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Pierre-Antoine Guihéneuf, Emmanuel Militon. Homotopic rotation sets for higher genus surfaces. 2022. hal-03537289v3

HAL Id: hal-03537289

https://hal.science/hal-03537289v3

Preprint submitted on 13 May 2022

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Homotopic rotation sets for higher genus surfaces

Pierre-Antoine Guihéneuf, Emmanuel Militon

May 13, 2022

Abstract

This paper states a definition of homotopic rotation set for higher genus surface homeomorphisms, as well as a collection of results that justify this definition. We first prove elementary results: we prove that this rotation set is star-shaped, we discuss the realisation of rotation vectors by orbits or periodic orbits and we prove the creation of new rotation vectors for some configurations.

Then we use the theory developped by Le Calvez and Tal in [LCT18a] to obtain two deeper results:
– If the homotopical rotation set contains the direction of a closed geodesic which has a self-intersection, then there exists a rotational horseshoe and hence infinitely many periodic orbits in many directions.
– If the homotopical rotation set contains the directions of two closed geodesics that meet, there exists infinitely many periodic orbits in many directions.

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1 Introduction

The key invariant in the study of circle homeomorphisms dynamics is Poincaré's rotation number, which measures the orbits' asymptotic mean speed of rotation around the circle. It leads to the celebrated Poincaré classification, which asserts – among others – that the rotation number is rational if and only if the homeomorphism possesses a periodic orbit.

The generalisation of this invariant to the two dimensional torus leads to the definition of rotation set, as the accumulation set of asymptotic mean speeds of orbits rotation around the torus. More formally, given a homeomorphism f of the torus $\mathbb{T}^2$ homotopic to identity, and $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ one of its lifts, the rotation set $\rho(\tilde{f})$ is the set of all possible limits of sequences $\frac{\tilde{f}^{n_k}(x_k)-x_k}{n_k}$, for n_k going to $+\infty$ and $x_k \in \mathbb{R}^2$. It is a compact convex subset of $\mathbb{R}^2$ [MZ89], invariant under conjugation by isotopically trivial homeomorphisms. As for the circle case, its shape is strongly related to the dynamics: for example, any point with rational coordinates in the interior of the rotation set is associated to a periodic point of the homeomorphism [Fra88].

The literature exploring the properties of this set is now quite consequent and makes use of a wide range of different techniques, from Brouwer theory and its improvements by Le Calvez [LC05], culminating to the Le Calvez-Tal recent works [LCT18a, LCT18b, Gui20], to Nielsen-Thurston classification [LM91], prime ends or Pesin theories (e.g. [AZ20])... Even if there are still quite a lot of open questions (e.g. whether there exists a torus homeomorphism having a rotation set with nonempty interior and smooth boundary), the subject is now mature and rich enough to attempt tackling similar issues in more complex situations.

A natural extension is to study rotation properties of homeomorphisms of higher genus (closed) surfaces. Let us point out that for the torus, first homotopy and homology groups coincide, while this is not the case for higher genus surfaces. Hence in this new setting one expects to get two distinct definitions of rotation sets, both in homotopical and homological senses. The latter, defined formally a long time ago [Sch57], regained attention in the last years.

Homological rotation sets

Let us recall one possible definition of the homological rotation set rotation set (see [Pol92, Sch57]).

Let S be a closed surface, and fix $f \in \mathrm{Homeo}_0(S)$. Let us denote by D the diameter of S. For any two points x and y of S, we choose a geodesic path $g_{x,y}$ of length lower than or equal to D which joins the point x to the point y. Fix an isotopy $(f_t)_{t\in[0,1]}$ between $f_0 = \mathrm{Id}_S$ and $f_1 = f$. As usual, we extend it to an isotopy $(f_t)_{t\in\mathbb{R}}$ by setting $f_t = f_{t-\lfloor t\rfloor} \circ f^{\lfloor t\rfloor}$, where $\lfloor t\rfloor$ denotes the lower integer part of t.

For any point x in S, we define $l_{n,x}$ as the loop obtained by concatenating the path $(f_t(x))_{t\in[0,n]}$ with the geodesic path $g_{f^n(x),x}$. This loop defines a cycle and we denote by $[l_{n,x}]_{H_1(S)}$ the class of this cycle in $H_1(S) = H_1(S,\mathbb{R})$.

Definition 1.1. The *homological rotation set* of $f \in \mathrm{Homeo}_0(S)$ is the set $\rho_{H_1}(f)$ of points $\rho \in H_1(S,\mathbb{R}) \simeq \mathbb{R}^{2g}$ such that there exists $(x_k)_k \in S$ and $(n_k)_k$ going to $+\infty$ such that $[l_{n_k,x_k}]_{H_1}/n_k$ tends to ρ.

As we divide by n_k in this definition, this set does not depend on the chosen geodesic paths $g_{x,y}$.

When $g(S) \geq 2$, this set does not depend on the chosen isotopy, as two such isotopies are homotopic with fixed endpoints. Indeed, the topological space $\mathrm{Homeo}_0(S)$ is contractible. If $g(S) = 1$, this set depends on the chosen isotopy but two such sets differ by an integral translation. Indeed, two isotopies between the identity and f are homotopic up to composition with an integral translation. Also, it is possible to associate a homological rotation vector to any f-invariant measure (see [Lel19] for more details).

