,N}(t), R_i^N(t), V_i^N(t), V_i^{r,N}(t))$$

Process $(\Lambda^N(t))$ takes values in

$$\mathcal{Y}^N = \{\Lambda \in \mathcal{P}(\chi), \Lambda_{(w,x,y,z)} \in \frac{\mathbb{N}}{N}, (w,x,y,z) \in \chi, \sum_{(w,x,y,z)\in\chi} (x+y+z)\Lambda_{(w,x,y,z)}N = M_N\}. \tag{19}$$

As process Z^N is not Markov, process $(\Lambda^N(t))$ is not Markov. The aim of this section is to prove the mean-field convergence for $(Z^N(t))$, i.e. that the sequence of processes $(\Lambda^N(t))$ converges in distribution to $(\overline{Z}(t))$. It means that, for any function f with finite support, the sequence of processes $(\Lambda^N(t)(f))$ converges in distribution to $(\mathbb{E}(f(\overline{Z}(t)))$.

Theorem 2 (Mean-field convergence)**.** *The sequence of empirical distribution process $(\Lambda^N(t))$ converges in distribution to a process $(\Lambda(t)) \in \mathcal{D}([0,T], \mathcal{P}(\chi))$ defined by, for f with finite support on χ,*

$$\Lambda(t)(f) = \mathbb{E}(f(\overline{Z}(t)))$$

with $(\overline{Z}(t))$ the unique solution of equation (15). Moreover, for any $k \ge 1$ and for $1 \le i_1 < \ldots < i_k \le N$, the sequence of finite marginals $(Z_{i_1}^N(t), \ldots, Z_{i_k}^N(t))$ converges in distribution to $(\overline{Z}_{i_1}(t), \ldots, \overline{Z}_{i_k}(t))$, where $(\overline{Z}_{i_1}(t))$,..., $(\overline{Z}_{i_k}(t))$ are independent random variables with the same distribution as $(\overline{Z}(t))$.

The last property is the propagation of chaos. The proof is presented in Section 4.3.

4.1. Evolution equations of the empirical measure.

Let us introduce the following notations. For $z \in \chi$, $f : \chi \to \mathbb{R}_+$ and $(e_i, 1 \le i \le 4)$ the canonical basis of $\mathbb{R}^4$ i.e. $e_1 = (1,0,0,0), \ldots, e_4 = (0,0,0,1)$,

$$\begin{aligned}
\Delta_{i,i+1}(f)(z) &= f(z - e_i + e_{i+1}) - f(z), \quad 1 \le i \le 3,\\
\Delta_1^+(f)(z) &= f(z + e_1) - f(z),\\
\Delta_4^-(f)(z) &= f(z - e_4) - f(z).
\end{aligned}$$

Let $\Sigma_K = \{(w,x,y,z) \in \mathbb{N}^4, w+x+y+z \leq K\}$. For $f : \Sigma_K \to \mathbb{R}_+$, straightforwardly by equations (6)-(9),

$$\begin{aligned}
df(Z_i^N(t)) =& \Delta_1^+(f)(Z_i^N(t)) \sum_{j=1}^N \mathbf{1}_{\{V_j^N(t^-)>0,\, S_i^N(t^-)<K\}}\, \mathcal{N}_{\lambda,j}^{U,N}(dt,\{i\}) \\
&+ \Delta_{1,2}(f)(Z_i^N(t)) \sum_{j=1}^N \left(\sum_{l=1}^\infty \mathbf{1}_{\{l \leq X_{j,i}^N(t^-)\}} \mathcal{N}_{\nu,j,i,l}(dt) \right) \\
&+ \Delta_{2,3}(f)(Z_i^N(t)) \sum_{l=1}^\infty \mathbf{1}_{\{l \leq R_i^N(t^-)\}} \mathcal{N}_{\mu,i,l}(dt) \\
&+ \Delta_{3,4}(f)(Z_i^N(t)) \sum_{j=1}^N \mathbf{1}_{\{V_i^N(t^-)>0,\, S_j^N(t^-)<K\}}\, \mathcal{N}_{\lambda,i}^{U,N}(dt,\{j\}) \\
&+ \Delta_4^-(f)(Z_i^N(t)) \sum_{j=1}^N \left(\sum_{l=1}^\infty \mathbf{1}_{\{l \leq X_{i,j}^N(t^-)\}} \mathcal{N}_{\nu,i,j,l}(dt) \right)
\end{aligned}$$

Thus, using the martingale decomposition for Poisson processes,

$$\begin{aligned}
df(Z_i^N(t)) =& \Delta_1^+(f)(Z_i^N(t)) \mathbf{1}_{\{S_i^N(t)<K\}} \sum_{j=1}^N \mathbf{1}_{\{V_j^N(t)>0\}} \frac{\lambda}{N} dt \\
&+ \Delta_{1,2}(f)(Z_i^N(t)) R_i^{r,N}(t) \nu dt + \Delta_{2,3}(f)(Z_i^N(t)) R_i^N(t) \mu dt \\
&+ \Delta_{3,4}(f)(Z_i^N(t)) \mathbf{1}_{\{V_i^N(t)>0\}} \sum_{j=1}^N \mathbf{1}_{\{S_j^N(t)<K\}}\, \frac{\lambda}{N} dt \\
&+ \Delta_4^-(f)(Z_i^N(t)) V_i^{r,N}(t) \nu dt + d\mathcal{M}_{f,i}^N(t)
\end{aligned} \tag{20}$$

where $(\mathcal{M}_{f,i}^N(t))$ is a martingale which will be detailed in Section 4.2. By summing equation (20) for i from 1 to N and dividing by N,

