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Improving Chaotic Features of Fractional Chaotic Maps

Chunxiao Yang*, Ina Taralova, Jean Jacques Loiseau*****

Laboratoire des Sciences du Numérique de Nantes
LS2N, UMR CNRS 6004
Ecole Centrale de Nantes
Nantes, France
*(*e-mail: chunxiao.yang@ls2n.fr)*
*(**e-mail: ina.taralova@ls2n.fr)*
*(***e-mail: jean-jacques.loiseau@ls2n.fr)*

Abstract: Two fractional chaotic maps, Lu and Chen fractional chaotic maps, have been analyzed using an innovative predictor-corrector method based on a non-uniform grid. An original chaotic law to control the grid size at each iteration has been introduced. The choice of the non-uniform grid was shown to play a fundamental role to obtain chaotic behavior for a larger fractional parameter range, while decreasing the computational time. The obtained results show that the complexity of the resulting fractional chaotic system in terms of Lyapunov exponents, Fourier transforms and chaos robustness is sensitive to the choice of the grid and can be considerably improved by the proposed method.

Keywords: fractional chaotic system, bifurcations, Predictor-Corrector approach, non-uniform grid, numerical calculation

1. INTRODUCTION

Nonlinear fractional maps have been mostly studied to describe and analyse the behaviour in electric and magnetic phenomena, Chua's memristors, fractional order electronic circuits (Petras, 2021), mechanics, thermodynamics and generally in many fields of physics (Uchaikin, 2013) etc. In mechanics of materials, much progress has been made to describe the viscoelasticity of complex materials by nonlinear fractional differential equations and to model composite materials behaviour (Krasnobrizha, et al., 2016).

At the same time, fractional chaotic maps have been less studied from a different perspective: taking advantage of the fractional derivative to achieve robust chaos, while increasing the parameter range for which chaotic behaviour exists. To deal with this issue, we need to define first the criteria to evaluate the features of the fractional chaotic map. From chaotic dynamics point of view, Lyapunov exponents (LEs) have been used as chaoticity quantifiers to evaluate the speed of divergence of two neighbouring trajectories initialised close to each other. It is considered that bigger Lyapunov exponent will give rise to higher complexity.

In the numerical implementation of fractional chaotic maps, the method to solve the fractional derivative is clearly of utmost importance. Here many related issues arise, from the choice of the numerical method for solving the fractional derivative equation, to the choice of the step size inside the method itself.

In this work, we deal with chaotic (i.e., nonlinear) fractional map, and the chosen approach has been the classical predictor-corrector approach (Diethelm, 2002), with the novelty here to use a variable step with very promising results as it will be shown hereafter.

Last but not the least, when comparing fractional and integer order chaotic maps, the fractional derivative plays the role of an additional parameter. This is an advantage for some applications, where increasing the number of parameters may be an interesting feature. For instance, in cryptography where the parameters constitute the encryption key, adding more parameters would increase the size of the key, and therefore improve the security of the whole system (longer the key, longer the time to break it).

It should be noticed that the precision of the numerical calculation is not the crucial issue in our work, the emphasis being given to find the biggest parameter range for which chaotic behaviour can exist.

2. PRELIMINARIES

2.1 Fractional calculus

Various definitions for fractional calculus exist (Riemann-Liouville definition, Grunwald-Leinikov definition and etc.) and they are equivalent under certain conditions (Poldubny, 1999).

To provide a general idea on this topic, we give here fractional integral under Riemann-Liouville (RL) definition and fractional derivatives under Caputo definition in equation (1) and (2) with fractional order α, respectively.

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \tag{1}$$

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$$D_*^{\alpha} f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau \tag{2}$$

a and t in equation (1) and (2) are the bounds of operators; n in equation (2) is the smallest integer greater than α; $\Gamma(\cdot)$ denotes the Euler Gamma function as follows,

$$\Gamma(\alpha) = \int_0^{\infty} \frac{t^{\alpha-1}}{e^t} dt. \tag{3}$$

2.2 Fractional chaotic system

Fractional systems are dynamic systems that can be modelled by fractional deferential equations. The system equation can be expressed as follows,

$$\begin{aligned} D_t^{\alpha_i} x_i(t) &= f_i(x_1(t), x_2(t), ..., x_n(t), t) \\ x_i(0) &= c_i, i = 1, 2, ..., n. \end{aligned} \tag{4}$$

where α_i denotes the fractional derivatives for i-th differential equation, $x_i(0)$ stands for the initial condition and f_i is in general a nonlinear function. The system is commensurate if all α_i hold the same value, otherwise, the system is incommensurate.