Let us describe a few known results about this homological rotation set:

- Entropy: Is it possible to get sufficient conditions on the homological rotation set for the homeomorphism to have positive topological entropy? Here are some known conditions:
  - If there exist $2g+1$ periodic points whose homological rotation vectors do not lie on a hyperplane of $H_1(S,\mathbb{R}) \simeq \mathbb{R}^{2g}$ [Pol92].
  - If f is a C^1-diffeomorphism, and if there exist $g+2$ periodic points whose homological rotation vectors form a $g+1$-nondegenerate simplex [Mat97].
  - If there exists two invariant probability measures whose homological rotation vectors have a nontrivial intersection [Lel19]. This result implies the two above results.
- Realisation of periodic points: in which cases some vectors of the homological rotation set are realised by periodic points? Such results were obtained under similar hypotheses to the ones for positiveness of entropy:
  - If there exist $2g+1$ periodic points whose homological rotation vectors do not lie on a hyperplane of $H_1(S,\mathbb{R}) \simeq \mathbb{R}^{2g}$, then any rational point in the interior of the simplex spanned by these rotation vectors is the rotation vector of some periodic point [Hay95].
  - If f is a C^1-diffeomorphism, and if there exist $g+2$ periodic points whose homological rotation vectors form a $g+1$-nondegenerate simplex, then any rational point in the interior of this simplex is the rotation vector of some periodic point [Mat97].

- Under the set of hypotheses called fully essential system of curves by the authors[1], any rational point in the interior of the rotation set is the rotation vector of some periodic point [AZdPJ21]. In this case, the authors also get convexity of the rotation set, uniform bounds on displacements, etc.
- If there exist two invariant probability measures μ and ν whose homological rotation vectors ρ_μ and ρ_ν have a nontrivial intersection, then any point of the simplex spanned by 0, ρ_μ and ρ_ν is accumulated by rotation vectors of periodic points [Lel19]. In this thesis the author also gets uniform bounds on displacements if 0 lies in the interior of the rotation set.

- Generic shape: for a generic homeomorphism, the rotation set is given by a union of at most 2^{5g-3} convex sets [ABP20].

Note also the work [KT18] which (among others) gives conditions under which the dynamics of an area preserving homeomorphism of S can be decomposed into dynamics of lower genus surface homeomorphisms.

Homotopical rotation sets

Note that some rotational information is lost when using the homological rotation set: for instance it does not see the difference between the trivial loop and a commutator (see the path α in Figure 1). This incites finding a practical definition of homotopical rotation set in the higher genus context.

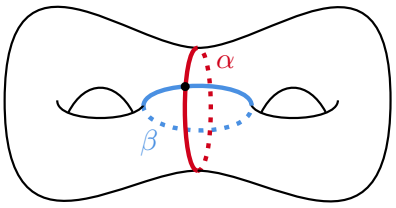

Figure 1: The rotation around the path α, which is homologically trivial (it is a commutator in the π_1) is not detected by the homological rotation set. In this paper, we get (among others) the existence of infinitely periodic orbits when there are rotation vectors in both directions α and β.

Unlike what we have seen in the homological context, there is no such commonly accepted definition of a homotopical rotation set. To our knowledge, the only known result is the one of Lessa [Les11], but it has no consequence on the initial surface homeomorphism dynamics.

In this paper, we propose a new notion of homotopical rotation set for higher genus surfaces homeomomorphisms. Let S be a closed surface of genus ≥ 2, and f a homeomorphism of S which is homotopic to the identity. The universal cover of S is the hyperbolic plane $\mathbb{H}^2$, that we equip with its canonical metric. Let $\tilde{f}$ be the unique lift of f to $\mathbb{H}^2$ that extends to identity to $\partial\mathbb{H}^2$. Remark that the set of geodesics of $\mathbb{H}^2$ can

[1] That is satisfied under some hypotheses on stable/unstable manifolds of periodic points if f is a $C^{1+\varepsilon}$ diffeomorphism; in particular it implies that the homological rotation set has nonempty interior.

be parametrized by the set of couples of distinct points of $\partial\mathbb{H}^2$. For any $(\alpha,\beta)\in(\partial\mathbb{H}^2)^2$ with $\alpha\neq\beta$ and any $v\in\mathbb{R}_+^*$, we will say that the triple (α,β,v) is a *rotation vector of* f if there exists a sequence $(x_k)_{k\in\mathbb{N}}$ of points of $\mathbb{H}^2$, and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers tending to infinity such that, if we denote by $\pi_{\alpha,\beta}$ the orthogonal projection[2] on the geodesic linking α to β,

$$\left(x_k\ ,\ \tilde{f}^{n_k}(x_k)\ ,\ \frac{d\Big(\pi_{\alpha,\beta}(x_k)\ ,\ \pi_{\alpha,\beta}\big(\tilde{f}^{n_k}(x_k)\big)\Big)}{n_k}\right)\underset{k\to+\infty}{\longrightarrow}(\alpha,\beta,v). \tag{1.1}$$

The (homotopical) *rotation set* $\rho(f)$ *of* f is then defined as the collection of rotation vectors of f, together with all the singletons $\{(\alpha,\beta,0)\}$ for all the geodesics (α,β) of $\mathbb{H}^2$ (to emphasize the fact that f has a contractible fixed point, by Lefschetz formula).

Note that this definition is a bit different from the one of Lessa [Les11].

Review of the results

In this whole paragraph, we consider an orientable closed surface[3] S of genus $g\geq 2$, and a homeomorphism f of S homotopic to the identity.

We will state quite a lot of different results, that in some sense give grounds for our definition of homotopical rotation set, some of them rather elementary, others more difficult. They are mainly of two different types: realisation of "rational" vectors by periodic orbits, and convexity-like results (the presence of some kinds of orbits forces the presence of others, which are "convex combinations" of the initial orbits). We will get our results from three different techniques. The first one consists in using the property of quasi-convexity of fundamental domains. The second one is also elementary, it uses geodesics in the universal cover and their images by the lift of the dynamics to get separating sets of the hyperbolic plane; it allows to get simple convexity-like results. The third and last main tool we use is the forcing theory of Le Calvez and Tal [LCT18a]; it gives much stronger results at the cost of longer and more difficult proofs.

