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Charlotte Perrin

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# Lagrangian approach to one-dimensional constrained systems

Charlotte Perrin  
joint work with Michael Westdickenberg

Institut für Mathematik  
RWTH Aachen University

Dispersive Hydrodynamics and oceanography  
August 2017

# Constrained systems in fluid mechanics

- partially free surface flows: flows in closed channels, floating structures, etc.
- mixtures

→ existence of a max. constraint (height of the flow, volume fraction, etc)  $\rho \leq \rho^*$

## typical system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = 0 \\ 0 \leq \rho \leq \rho^*, (\rho^* - \rho)p = 0, p \geq 0 \end{cases}$$

in the following:  $d = 1, \rho^* = 1$

## two-phase flow

- pressureless equations in the free domain
- $\operatorname{div} u \geq 0$  on  $\rho = \rho^*$
- activation of the pressure  $p$  on  $\rho = \rho^*$   
Lagrange multiplier associated to the constraint

# Modelling of immersed granular flows

- **mixture: solids particles / liquid**

- ▶ volume fraction of grains



- **constitutive laws in the congested zones ?**

- ▶ 1d model with [memory effects](#) proposed by Lefebvre-Lepot & Maury (2012)

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \rho f \\ \partial_t \gamma + u \partial_x \gamma = -p \\ 0 \leq \rho \leq 1, (1 - \rho) \gamma = 0, \gamma \leq 0 \end{array} \right.$$

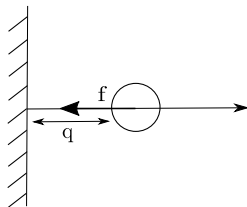
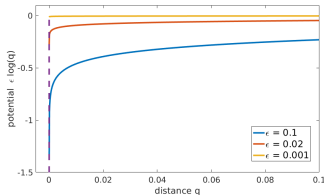
# Memory effects, case of a single particle (Maury, 2007)

vanishing liquid viscosity  $\nu = \varepsilon$

$$\ddot{q}_\varepsilon + \varepsilon \frac{\dot{q}_\varepsilon}{q_\varepsilon} = f$$

$$\varepsilon \rightarrow 0$$

$$\varepsilon \ln q_\varepsilon \rightarrow \gamma$$



$$\begin{cases} \dot{q} + \gamma = u^0 + \int_0^t f(s) ds \\ \gamma \leq 0, q \geq 0, q\gamma = 0 \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} \ddot{q} - \mu = f \\ \dot{\gamma} = -\mu \\ \text{spt}(\mu) \in \{t, q(t) = 0\} \\ \dot{q}(t^+) = P_{C_{q,\gamma}} \dot{q}(t^-) \\ \gamma \leq 0, q \geq 0 \end{cases}$$

# Macroscopic models

- Lefebvre-Lepot, Maury (2012): Micro. variable  $q \rightarrow$  Macro. variable  $1 - \rho$

## suspension

compressible with  
singular viscosity

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0 \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) \\ \quad - \partial_x(\lambda_\varepsilon(\rho_\varepsilon) \partial_x u_\varepsilon) = \rho f \\ \lambda_\varepsilon(\rho_\varepsilon) = \frac{\varepsilon}{1 - \rho_\varepsilon} \end{cases}$$

FORMAL  
 $\xrightarrow{\varepsilon \rightarrow 0}$

## granular

two-phase flow with  
adhesion potential

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \rho f \\ \partial_t \gamma + u \partial_x \gamma = -p \\ 0 \leq \rho \leq 1, (1 - \rho)\gamma = 0, \gamma \leq 0 \end{cases}$$

Issue: existence of solutions to the granular model

## Plan of the talk:

- 1 Lagrangian approach to pressureless gas equations
- 2 addition of the maximal density constraint
- 3 granular system with memory effects

# Lagrangian approach to pressureless gas eq.

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0 \end{cases} \quad \text{Bouchut (1994)}$$

- concentration phenomena on the density

- some references:

- ▶ approx. by sticky particles: Brenier, Grenier (1998), Natile, Savaré (2009)
- ▶ viscous regularization: Sobolevskii (1997), Boudin (2000)
- ▶ hydrodynamical limit: Jabin, Rey (2016)

# Lagrangian approach to pressureless gas eq.

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0 \end{cases} \quad \begin{array}{l} \text{Bouchut (1994)} \\ \bullet \text{ concentration phenomena on the density} \end{array}$$

- some references:
  - ▶ approx. by sticky particles: Brenier, Grenier (1998), Natile, Savaré (2009)
    - Lagrangian point of view via the monotone rearrangement
  - ▶ viscous regularization: Sobolevskii (1997), Boudin (2000)
  - ▶ hydrodynamical limit: Jabin, Rey (2016)



# Dynamics of sticky particles

- $N$  particles  $P_i = (m_i, x_i, u_i)$  such that  $\sum_{i=1}^N m_i = 1, x_1 < x_2 < \dots < x_N$

- free motion ...

$$x_i(t) = x_i(0) + tu_i(0), \quad u_i(t) = u_i(0)$$

... until the first time of collision  $t^1$  when  $x_j(t^1) = x_{j+1}(t^1) = \dots = x_k(t^1)$

particles  $P_j, P_{j+1}, \dots, P_k$  form a new particle  $P$  with

- ▶ mass  $m = m_j + \dots + m_k$

- ▶ velocity  $u = \frac{m_j u_j(t^1) + m_{j+1} u_{j+1}(t^1) + \dots + m_k u_k(t^1)}{m}$

then  $P$  evolves freely with velocity  $u$  until the next collision

- measure description:  $\rho_t = \sum_{i=1}^{N(t)} m_i \delta_{x_i(t)}, (\rho u)_t = \sum_{i=1}^{N(t)} m_i u_i \delta_{x_i(t)}$  satisfy

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0 \end{cases} \quad \text{in } \mathcal{D}'$$

# Lagrangian representation of the sticky particles dynamics

- use of the **monotone rearrangement**  $X_\rho : (0, 1) \rightarrow \mathbb{R}$   
(pseudo inverse of the cumulative distributive function)

$$X_\rho(w) := \inf \{x \in \mathbb{R}, \rho((-\infty, x]) > w\}$$

$\rightarrow X_\rho$  is nondecreasing and right-continuous

- it pushes the Lebesgue measure on  $(0, 1)$  onto  $\rho$

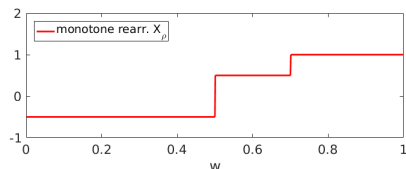
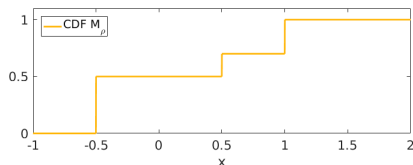
$$\rho = (X_\rho)_\# \mathcal{L}^1_{|(0,1)} \quad \text{i.e.}$$

$$\int_{\mathbb{R}} \varphi(x) d\rho(x) = \int_0^1 \varphi(X_\rho(w)) dw$$

$$\rho_t \in \mathcal{P}_2(\mathbb{R}) \iff X_t \in K$$

$K$  = cone of non-decreasing functions of  $L^2(0, 1)$

$$\rho = 0.5 \delta_{\{-0.5\}} + 0.2 \delta_{\{0.5\}} + 0.3 \delta_{\{1\}}$$



- velocity  $U_t = u_t \circ X_t \in \mathcal{H}_{X_t}$ ,  $\mathcal{H}_{X_t} = \left\{ U = u \circ X_t \text{ for some } u \in L^2_{\rho_t}(\mathbb{R}) \right\}$