$$\begin{aligned}
\Lambda^N(t)(f) =& \Lambda^N(0)(f) + \mathcal{M}_f^N(t) \\
&+ \lambda \int_0^t \Lambda^N(s)(\Sigma_K \cap \{y>0\}) \Lambda^N(s)(\Delta_1^+(f) \mathbf{1}_{\Sigma_{<K}}) ds \\
&+ \nu \int_0^t \Lambda^N(s)(\Delta_{1,2}(f) p_1) ds + \mu \int_0^t \Lambda^N(s)(\Delta_{2,3}(f) p_2) ds \\
&+ \lambda \int_0^t \Lambda^N(s)(\Sigma_{<K}) \Lambda^N(s)(\Delta_{3,4}(f) \mathbf{1}_{\{y>0\}}) ds + \nu \int_0^t \Lambda^N(s)(\Delta_4^-(f) p_4) ds
\end{aligned} \tag{21}$$

where

$$\mathcal{M}_f^N(t) = \frac{1}{N}(\mathcal{M}_{f,1}^N(t) + \mathcal{M}_{f,2}^N(t) \ldots + \mathcal{M}_{f,N}^N(t)), \tag{22}$$

$$\Sigma_{<K} = \{(w,x,y,z) \in \mathbb{N}^4, w+x+y+z < K\} \tag{23}$$

and

$$p_i : \mathbb{N}^4 \to \mathbb{N} \text{ is the } i-th \text{ projection (for example } p_1(w,x,y,z) = w). \tag{24}$$

4.2. **The martingale term.** The martingale term is $\mathcal{M}_f^N(t)$ given by equation (22) where, for $1 \leq i \leq N$,

$$\begin{aligned}
d\mathcal{M}_{f,i}^N(t) =& \Delta_1^+(f)(Z_i^N(t)) \sum_{j=1}^N \mathbf{1}_{\{V_j^N(t^-)>0,\ S_i^N(t^-)<K\}} \ (\mathcal{N}_{\lambda,j}^{U,N}(dt,\{i\}) - \frac{\lambda}{N}dt) \\
&+ \Delta_{1,2}(f)(Z_i^N(t)) \sum_{j=1}^N \sum_{l=1}^\infty \mathbf{1}_{\{l\leq X_{j,i}^N(t^-)\}} (\mathcal{N}_{\nu,j,i,l}(dt) - \nu dt) \\
&+ \Delta_{2,3}(f)(Z_i^N(t)) \sum_{l=1}^\infty \mathbf{1}_{\{l\leq R_i^N(t^-)\}} (\mathcal{N}_{\mu,i,l}(dt) - \mu dt) \\
&+ \Delta_{3,4}(f)(Z_i^N(t)) \sum_{j=1}^N \mathbf{1}_{\{V_i^N(t^-)>0,\ S_j^N(t^-)<K\}} \ (\mathcal{N}_{\lambda,i}^{U,N}(dt,\{j\}) - \frac{\lambda}{N}dt) \\
&+ \Delta_4^-(f)(Z_i^N(t)) \sum_{j=1}^N \sum_{l=1}^\infty \mathbf{1}_{\{l\leq X_{i,j}^N(t^-)\}} (\mathcal{N}_{\nu,i,j,l}(dt) - \nu dt)
\end{aligned}$$

whose increasing process is expressed as follows

$$\langle \mathcal{M}_f^N \rangle(t) = \frac{1}{N^2}(\lambda I_1^N(t) + \mu I_2^N(t) + \nu I_3^N(t))$$

where, from careful calculations,

$$\begin{aligned}
I_1^N(t) =& \sum_{i=1}^N \int_0^t \Big(\left(\Delta_1^+(f)(Z_i^N(s))\right)^2 \mathbf{1}_{\{S_i^N(s)<K\}} \Lambda^N(s)(\Sigma_K \cap \{y>0\}) \\
&+ \left(\Delta_{3,4}(f)(Z_i^N(s))\right)^2 \mathbf{1}_{\{V_i^N(s)>0\}} \Lambda^N(s)(\Sigma_{<K}) \\
&+ \frac{2}{N}\Delta_1^+(f)(Z_i^N(s))\ \Delta_{3,4}(f)(Z_i^N(s)) \mathbf{1}_{\{V_i^N(s)>0,\ S_i^N(s)<K\}} \Big) ds \\
I_2^N(t) =& \sum_{i=1}^N \int_0^t \left(\Delta_{2,3}(f)(Z_i^N(s))\right)^2 R_i^N(s) ds \\
I_3^N(t) =& \sum_{i=1}^N \int_0^t \Big(\left(\Delta_{1,2}(f)(Z_i^N(s))\right)^2 R_i^{r,N}(s) + \left(\Delta_4^-(f)(Z_i^N(s))\right)^2 V_i^{r,N}(s) \\
&+ \frac{2}{N}\Delta_{1,2}(f)(Z_i^N(s))\ \Delta_4^-(f)(Z_i^N(s)) X_{i,i}(s) \Big) ds.
\end{aligned}$$

The term with $X_{i,i}^N(s)$ comes from the fact that only sequences $(\mathcal{N}_{\nu,i,j,.}, j \neq i)$ and $(\mathcal{N}_{\nu,j,i,.}, j \neq i)$ are independent. See Remark 1. It yields then straightforwardly that there exist $C_0,\ C_1 > 0$ such that, for $1 \leq i \leq 3$,

$$\|I_i^N\|_{\infty,T} \leq (C_0 N + C_1) T \|f\|_\infty^2.$$

Thus, for

$$\|\langle \mathcal{M}_f^N \rangle\|_{\infty,T} \leq (\lambda+\mu+\nu)\left(\frac{C_0}{N} + \frac{C_1}{N^2}\right) T\|f\|_\infty^2. \tag{25}$$

Thus, applying Cauchy-Schwarz then Doob's inequalities,

$$\begin{aligned}\left(\mathbb{E}\left(\sup_{0\le s\le T}|\mathcal{M}_f^N(s)|\right)\right)^2 &\le \mathbb{E}\left(\sup_{0\le s\le T}|\mathcal{M}_f^N(s)|^2\right)\\ &\le 4\mathbb{E}((\mathcal{M}_f^N)^2(T))\\ &= 4\mathbb{E}(\langle\mathcal{M}_f^N\rangle(T)),\end{aligned}$$

and the martingale $(\mathcal{M}_f^N(t))$ converges in distribution to 0 when N tends to ∞.

