

Congestion phenomena in fluid equations

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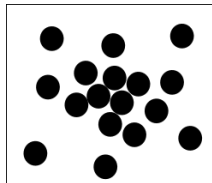
Workshop on Kinetic and Fluid PDEs

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Dense media, congestion phenomena

- microscopic packing constraint
→ finite nb of particles in a given volume
- macroscopic maximal density

$$0 \leq \rho \leq \rho^*$$



collective motion



immersed granular flow
(mud-flows, avalanches, etc.)

Two modelling approaches

soft congestion models

- repulsive forces that prevent the contact between particles
- singular constitutive laws at the macroscopic scale

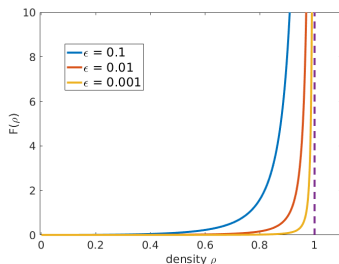
$$F(\rho) \xrightarrow{\rho \rightarrow \rho^*} +\infty$$

hard congestion models

- network of contacts between particles in congested regions
- restriction of admissible velocities
$$\operatorname{div} u \geq 0$$
+ additional potential in $\{\rho = \rho^*\}$

link between the two approaches

$$F(\rho) = \frac{\varepsilon}{(\rho^* - \rho)^\gamma}, \quad \varepsilon \rightarrow 0$$



Toy models

soft congestion model

compressible NS eq. with sing. pressure

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{u}_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla p_\varepsilon(\rho_\varepsilon) \\ \quad - \nabla(\lambda \operatorname{div} \mathbf{u}_\varepsilon) - 2 \operatorname{div}(\mu \mathbf{D}(\mathbf{u}_\varepsilon)) = 0 \\ 0 \leq \rho < 1 \end{cases} \xrightarrow{\varepsilon \rightarrow 0}$$

hard congestion model

compressible/incompressible NS eq.

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p \\ \quad - \nabla(\lambda \operatorname{div} \mathbf{u}) - 2 \operatorname{div}(\mu \mathbf{D}(\mathbf{u})) = 0 \\ 0 \leq \rho \leq 1, (1 - \rho)p = 0, p \geq 0 \\ \operatorname{div} \mathbf{u} = 0 \quad \text{on } \{\rho = 1\} \end{cases}$$

- singular pressure $p_\varepsilon(\rho) = \varepsilon \left(\frac{\rho}{1 - \rho} \right)^\gamma$
→ represents the repulsive microscopic forces between grains
- viscosity coefficients are constant such that $\mu > 0, \lambda + \frac{2}{3}\mu \geq 0$

hard congestion system: Lions & Masmoudi (1999), Berthelin $\mu = \lambda = 0$ (2002, 2017)

limit $\varepsilon \rightarrow 0$: Perrin & Zatorska (2015)

Suspension models, adhesion potential

soft congestion model

compressible NS eq. with sing viscosity

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) \\ \quad - \partial_x(\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon) = \rho f \\ 0 \leq \rho < 1 \end{cases}$$

FORMAL
 $\xrightarrow{\varepsilon \rightarrow 0}$

hard congestion model

two-phase flow with **adhesion potential**

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \rho f \\ \partial_t \gamma + u \partial_x \gamma = -p \\ 0 \leq \rho \leq 1, (1 - \rho)\gamma = 0, \gamma \leq 0 \end{cases}$$

- singular bulk viscosity $\lambda_\varepsilon(\rho_\varepsilon) = \frac{\varepsilon}{1 - \rho_\varepsilon}$

→ represents the repulsive lubrication forces

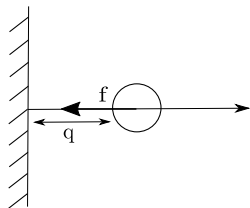
ref: Lefebvre-Lepot, Maury (2012)