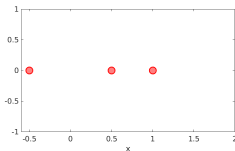
$\rightarrow U_t$  is constant where  $X_t$  is constant

**Property of sticky particles**  $\mathcal{H}_{X_t} \subset \mathcal{H}_{X_s}$ ,  $U_t = P_{\mathcal{H}_{X_t}}(U_s) \quad \forall s \leq t$

## Lagrangian representation

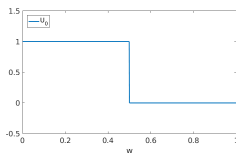
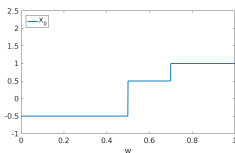
The sticky particles system satisfies  $X_t = P_K(X_0 + tU_0)$

- $X_0 + tU_0$  corresponds to the "collision free" motion



ref: Natile, Savaré (2009)

two other Lagr. characterizations  
(differential inclusions) are derived



## Direct construction of weak sol. to the pressureless eq.

- definition of the transport  $X_t := P_K(X_0 + tU_0)$ 
  - $\implies t \mapsto X_t$  is Lipschitz continuous
  - $\implies$  definition of the Lagrangian velocity  $U_t := \frac{d}{dt}X_t$  a.e.  $t$

- proof of

$$U_t = P_{\mathcal{H}_{X_t}}(U_0) \quad \text{a.e. } t$$

$\rightarrow U_t$  is constant where  $X_t$  is constant

$\rightarrow \exists u_t \in L^2_{\rho_t}$  such that  $U_t = u_t \circ X_t$

- the Eulerian variables satisfy the weak formulation of the pressureless system

**Remark:** one does not use the sticky properties

$$(\mathcal{H}_{X_t} \subset \mathcal{H}_{X_s}, U_t = P_{\mathcal{H}_{X_t}}(U_s) \quad \forall s \leq t)$$

ref: Cavalletti, Sedjro, Westdickenberg (2015)

# Constrained pressureless system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0 \\ 0 \leq \rho \leq 1, (1 - \rho)p = 0, p \geq 0 \end{cases}$$

Bouchut, Brenier, Cortes, Ripoll (2000)

- no more concentration phenomena

- some references

- ▶ theoretical existence results: Berthelin in 1d (2002), multi-d (2016)
- ▶ numerical studies: Degond, Hua, Navoret (2011), Preux (2016)

# Max. density constraint from the transport point of view

$$\rho_t := (X_t)_\# \mathcal{L}^1_{|(0,1)}$$

**Proposition**  $\rho_t \ll \mathcal{L}^1$  if and only if  $\partial_w X_t > 0$  a.e. on  $(0, 1)$  and in this case

$$\rho_t(X_t(w)) = \frac{1}{\partial_w X_t(w)} \quad \text{a.e. on } (0, 1)$$

**Consequence**

$$\rho_t \leq 1 \iff X_t \in \tilde{K} = K + \text{id}$$

$\tilde{K}$  is a convex subset of  $L^2(0, 1)$

in the following  $X_t = S_t + \text{id}$ ,  $S_t \in K$

**Congested domain**

$$\Omega_t = \left\{ w \in (0, 1), S_t \text{ is constant on a neighborhood of } w \right\}$$

# Sticky blocks

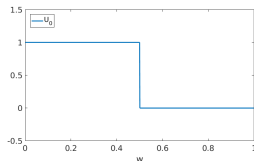
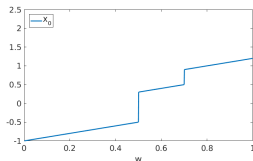
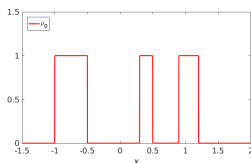
- particles  $P_i = (m_i, x_i, u_i) \rightarrow$  blocks  $B_i = (\ell_i, x_i, u_i)$