To start with, we prove quasi-convexity of fundamental domains. This result was already known for the torus [MZ89], we extend the proof to the higher genus case: there exists $R=R(S)>0$ such that for any path connected fundamental domain D of S in its universal cover $\widetilde{S}$, and any point x in the convex hull $\mathrm{conv}(D)$ of D, we have $B(x,R)\cap D\neq\emptyset$ (Proposition 2.2).

As a consequence, we get that the rotation set $\rho(f)$ is star-shaped (Theorem 3.5).

Theorem A. *For any $(\alpha,\beta,v)\in\rho(f)$, and any $v'\in[0,v]$, one has $(\alpha,\beta,v')\in\rho(f)$.*

As a byproduct of this theorem's proof, we get that rotation vectors are realised by segments of orbits whose endpoints stay at a bounded distance to the corresponding geodesic (Proposition 3.6).

Section 4 is devoted to other realisation results for rotation vectors associated to closed geodesics. These rotation vectors are directly related to the rotation set of the annulus homeomorphism obtained by quotienting the universal cover $\widetilde{S}$ of S by this closed geodesic. Note that this is where the fact that in our definition of rotation set, speeds

[2] In other words, the projection to the closest point of the geodesic (α,β).

[3] In the core of the article, we will mention when our results trivially generalize to the case of a non compact orientable surface of finite type; in this introduction we will simply give the statements in the compact case.

are measured by means of projections on geodesics, is crucial. As a direct consequence, an application of already known results for rotation sets of annulus homeomorphisms leads to realisation of rotation vectors by periodic orbits, under "classical" conditions (Proposition 4.1). As an application of [Les11], we also get that (still in the closed geodesic case) the extremal rotation vector is realised by an orbit whose lift to $\widetilde{S}$ stays at sublinear distance from the geodesic (Proposition 4.3).

We then get to forcing results. The ones of Section 5 use only elementary arguments. To begin with, we consider geodesics of the surface with auto-intersection (see Proposition 5.1 for a more formal statement).

Proposition B. *Let $\tilde{\gamma}$ be a geodesic of $\widetilde{S}$ projecting to a geodesic γ of S which auto-intersects. Let γ' be the geodesic of S obtained as a "shortcut" of the geodesic γ.*

If $(\tilde{\gamma}, v) \in \rho(f)$, then $(\tilde{\gamma}', v) \in \rho(f)$.

The general case of two geodesics intersecting is treated in Proposition 5.4, with weaker conclusions.

Proposition C. *Let $(\alpha_1, \beta_1, v_1) \in \rho(f)$, with $v_1 > 0$. Let also (α_2, β_2) be a geodesic of $\mathbb{H}^2$ that intersects (α_1, β_1), and such that there exists $(y_k) \in \mathbb{H}^2$ and $u_k \in \mathbb{N}$ such that $y_k \to \alpha_2$ and $\tilde{f}^{u_k}(y_k) \to \beta_2$. Then, there exist $v', v'' \geq 0$ satisfying $v' + v'' = v_1$ such that:*

(i) either $(\alpha_1, \beta_2, v') \in \rho(f)$ or $(\alpha_1, \alpha_2, v') \in \rho(f)$;

(ii) either $(\beta_2, \beta_1, v'') \in \rho(f)$, or $(\alpha_2, \beta_1, v'') \in \rho(f)$.

The proofs of these results are heavily inspired by the forcing theory [LCT18a], where geodesic play the role of leaves of Brouwer-Le Calvez foliations.

This last proposition is used in Section 6 to study what we call *almost annular* homeomorphisms (Proposition 6.1).

Proposition D. *Suppose that the only nonzero rotation vectors of f are associated to the lifts of a single geodesic γ of S. Then γ has no self-intersection.*

After exposing some examples in Section 7, we study in more detail the creation of new rotation vectors for closed geodesics.

As a first step, in Section 8, we get weak consequences when the homotopical rotation set contains two vectors associated to two closed geodesics which intersect, as in Figure 1 (Proposition 8.10 and Corollary 8.11; in particular we get positive entropy). These statements rely on the notion of *covering map associated to two distinct closed geodesics* (Definition 8.3). In our case, the covering surface is a single punctured torus, and the homological consequences we mentioned are stated in terms of rotation set of the lift of the initial homeomorphism to this torus.

This formalism is used in Section 9 to get the existence of a rotational horseshoe (see Definition 9.15) when f has a rotation vector associated to a closed geodesic with auto-intersection (Theorem 9.27).

Theorem E. *Let γ be a closed geodesic with a geometric auto-intersection (as in Figure 2) associated to the deck transformation T_1 (in the sense of Definition 8.1). Denote T_2 the deck transformation such that $T = T_1T_2$ is the deck transformation associated to the closed geodesic γ.*

Suppose that $(\gamma, \ell(\gamma)) \in \rho(f)$. Then, f^7 has a topological horseshoe associated to the deck transformations T_1, T_1^2, T_2, T_1T_2, T_2T_1 and $T_1T_2T_1$.

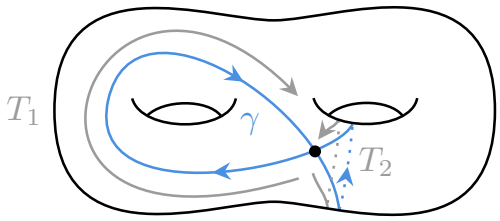

Figure 2: A possible configuration for Theorem E: the geodesic γ on the surface has a geometric auto-intersection.