4.3. **Convergence of the empirical measure process.** This section is devoted to the proof of the mean-field convergence theorem (Theorem 2). The proof is based on tightness and uniqueness arguments.

Proposition 1 (Tightness of the empirical measure process)**.** *The sequence $(\Lambda^N(t))$ is tight with respect to the convergence in distribution in $\mathcal{D}(\mathbb{R}_+,\mathcal{Y}^N)$, $\mathcal{Y}^N$ defined by equation* (19)*. Any limiting point $(\Lambda(t))$ is a continuous process with values in*

$$\mathcal{Y}=\{\Lambda\in\mathcal{P}(\Sigma_K),\ \sum_{(w,x,y,z)\in\Sigma_K}(x+y+z)\Lambda_{(w,x,y,z)}=s\},\tag{26}$$

solution of

$$\begin{aligned}\Lambda(t)(f)=\Lambda(0)(f)&+\lambda\int_0^t\Lambda(s)(\Sigma_K\cap\{y>0\})\Lambda(s)(\Delta_1^+(f)\mathbf{1}_{\Sigma_{<K}})ds\\ &+\nu\int_0^t\Lambda(s)(\Delta_{1,2}(f)p_1)ds+\mu\int_0^t\Lambda(s)(\Delta_{2,3}(f)p_2)ds\\ &+\lambda\int_0^t\Lambda(s)(\Sigma_{<K})\Lambda(s)(\Delta_{3,4}(f)\mathbf{1}_{\{y>0\}})ds+\nu\int_0^t\Lambda(s)(\Delta_4^-(f)p_4)ds\end{aligned}\tag{27}$$

for any function f on $\Sigma_K=\{(w,x,y,z)\in\mathbb{N}^4, w+x+y+z\le K\}$ and $\Sigma_{<K}$ and the p_i's defined by equations (23) *and* (24)*.*

Proof. This amounts to proving that, for any function f on Σ_K, the sequence of processes $(\Lambda^N(t)(f))$ is tight with respect to the topology of the uniform norm on compact sets. For this, using the the modulus of continuity criterion (see Billinsley [2]), it suffices to prove that, for T, ε, $\eta>0$, there exist $\delta_0>0$ and $N_0\in\mathbb{N}$ such that, for all $\delta<\delta_0$ and all $N\ge N_0$,

$$\mathbb{P}\left(\sup_{\substack{0\le s\le t\le T\\ |s-t|<\delta}}|\Lambda^N(t)(f))-\Lambda^N(s)(f))|>\eta\right)<\varepsilon.\tag{28}$$

Let $T>0$, $\varepsilon>0$, $\eta>0$, $\delta>0$ and $s,t\in[0,T]$ such that $|s-t|<\delta$ be fixed. For the third term of the right-hand side of equation (21), there exists $C_2>0$ such that

$$\left|\int_s^t\Lambda^N(u)(\Sigma_K\cap\{y>0\})\Lambda^N(u)(\Delta_1^+(f)\mathbf{1}_{\{\Sigma_{<K}\}})du\right|\le\delta C_2\|f\|_\infty$$

and the same holds for the other terms. Thus there exists $C_3>0$ such that

$$|\Lambda^N(t)(f))-\Lambda^N(s)(f))|\le\delta C_3\|f\|_\infty+|\mathcal{M}_f^N(t)-\mathcal{M}_f^N(s)|.$$

Using equation (25) for the martingale term, it holds that there exists $C_4>0$ such that

$$\mathbb{E}\left(\sup_{\substack{0\le s\le t\le T\\ |s-t|<\delta}}|\Lambda^N(t)(f))-\Lambda^N(s)(f))|\right)\le\delta C_4\|f\|_\infty+2\mathbb{E}\left(\sup_{0\le t\le T}|\mathcal{M}_f^N(t)|\right).$$

Thus, as the martingale term converges in distribution to 0 (see Section), there exist $\delta_0 > 0$ and $N_0 \in \mathbb{N}$ such that, for all $\delta < \delta_0$ and all $N \geq N_0$, the left-hand side of the previous equation is less than ε. Then, using Markov's inequality, the sequence of processes $(\Lambda^N(t)(f))$ satisfies equation (28) thus is tight. Therefore, if Λ is a limiting point of Λ^N, using again equation (21), as $(\mathcal{M}_f^N(t))$ converges in distribution to 0, equation (27) holds. As function f has finite support, all the terms of the right-hand side are straightforwardly continuous on t. □

The following proposition gives the uniqueness of a limiting point of $(\Lambda^N(t))$.