Adhesion potential, case of a single particle (Maury, 2007)

vanishing liquid viscosity $\nu = \varepsilon$

$$\ddot{q}_\varepsilon + \varepsilon \frac{\dot{q}_\varepsilon}{q_\varepsilon} = f \quad \xrightarrow{\varepsilon \rightarrow 0} \quad \begin{cases} \dot{q} + \gamma = u^0 + \int_0^t f(s) ds \\ \gamma \leq 0, q \geq 0, q\gamma = 0 \end{cases}$$

$$\iff \begin{cases} \ddot{q} - \mu = f \\ \dot{\gamma} = -\mu, \text{ spt}(\mu) \in \{t, q(t) = 0\} \\ \dot{q}(t^+) = P_{C_{q,\gamma}} \dot{q}(t^-) \\ \gamma \leq 0, q \geq 0 \end{cases}$$



macroscopic extension

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) \\ - \partial_x(\lambda_\varepsilon(\rho_\varepsilon) \partial_x u_\varepsilon) = \rho f \end{cases} \quad \xrightarrow{\varepsilon \rightarrow 0} \quad \begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \rho f \\ \partial_t \gamma + u \partial_x \gamma = -p \\ 0 \leq \rho \leq 1, (1 - \rho)\gamma = 0, \gamma \leq 0 \end{cases}$$

ref: Perrin & Westdickenberg (2017)

- adhesion potential
 - ▶ consequence of the singular bulk viscosity

$$\partial_t \gamma_\varepsilon(\rho) + \partial_x(\gamma_\varepsilon(\rho)u) = \lambda_\varepsilon(\rho_\varepsilon)\partial_x u \rightarrow \partial_t \gamma + u\partial_x \gamma = -p$$

- justification of the singular limit in 2d with additional singular shear viscosity such that $\lambda_\varepsilon(\rho) = 2(\mu'_\varepsilon(\rho)\rho - \mu_\varepsilon(\rho))$

$$\hookrightarrow \partial_t \mu_\varepsilon(\rho) + \operatorname{div}(\mu_\varepsilon(\rho)u) = -\frac{1}{2}\lambda_\varepsilon(\rho)\operatorname{div} u$$

ref: Perrin (2016)

How do the pressure and the bulk viscosity interact in highly dense regimes ?

Quasi-stationary model with singular $p(\rho)$ and $\lambda(\rho)$

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \nabla p_\varepsilon(\rho_\varepsilon) - \nabla(\lambda_\varepsilon(\rho_\varepsilon) \operatorname{div} u_\varepsilon) - 2 \operatorname{div}(\mu D(u_\varepsilon)) = f \end{cases}$$

$$p_\varepsilon(\rho) = \varepsilon \left(\frac{\rho}{1-\rho} \right)^\gamma \quad \lambda_\varepsilon(\rho) = \varepsilon \left(\frac{\rho}{1-\rho} \right)^\beta \quad \mu > 0$$

Plan

- 1 uniform estimates wrt ε
- 2 limit $\varepsilon \rightarrow 0$, hierarchy of limit systems (depending on γ, β)
- 3 some ideas for the construction of weak solutions $(\rho_\varepsilon, u_\varepsilon)$

ref: Bresch, Necasova, Perrin

Preliminary remarks

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \nabla p(\rho) - \nabla(\lambda(\rho) \operatorname{div} u) - 2 \operatorname{div}(\mu D(u)) = f \end{cases}$$

- domain $\Omega = \mathbb{T}^3$ → addition of the constraint $\int_{\mathbb{T}^3} u = 0$ for all $t \geq 0$

- effective viscous flux $F = (2\mu + \lambda) \operatorname{div} u - p$

$$\Delta F = \operatorname{div} f$$

- if $f = 0$ → relation between pressure and viscosity

$$(2\mu + \lambda(\rho)) \operatorname{div} u = p(\rho) + \langle \lambda(\rho) \operatorname{div} u - p(\rho) \rangle$$

$$\langle \lambda(\rho) \operatorname{div} u \rangle = \langle p(\rho) \rangle - \frac{\int \frac{p(\rho)}{2\mu + \lambda(\rho)}}{\int \frac{1}{2\mu + \lambda(\rho)}}$$