The fractional chaotic systems on the other hand are the dynamic systems with fractional derivatives which exhibit chaotic behaviour. It is found that the chaotic behaviour of many integer-order derivative chaotic systems can be preserved after extending them to fractional orders (Petráš, 2011). Hence, in our following work, the fractional chaotic systems adopted are obtained though extension from classical integer order derivative chaotic functions.

3. PROPOSED MODIFIED CORRECTOR-PREDICTOR METHOD BASED ON NON-UNIFORM GRID

The proposed numerical calculation method for fractional systems is based on ABM fractional Corrector and Predictor method (Diethelm, 2002). The original method makes use of the equivalence between the analytical property of fractional differential equations with initial condition problem of Caputo type and Volterra integral equation. The numerical approximated solutions $x_h(t_j)(j=1,2,\ldots n)$ are calculated assuming a uniform grid $\{ t_n = nh: n= 0, 1, \ldots, N\}$ with some integer N and $h := T / N$ employed.

Unlike the classical approach which is calculated with a fixed step size h, we propose an original non-uniform grid to vary the step size in each iteration. We do not go into details of the formula deducing, since it is the same as the classical ABM method which can be found in the paper of Diethelm and only give out the deduced simplified next states calculation equations in (5)-(8). Bearing in mind that the fractional derivatives in the equations are smaller than 1.

$$\begin{aligned} \mathrm{X}(n+1) = \mathrm{X}(0) &+ \frac{h(n)^{\alpha}}{\Gamma(\alpha+2)} f\left(\mathrm{X}^{\mathrm{Pr}}(n+1)\right) \\ &+ \frac{h(n)^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(\mathrm{X}(j)), \end{aligned} \tag{5}$$

In equation (5), $h(n)$ is the non-uniform grid space (step size) taking a value from 0.001 to 0.005 with a gap of 0.001; f is the system function, $\Gamma(\cdot)$ is the gamma function given in (3), and parameter a takes the form as follows,

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & \text{if } 1 \le j \le n, \\ 1, & \text{if } j = n+1. \end{cases} \tag{6}$$

$\mathrm{X}^P(n+1)$ in equation (5) denotes the predicted value of $\mathrm{X}(n+1)$ and is formulated as,

$$\mathrm{X}^P(n+1) = \mathrm{X}(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1}^1 f(\mathrm{X}(j)), \quad 0 < \alpha < 1. \tag{7}$$

where parameter b holds the following form,

$$b_{j,n+1} = \frac{h(n)^{\alpha}}{\alpha} \left((n+1-j)^{\alpha} - (n-j)^{\alpha} \right). \tag{8}$$

It is worth mentioning that in all the above equations, $\mathrm{X}(n+1)$ and $\mathrm{X}^P(n+1)$ stand for the state vectors with state components. The number of components is equal to the dimension of the system adopted.

To determine the variable step, we introduce an external map to construct a switching mechanism. We choose the chaotic skew tent map which holds the form as in (9) due to the fact that the states of the map are uniformly distributed in the range of (0,1) and exhibit chaotic behaviour (More information on this map can be find in paper of Qiao, et al., 2020). The phase space diagram of skew tent map with control parameter equal to 0.4 is given in Fig 1.

$$\mathrm{Xst}(n) = \begin{cases} \dfrac{\mathrm{Xst}(n-1)}{p}, & 0 < \mathrm{Xst}(n-1) \le p \\ \dfrac{1-\mathrm{Xst}(n-1)}{1-p}, & p < \mathrm{Xst}(n-1) < 1 \\ \mathrm{Xst}(n-1) - 0.05, & \text{otherwise} \end{cases} \tag{9}$$

The variable step $h(n)$ is then determined by the following equation,

$$\begin{aligned} h(n) = h \times (i+1) \ \text{ if } \ \mathrm{Xst}(n) \in [0.2 \times i,\ 0.2 \times (i+1)[, \\ i = 1, 2, ..., 5. \end{aligned} \tag{10}$$

In the following section, our proposed variable step size method is applied to numerically calculate two 3D fractional system. The chaoticity of the maps is discussed.

