

Handling congestion in fluid equations

Charlotte Perrin (CNRS, Institut de Mathématiques de Marseille)

joint work with Anne-Laure Dalibard

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Recent developments in fluid dynamics - MSRI



INSTITUT de MATHÉMATIQUES
de MARSEILLE



Fluid equations under a maximal density constraint

- free boundary pb: free domain $\{\rho < \rho^*\}$ / congested domain $\{\rho = \rho^*\}$
- free-congested Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p + \nabla P(\rho) - \operatorname{div} \mathbb{S} = f \\ 0 \leq \rho \leq \rho^*, (\rho^* - \rho)p = 0, p \geq 0 \end{cases}$$

Outline

- motivation, applications in physics, geophysics and biology
- overview of theoretical results
- 1D case: existence and stability of partially congested TW

A quick overview of congestion phenomena in fluids

Modeling of crowd motion

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ u = P_{C_\rho}(U_{\text{des}}) \end{cases}$$

one should ensure that “ $\operatorname{div} u \geq 0$ where $\rho = 1$ ”

\rightsquigarrow space of adm. velocities $C_\rho = \left\{ v \in L^2, \int v \cdot \nabla q \leq 0, \forall q \in H_\rho^1 \right\}$

\rightsquigarrow space of pressures $H_\rho^1 = \{ q \in H^1, q \geq 0 \text{ a.e.}, q = 0 \text{ a.e. in } \{\rho < 1\} \}$

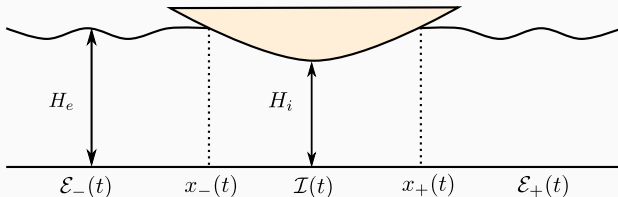
saddle-point formulation find $(u, p) \in L^2 \times H_\rho^1$ such that

$$\begin{cases} u + \nabla p = U_{\text{des}} \\ \int u \cdot \nabla q \leq 0 \quad \forall q \in H_\rho^1 \end{cases}$$

ref: Maury, Roudneff-Chupin, Santambrogio '10

Applications in physics - wave-structure interaction

- mixed flows in closed channels, floating structures



shallow water approximation

$$\begin{cases} \partial_t h + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{gh^2}{2} \right) = -\frac{1}{\rho} h \partial_x p \\ 0 \leq h \leq h_{\text{sol}}, (h_{\text{sol}} - h)(p - p_{\text{atm}}) = 0 \end{cases}$$

ref: Lannes '17, Godlewski et al. '18

Applications in physics - modeling of two-phase flows

- 1D biphasic eq. for liquid-gas mixtures, α volume fraction of the liquid

$$\begin{cases} \partial_t(\alpha\rho_l) + \partial_x(\alpha\rho_l u_l) = 0 \\ \partial_t((1-\alpha)\rho_g) + \partial_x((1-\alpha)\rho_g u_g) = 0 \\ \partial_t(\alpha\rho_l u_l) + \partial_x(\alpha\rho_l u_l^2) + \alpha\partial_x p + \tau_l = M \\ \partial_t((1-\alpha)\rho_g u_g) + \partial_x((1-\alpha)\rho_g u_g^2) + (1-\alpha)\partial_x p + \tau_g = -M \end{cases}$$

$$M = \mu\alpha(1-\alpha)\rho_l(u_g - u_l),$$

$$\rho_l = \text{cst}, \quad p = a\rho_g^\gamma = a\bar{\rho}_g^\gamma (\tilde{\rho}_g)^\gamma \quad \text{where } \tilde{\rho}_g(t, x) = \frac{\rho_g(t, x)}{\bar{\rho}_g}$$

- formal limit system as $\varepsilon = \frac{\bar{\rho}_g}{\rho_l} \rightarrow 0$, $u_g = u_l + \varepsilon w$

$$\begin{cases} \partial_t\alpha + \partial_x(\alpha u_l) = 0 \\ \partial_t(\alpha u_l) + \partial_x(\alpha u_l^2) = 0 \end{cases}$$

pressureless equations: α can exceed 1 !!