Note that our definition of rotational horseshoe is different from the one of [PPS18] and [LCT18b], as it is stated in terms of Markovian intersection and not of semi-conjugation to a shift (we get this semi-conjugacy as a consequence in Proposition 9.16). In fact, this kind of rotational horseshoes appears as soon as the homeomorphism has a periodic trajectory under the isotopy to identity which auto-intersects geometrically (see also Proposition 9.18).

The proof of this theorem is much longer than the previous ones and based on the recent forcing theory [LCT18a, LCT18b]. This is also the case for our last result (Theorem 10.1 and Corollary 10.2). For any deck transformation T of the universal cover $\widetilde{S} \to S$, we denote by $\tilde{\gamma}(T)$ its axis.

Theorem F. *Let γ_1 and γ_2 be two closed geodesics of S, that lift to $\widetilde{S}$ to geodesics $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ that cross (they can be for example the curves α and β of Figure 1, note also that they can have auto-intersections). Let T_1 and T_2 be the deck transformations associated to the respective closed geodesics γ_1 and γ_2 and which respectively preserve $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$.*

Suppose that there exist nonzero rotation vectors of directions $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in $\rho(f)$. Then, for any element w in $\langle T_1, T_2\rangle_+ \setminus (\langle T_1\rangle_+ \cup \langle T_2\rangle_+)$, there are nonzero vectors of direction $\tilde{\gamma}(w)$ in $\rho(f)$ which are realised by periodic orbits.

The proof of this theorem is quite long and divided in numerous sub-cases. One of the difficulties is that the transverse paths associated to the trajectories realising the rotation vectors do not need to have an $\mathcal{F}$-transverse intersection (see Figure 37 for such an example, due to Lellouch [Lel19]).

Some open questions

We state here some questions about homotopical rotation sets in higher genus that are still open.

1) Clarify the links between homotopical and homological rotation sets. In particular, when does a homological rotation vector gives birth to a homotopical rotation vector? This would certainly bring into play hyperbolic geometry as in [Les11].

2) Get more realisation results: is every rotation vector realised by a single orbit of the homeomorphism? What can be the sets of times n_k appearing in (1.1)?

3) Get more forcing results, for example: if $(\alpha_1, \beta_1, v_1), (\alpha_2, \beta_2, v_2) \in \rho(f)$, with $v_1, v_2 > 0$, and if the geodesics (α_1, β_1) and (α_2, β_2) cross, do we have $(\alpha_1, \beta_2, v') \in$

$\rho(f)$ for some $v' > 0$? A first step may be to get such results under some recurrence hypotheses about the geodesics, or to get it for a single (non closed) geodesic with auto-intersection.

4) Obtain a wider collection of examples to illustrate the diversity of possible behaviours.

5) Explore more the notion of almost annular homeomorphisms.

6) If $\tilde{f}$ is transitive, what can be said about the $\rho(f)$ (see [Tal12])?

7) What is the shape of the rotation set of a generic homeomorphism?

Acknowledgements

We warmly thank Erwann Aubry for insightful discussions about Proposition 2.2, as well as Maxime Wolff and Frédéric Le Roux. We are indebted to Alejo García who has read a first version of this article in detail and suggested numerous corrections. The reference [Hem72] was suggested to us by Indira Chatterji, and the references [Les12, Kra81, IIY03] by Pablo Lessa.

P.-A. Guihéneuf was supported by a PEPS/CNRS grant. E. Militon was supported by the ANR project Gromeov ANR-19-CE40-0007.

2 Quasi convexity of fundamental domains

A now well known result is the quasi-convexity of 2-torus fundamental domains, whose first proof was given in [MZ89], with an argument due to Douady. Based on the index of a curve, it can be replaced by a very elementary one. Here, we adapt this elementary proof to higher genus surfaces[4].

Definition 2.1. We say that a set $X \subset \mathbb{H}^2$ is *R-quasi convex* if for any point x of the hyperbolic convex hull $\mathrm{conv}(X)$ of X, one has $B(x, R) \cap X \neq \emptyset$.

In what follows, we identify $\overline{\mathbb{H}^2}$ with the unit closed disk in the complex plane $\mathbb{C}$. In particular, the complex numbers i and $-i$ are identified with points of $\partial\mathbb{H}^2$. We endow $\partial\mathbb{H}^2$ with the distance induced by the euclidean distance on $\mathbb{C}$. For any two distinct points of the boundary $\alpha, \beta \in \partial\mathbb{H}^2$, we denote (α, β) the oriented geodesic of $\mathbb{H}^2$ having α as α-limit and β as ω-limit.

Proposition 2.2. *For any orientable closed surface S of genus $g \geq 2$ there exists $R = R(S) > 0$ such that any path connected fundamental domain $D \subset \mathbb{H}^2$ of S is R-quasi convex.*

Lemma 2.3. *Let S be a closed surface of genus $g \geq 2$ and K a compact subset of $\mathbb{H}^2$. Then, there exists a finite set $F \subset \pi_1(S)$ such that for any oriented geodesic $\tilde{\gamma}$ of $\mathbb{H}^2$ passing through K, there exists $T \in F$ such that the right of $T\tilde{\gamma}$ is a strict subset of the right of the segment $(-i, i)$, where $(-i, i)$ is oriented from $-i$ to i.*

[4] A. Passeggi informed us in a private communication that he had a proof of this result, but it stayed unpublished.