Proposition 2 (Uniqueness)**.** *For every probability Λ_0 on Σ_K, equation* (27) *has at most one solution $(\Lambda(t))$ in $\mathcal{D}(\mathbb{R}_+, \mathcal{P}(\Sigma_K))$ with initial condition Λ_0.*

Proof. Let $\Lambda^1(t))$ and $\Lambda^2(t))$ in $\mathcal{D}(\mathbb{R}_+, \mathcal{P}(\Sigma_K))$ be two solutions of equation (27) with initial condition Λ_0. For f function on Σ_K and $t \geq 0$,

$$
\begin{aligned}
&\Lambda^1(t)(f) - \Lambda^2(t)(f) \\
&= \lambda \int_0^t (\Lambda^1(s) - \Lambda^2(s))(\Sigma_K \cap \{y > 0\})\Lambda^1(s)(\Delta_1^+(f)\mathbf{1}_{\Sigma_{<K}})ds \\
&+ \lambda \int_0^t \Lambda^2(s)(\Sigma_K \cap \{y > 0\})(\Lambda^1(s) - \Lambda^2(s))(\Delta_1^+(f)\mathbf{1}_{\Sigma_{<K}})ds \\
&+ \nu \int_0^t (\Lambda^1(s) - \Lambda^2(s))(\Delta_{1,2}(f)p_1 + \Delta_4^-(f)p_4)ds + \mu \int_0^t (\Lambda^1(s) - \Lambda^2(s))(\Delta_{2,3}(f)p_2)ds \\
&+ \lambda \int_0^t (\Lambda^1(s) - \Lambda^2(s))(\Sigma_{<K})\Lambda^1(s)(\Delta_{3,4}(f)\mathbf{1}_{\{y>0\}})ds \\
&+ \lambda \int_0^t \Lambda^2(s)(\Sigma_{<K})(\Lambda^1(s) - \Lambda^2(s))(\Delta_{3,4}(f)\mathbf{1}_{\{y>0\}})ds.
\end{aligned}
$$

Recall that, for a signed measure π on Σ_K,

$$\|\pi\|_{TV} = \sup\{|\pi(f)|, f : \Sigma_K \to \mathbb{R}, \|f\|_\infty \leq 1\}.$$

From the previous equation, it holds that

$$\|\Lambda^1(t) - \Lambda^2(t)\|_{TV} \leq (8\lambda + 4\nu + 2\mu) \int_0^t \|\Lambda^1(s) - \Lambda^2(s)\|_{TV} ds.$$

Applying Grönwall's lemma,

$$\|\Lambda^1(t) - \Lambda^2(t)\|_{TV} = 0.$$

It ends the proof. □

Proof of Theorem 2. Let $(w, x, y, z) \in \Sigma_K$ and $\Lambda_0 = \delta_{(w,x,y,z)}$. If $(\overline{Z}(t))$ is the unique solution of equation (15) and the measure valued process $(\Lambda(t))$ is defined, for f function on Σ_K, by

$$\Lambda(t)(f) = \mathbb{E}(f(\overline{Z}(t)))$$

then it is easy to check that $(\Lambda(t))$ is a solution of equation (27). The convergence of $(\Lambda^N(t))$ follows from Propositions 1 and 2. See Sznitman [13, Proposition 2.2] for the propagation of chaos property.

4.4. **Probabilistic interpretation of the asymptotic process.** Note that the Fokker–Planck equation (1) is the functional form of the stochastic equation (27). Recall that, in equation (1), equalities in distribution hold, as

$$\iint_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{0 \leq h \leq \overline{R^r}(s^-)\}} \overline{\mathcal{N}}_\nu(ds, dh) = \int_0^t \sum_{l=0}^{\infty} \mathbf{1}_{\{l \leq \overline{R^r}(s^-)\}} \mathcal{N}_{\nu,l}(ds).$$

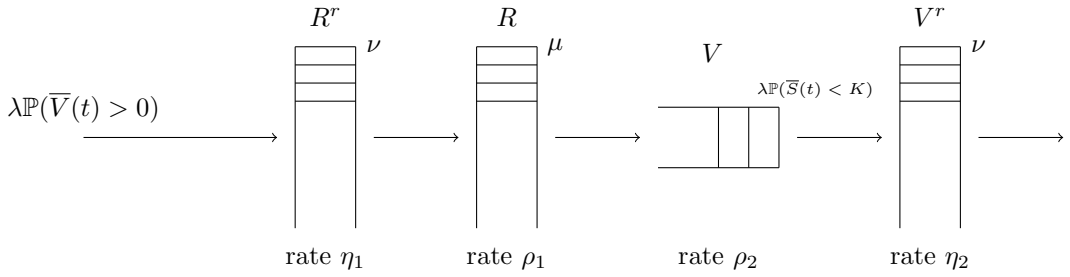

FIGURE 1. Dynamics of $(\overline{Z}(t))$ as a tandem of four queues. The *vertical* queues are $M/M/\infty$ queues, the *horizontal* is a $M/M/1$ queue. The overall capacity is K.

Thus the non-homogeneous Markov process $(\overline{Z}(t))$ can be seen as the state process of four queues in tandem, with overall capacity K. This means that $\overline{R^r}(t)$, $\overline{R}(t)$, $\overline{V}(t)$ and $\overline{V^r}(t)$ are the numbers of customers in respectively

- the first queue, an infinite-server queue with service rate ν,
- the second one, an infinite-server queue with service rate μ,
- the third one, a one-server queue with variable service rate $\lambda\mathbb{P}(\overline{S}(t) < K)$ at time t
- and the last one, an infinite-server queue with service rate ν,

while the arrival process is an non-homogeneous Poisson process with intensity $\lambda\mathbb{P}(\overline{V}(t) > 0)dt$.