ref: Bouchut, Brenier, Cortes, Ripoll (2001)

Applications in physics - modeling of two-phase flows

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$$M = \mu\alpha(1-\alpha)\rho_l(u_g - u_l),$$

$$\rho_l = \text{cst}, \quad p = a\rho_g^\gamma = a\tilde{\rho}_g^\gamma (\tilde{\rho}_g)^\gamma \quad \text{where } \tilde{\rho}_g(t, x) = \frac{\rho_g(t, x)}{\bar{\rho}_g}$$

- conject.: as $\varepsilon = \frac{\bar{\rho}_g}{\rho_l} \rightarrow 0$, $u_g = u_l + \varepsilon w$, $\varepsilon\tilde{\rho}_g^{\gamma-1}p \rightarrow \pi$ with $(1-\alpha)\pi = 0$

$$\begin{cases} \partial_t\alpha + \partial_x(\alpha u_l) = 0 \\ \partial_t(\alpha u_l) + \partial_x(\alpha u_l^2) + \partial_x\pi = 0 \end{cases}$$

π ensures the constraint $\alpha \leq 1$

ref: Bouchut, Brenier, Cortes, Ripoll (2001)

hard congestion models

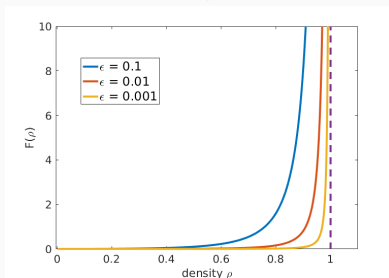
- contacts between grains in dense regime
Signorini's conditions at micro. level
- restriction of adm. velocities $\operatorname{div} u \geq 0$
→ macro. potential $p \geq 0$ in $\{\rho = 1\}$

soft congestion models

- repulsive forces that prevent the contact between the grains
- $\rho < 1$ using singular constitutive laws
 $F(\rho) \xrightarrow{\rho \rightarrow 1} +\infty$

link between the two approaches

$$F(\rho) = \varepsilon \left(\frac{\rho}{1 - \rho} \right)^\gamma, \quad \varepsilon \rightarrow 0$$



ref: Maury '12

Congestion phenomena - some mathematical issues

soft congestion model

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) \\ \quad - \nabla(\lambda \operatorname{div} u_\varepsilon) - 2 \operatorname{div}(\mu D(u_\varepsilon)) = f \\ 0 \leq \rho < 1 \end{cases} \xrightarrow{\varepsilon \rightarrow 0}$$

hard congestion model

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p \\ \quad - \nabla(\lambda \operatorname{div} u) - 2 \operatorname{div}(\mu D(u)) = f \\ 0 \leq \rho \leq 1, (1 - \rho)p = 0, p \geq 0 \end{cases}$$

Existence and properties of solutions to the hard and soft congestion systems

- mechanisms preventing ρ to exceed 1;
- regularity of the solutions, uniqueness;
- justification of the singular limit $\varepsilon \rightarrow 0$
- explicit partially congested solutions;

Free-congested Navier-Stokes equations

soft congestion model

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) \\ \quad - \nabla(\lambda \operatorname{div} u_\varepsilon) - 2 \operatorname{div}(\mu D(u_\varepsilon)) = 0 \\ 0 \leq \rho < 1 \end{cases}$$

hard congestion model

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p \\ \quad - \nabla(\lambda \operatorname{div} u) - 2 \operatorname{div}(\mu D(u)) = 0 \\ 0 \leq \rho \leq 1, (1 - \rho)p, p \geq 0 \end{cases}$$

$$\xrightarrow{\varepsilon \rightarrow 0}$$

- $p_\varepsilon(\rho) = \varepsilon \left(\frac{\rho}{1 - \rho} \right)^\gamma$ with $\gamma > 5/2$ (hyp for the construction of $(\rho_\varepsilon, u_\varepsilon)$)

Theorem (P., Zatorska 2015)