- sticky dynamics  $\rightarrow \mathcal{H}_{S_t} \subset \mathcal{H}_{S_\tau} \quad \forall \tau \leq t$

property of the Lagrangian velocity  $U_t = P_{\mathcal{H}_{S_t}}(U_\tau) \quad \forall \tau \leq t$

## Lagrangian representation

The sticky blocks system satisfies  $X_t = P_{\tilde{\mathcal{K}}}(X_0 + tU_0)$



## Resulting pressure

- define  $\Gamma_t$  such that  $\partial_w \Gamma_t := U_t - U_0 = P_{\mathcal{H}_{S_t}}(U_0) - U_0$

**properties:**  $\text{spt } \Gamma_t \subset \Omega_t, \Gamma_t \leq 0$

- define  $P_t := -\frac{d}{dt}\Gamma_t$  in  $\mathcal{D}' \rightarrow \text{spt } P_t \subset \Omega_t$

**sticky property**  $U_t = P_{\mathcal{H}_{S_t}}(U_\tau) \quad \forall \tau \leq t \rightarrow P_t \geq 0$

- the associated Eulerian variables:  $\gamma_t = \Gamma_t \circ X_t^{-1}, p_t = P_t \circ X_t^{-1}$  satisfy in  $\mathcal{D}'$

$$\partial_t \gamma + u \partial_x \gamma = -p \quad \text{on } \{\rho = 1\}$$



## Theorem (Perrin, Westdickenberg)

Let  $\rho_0 \in \mathcal{P}_2(\mathbb{R})$  with  $\rho_0 \leq 1$  a.e.,  $u_0 \in L^2_{\rho_0}(\mathbb{R})$ , there exists a global weak sol. to

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0 \\ 0 \leq \rho \leq 1, \quad p \geq 0, \quad \text{spt } p \subset \{\rho = 1\} \end{cases}$$

**remark:** Berthelin's hypothesis  $\rho_0 \in L^1(\mathbb{R})$ ,  $0 \leq \rho_0 \leq 1$ ,  $u_0 \in L^\infty(\mathbb{R})$ .

# Granular system with memory effects

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \rho f \\ \partial_t \gamma + u \partial_x \gamma = -p \\ 0 \leq \rho \leq 1 \\ (1 - \rho)\gamma = 0, \gamma \leq 0 \end{array} \right. \quad \begin{array}{l} \bullet \text{ no sticky dynamics due to the external force } f \\ \bullet \text{ no sign on the pressure } p \end{array}$$

- introduction of system by Lefebvre, Maury (2012) but no existence result
- similar system with additional viscosity: C.P. (2016)
  - ▶ weak solutions obtained by a singular limit from compressible NS eq.

**Strategy:** construct directly weak sol. without any discrete approximation

# Definition of the transport

- set of admissible transports

maximal density constraint  $\rho_t \leq 1 \implies X_t = S_t + \text{id} \in \tilde{K}, S_t \in K$

- free trajectory and effective transport

▶ free velocity  $U_t^{\text{free}} = U_0 + \int_0^t f(s, X_s) ds$

▶ associated free trajectory  $X_0 + \int_0^t U_s^{\text{free}} ds$

we would like to define

$$X_t = P_{\tilde{K}} \left( X_0 + \int_0^t U_s^{\text{free}} ds \right)$$

→ conditions on  $f$  to ensure the existence and uniqueness of  $X_t \forall t$

# Velocity and adhesion potential in Lagrangian coordinates

- $t \mapsto X_t$  is Lipschitz continuous

- definition of the Lagrangian velocity  $U_t := \frac{d}{dt}X_t$  a.a.  $t$

$$\implies \boxed{U_t = P_{\mathcal{H}_{S_t}}(U_t^{\text{free}})} \quad \text{where } \mathcal{H}_{S_t} := \{u \circ S_t, u \in L^2_{\rho_t}\}$$