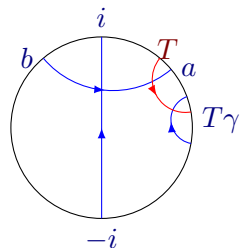

Figure 3: Configuration of the proof of Lemma 2.3.

Proof of Lemma 2.3. We will use the following classical fact: *For any point $x \in \partial\mathbb{H}^2$, there exists a sequence of geodesic axis of deck transformations whose endpoints both tend to x (and x is between these two endpoints).* It comes from the following: there exists a constant θ_0 such that any geodesic $\tilde{\gamma}$ of $\mathbb{H}^2$ whose projection to S is non-closed, crosses axis of deck transformations, with an angle $\geq \theta_0$, syndetically. To see this, consider a fundamental domain of S with boundary made of deck transformations axis.

First, by applying some iterate of some $T_0 \in \pi_1(S)$ with one axis endpoint on the right of $(-i, i)$ if necessary, one can suppose that K is contained in the right of $(-i, i)$ and moreover that the Hausdorff distance between these two sets is at least 1.

Now, take a geodesic $\tilde{\gamma}$ that crosses K. It has to have an endpoint a at the right of $(-i, i)$, and moreover, if we denote by b the other endpoint of this geodesic, the distances[5] $d(a, b)$, $d(a, -i)$ and $d(a, i)$ are bigger than some $d_0 > 0$ which only depends on K and S. Take F_0 a finite subset of $\pi_1(S)$ such that for any $c \in \partial\mathbb{H}^2$, there exists an element of F_0 whose axis endpoints belong to respectively $]c - d_0, c[$ and $]c, c + d_0[$. It exists by the above fact. Moreover, we can take F_0 finite by compactness of $\partial\mathbb{H}^2$.

Then, still by compactness of $\partial\mathbb{H}^2$, there exists $N \in \mathbb{N}$, depending only on F_0 and d_0, some $n \in \mathbb{Z}$ with $|n| \leq N$ and $T \in F_0$, such that both points $T^n(a)$ and $T^n(b)$ lie on the right of $(-i, i)$ (and thus $T^n\tilde{\gamma}$ is entirely contained in the right of $(-i, i)$), and that the right of $T^n\tilde{\gamma}$ is contained in the right of $(-i, i)$. This proves the lemma for

$$F = \left\{T^n \mid T \in F_0, |n| \leq N\right\}.$$

□

Proof of Proposition 2.2. In this proof, d denotes the distance in $\mathbb{H}^2$. Denote by r_0 a positive number such that any half-ball of radius r_0 in $\mathbb{H}^2$ contains some fundamental domain of S. Let B be the closure of the connected component of $B(0, r_0) \setminus (-i, i)$ on the right of $(-i, i)$. Let $F \subset \pi_1(S)$ be given by Lemma 2.3 applied to $K = B$, and set

$$R = \max\left\{d(0, Tx) \mid T \in F, x \in B\right\}.$$

Now, take D a path connected fundamental domain of S, and $x \in \mathrm{conv}(D)$. Suppose for a contradiction that $B(x, R) \cap D = \emptyset$.

As $x \in \mathrm{conv}(D)$, there exists $a, b \in D$ such that $x \in [a, b]$. The property of quasi-convexity being invariant under isometry, one can suppose that $x = 0$ is the center of the Poincaré disk, and that the geodesic line (a, b) is $(-i, i)$.

[5] We have endowed the Poincaré circle $\partial\mathbb{H}^2$ with its canonical distance.

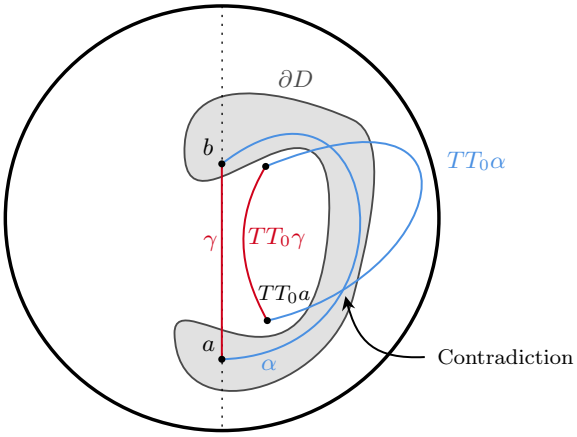

Figure 4: Proof of Proposition 2.2. Here, $\beta = \alpha \cup \gamma$.

As D is path connected, there exists a path α contained in D whose endpoints are a and b. By taking a subpath and changing a and b in γ if necessary, one can suppose that α does not meet γ on its interior. Moreover, by applying a symmetry with respect to γ if necessary, one can suppose that the interior of α is included in the right of γ. Let β be the Jordan curve formed by the union of α with the geodesic segment $[a, b]$.

As B contains a fundamental domain, there exists $T_0 \in \pi_1(S)$ such that $T_0(a) \in B$. By Lemma 2.3, there exists $T \in F$ such that the right of $TT_0(-i, i)$ is included in the right of $(-i, i)$.