Assume that initially $0 \leq \rho_\varepsilon^0 < 1$ a.e. and

$$\langle \rho_\varepsilon^0 \rangle \leq M^0 < 1, \quad \int_{\Omega} \left[\frac{\rho_\varepsilon^0 |u_\varepsilon^0|^2}{2} + H_\varepsilon(\rho_\varepsilon^0) \right] \leq E^0.$$

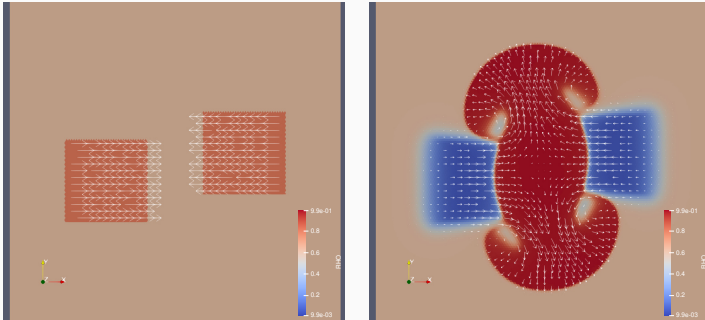
Then

- there exists a global weak solution $(\rho_\varepsilon, u_\varepsilon)$ to the soft congestion system;
- as $\varepsilon \rightarrow 0$, there exists a subsequence $(\rho_\varepsilon, u_\varepsilon, p_\varepsilon(\rho_\varepsilon))$ which converges weakly to a global weak solution (ρ, u, p) to the free-congested NS eq.

see also Bianchini, P. '20 in the 1D inviscid case

Additional remarks

- incompressible dynamics $\operatorname{div} \boldsymbol{u} = 0$ in the congested domain
→ analogy with the low Mach nb limit for the design of num. schemes



numerical simulations performed with the software CALIF³S
developed at the french institute IRSN

ref: P., Saleh '21

Additional remarks

- low regularity of the limit pressure: $\|p_\varepsilon(\rho_\varepsilon)\|_{L^1} \leq C, p_\varepsilon(\rho_\varepsilon) \rightarrow p$ in \mathcal{M}_+
- no info. on the interface between the free and the congested domains
- local well-posedness result for the free-congested NS eq. ?
 - results for compressible-incompressible NS eq with closed interface
ref: Denisova, Solonnikov '16, Shibata '17
 - local well-posedness of the floating body pb
ref: Iguchi, Lannes '19

Existence and stability of 1D partially congested fronts

1D free-congested Navier-Stokes eq. in Lagrangian coord.

$$\begin{cases} \partial_t v - \partial_x u = 0 \\ \partial_t u + \partial_x p - \mu \partial_x \left(\frac{1}{v} \partial_x u \right) = 0 \\ v \geq 1, \quad (v - 1)p = 0, \quad p \geq 0 \end{cases}$$

$v = 1/\rho$: specific volume, u : velocity, p : pressure, $\mu > 0$: viscosity

Free boundary problem:

- $\{v = 1\}$: $p \geq 0$ and incompressible dynamics;
- $\{v > 1\}$: $p = 0$, pressureless dynamics.

Explicit free-congested front $(\bar{v}, \bar{u}, \bar{p})(t, x) = (\bar{v}, \bar{u}, \bar{p})(x - st)$

$$\begin{cases} -s\bar{v}' - \bar{u}' = 0 \\ -s\bar{u}' + \bar{p}' - \mu \left(\frac{\bar{u}'}{\bar{v}} \right)' = 0 \end{cases} \quad + \text{ far field cond. } (\bar{v}, \bar{u}) \xrightarrow{\pm\infty} (v_{\pm}, u_{\pm})$$

we set $v_- = 1 < v_+$, $u_+ < u_-$, we look for a profile which is such that

- congested on $] -\infty, x^* [$, free on $] x^*, +\infty [$;
- \bar{u} , \bar{v} and $\bar{p} - \mu \frac{\bar{u}'}{\bar{v}}$ continuous at x^*

Lemma

Let $s = \frac{u_- - u_+}{v_+ - 1} > 0$, there exists a unique (up to a shift) propagation front:

$$\bar{v}(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ \frac{v_+}{1 + (v_+ - 1) \exp(-\frac{sv_+}{\mu} x)} & \text{if } x > 0 \end{cases} \quad \bar{p}(x) = \begin{cases} s^2(v_+ - 1) & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$
$$\bar{u}(x) = u_+ + sv_+ - s\bar{v}(x)$$

→ what happens if we perturb this profile ?