Uniform estimates

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \nabla p_\varepsilon(\rho_\varepsilon) - \nabla(\lambda_\varepsilon(\rho_\varepsilon) \operatorname{div} u_\varepsilon) - 2 \operatorname{div}(\mu \mathbf{D}(u_\varepsilon)) = 0 \end{cases}$$

- energy inequality $\rho H_\varepsilon''(\rho) = p_\varepsilon'(\rho) \rightsquigarrow H_\varepsilon(\rho) \underset{1}{\sim} \frac{C_\varepsilon}{(1-\rho)^{\gamma-1}}$

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^3} H_\varepsilon(\rho_\varepsilon) + \int_0^T \int_{\mathbb{T}^3} (2\mu |\mathbf{D}(u_\varepsilon)|^2 + \lambda_\varepsilon(\rho_\varepsilon) (\operatorname{div} u_\varepsilon)^2) \leq E_\varepsilon^0$$

typical difficulty: **no uniform control of the singular laws $p_\varepsilon, \lambda_\varepsilon$**

- we introduce Λ_ε s.t. $\partial_t \Lambda_\varepsilon(\rho_\varepsilon) + \operatorname{div}(\Lambda_\varepsilon(\rho_\varepsilon) u_\varepsilon) = -\lambda_\varepsilon(\rho_\varepsilon) \operatorname{div} u_\varepsilon$

$$\implies \frac{d}{dt} \int_{\mathbb{T}^3} \Lambda_\varepsilon(\rho_\varepsilon) + \int_{\mathbb{T}^3} p_\varepsilon(\rho_\varepsilon) = \frac{\int_{\mathbb{T}^3} \frac{\rho_\varepsilon(\rho_\varepsilon)}{2\mu + \lambda_\varepsilon(\rho_\varepsilon)}}{\int_{\mathbb{T}^3} \frac{1}{2\mu + \lambda_\varepsilon(\rho_\varepsilon)}}$$

Uniform estimates

• if $\langle \rho_\varepsilon^0 \rangle \leq M < 1$ then $\int_{\mathbb{T}^3} \frac{1}{2\mu + \lambda_\varepsilon(\rho_\varepsilon)} \geq C$

$$\frac{d}{dt} \int_{\mathbb{T}^3} \Lambda_\varepsilon(\rho_\varepsilon) + \int_{\mathbb{T}^3} p_\varepsilon(\rho_\varepsilon) \leq C \int_{\mathbb{T}^3} \frac{p_\varepsilon(\rho_\varepsilon)}{2\mu + \lambda_\varepsilon(\rho_\varepsilon)}$$

▶ if $\gamma \leq \beta \rightarrow \int_{\mathbb{T}^3} \frac{p_\varepsilon(\rho_\varepsilon)}{2\mu + \lambda_\varepsilon(\rho_\varepsilon)} \leq C$

▶ if $\gamma > \beta \rightarrow \int_{\mathbb{T}^3} \frac{p_\varepsilon(\rho_\varepsilon)}{2\mu + \lambda_\varepsilon(\rho_\varepsilon)} \leq C + C\varepsilon^\alpha \int_{\mathbb{T}^3} p_\varepsilon(\rho_\varepsilon)$ for some $\alpha > 0$

$$\|\Lambda_\varepsilon(\rho_\varepsilon)\|_{L^\infty L^1} + \|p_\varepsilon(\rho_\varepsilon)\|_{L^1 L^1} \leq C + \|\Lambda_\varepsilon(\rho_\varepsilon^0)\|_{L^1}$$

• $\|p_\varepsilon(\rho_\varepsilon)\|_{L^1 L^1} \leq C \implies \lambda_\varepsilon(\rho_\varepsilon) \operatorname{div} u_\varepsilon$ bounded in $L^1((0, T) \times \mathbb{T}^3)$