Let η_1, ρ_1, ρ_2 and η_2 be the arrival-to-service rate ratios for the four queues from the left to the right (as shown on Figure 1). By definition, one gets

$$\begin{cases} \eta_1(t) &= \frac{\lambda}{\nu}\mathbb{P}(\overline{V}(t) > 0), \quad \rho_1(t) = \frac{\lambda}{\mu}\mathbb{P}(\overline{V}(t) > 0), \\ \rho_2(t) &= \dfrac{\mathbb{P}(\overline{V}(t) > 0)}{\mathbb{P}(\overline{S}(t) < K)}, \quad \eta_2(t) = \frac{\lambda}{\nu}\mathbb{P}(\overline{V}(t) > 0). \end{cases} \tag{29}$$

5. Equilibrium of the asymptotic process

The quadruplet of the numbers of customers in such four queues in tandem with fixed arrival-to-service rate ratios η_1, ρ_1, ρ_2 and η_2 and finite overall capacity K is an ergodic Markov process as an irreducible Markov process on a finite state space. Moreover, the unique invariant probability measure has a well-known product form given by

$$\pi_{j,k,l,m}(\rho) = \frac{1}{Z(\rho)} \frac{\eta_1^j \rho_1^k}{j!k!} \rho_2^l \frac{\eta_2^m}{m!} \tag{30}$$

where, to shorten the notations, $(\eta_1, \rho_1, \rho_2, \eta_2)$ is denoted by ρ and the normalizing constant is

$$Z(\rho) = \sum_{j+k+l+m \le K} \frac{\eta_1^j \rho_1^k}{j!k!} \rho_2^l \frac{\eta_2^m}{m!}. \tag{31}$$

If the process $(\overline{Z}(t))$ of the number of customers in the four queues in tandem is at equilibrium then, denoting by $L_{\rho(t)}$ its generator, it holds that

$$0 = \pi(\rho(t)) L_{\rho(t)}$$

where $\rho(t) = (\eta_1, \rho_1, \rho_2, \eta_2)(t)$ is given by equation (29). It means that the equilibrium point is the probability measure $\pi(\rho)$ on $\mathcal{Y}$ defined by (26) where $\rho = (\eta_1, \rho_1, \rho_2, \eta_2)$ satisfies

$$\eta_1 = \frac{\lambda}{\nu}(1 - \pi_{0V}(\rho)), \tag{32}$$

$$\rho_1 = \frac{\lambda}{\mu}(1 - \pi_{0V}(\rho)), \tag{33}$$

$$\rho_2 = \frac{1 - \pi_{0V}(\rho)}{1 - \pi_S(\rho)}, \tag{34}$$

$$\eta_2 = \frac{\lambda}{\nu}(1 - \pi_{0V}(\rho)) \tag{35}$$

with $\pi_{0V}(\rho) = \sum_{j+l+m\leq K} \pi_{j,0,l,m}(\rho)$ and $\pi_S(\rho) = \sum_{j+k+l+m=K} \pi_{j,k,l,m}(\rho)$. Viewing the tandem of four queues as a station, $\pi_{0V}(\rho)$ is the probability that there is no car available and $\pi_S(\rho)$ the probability that the station is saturated.
Because $\pi(\rho)$ has support on $\mathcal{Y}$, it holds that

$$s = \sum_{j+k+l+m\leq K} (k + l + m)\pi_{j,k,l,m}(\rho). \tag{36}$$

Theorem 3. *If ν is enough large then there exists a unique equilibrium point for the Fokker–Planck equation* (27) *which is $\pi(\rho)$ defined by equation* (30) *where $\rho = (\frac{\mu}{\nu}\rho_1, \rho_1, \rho_2, \frac{\mu}{\nu}\rho_1)$ and (ρ_1, ρ_2) is the unique solution of*

$$\rho_1 = \frac{\lambda}{\mu}(1 - \pi_{0V}(\rho)), \tag{37}$$

$$s = \sum_{j+k+l+m\leq K} (k + l + m)\pi_{j,k,l,m}(\rho). \tag{38}$$

Remark 2. *The assumption that ν is enough large is a technical assumption for the proof. It seems that the result is true for all $\nu > 0$ but the proof is for the moment out of reach.*

Let us prove Theorem 3 in five steps. Steps 2, 3 and 4 are devoted to the special case where ν tends to infinity. This case is called the simple reservation case. Indeed, when ν gets large, it turns out that the model corresponds to the case where the car is not reserved in advance, and the parking space is just reserved when the user takes the car. This model is studied in [8]. Step 1 is here to present the framework for the simple reservation case.

STEP 1: THE SIMPLE RESERVATION CASE. For the model with simple reservation, the problem of existence and uniqueness of an equilibrium point amounts to finding (ρ_1, ρ_2) such that

$$\rho_1 = \frac{\lambda}{\mu}(1 - \pi_{.,0}(\rho_1, \rho_2)), \tag{39}$$

$$\rho_2 = \frac{1 - \pi_{.,0}(\rho_1, \rho_2)}{1 - \pi_S(\rho_1, \rho_2)}, \tag{40}$$

$$s = \sum_{i+j\leq K} (i + j)\pi_{i,j}(\rho_1, \rho_2). \tag{41}$$

where the invariant probability measure is now

$$\pi_{i,j}(\rho_1, \rho_2) = \frac{1}{Z(\rho_1, \rho_2)} \frac{\rho_1^i}{i!}\rho_2^j$$

with $Z(\rho_1, \rho_2) = \sum_{i+j\leq K} \pi_{i,j}(\rho_1, \rho_2)$, $\pi_{.,0} = \sum_{i=0}^{K} \pi_{i,0}$ and $\pi_S = \sum_{i+j=K} \pi_{i,j}$.