Study of approximate propagation fronts $(\bar{v}_\varepsilon, \bar{u}_\varepsilon)$

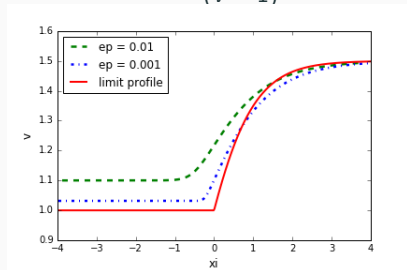
soft congestion eq. with $v_- = v_-^\varepsilon = 1 + \varepsilon^{1/\gamma} < v_+$, $p_\varepsilon(v) = \frac{\varepsilon}{(v-1)^\gamma}$

$$p_\varepsilon(v_-^\varepsilon) = 1, \quad s_\varepsilon = \sqrt{-\frac{p_\varepsilon(v_+) - 1}{v_+ - v_-^\varepsilon}}$$

$$\bar{u}'_\varepsilon = -s_\varepsilon \bar{v}'_\varepsilon$$

$$\bar{v}'_\varepsilon = \frac{\bar{v}_\varepsilon}{\mu s_\varepsilon} (s_\varepsilon^2 (v_+ - \bar{v}_\varepsilon) + p_\varepsilon(v_+) - p_\varepsilon(\bar{v}_\varepsilon))$$

fix the shift by imposing $\bar{v}_\varepsilon(0) = 1 + \varepsilon^{\frac{1}{\gamma+1}}$



Lemma

As $\varepsilon \rightarrow 0$, up to the extraction of a subsequence, we have

$$\bar{v}_\varepsilon \rightarrow \bar{v} \quad \text{in } C(-R, R) \quad \forall R > 0 \quad \text{and weakly-* in } W^{1,\infty}(\mathbb{R})$$

fine description of the transition zone available in [Dalibard, P. '20].

Stability of the profiles $(\bar{v}_\varepsilon, \bar{u}_\varepsilon)$

Theorem (Dalibard, P. '20)

The profiles $(\bar{v}_\varepsilon, \bar{u}_\varepsilon)$ are asymptotically stable under suitably small initial perturbations with zero integral.

idea: introduction of the **effective velocity** $w = u - \mu \partial_x \ln v$ and use of the **integrated variables** (V, W) s.t. $v - \bar{v}_\varepsilon = \partial_x V$, $w - \bar{w}_\varepsilon = \partial_x W$ to derive **weighted energy estimates**

Nonlinear stability of $(\bar{v}_\varepsilon, \bar{w}_\varepsilon = \bar{u}_\varepsilon - \mu \partial_x \ln \bar{v}_\varepsilon)$

- 1d compressible NS equations can be rewritten with $w = u - \mu \partial_x \ln v$

$$\begin{cases} \partial_t v - \partial_x w - \mu \partial_{xx} \ln v = 0 \\ \partial_t w + \partial_x p_\varepsilon(v) = 0 \end{cases}$$

→ diffusion on v ref: Shelukhin'84, Bresch, Desjardins '06, Haspot '14

- if $(v - \bar{v}_\varepsilon)(t) \in L_0^1(\mathbb{R})$, $(w - \bar{w}_\varepsilon)(t) \in L_0^1(\mathbb{R})$, define

$$V(t, x) = \int_{-\infty}^x (v(t, z) - \bar{v}_\varepsilon(t, z)) dz, \quad W(t, x) = \int_{-\infty}^x (w(t, z) - \bar{w}_\varepsilon(t, z)) dz$$

and consider the system

$$\begin{cases} \partial_t V - \partial_x W - \mu \partial_x \ln \left(1 + \frac{\partial_x V}{\bar{v}_\varepsilon} \right) = 0 \\ \partial_t W + p_\varepsilon(\bar{v}_\varepsilon + \partial_x V) - p_\varepsilon(\bar{v}_\varepsilon) = 0 \end{cases}$$
$$\rightsquigarrow \begin{cases} \partial_t V - \partial_x W - \mu \partial_x \left(\frac{\partial_x V}{\bar{v}_\varepsilon} \right) = G_\varepsilon(\partial_x V) \\ \partial_t W + p'_\varepsilon(\bar{v}_\varepsilon) \partial_x V = F_\varepsilon(\partial_x V) \end{cases}$$