By the definition of R, one has $TT_0(a) \in B(0, R)$. Then, the hypothesis $B(0, R) \cap D = \emptyset$ implies that $TT_0(a)$ belongs to the Jordan domain bounded by β. Let $m \in \alpha$ be such that $d(m, (-i, i)) = \max_{y \in \alpha} d(y, (-i, i))$. As $TT_0(-i, i)$ lies on the right of $(-i, i)$, and as TT_0 is an isometry, one has that

$$d(TT_0(m), (-i, i)) = d(m, (TT_0)^{-1}(-i, i)) > d(m, (-i, i));$$

so $TT_0(m)$ does not belong to the Jordan domain defined by β. By continuity, there exists a point $x_0 \in \alpha$ such that $TT_0(x_0) \in \beta$. But $TT_0(x_0)$ belongs to the right of $TT_0(-i, i)$, which is included in the right of $(-i, i)$, so $TT_0(x_0) \notin (-i, i)$. This implies that $TT_0(x_0) \in \alpha$. We have found a point x_0 and a deck transformation $TT_0 \neq \mathrm{Id}$ such that x_0 and $TT_0(x_0)$ both lie in D. This is a contradiction, thus $B(0, R) \cap D \neq \emptyset$. $\square$

3 Rotation sets: definition and star shape

We fix a distance on $\partial\mathbb{H}^2$, given by the Euclidean distance on the circle in the Poincaré disk model. We denote by Δ the diagonal in $(\partial\mathbb{H}^2)^2$, that is to say

$$\Delta = \left\{(x, x) \mid x \in \partial\mathbb{H}^2\right\}.$$

In the whole paper, S will be an orientable surface of negative Euler characteristic of finite type. We will specify in each statement when the additional assumption of compactness of S is necessary.

Let $f \in \mathrm{Homeo}_0(S)$ (where $\mathrm{Homeo}_0(S)$ denotes the set of homeomorphisms of S that are homotopic to identity). We denote by $\tilde{f}$ the lift of f to $\mathbb{H}^2$ which is isotopic to the

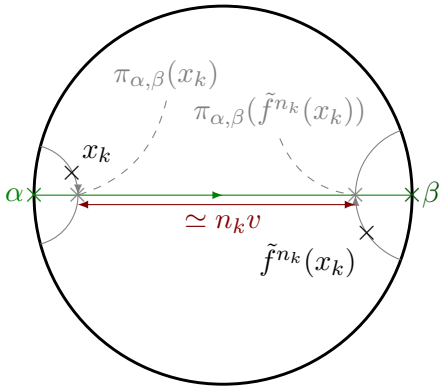

Figure 5: Definition of the rotation set.

identity (and thus it extends to the circle $\partial\mathbb{H}^2$ by the identity by Lemma 3.8 p.53 in [CB88]). It is well-known that the homeomorphism $\tilde{f}$ has a fixed point: otherwise, by associating to each point $\tilde{x} \in \mathbb{H}^2$ the vector at $\tilde{x}$ pointing towards $\tilde{f}(\tilde{x})$, we would obtain a nowhere vanishing vector field on our surface S, a contradiction.

Definition 3.1. A point $(\alpha, \beta, v) \in ((\partial\mathbb{H}^2)^2 \setminus \Delta) \times \mathbb{R}_+^*$ is a *rotation vector of f* if there exists a sequence $(x_k)_{k\in\mathbb{N}}$ of points of $\mathbb{H}^2$, and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers tending to infinity such that, if we denote by $\pi_{\alpha,\beta}$ the orthogonal projection[6] on the geodesic linking α to β,

$$\left(x_k \ , \ \tilde{f}^{n_k}(x_k) \ , \ \frac{d\Big(\pi_{\alpha,\beta}(x_k) \ , \ \pi_{\alpha,\beta}\big(\tilde{f}^{n_k}(x_k)\big)\Big)}{n_k} \right) \underset{k\to+\infty}{\longrightarrow} \big(\alpha, \beta, v\big). \tag{1.1}$$

The *rotation set of f* is the union of rotation vectors of f, together with the singleton $\{(\alpha, \beta, 0)\}$, and quotiented by the relation

$$(\alpha, \beta, 0) \sim (\alpha', \beta', 0).$$

We add the point $\{(\alpha, \beta, 0)\}$ to the rotation set to stress out the fact that the homeomorphism $\tilde{f}$ has a fixed point (by Lefschetz formula), hence an orbit with speed 0.

Note that the first two elements of Equation (1.1) define a geodesic of $\mathbb{H}^2$, and the last element corresponds to a speed. Hence, a rotation vector is made of an asymptotic direction and an asymptotic speed.

Be careful, this set is not necessarily closed (see Subsection 7.3).

Remark also that, as isotopically trivial homeomorphisms of higher genus surfaces have a canonical lift (the one extending to the identity on $\partial\mathbb{H}^2$), the rotation set is uniquely defined for the homeomorphism. It contrasts with rotation sets of torus or annulus homeomorphisms, which depend on the choice of the lift to the universal cover.

Proposition 3.2. *If $f \in \mathrm{Homeo}_0(S)$, then*

$$\rho(f^{-1}) = \big\{(\beta, \alpha, v) \mid (\alpha, \beta, v) \in \rho(f)\big\},$$

[6] In other words, the projection to the closest point of the geodesic (α, β).

and for any $n \geq 1$,

$$\rho(f^n) = n\rho(f) \doteq \big\{(\alpha, \beta, nv) \mid (\alpha, \beta, v) \in \rho(f)\big\}.$$

For any deck transformation T of S, if $(\alpha, \beta, v) \in \rho(f)$, then $(T\alpha, T\beta, v) \in \rho(f)$.

Proof. The first part is immediate.

For the second part, the inclusion $\rho(f^n) \subset n\rho(f)$ is trivial. For the other inclusion, it suffices to remark that any $k \in \mathbb{N}$ can be written as $k = nq + r$, with $0 \leq r < n$, and that there exists $C > 0$ such that $d(\tilde{f}^r, \mathrm{Id}) \leq C$ for any $0 \leq r < n$. Hence, for any $x \in \mathbb{H}^2$, one has $d(\tilde{f}^k(x), (\tilde{f}^n)^q(x)) \leq C$.