STEP 2: EQUATIONS (39)**-**(41) **AMOUNT TO EQUATIONS** (39) **and** (41)**.** Note that any ρ solution of (39) and (40) lies in $\Gamma = [0, \lambda/\mu[\times \mathbb{R}^+$. For any $\rho \in \Gamma$, ρ is solution of equation (40)

because

$$\rho_2(1-\pi_S(\rho_1,\rho_2)) = \rho_2\frac{1}{Z(\rho_1,\rho_2)}\sum_{i+j<K}\frac{\rho_1^i}{i!}\rho_2^j = \frac{1}{Z(\rho_1,\rho_2)}\sum_{j>0,i+j\leq K}\frac{\rho_1^i}{i!}\rho_2^j = 1-\pi_{\cdot,0}(\rho_1,\rho_2).$$

STEP 3: DIFFEOMORPHISM FROM EQUATION (39)**.**

Lemma 1 (Diffeomorphism)**.** *There exists a strictly increasing diffeomorphism* $\phi :]0,\lambda/\mu[\to]0,\infty[$*, and* $\psi=\phi^{-1}$*, such that* (ρ_1,ρ_2) *is a solution of equation* (39) *if and only if* $\rho_2=\phi(\rho_1)$*.*

Proof. First, $(\rho_1,\rho_2)\in\Gamma$ is solution of equation (39) if and only if $f(\rho_1,\rho_2)=0$ where f is the $\mathcal{C}^\infty$ function defined by

$$f(\rho_1,\rho_2) = \left(\frac{\lambda}{\mu}-\rho_1\right)Z(\rho_1,\rho_2) - \frac{\lambda}{\mu}\sum_{i=0}^{K}\frac{\rho_1^i}{i!}. \tag{42}$$

To prove the existence of a strictly increasing diffeomorphism ϕ which maps ρ_1 to ρ_2, we apply the global inverse function theorem to auxiliary function h defined on Γ by

$$h(\rho_1,\rho_2) = (\rho_1, f(\rho_1,\rho_2)).$$

Indeed, h is injective if and only if, for any $\rho_1\in]0,\lambda/\mu[$ and $\rho_2,\ \rho_2'\in]0,+\infty[$, $f(\rho_1,\rho_2)=f(\rho_1,\rho_2')$ implies that $\rho_2=\rho_2'$. As f is C^1 on $]0,\lambda/\mu[\times]0,+\infty[$, it is sufficient to prove that $\partial f/\partial\rho_2\neq 0$ on $]0,\lambda/\mu[\times]0,+\infty[$ to obtain that $\rho_2\mapsto f(\rho_1,\rho_2)$ is strictly monotone thus injective.

Note that it is easy to see that $\partial f/\partial\rho_2$ is positive on $]0,\lambda/\mu[\times]0,+\infty[$. Indeed, since $\rho_1<\lambda/\mu$ and $\rho_2\mapsto Z(\rho_1,\rho_2)$ is non decreasing,

$$\frac{\partial f}{\partial\rho_2}(\rho_1,\rho_2) = \left(\frac{\lambda}{\mu}-\rho_1\right)\frac{\partial Z}{\partial\rho_2}(\rho_1,\rho_2) > 0.$$

As a consequence, h is injective and C^1 on $]0,\lambda/\mu[\times]0,+\infty[$. By global inversion function theorem, h^{-1} is C^1 on $h\big(]0,\lambda/\mu[\times]0,+\infty[\big)$ and it exists ϕ C^1 defined on $]0,\lambda/\mu[$ by

$$h^{-1}(\rho_1,0) = (\rho_1,\phi(\rho_1)).$$

Thus, to prove that diffeomorphism ϕ is strictly increasing on $]0,\lambda/\mu[$, it amounts to showing that, for all $\rho_1\in]0,\lambda/\mu[$, one gets

$$\frac{\partial f}{\partial\rho_1}(\rho_1,\phi(\rho_1)) < 0,$$

or equivalently that, for all $(\rho_1,\rho_2)\in]0,\lambda/\mu[\times]0,+\infty[$ such that $f(\rho_1,\rho_2)=0$,

$$\frac{\partial f}{\partial\rho_1}(\rho_1,\rho_2) < 0.$$

First, from equation (42), one obtains the first partial derivative of function f,

$$\frac{\partial f}{\partial\rho_1}(\rho_1,\rho_2) = -Z(\rho_1,\rho_2) + \left(\frac{\lambda}{\mu}-\rho_1\right)\sum_{i+j\leq K-1}\frac{\rho_1^i}{i!}\rho_2^j - \frac{\lambda}{\mu}\sum_{i=0}^{K-1}\frac{\rho_1^i}{i!} \tag{43}$$

thus, subtracting equation (43) to equation (42) yields

$$f(\rho_1,\rho_2) - \frac{\partial f}{\partial\rho_1}(\rho_1,\rho_2) = Z(\rho_1,\rho_2) + \left(\frac{\lambda}{\mu}-\rho_1\right)\sum_{j=0}^{K}\frac{\rho_2^j\rho_1^{K-j}}{(K-j)!} - \frac{\lambda}{\mu}\frac{\rho_1^K}{K!}. \tag{44}$$

By equation (42), $f(\rho_1,\rho_2)=0$ can be rewritten

$$\frac{\lambda}{\mu}-\rho_1 = \frac{\lambda}{\mu Z(\rho_1,\rho_2)}\sum_{i=0}^{K}\frac{\rho_1^i}{i!}.$$

Thus, the second term of the right-hand side of equation (44) is

$$\left(\frac{\lambda}{\mu}-\rho_1\right)\sum_{j=0}^{K}\frac{\rho_2^j\rho_1^{K-j}}{(K-j)!} = \frac{\lambda}{\mu}\frac{1}{Z(\rho_1,\rho_2)}\sum_{i=0}^{K}\frac{\rho_1^i}{i!}\sum_{j=0}^{K}\frac{\rho_2^j\rho_1^{K-j}}{(K-j)!} \tag{45}$$

$$= \frac{\lambda}{\mu}\rho_1^K\frac{1}{Z(\rho_1,\rho_2)}\sum_{i,j=0}^{K}\frac{\rho_1^{i-j}\rho_2^j}{i!(K-j)!} \geq \frac{\lambda}{\mu}\rho_1^K\frac{1}{Z(\rho_1,\rho_2)}\sum_{0\leq j\leq i\leq K}\frac{\rho_1^{i-j}\rho_2^j}{i!(K-j)!}$$

using that all the terms are positive. For the sum of the right-hand side of (45), it holds that

$$\sum_{0\leq j\leq i\leq K}\frac{\rho_1^{i-j}\rho_2^j}{i!(K-j)!} = \frac{1}{K!}\sum_{j+k\leq K}\frac{K!\rho_1^k\rho_2^j}{(j+k)!(K-j)!} \geq \frac{Z(\rho_1,\rho_2)}{K!}$$

using that, for any $j,k\in\mathbb{N}, j+k\leq K$,

$$\frac{K!}{(j+k)!(K-j)!} = \frac{1}{k!}\frac{K!}{(K-j)!}\frac{k!}{(j+k)!} = \frac{1}{k!}\prod_{i=0}^{j-1}\frac{K-i}{j+k-i} \geq \frac{1}{k!}.$$