↪ good structure of the diffusion term for global-in-time estimates

Sketch of the proof - structure of the linearized system

$$\begin{cases} \partial_t W + p'_\varepsilon(\bar{v}_\varepsilon) \partial_x V = S_W \\ \partial_t V - \partial_x W - \mu \partial_x \left(\frac{\partial_x V}{\bar{v}_\varepsilon} \right) = S_V \end{cases}$$

Lemma (Weighted energy estimate)

Let (V, W) and (S_V, S_W) , regular enough, satisfying the previous equations.

Then

$$\begin{aligned} \sup_{t \geq 0} \int_{\mathbb{R}} \left[\frac{-|W|^2}{2p'_\varepsilon(\bar{v}_\varepsilon)} + \frac{|V|^2}{2} \right] + s_\varepsilon \int_{\mathbb{R}_+} \int_{\mathbb{R}} \frac{p''_\varepsilon(\bar{v}_\varepsilon)}{(p'_\varepsilon(\bar{v}_\varepsilon))^2} \partial_x \bar{v}_\varepsilon \frac{|W|^2}{2} + \mu \int_{\mathbb{R}_+} \int_{\mathbb{R}} \frac{(\partial_x V)^2}{\bar{v}_\varepsilon} \\ \leq \int_{\mathbb{R}} \left[\frac{-|W_0|^2}{2p'_\varepsilon(\bar{v}_\varepsilon)} + \frac{|V_0|^2}{2} \right] + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left[S_W \frac{W}{-p'_\varepsilon(\bar{v}_\varepsilon)} + S_V V \right] \end{aligned}$$

- weight $\frac{-1}{p'_\varepsilon(\bar{v}_\varepsilon)} = \frac{(\bar{v}_\varepsilon - 1)^{\gamma+1}}{\gamma \varepsilon}$ on $|W|^2 \rightarrow$ loss of info. in the congested domain
- $\frac{p''_\varepsilon(\bar{v}_\varepsilon)}{(p'_\varepsilon(\bar{v}_\varepsilon))^2} = \frac{C(\gamma)}{p_\varepsilon(\bar{v}_\varepsilon)} \geq C(\gamma) > 0$
- we want to iterate this estimate for $(\partial_x^k V, \partial_x^k W)$, $k = 1, 2$

Sketch of the proof - structure of the linearized system

$k = 1$

$$\begin{cases} \partial_t \partial_x W + p'_\varepsilon(\bar{v}_\varepsilon) \partial_{xx} V = \partial_x S_W - p''_\varepsilon(\bar{v}_\varepsilon) \partial_x \bar{v}_\varepsilon \partial_x V \\ \partial_t \partial_x V - \partial_{xx} W - \mu \partial_x \left(\frac{\partial_x (\partial_x V)}{\bar{v}_\varepsilon} \right) = \partial_x S_V - \mu \partial_x \left(\frac{\partial_x \bar{v}_\varepsilon}{\bar{v}_\varepsilon^2} \partial_x V \right) \end{cases} \quad \text{where } \frac{p''_\varepsilon(\bar{v}_\varepsilon)}{|p'_\varepsilon(\bar{v}_\varepsilon)|} \lesssim \varepsilon^{-\frac{1}{\gamma}}$$