4. SIMULATOINS AND ANALYSIS

4.1 Fractional chaotic systems adopted for simulation

Fractional chaotic Chen and Lu systems are described by the

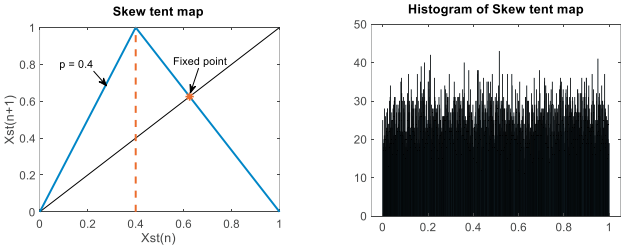


Fig. 1. Phase portrait and histogram of skew tent map p equals 0.4, initial condition for histogram is Xst(0) = 0.3

following equations:

$$\begin{cases} D^{\beta_c} x_1(t) = a_c\left(x_2(t) - x_1(t)\right) \\ D^{\beta_c} x_2(t) = \left(c_c - a_c\right) x_1(t) - x_1(t) x_3(t) + c_c x_2(t) \\ D^{\beta_c} x_3(t) = x_1(t) x_2(t) - b_c x_3(t) \end{cases} \tag{11}$$

$$\begin{cases} D^{\beta_l} x_1(t) = a_l\left(x_2(t) - x_1(t)\right) \\ D^{\beta_l} x_2(t) = -x_1(t) x_3(t) + c_l x_2(t) \\ D^{\beta_l} x_3(t) = x_1(t) x_2(t) - b_l x_3(t) \end{cases} \tag{12}$$

In the above equations, β_c and β_l are the commensurate fractional derivatives of fractional Chen and Lu system smaller than 1, respectively; (a_c, b_c, c_c) and (a_l, b_l, c_l) are the parameters of the systems.

The phase portrait of the two systems are generated applying the non-uniform grid discussed in the previous section and their phase space diagrams are given in Fig. 2 with fractional derivative order 0.9. The control parameter p and initial condition Xst(0) for the skew tent map are set to 0.4 and 0.3 respectively to acquire the variable step size $h(n)$. For the fractional Chen map, the parameters are set to p_c = (35, 3.2, 28), and the initial condition is fixed at (-9,-5,14). Whereas for the fractional Lu map, the parameters and initial condition are chosen as p_l = (a_l, b_l, c_l) = (36, 3, 20), and ($x_1(0)$, $x_2(0)$, $x_3(0)$) = (0.2,0.5,0.3).

In the following, the chaoticity of the fractional maps is discussed and compared in terms of bifurcations, Lyapunov Exponents (LEs) (obtained through iterations as given in paper of Danca, et al, 2018) and etc.

4.2 Analysis of singularities

To begin with, we calculate from analytical point of view the fixed points of the systems. Knowing that equilibrium of the fractional systems and their stability can be preserved with fractional derivative orders, the equilibria of fractional systems can be obtained using the same method as used in the case of classical integer order derivative system. With the given parameters p_c and p_l, we obtained three equilibrium points for Chen system $E_c^1 = (0,\ 0,\ 0)$, $E_c^2 = (-8.1976,\ -8.1976,\ 21)$, $E_c^3 = (8.1976,\ 8.1976,\ 21)$ and three for Lu system $E_l^1 = (0,\ 0,\ 0)$, $E_l^2 = (-7.7460,\ -7.7460,\ 20)$ and $E_l^3 = (7.7460,\ 7.7460,\ 20)$.

The singularity of the equilibria can also been acquired through conventional method by calculating the eigenvalues of the Jacobian matrix of the righthand sides of the system equations at the equilibria, and the results are given in Table 1. λ_c and λ_l in the table stand for the eigenvalues of the two systems. (Parameters p_l = (a_l, b_l, c_l) = (36, 3, 20) for Lu system and p_c = (35, 3.2, 28)) for Chen system as adopted before).