- adhesion potential

$$\boxed{\partial_w \Gamma_t := U_t - U_t^{\text{free}} + \partial_w \Gamma_0}$$

$$\begin{cases} \text{spt } \Gamma_0 \in \Omega_0 \\ \Gamma_0 \leq 0 \end{cases} \implies \begin{cases} \text{spt } \Gamma_t \in \Omega_t \\ \Gamma_t \leq 0 \end{cases}$$

## Reformulation of the granular system

we remove the pressure from the eq. by use of the **effective velocity**  $v = u - \partial_x \gamma$

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \rho f \\ \partial_t \gamma + u \partial_x \gamma = -p \\ 0 \leq \rho \leq 1, (1 - \rho)\gamma = 0, \gamma \leq 0 \end{array} \right. \rightarrow \left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho v) + \partial_x(\rho u v) = \rho f \\ v = u - \partial_x \gamma \\ 0 \leq \rho \leq 1, (1 - \rho)\gamma = 0, \gamma \leq 0 \end{array} \right.$$

## Theorem (Perrin, Westdickenberg (ArXiv))

Assume that

- $f \in L^\infty(0, T; \text{Lip} \cap L^\infty(\mathbb{R}))$
- $\rho_0 \in \mathcal{P}_2(\mathbb{R})$  with  $\rho_0 \leq 1$  a.e.
- $(u_0, \gamma_0) \in L^2_{\rho_0}(\mathbb{R}) \times W^{1,2}(\mathbb{R})$  with  $(1 - \rho_0)\gamma_0 = 0$ ,  $\gamma_0 \leq 0$  a.e.

Let  $X_0$  be such that  $\rho_0 = (X_0)_\# \mathcal{L}^1_{(0,1)}$  and  $U_0 := u_0 \circ X_0$ ,  $\Gamma_0 := \gamma_0 \circ X_0$ .

One can then define

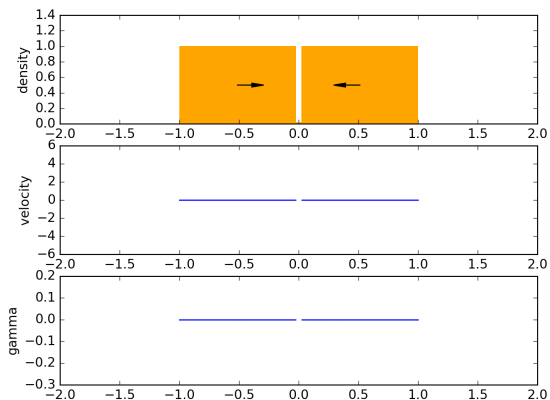
$$U_\tau^{\text{free}} := U_0 + \int_0^\tau f(s, X_s) ds, \quad X_t := P_{\tilde{K}} \left( X_0 + \int_0^t U_\tau^{\text{free}} d\tau \right) \quad \forall t$$
$$U_t := \frac{d}{dt} X_t, \quad \partial_w \Gamma_t := U_t - U_t^{\text{free}} + \partial_w \Gamma_0 \quad \text{a.e. } t$$

and the associated family of Eulerian variables  $(\rho_t, u_t, \gamma_t)$  is a weak solution of

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho(u - \partial_x \gamma)) + \partial_x(\rho u(u - \partial_x \gamma)) = \rho f \\ 0 \leq \rho \leq 1, (1 - \rho)\gamma = 0, \gamma \leq 0 \end{cases}$$

# Numerical simulation

- minimization at each time step of  $\|X_0 + \int_0^{t_h} U_s^{\text{free}} ds - X\|_2^2$   
under the constraint  $X - \text{id} \in K$
- Python software for convex optimization CVXOPT



# Conclusion

- Lagrangian approach
  - ▶ Euler under max. constraint  $\rightarrow$  improvement of the existing theoretical result
  - ▶ system with memory effects  $\rightarrow$  first existence result
  - ▶ easy implementation
- direct extension to the heterogeneous case  $\rho \leq \rho^*(t, x)$  with

$$\partial_t \rho^* + u \partial_x \rho^* = 0$$