Lemma

Let (V, W) and (S_V, S_W) , regular enough, satisfying the previous equations. Then there exists C depending only on μ, v_+ and γ such that for all $\delta > 0$

$$\begin{aligned} & \sup_{t \geq 0} \int_{\mathbb{R}} \left[\frac{-|\partial_x W|^2}{2p'_\varepsilon(\bar{v}_\varepsilon)} + \frac{|\partial_x V|^2}{2} \right] + s_\varepsilon \int_{\mathbb{R}_+} \int_{\mathbb{R}} \frac{p''_\varepsilon(\bar{v}_\varepsilon)}{(p'_\varepsilon(\bar{v}_\varepsilon))^2} \partial_x \bar{v}_\varepsilon \frac{|\partial_x W|^2}{2} + \mu \int_{\mathbb{R}_+} \int_{\mathbb{R}} \frac{(\partial_x^2 V)^2}{\bar{v}_\varepsilon} \\ & \leq \int_{\mathbb{R}} \left[\frac{-|\partial_x W_0|^2}{2p'_\varepsilon(\bar{v}_\varepsilon)} + \frac{|\partial_x V_0|^2}{2} \right] + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left[\partial_x S_W \frac{\partial_x W}{-p'_\varepsilon(\bar{v}_\varepsilon)} + S_V \partial_x V \right] \\ & \quad + \eta \int_{\mathbb{R}_+} \int_{\mathbb{R}} \partial_x \bar{v}_\varepsilon |\partial_x W|^2 + \frac{C}{\eta} \varepsilon^{-\frac{2}{\gamma}} \int_{\mathbb{R}_+} \int_{\mathbb{R}} |\partial_x V|^2 \end{aligned}$$

→ multiply by $\varepsilon^{\frac{2}{\gamma}}$ and add the estimate $k = 0$ to use the dissipation involving $|\partial_x V|^2$

Sketch of the proof - fixed point argument

Define

$$E_k(t; V, W) := \int_{\mathbb{R}} \left[\frac{-|\partial_x^k W|^2}{p'_\varepsilon(\bar{v}_\varepsilon)} + |\partial_x^k V|^2 \right], \quad D_k(t; V, W) := \int_{\mathbb{R}} \left[\partial_x \bar{v}_\varepsilon |\partial_x^k W|^2 + |\partial_x^{k+1} V|^2 \right]$$

- the functional space

$$\mathcal{X} := \{(W, V) \in (L^\infty(\mathbb{R}_+; H^2(\mathbb{R})))^2; D_k(\cdot; v, W) \in L^1(\mathbb{R}_+)\}$$

endowed with the norm

$$\|(W, V)\|_{\mathcal{X}}^2 := \sup_{t \in \mathbb{R}_+} \sum_{k=0,1,2} c^k \varepsilon^{\frac{2k}{\gamma}} \left[E_k(t, V, W) + \int_0^t D_k(s, V, W) ds \right]$$

where $0 < c \leq c_0$ for some c_0 depending only on μ, v_+, γ .

- the application $\mathcal{A}_\varepsilon : (W_1, V_1) \in \mathcal{X} \mapsto (W_2, V_2) \in \mathcal{X}$ where

$$\partial_t \begin{pmatrix} W_2 \\ V_2 \end{pmatrix} + \mathcal{L}_\varepsilon \begin{pmatrix} W_2 \\ V_2 \end{pmatrix} = \begin{pmatrix} F_\varepsilon(W_1) \\ G_\varepsilon(V_1) \end{pmatrix}, \quad \begin{pmatrix} W_2 \\ V_2 \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} W_0 \\ V_0 \end{pmatrix}$$

Sketch of the proof - fixed point argument

Proposition

Assume that there exists a constant δ_0 such that

$$E_0(0; W_0, V_0) + \varepsilon^{\frac{2}{\gamma}} E_1(0; W_0, V_0) + \varepsilon^{\frac{4}{\gamma}} E_2(0; W_0, V_0) \leq \delta_0 \varepsilon^{\frac{5}{\gamma}}$$

Then for δ, δ_0, c small enough

- the ball $B_\delta = \{(W, V) \in \mathcal{X}, \|(W, V)\|_{\mathcal{X}} \leq \delta \varepsilon^{\frac{5}{2\gamma}}\}$ is stable by \mathcal{A}_ε ;
- the application \mathcal{A}_ε is a contraction on B_δ .