4.3 Lyapunov exponent results

We analyzed first the LEs of fractional Chen system for different fractional derivative orders applying classical uniform grid ABM corrector predictor, and compared with our proposed algorithm. The map has been evaluated on 55 values ranging from 0.45 to 1 with a discrepancy of 0.01. For each fractional derivative orders, 10^6 states have been generated and the LEs have been calculated throughout the iterations. The LE values for the last iteration were processed and combined together to form the LE spectrum curve given

Table 1. Equilibria of Chen and Lu system and their singularity

System	Equilibrium	Eigenvalue			Types of singularity
		λ_1	λ_2	λ_3	
Fractional Chen system	(0,0,0)	-35	28	-3.2	Saddle
	(-8.1976,-8.1976,21)	-18.7738	4.2869+15.2376i	4.2869-15.2376i	Saddle Focus
	(8.1976, 8.1976,21)	-18.7738	4.2869+15.2376i	4.2869-15.2376i	Saddle Focus
Fractional Lu system	(0,0,0)	-36	20	-3	Saddle
	(-7.746,-7.746,20)	-22.6516	1.8258+13.6887i	1.8258-13.6887i	Saddle Focus
	(7.746, 7.746,20)	-22.6516	1.8258+13.6887i	1.8258-13.6887i	Saddle Focus

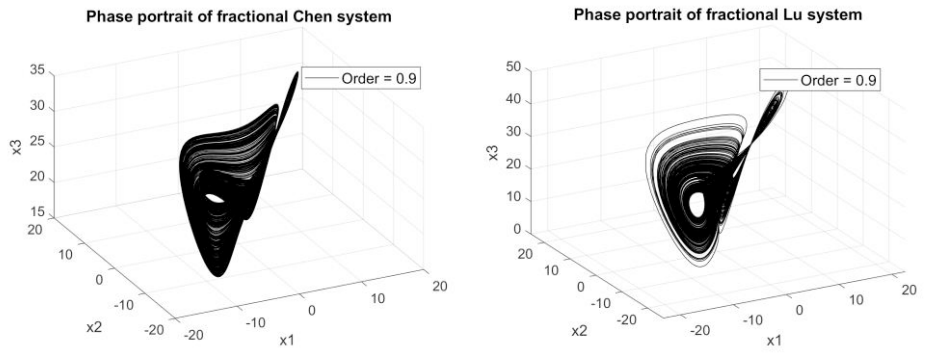


Fig. 2. Phase space of fractional Chen and Lu systems with fractional derivative order 0.9

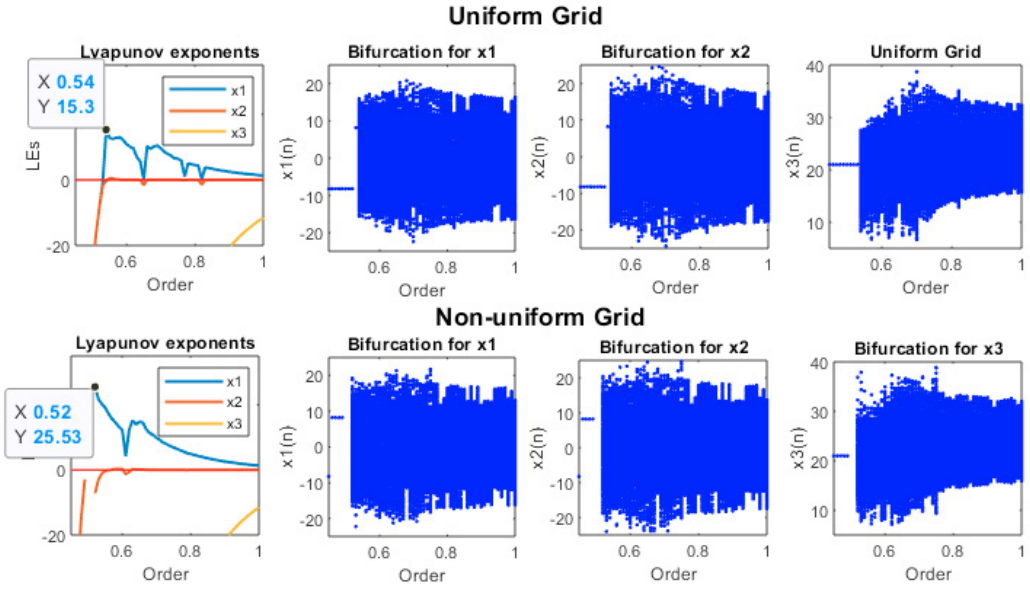


Fig. 3. Lyapunov exponent and Bifurcations for Chen maps with fractional order β_c varying from 0.45 to 1 applying classical method and proposed non-uniform grid method

in Fig. 3.