In conclusion, equation (45) gives that

$$\left(\frac{\lambda}{\mu}-\rho_1\right)\sum_{k=0}^{K}\frac{\rho_2^K\rho_1^{K-k}}{(K-k)!} \geq \frac{\lambda}{\mu}\frac{\rho_1^K}{K!}. \tag{46}$$

Plugging equation (46) in equation (44) and using that $Z(\rho_1,\rho_2)>0$, it turns out that

$$f(\rho_1,\rho_2) - \frac{\partial f}{\partial\rho_1}(\rho_1,\rho_2) > 0.$$

Therefore, as $f(\rho_1,\rho_2)=0$,

$$\frac{\partial f}{\partial\rho_1}(\rho_1,\rho_2) < 0$$

for all $(\rho_1,\rho_2)\in]0,\lambda/\mu[\times]0,+\infty[$ such that $f(\rho_1,\rho_2)=0$.

□

Lemma 1 means that the solutions of equation (39) can be expressed with only one parameter. Note that it will be useful for the study of the equilibrium, especially in calculations to obtain asymptotics. This concludes the third step of the proof.

STEP 4: A MONOTONICITY ARGUMENT TO CONCLUDE SIMPLE RESERVATION CASE. After equations (39) and (40), let us focus now on equation (41). The idea is to prove that the mean number in a tandem of two queues with total capacity K is a strictly increasing function of both arrival-to-service rates. It generalises the monotonicity argument in the similar proof in [8, Section 3.1]. Then using Lemma 1, it concludes to the existence and uniqueness of the equilibrium point.

Lemma 2 (Monotonicity)**.** *The average number $\mathbb{E}(R+V)$ of vehicles and reserved spaces per station, where (R,V) is a random variable with distribution $\pi(\rho_2,\rho_1)$, is a strictly increasing function of both ρ_2 and ρ_1.*

Proof. Let $(\rho_1,\rho_2)\mapsto\mathbb{E}(R+V)$ be denoted by g_K. It is sufficient to prove that, for all $(\rho_1,\rho_2)\in\Gamma$,

$$\frac{\partial g_K}{\partial\rho_1}(\rho_1,\rho_2)>0 \text{ and } \frac{\partial g_K}{\partial\rho_2}(\rho_1,\rho_2)>0$$

by induction on K.

By a change of indexes, g_K can be rewritten

$$g_K = \frac{\sum\limits_{i+j\leq K}(i+j)\frac{\rho_1^i}{i!}\rho_2^j}{\sum\limits_{i+j\leq K}\frac{\rho_1^i}{i!}\rho_2^j} = \frac{\sum\limits_{k=0}^{K} kp_k}{\sum\limits_{k=0}^{K} p_k}$$

where, by definition,

$$p_k = \sum_{j=0}^{k}\rho_2^j\frac{\rho_1^{k-j}}{(k-j)!}.$$

Define also, for $(k,l)\in\mathbb{N}^2$,

$$r_{l,k} = \frac{p_l}{p_k}.$$

Let $k>0$ be fixed. We first show that $r_{k,k-1}$ is an increasing function of both ρ_1 and ρ_2. Indeed

$$r_{k,k-1} = \frac{p_k}{p_{k-1}} = \frac{\rho_2 p_{k-1}+\rho_1^k/k!}{p_{k-1}} = \rho_2 + \frac{\rho_1^k}{k!}\frac{1}{p_{k-1}},$$

thus

$$\frac{\partial r_{k,k-1}}{\partial \rho_2} = 1 - \frac{\rho_1^k}{k!}\frac{\partial p_{k-1}}{\partial \rho_2}\frac{1}{p_{k-1}^2}. \tag{47}$$

But, by definition of p_k,

$$\rho_1^k\frac{\partial p_{k-1}}{\partial \rho_2} = \sum_{j=1}^{k-1} j\rho_2^{j-1}\frac{\rho_1^{2k-j-1}}{(k-j-1)!} = \sum_{j=0}^{k-2}(j+1)\rho_2^j\frac{\rho_1^{2k-j-2}}{(k-j-2)!}. \tag{48}$$

And, using that all terms of the sum in the following equation are positive,

$$\begin{aligned} k!p_{k-1}^2 &= k!\sum_{1\leq u,v\leq k}\rho_2^{u+v-2}\frac{\rho_1^{2k-u-v}}{(k-u)!(k-v)!} \\ &> \sum_{i=0}^{k-2}\rho_2^i\rho_1^{2k-i-2}\sum_{j=1}^{i+1}\frac{k!}{(k-2-i+j)!(k-j)!}. \end{aligned} \tag{49}$$

For all j, $0\leq j\leq i+1$, while $k-2-i+j<k$, it holds that

$$\frac{k!}{(k-2-i+j)!(k-j)!} > \frac{1}{(k-i-2)!}$$

and then

$$\sum_{j=1}^{i+1}\frac{k!}{(k-2-i+j)!(k-j)!} > \frac{i+1}{(k-i-2)!}.$$

Plugging in equation (49) and comparing with equation (48), it gives

$$k!p_{k-1}^2 > \rho_1^k\frac{\partial p_{k-1}}{\partial \rho_2}.$$

Therefore, using (47), it allows to conclude that

$$\frac{\partial r_{k,k-1}}{\partial \rho_2} > 0.$$

Moreover,

$$\frac{\partial r_{k,k-1}}{\partial \rho_1} = \frac{\rho_1^{k-1}}{(k-1)!p_{k-1}}\left(1-\frac{\rho_1}{kp_{k-1}}\frac{\partial p_{k-1}}{\partial \rho_1}\right) > 0. \tag{50}$$

because it is easily checked that $\partial p_{k-1}/\partial \rho_1 = p_{k-2}$ and then $k p_{k-1} > \rho_1 p_{k-2}$.