Remark on the size of the perturbations

$$E_k(t; V, W) := \int_{\mathbb{R}} \left[\frac{-|\partial_x^k W|^2}{p'_\varepsilon(\bar{v}_\varepsilon)} + |\partial_x^k V|^2 \right], \quad D_k(t; V, W) := \int_{\mathbb{R}} \left[\partial_x \bar{v}_\varepsilon |\partial_x^k W|^2 + |\partial_x^{k+1} V|^2 \right]$$

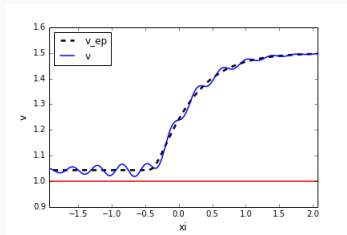
$$\|(W, V)\|_{\mathcal{X}}^2 := \sup_{t \in \mathbb{R}_+} \sum_{k=0,1,2} c^k \varepsilon^{\frac{2k}{\gamma}} \left[E_k(t, V, W) + \int_0^t D_k(s, V, W) ds \right]$$

- for $(W, V) \in B_\delta = \{(W, V) \in \mathcal{X}, \|(W, V)\|_{\mathcal{X}} \leq \delta \varepsilon^{\frac{5}{2\gamma}}\}$

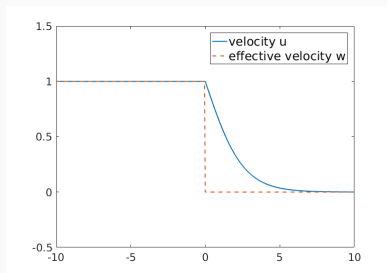
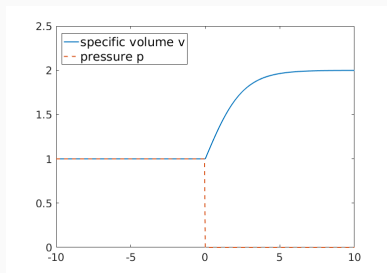
$$\begin{aligned} \|\partial_x V\|_{L_{t,x}^\infty} &\leq C \|\partial_x V\|_{L_t^\infty L_x^2}^{1/2} \|\partial_x^2 V\|_{L_t^\infty L_x^2}^{1/2} \\ &\leq C \varepsilon^{-\frac{3}{2\gamma}} \|(W, V)\|_{\mathcal{X}} \\ &\leq C \delta \varepsilon^{\frac{1}{\gamma}} \end{aligned}$$

so that for δ small enough

$$\|v - \bar{v}_\varepsilon\|_{L_{t,x}^\infty} < \varepsilon^{\frac{1}{\gamma}}$$



Perturbations of the limit profile (\bar{v}, \bar{u})



- the previous estimates degenerate as $\varepsilon \rightarrow 0$
- perturb only the free zone $\{x > 0\}$ and study the initial boundary value pb
→ necessary compatibility condition at $t = 0, x = 0$

Perturbations of the limit profile (\bar{v}, \bar{u})

$$\begin{cases} \partial_t v - \partial_x u = 0 \\ \partial_t u + \partial_x p - \mu \partial_x \left(\frac{1}{v} \partial_x u \right) = 0 \\ (v^0, u^0) = (\bar{v}, \bar{u}) + H^3(\mathbb{R}_+) \end{cases}$$

we look for a solution which is congested for $x < \tilde{x}(t)$, free for $x > \tilde{x}(t)$

Theorem (Dalibard, P. '21)

Assuming some compatibility and non-degeneracy conditions on the initial data, assuming that $\partial_x w^0$ is a decreasing function, then there exists a unique solution (v, u, p, \tilde{x}) on short time intervals.