It can be seen from the figure that for both methods, using uniform and non-uniform grid, among the three LEs, the LE for x_1 direction is greater than 0, indicating that the trajectories are expanding along this direction. From the obtained LE values, it can be concluded that for the fractional derivative ranging from 0.52 to 1(non-uniform grid) and 0.54 to 1(uniform grid), the system remains chaotic. One can also notice that, the LE obtained through our proposed method exceeds zero at a smaller fractional derivative order compared to that of the classical uniform grid predictor-corrector method. This indicates that with our proposed method, the fractional derivative range for the map to be chaotic is enlarged. The same conclusion can also be detected from the bifurcation diagrams. In the diagram, the fixed focus can be observed with fractional order smaller than 0.52 for our proposed method and 0.54 for uniform grid calculation method. After order 0.52 (0.54 for uniform grid), the bifurcation reveals the chaotic behavior.

It should be noticed here that our concern in this work was not to obtain a better approximation of the original fractional chaotic system (eq. (11) and eq. (12)) but to select a method allowing to achieve chaotic behaviors for a larger parameter range, which is an interesting feature in some applications such as cryptography or pseudo-chaotic number generators.

We also give here the simulation results of fractional Lu map for a clearer comparison between the two calculation methods. The LEs for fractional order from 0.55 to 1 with a discrepancy of 0.01 are calculated and plotted for both methods in Fig.4(a). A clear gap can be observed between the red line (uniform grid) and blue line (non-uniform grid) where the latter crosses the horizontal line (LE equals 0) between fractional derivatives orders of 0.56 and 0.57, while the former crosses the line between 0.6 and 0.61. This is also in accordance with our previous findings which indicates that the implementation of non-uniform would enable to enlarge the fractional order parameter range for which the system preserves its chaotic properties.

Additional to the fractional derivatives, we have also analysed the impact of the other control parameters. Comparative results on the LEs obtained by our proposed non-uniform grid calculation method and by the classical fractional ABM Predictor Corrector method are presented in Table 2 for Chen and Lu systems. For fractional Chen system, we have evaluated 50 successive values in the range of [20, 45], [1, 11], [20, 45] for its control parameters a_c, b_c, and c_c, respectively (Change only one at a time, holding the others unchanged). The number of values whose LEs are greater than 0 among the 50 evaluated values have been acquired. For fractional Lu map, evaluation of 50 successive values in the range of [20, 45], [0, 10], [20, 45] for its control parameters a_l, b_l, and c_l has also been conducted. It can be observed from the table that our proposed method possesses more LEs greater than 0 for all the parameters as well as the fractional orders. This indicates that our proposed calculation method also introduces extra control parameter range for the system to be chaotic.

Knowing that the applied LE calculation method is only a qualitative measurement of chaotic properties for fractional order systems, we also calculate the percentage of the number of LEs obtained applying our method which exceeds that of the classical approach for a rough idea of the enhancement of chaoticity. The results are also given in Table 2. It can be concluded that for the evaluated parameter values who have LE greater than 0, the proposed method gives a LE which exceeds that of the classical method no less than 70% of the time.

4.4 Time response and other results

To further evaluate the calculation methods and justify LE results from another perspective, we 'synchronized' the states

Table 2. LE results for Chen and Lu system with different parameters and fractional orders

Systems	Parameter and Range		Number of value estimated	Number of LEs greater than 0 (% in total value estimated values)		Proposed > Classical (greater in LE value) (%)
				Uniform	Non-uniform	
Chen map	β_c	0.45 to 1	55	46(84%)	48(87%)	89%
	a_c	20 to 45	50	48(96%)	49(98%)	80%
	b_c	1 to 11	50	39(78%)	43(86%)	78%
	c_c	20 to 45	50	49(98%)	50(100%)	70%
Lu map	β_l	0.55 to 1	45	38 (84%)	42(93%)	89%
	a_l	20 to 45	50	41(82%)	45(90%)	92%
	b_l	0 to 10	50	48 (96%)	49(98%)	90%
	c_l	20 to 45	50	37(74%)	39(78%)	97%