The last property comes from the fact that $\tilde{f}$ commutes with deck transformations. □

Remark 3.3. One can translate our definition of rotation set in the case of the torus $\mathbb{T}^2$; it gives a rotation set which is formed of couples (direction, speed), where the direction is a line of $\mathbb{R}^2$ passing through the origin (which is the same as fixing an angle in $\mathbb{R}/2\pi\mathbb{Z}$) and the speed of orbits of a lift of our homeomorphism is measured *via* an orthogonal projection on this line. From this definition of rotation set, it is possible to recover the classical rotation set for torus homeomorphisms (and reciprocally): fixing one direction D (a line through the origin) in $\mathbb{R}^2$, one can consider the set of points whose projection on D is v, where (D, v) is in the rotation set. This set of points is a band of $\mathbb{R}^2$, and the intersection of such bands over all directions D of $\mathbb{R}^2$ gives us the classical rotation set.

Remark 3.4 (Dependence of the rotation set on the hyperbolic metric)**.** Fix $f \in \mathrm{Homeo}_0(S)$. Denote by $\tilde{S}$ the universal cover of S. We call set of directions of the rotation set of f the set

$$\mathcal{D}(f) = \Big\{(\alpha, \beta) \in \partial\tilde{S} \times \partial\tilde{S} \mid \exists v > 0,\ (\alpha, \beta, v) \in \rho(f)\Big\}.$$

This set does not depend on the hyperbolic metric we chose on the surface S. To understand this better, we describe below another way to see $\mathcal{D}(f)$. We see the fundamental group $\pi_1(S)$ of S as the group of deck transformations of $\tilde{S}$. We embed $\pi_1(S)$ in $\tilde{S}$ by taking one orbit of its action on $\tilde{S}$. The Svarč-Milnor lemma (see Lemma 8.9) ensures that this embedding is a quasi-isometry so that it defines an isomorphism between the Gromov boundary $\partial\pi_1(S)$ of the fundamental group of S and the boundary $\partial\tilde{S}$ of $\tilde{S}$. Denote by $T : \tilde{S} \to \pi_1(S)$ a quasi-inverse of this quasi-isometry and endow $\pi_1(S)$ with a distance induced by wordlength. Then, via this isomorphism, the set of directions of $\rho(f)$ is the set of $(\alpha, \beta) \in \partial\pi_1(S) \times \partial\pi_1(S) \setminus \Delta$ such that there exists a sequence $(x_k)_{k\geq 0}$ of points of $\tilde{S}$ a sequence of intergers $n_k \to +\infty$ such that the sequence $(T(x_k))_k$ converges to α, the sequence $T(\tilde{f}^{n_k}(x_k))$ converges to β and

$$\liminf_{k\to+\infty} \frac{1}{n_k} d(T(\tilde{f}^{n_k}(x_k)), T(x_k)) > 0.$$

Changing the hyperbolic metric on S changes the map T into a map T' such that $d(T(x), T'(x))$ is bounded. Hence it does not change the above set.

Moreover, for any direction $(\alpha, \beta) \in \mathcal{D}(f)$ which corresponds to a closed geodesic on S, we understand how the associated speeds change when we change the hyperbolic metric on S thanks to Lemma 4.2. Indeed, take two hyperbolic metrics g_1 and g_2 on S

and, for $i = 1, 2$, denote by $\rho_i(f)$ the rotation set of f with respect to the hyperbolic metric g_i. Fix an isotopy class of loops on S. For $i = 1, 2$, denote by γ_i the unique closed geodesic in this homotopy class and by (α_i, β_i) a lift of γ_i to $\tilde{S}$. Then, by Lemma 4.2,

$$\left\{ \frac{v}{\ell(\gamma_1)} \mid (\alpha_1, \beta_1, v) \in \rho_1(f) \right\} = \left\{ \frac{v}{\ell(\gamma_2)} \mid (\alpha_2, \beta_2, v) \in \rho_2(f) \right\}.$$

Note that some our results of Section 8 and the next ones only concern closed geodesics: how these results change after a metric modification is completely explicit.

For nonclosed geodesics, how speeds change in directions which correspond to nonclosed geodesics after a modification of the hyperbolic metric depends on the restriction of this metric to the geodesic. However, the general picture of these changes remains quite mysterious to the authors.

For the torus case, a consequence of the quasi-convexity of fundamental domains is the convexity of rotation sets (see [MZ89]). In the case of negatively curved surfaces, the outcome is weaker: the rotation set is star-shaped with respect to 0 (Theorem A of the introduction).

Theorem 3.5. *If S is closed, then the rotation set of any $f \in \mathrm{Homeo}_0(S)$ is star-shaped: for any $(\alpha, \beta, v) \in \rho(f)$, and any $v' \in [0, v]$, one has $(\alpha, \beta, v') \in \rho(f)$.*

As a byproduct of the proof of Theorem 3.5, we obtain the realisation of rotation vectors by pieces of orbits whose extremities stay at a finite distance to the geodesic.

Proposition 3.6. *Suppose that S is closed. Let $f \in \mathrm{Homeo}_0(S)$ and $(\alpha, \beta, v) \in \rho(f)$ such that $v > 0$. Then, there exists a sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ of points of $\mathbb{H}^2$, and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers tending to infinity such that (1.1) holds, and moreover,*

$$\max\Big(d\big(\mathbf{x}_k, (\alpha, \beta)\big),\, d\big(\tilde{f}^{n_k}(\mathbf{x}_k), (\alpha, \beta)\big)\Big) \leq R + 1 + \delta,$$

where R is the constant of Proposition 2.2, and δ the smallest diameter of fundamental domains of S in $\mathbb{H}^2$.