$p_\varepsilon(\rho_\varepsilon) \rightharpoonup p$ in $\mathcal{M}_+((0, T) \times \mathbb{T}^3)$, $\lambda_\varepsilon(\rho_\varepsilon) \operatorname{div} u_\varepsilon \rightharpoonup \Pi$ in $\mathcal{M}((0, T) \times \mathbb{T}^3)$

Dominant pressure case $\gamma > \beta$, extension to $\gamma > \beta - 1$

- additional control of the pressure: for some $\theta > 0$ (depending on γ, β)

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^3} \Lambda_\varepsilon(\rho_\varepsilon) (\rho_\varepsilon(\rho_\varepsilon))^\theta + \int_0^T \int_{\mathbb{T}^3} (\rho_\varepsilon(\rho_\varepsilon))^{\theta+1} \leq C$$

- $\lambda_\varepsilon(\rho_\varepsilon) \leq C \varepsilon^{1-\beta/\gamma} (\rho_\varepsilon(\rho_\varepsilon))^{\beta/\gamma} \implies \lambda_\varepsilon(\rho_\varepsilon) \rightarrow 0$ and $\Pi = \overline{\lambda(\rho) \operatorname{div} u} = 0$

limit system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \nabla p - 2 \operatorname{div}(\mu D(u)) = f \\ 0 \leq \rho \leq 1, (1 - \rho)p = 0, p \geq 0 \end{cases}$$

- if $\beta - 1 < \gamma \leq \beta$ (+ $\beta > \beta^*$) \rightarrow same limit system

- ▶ $\beta - 1 < \gamma \implies \Lambda_\varepsilon(\rho_\varepsilon) \rightarrow 0$

- ▶ passing to the limit in

$$\partial_t \Lambda_\varepsilon + \operatorname{div}(\Lambda_\varepsilon u_\varepsilon) = -\lambda_\varepsilon \operatorname{div} u_\varepsilon \implies \Pi = \overline{\lambda(\rho) \operatorname{div} u} = 0$$

Dominant bulk viscosity $\beta - 1 \geq \gamma$

$$\|\Lambda_\varepsilon(\rho_\varepsilon)\|_{L^\infty L^1} + \|\rho_\varepsilon(\rho_\varepsilon)\|_{L^1 L^1} + \|\lambda_\varepsilon(\rho_\varepsilon) \operatorname{div} u_\varepsilon\|_{L^1 L^1} \leq C$$

- additional control of $\Lambda_\varepsilon(\rho_\varepsilon)$ for some $\theta > 0$ (depending on $\beta > \beta^*$)

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^3} (\Lambda_\varepsilon(\rho_\varepsilon))^{1+\theta} + \int_0^T \int_{\mathbb{T}^3} (\Lambda_\varepsilon(\rho_\varepsilon))^\theta \rho_\varepsilon(\rho_\varepsilon) \leq C$$

- no control on $\lambda_\varepsilon(\rho_\varepsilon)$!

$$\boxed{\gamma < \beta - 1}$$

$$\rho_\varepsilon(\rho_\varepsilon) \leq C \varepsilon^{1 - \frac{\gamma}{\beta-1}} (\Lambda_\varepsilon(\rho_\varepsilon))^{\frac{\gamma}{\beta-1}} \rightarrow 0$$

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \nabla \Pi - 2 \operatorname{div}(\mu D(u)) = f \\ \partial_t \Lambda + \operatorname{div}(\Lambda u) = \Pi \\ 0 \leq \rho \leq 1, (1 - \rho)\Lambda = 0, \Lambda \geq 0 \end{cases}$$

$$\boxed{\gamma = \beta - 1}$$

$$\rho_\varepsilon(\rho_\varepsilon) = (\beta - 1)\Lambda_\varepsilon(\rho_\varepsilon) \rightarrow (\beta - 1)\Lambda$$

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ (\beta - 1)\nabla \Lambda + \nabla \Pi - 2 \operatorname{div}(\mu D(u)) = f \\ \partial_t \Lambda + \operatorname{div}(\Lambda u) = \Pi \\ 0 \leq \rho \leq 1, (1 - \rho)\Lambda = 0, \Lambda \geq 0 \end{cases}$$