of the Lu system calculated through the two methods by identifying and matching the states with identical time stamps (199961 'synchronized' states out of 10^6 iterated states). The last 150 'synchronized' states are plotted out in Fig. 4(b), (c) and (e) which show in the time domain the evolution of the different system dynamics according to the grid choices. These figures confirm the analysis based on the LE (Fig. 4(a)), for which with uniform grid, the system is not chaotic when the fractional derivative order β_l is smaller than 0.61, whereas with our proposed non-uniform gird, the system states exhibit chaotic behaviour at fractional derivative order starting at 0.57. One can also observe from the y-coordinates of red and blue lines in Fig. 4(b) and 4(c) that for order 0.56, both methods reach at the same point (-7.746, -7.746, 20), as well as the states obtained through uniform grid calculation method for order 0.57 (red line in Fig. 4(c)). This indicates that under the given initial conditions and parameters p_l = (36, 3, 20), ($x_1(0)$, $x_2(0)$, x_3 (0)) = (0.2,0.5,0.3), at fractional orders where there is no sign of chaos, after sufficient iterations, the trajectories converge towards the fixed points E_l^2 as obtained from analytical study. The phase portrait of the attractors in Fig. 4(e) and (f) also confirms the coherence. There is only a fixed point for order 0.57 for the uniform grid method whereas for order 0.61 both methods possess LEs greater than 0, the phase portraits are chaotic and exhibit similar shape.

After matching the states for the same time instants from the uniform and non-uniform grid, we acquired the difference between the two attractors obtained for Lu system. The time response of the difference (last 150 states) and the phase portrait (last 10000 states) for β_l = 0.7 are given in Fig. 5(b) and (c). For this fractional order the system behavior is always chaotic independently of the grid choice. A new distinguished pattern is generated which could be considered as a new system possessing chaotic properties. The fast Fourier Transformation of the system is also acquired by employing the nonuniform FFT function integrated in MATLAB. And the results for order 0.7 is given in Fig. 5(a).

4.5 Computational time

The computation time applying both methods for fractional Chen and Lu system are recorded and shown in the table (run with MATLAB R2018b). It can be observed that our proposed method takes less computational time with respect to the classical one (10^6 iterations).

Note that achieving higher precision in the numerical calculation was not the objective of this work. The purpose was to investigate the robust chaos appearing for a large range of parameter values with a higher chaoticity. As for the applications, the maps obtained through our proposed non-uniform grid have been tested in a coupling with another fractional double-humped logistic map for the design of fractional pseudo chaotic random generator in paper Yang et al. (2020). Very good statistical results in terms of randomness have been achieved.

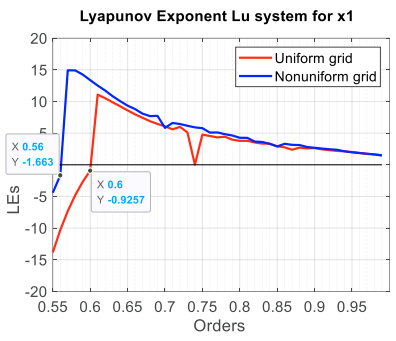
5. CONCLUSION

In this paper, we proposed a non-uniform grid calculation method based on classical fractional ABM corrector-predictor method using variable step sizes obtained from an external skew tent map. The proposed method based on chaotic threshold for fractional differential equations is not limited to the specific examples of fractional Lu and Chen systems. It may be generalized to other linear or nonlinear fractional function *f*.

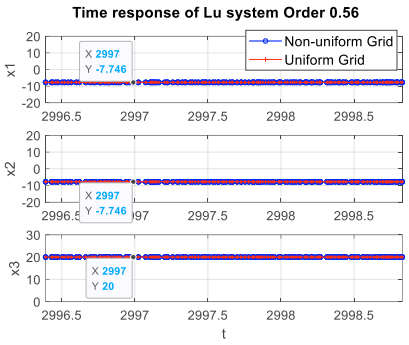
The advantages of the proposed method are that it is faster and easily implementable, and it is not more complex for application than the original uniform grid predictor-corrector method. At the same time, the parameter and fractional derivative ranges for which chaos can be observed have been increased, and the chaoticity in terms of LE is higher. The expected potential applications are numerous such as designing systems requiring robust chaos, or chaos-based encryption schemes with larger set of parameters required to achieve higher security. Further analysis based on coupled Lu and Chen fractional chaotic systems obtained by non-uniform grid are currently under investigation.