Strategy

- (v, u) should be constant on $x < \tilde{x}(t) \rightarrow$ we only study (v, u) on $x > \tilde{x}(t)$

let $(v_s, u_s) = (v, u)(t, x - \tilde{x}(t))$

$$\begin{cases} \partial_t v_s - \tilde{x}'(t) \partial_x v_s - \partial_x u_s = 0 \\ \partial_t u_s - \tilde{x}'(t) \partial_x u_s - \mu \partial_x \left(\frac{1}{v_s} \partial_x u_s \right) = 0 \end{cases} \quad \text{for } t > 0, x > 0$$

boundary cond. $[v_s]_{x=0} = 1, [u_s]_{x=0} = u_-, \lim_{x \rightarrow +\infty} (v, u) = (v_+, u_+)$

- **dynamics of the interface** coupled with the dynamics of (v_s, u_s)

$$\tilde{x}'(t) = - \left[\frac{\partial_x u_s}{\partial_x v_s} \right]_{|x=0^+}$$

- effective velocity $w = u - \mu \partial_x \ln v$

$$\rightsquigarrow \partial_t w_s - \tilde{x}'(t) \partial_x w_s = 0 \implies w_s(t, x) = w^0(x + \tilde{x}(t)) \text{ provided } \tilde{x}' > 0$$

$$\rightsquigarrow \partial_t v_s - \tilde{x}'(t) \partial_x v_s - \mu \partial_x^2 \ln v_s = \partial_x w^0(x + \tilde{x}(t))$$

Strategy - Fixed point argument

given $\tilde{y} \in H_{loc}^2(\mathbb{R}_+)$ such that $\tilde{y}(0) = 0$, $\tilde{y}'(0) = - \left[\frac{\partial_x u_s}{\partial_x v_s} \right]_{x=0^+}$

we look for a fixed point of $\mathcal{A} : H_{loc}^2(\mathbb{R}_+) \longrightarrow H_{loc}^2(\mathbb{R}_+)$

$$\tilde{y} \mapsto \tilde{z}$$

where \tilde{z} is defined through the following steps:

1. define and find high order energy estimates on v_s solution to:

$$\begin{cases} \partial_t v_s - \tilde{y}'(t) \partial_x v_s - \mu \partial_x^2 \ln v_s = \partial_x w^0(x + \tilde{y}(t)) & \text{for } t > 0, x > 0 \\ [v_s]_{x=0} = 1, \quad \lim_{x \rightarrow +\infty} v = v_+, \quad v_s(t=0) = v^0 \end{cases}$$

in particular show that $v_s(t, x) > 1$ for $x > 0$

2. define and find high order energy estimates on u_s solution to:

$$\begin{cases} \partial_t u_s - \tilde{y}'(t) \partial_x u_s - \mu \partial_x \left(\frac{1}{v_s} \partial_x u_s \right) = 0 & \text{for } t > 0, x > 0 \\ [u_s]_{x=0} = u_-, \quad \lim_{x \rightarrow +\infty} u = u_+, \quad u_s(t=0) = u^0 \end{cases}$$

3. set

$$\tilde{z}(t) = - \int_0^t \frac{\partial_x u_s(\tau, 0)}{\partial_x v_s(\tau, 0)} d\tau = -\mu \int_0^t \frac{\partial_x u_s(\tau, 0)}{u_- - w^0(\tilde{y}(\tau))} d\tau$$

- about the regularity $H^2(0, T)$ on \tilde{x}
- regularity of the solution

$$v_s - \bar{v}, u_s - \bar{u} \in L^\infty([0, T[; H^3(\mathbb{R}_+))$$

$$\partial_t(v_s - \bar{v}), \partial_t(u_s - \bar{u}) \in L^\infty(0, T; H^1(\mathbb{R}_+)) \cap L^2(0, T; H^2(\mathbb{R}_+))$$

- extension of the solution (v_s, u_s) by setting $(v_s, u_s) = (1, u_-)$ on \mathbb{R}_-
 $p_s(t, x) := p_-(t) = -\mu[\partial_x u_s]_{|x=0^+}$ for $x < 0$, and $p_- \in H^1(0, T)$

- analysis of fluid equations under a maximal density constraint
 - different modeling approaches (singular potentials or not) and link between them (multiscale analysis)
 - towards more complex flows (e.g. sing. viscosities, heterogeneities)
- analysis of 1D partially congested traveling waves
 - explicit fronts $(\bar{v}_\varepsilon, \bar{u}_\varepsilon)$, (\bar{v}, \bar{u})
 - asympt. stability of $(\bar{v}_\varepsilon, \bar{u}_\varepsilon)$ under small perturbations (quant. in ε)
 - local-in-time existence and uniqueness of partially congested solutions