Construction of global weak solutions for ε fixed

- truncation of the singular laws at $\bar{\rho} = 1 - \delta$
 - ▶ existence of global weak sol. at δ fixed
- uniform estimates wrt δ ($\beta > \beta^*$ if $\beta - 1 \geq \gamma$)
 - ▶ uniform control of $\rho_\delta, \lambda_\delta$ in $L^q, q > 1$
 - ▶ max. bound on ρ_δ (possibly larger than 1)

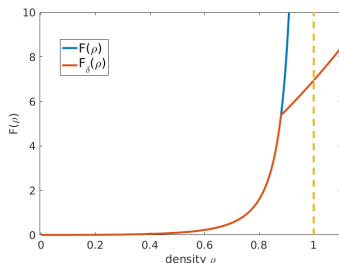
$$\text{meas } \{\rho_{\varepsilon, \delta} \geq 1 - \delta\} \leq C(\varepsilon) \delta^{\max\{\gamma, \beta\}}$$

- limit $\delta \rightarrow 0$ in the equations

- ▶ weak convergence of ρ_δ to ρ_ε such that $\text{meas } \{\rho_\varepsilon \geq 1\} = 0$
- ▶ **strong convergence of (ρ_δ)** : control of the oscillations

↳ identification of the limit pressure and bulk viscosity

$$p_{\varepsilon, \delta}(\rho_{\varepsilon, \delta}) \longrightarrow p_\varepsilon(\rho_\varepsilon) \quad \lambda_{\varepsilon, \delta}(\rho_{\varepsilon, \delta}) \longrightarrow \lambda_\varepsilon(\rho_\varepsilon)$$



Theorem (Bresch, Necasova, Perrin)

Let $\gamma, \beta > 1$ with $\beta > \beta^*$,

$$\beta^* = \begin{cases} \frac{3+\sqrt{5}}{2} & \text{if } \beta - 1 \geq \gamma, \\ \frac{5}{4} & \text{if } \beta - 1 < \gamma. \end{cases}$$

Assume that

$$0 \leq \rho_\varepsilon^0 < 1 \text{ a.e.}, \quad \langle \rho_\varepsilon^0 \rangle \leq M^0 < 1, \quad \int_{\mathbb{T}^3} \frac{\varepsilon}{(1-\rho)^q} \leq E^0$$

- there exists a global weak solution $(\rho_\varepsilon, u_\varepsilon)$ to

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0 \\ \nabla p_\varepsilon(\rho_\varepsilon) - \nabla(\lambda_\varepsilon(\rho_\varepsilon) \operatorname{div} u_\varepsilon) - 2 \operatorname{div}(\mu D(u_\varepsilon)) = f \\ (+) \quad \partial_t \Lambda_\varepsilon(\rho_\varepsilon) + \operatorname{div}(\Lambda_\varepsilon(\rho_\varepsilon) u_\varepsilon) = -\lambda_\varepsilon(\rho_\varepsilon) \operatorname{div} u_\varepsilon \end{cases}$$

- as $\varepsilon \rightarrow 0$, there exists a subsequence converging weakly to $(\rho, u, p, \Pi, \Lambda)$ a global weak solution of

if $\gamma > \beta - 1$

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \nabla p - 2 \operatorname{div}(\mu D(u)) = f \\ 0 \leq \rho \leq 1, (1-\rho)p = 0, p \geq 0 \end{cases}$$

if $\gamma < \beta - 1$, (=)

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \nabla \Pi + (\beta - 1) \nabla \Lambda - 2 \operatorname{div}(\mu D(u)) = f \\ \partial_t \Lambda + \operatorname{div}(\Lambda u) = \Pi \\ 0 \leq \rho \leq 1, (1-\rho)\Lambda = 0, \Lambda \geq 0 \end{cases}$$

Works in progress, perspectives

- theoretical and numerical study of stationary partially congested states
- extension to other singular constitutive laws (drag for instance) ?