Therefore, if $l > k$, $r_{l,k} = \prod_{i=k+1}^{l} r_{i,i-1}$ is an increasing function of ρ_2 and ρ_1. This gives that u_K defined by

$$u_K = \frac{p_K}{\sum_{k=0}^{K} p_k} = \frac{1}{\sum_{k=0}^{K} r_{k,K}}$$

is non decreasing in $x \in \{\rho_1, \rho_2\}$, because $r_{k,K} = 1/r_{K,k}$ is non increasing in x.

Note that g_0 is constant with x and that

$$g_K = (1 - u_K) g_{K-1} + K u_K,$$

which yields that

$$\frac{\partial g_K}{\partial x} = (K - g_{K-1}) \frac{\partial u_K}{\partial x} + (1 - u_K) \frac{\partial g_{K-1}}{\partial x}.$$

Since $K - g_{K-1} > 0$, $u_K < 1$ and $\partial u_K/\partial x > 0$, by induction we can conclude that $\partial g_K/\partial x > 0$ for all $K \geq 1$. It ends the proof. □

By step 2, it remains to prove that, for any $s > 0$, there exists a unique (ρ_1, ρ_2) solution of both equations (39) and (41). By Lemma 1, equation (39) can be rewritten $\rho_1 = \psi_{\rho_2}(\rho_2)$ with $\rho_2 \geq 0$. Then equation (41) becomes

$$s = g_K(\psi_{\rho_2}(\rho_2), \rho_2). \tag{51}$$

Let us denote the right-hand side of equation (51) by function s_K defined on $[0, +\infty[$ which maps ρ_2 to $g_K(\psi_{\rho_2}(\rho_2), \rho_2) = \mathbb{E}(R+V)$. It is sufficient to prove that s_K is strictly increasing. It is true, using that

$$s'_K = \frac{\partial g_K}{\partial \rho_2} + \frac{\partial g_K}{\partial \rho_1} \psi'_{\rho_2}$$

as equation (51) holds with ψ_{ρ_2} at ρ_2 but also in a neighborhood of ρ_2 and using both Lemmas 1 and 2. To conclude the proof, one can easily check that s_K covers the whole interval $[0, +\infty[$. Indeed if $\rho_2 = 0$, the only ρ_1 solution of $f(\rho_1, \rho_2) = 0$ is 0 and $E(R+V) = 0$. Similarly, when ρ_2 tends to infinity, ρ_1 has to tend to λ/μ to keep $f(\rho_1, \rho_2) = 0$ and $E(R+V)$ tends to $+\infty$. It ends the proof for the single reservation case.

STEP 5: THE DOUBLE RESERVATION CASE. Straightforwardlly equations (32) and (35) leads to

$$\eta_1 = \eta_2 = \frac{\mu}{\nu} \rho_1.$$

Then the main argument is that equations (33) and (34) can be rewritten as equations (39) and (40) where ρ_1 is replaced by $\tilde{\rho_1} = (1 + 2\mu/\nu)\rho_1$ and λ/μ by $a = \lambda/\mu(1 + 2\mu/\nu)$. Indeed, by straightforward algebra, it holds that $\pi_{0V}(\rho) = \pi_{.,0}(\tilde{\rho_1}, \rho_2)$ and $\pi_S(\rho) = \pi_S(\tilde{\rho_1}, \rho_2)$.

Unfortunately, (36) is not rewritten as (41) with the previous change of variables $\tilde{\rho_1} = (1 + 2\mu/\nu)\rho_1$ but, with careful calculations, as

$$s = \sum_{i+j \leq K} \left(\frac{1 + \mu/\nu}{1 + 2\mu/\nu} i + j \right) \pi_{i,j}(\tilde{\rho_1}, \rho_2). \tag{52}$$

To complete the proof for any $\nu > 0$, it amounts to proving that, for $a > 0$ and $K \in \mathbb{N}$,

$$S(x, y) = \frac{1}{Z(x, y)} \sum_{i+j \leq K} (i + 2j) \frac{x^i}{i!} y^j$$

is a strictly increasing function of $x \in [0, a[$ under $f(x, y) = 0$, where

$$f(x,y) = (a-x)Z(x,y) - a\sum_{i=0}^{K} \frac{x^i}{i!}, \quad Z(x,y) = \sum_{i+j\leq K} \frac{x^i}{i!} y^j.$$

Indeed, the ratio in the right-hand side of equation (52) is a non increasing function from $(0, +\infty)$ to $(1/2, 1)$. By linearity of differentiability and multiplication by a scalar, it suffices to prove that the right-hand side of equation (52) is non-increasing as a function of $\tilde{\rho_1}$, for $\rho_2 = \phi(\rho_1)$ defined in Lemma 1 for the two values $1/2$ and 1 of this ratio. For value 1, it is exactly Lemma 2. It remains to prove it for value $1/2$, which has been asserted previously.

Nevertheless, due to simple reservation case (steps 2 to 4), by continuity, the proof is complete for ν enough large.

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