Table 3. Computational time

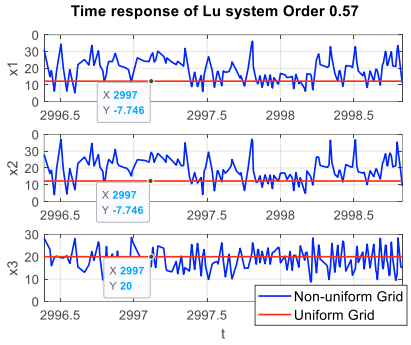
System	Computation time (s) for 10^6 iterations	
	Uniform Grid	Non-uniform Grid
Chen	1171.511	1084.5942
Lu	1142.0571	1029.2975



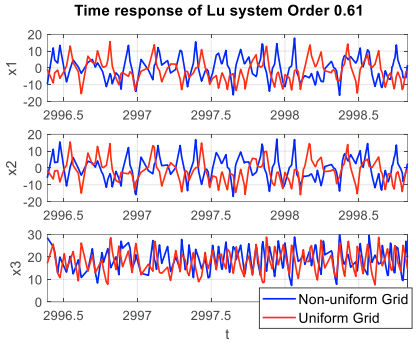
(a) LEs comparision for x_1 component



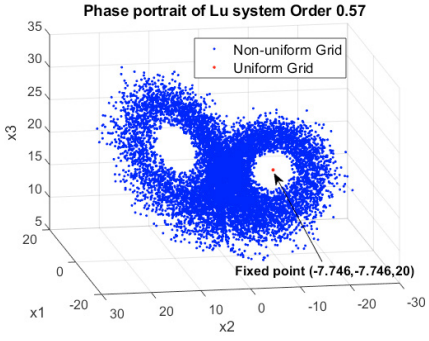
(b) Time response of last 150 points (β_l=0.56)



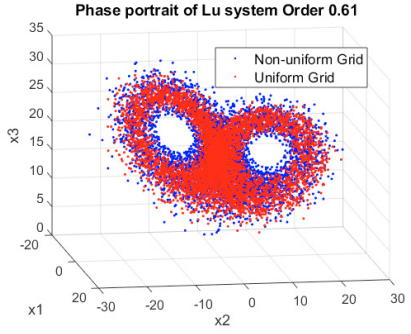
(c) Time response of last 150 points (β_l=0.57)



(d) Time response of last 150 points (β_l =0.61)

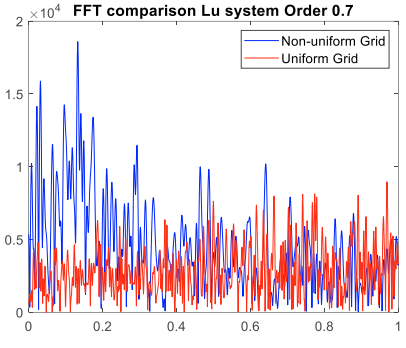


(e) Attractors of last 10000 states (β_l =0.57)

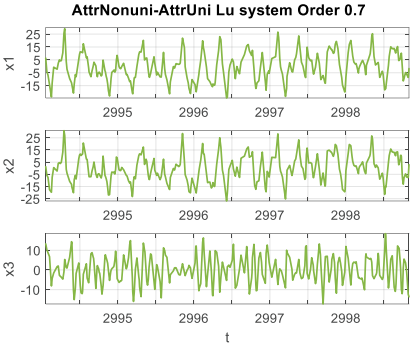


(f) Attractors of last 10000 states (β_l =0.61)

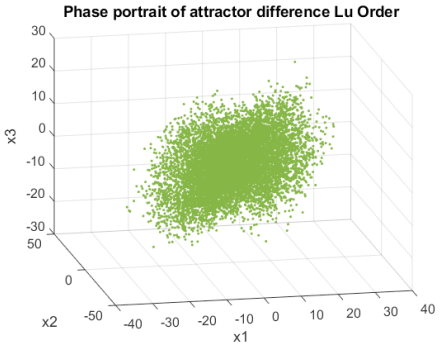
Fig. 4. Comparison of LEs for x_1 component; time response for Lu system with fractional order 0.56, 0.57 and 0.61; Phase portrait of the attractors obtained from proposed non-uniform method and classical uniform approach



(a) FFT results for Lu system (order 0.7)



(b) Time response of difference between two methods



(c) Phase portrait of the difference

Fig. 5. Simulation results for Lu system with fractional derivative order 0.7

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