Proof of Theorem 3.5. We fix a fundamental domain D of S in $\mathbb{H}^2$, and choose $(\alpha, \beta, v) \in \rho(f)$, with $v > 0$. Let $v' \in (v/2, v)$.

By definition, there exists two sequences $x_k \in \mathbb{H}^2$ and $n_k \in \mathbb{N}$, with $\lim n_k = +\infty$, such that Equation (1.1) holds.

Fix one point a_0 on the geodesic defined by α and β, and $\varepsilon > 0$ such that $v/2 + \varepsilon < v'$. Then, by taking a subsequence if necessary, at least one of the two sequences $d(a_0, \pi_{\alpha,\beta}(x_k))/n_k$ and $d(a_0, \pi_{\alpha,\beta}(\tilde{f}^{n_k}(x_k)))/n_k$ is eventually bigger than or equal to $v/2-\varepsilon$. By taking f^{-1} instead of f and applying Proposition 3.2 if necessary, one can suppose that it is the second one. Moreover, by taking a subsequence again, one can suppose that $d(a_0, \pi_{\alpha,\beta}(x_k))/n_k$ is eventually smaller than or equal to $v/2 + \varepsilon$.

Let T_k be a deck transformation such that $x_k \in T_k(D)$. Then, there exists $p_k \in T_k(D)$ which is a fixed point of $\tilde{f}$. As deck transformations are isometries, the distance $d(x_k, p_k)$ is uniformly bounded (by $\mathrm{diam}(D)$). In particular p_k tends to α, and the distance between $\pi_{\alpha,\beta}(p_k)$ and $\pi_{\alpha,\beta}(x_k)$ is bounded by $\mathrm{diam}(D)$.

Hence, the fundamental domain $D_k = \tilde{f}^{n_k}(T_k D)$ contains both points $p_k = \tilde{f}^{n_k}(p_k)$ and $\tilde{f}^{n_k}(x_k)$. By Proposition 2.2, this fundamental domain is R-quasi convex for some fixed $R > 0$: for any $y \in [p_k, \tilde{f}^{n_k}(x_k)]$, one has $B(y, R) \cap D_k \neq \emptyset$.

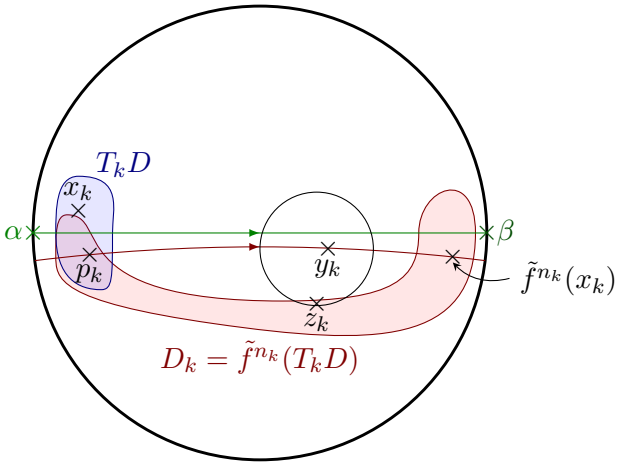

Figure 6: Proof of Theorem 3.5.

Let us choose $y_k \in [p_k, \tilde{f}^{n_k}(x_k)]$ such that $d(\pi_{\alpha,\beta}(p_k), \pi_{\alpha,\beta}(y_k)) = n_k v'$, and $z_k \in B(y_k, R) \cap D_k$. This implies that $\lim d(\pi_{\alpha,\beta}(p_k), \pi_{\alpha,\beta}(z_k))/n_k = v'$. As $\lim d(\pi_{\alpha,\beta}(p_k), a_0)/n_k \leq v/2 + \varepsilon$ (because p_k is at a bounded distance of x_k), we have $\lim d(a_0, \pi_{\alpha,\beta}(z_k))/n_k \geq v' - v/2 - \varepsilon > 0$, so the sequence z_k tends to β (this is here where we need $v' > \frac{v}{2}$).

Moreover, $\tilde{f}^{-n_k}(z_k) \in T_kD$ is at a bounded distance of x_k, so it tends to α, and $\lim d(\pi_{\alpha,\beta}(\tilde{f}^{-n_k}(z_k)), \pi_{\alpha,\beta}(z_k))/n_k = v'$.

We have proved that for any $v' \in (v/2, v]$, one has $(\alpha, \beta, v') \in \rho(f)$. The theorem follows easily from an induction using this property (in fact, one only has to use this property twice: once for f and once for f^{-1}). □

Proof of Proposition 3.6. We fix a fundamental domain D of S in $\mathbb{H}^2$, with minimal diameter δ, and choose $(\alpha, \beta, v) \in \rho(f)$, with $v > 0$. Let (x_k) and (n_k) be such that (1.1) holds. Fix one point a_0 on the geodesic defined by α and β.

Let T_k be a deck transformation such that $x_k \in T_k(D)$. Then, there exists $p_k \in T_k(D)$ which is a fixed point of $\tilde{f}$. The distance $d(x_k, p_k)$ is uniformly bounded (by $\mathrm{diam}(D) = \delta$). In particular p_k tends to α, and the distance between $\pi_{\alpha,\beta}(p_k)$ and $\pi_{\alpha,\beta}(x_k)$ is bounded by δ.

Hence, the fundamental domain $D_k = \tilde{f}^{n_k}(T_kD)$ contains both points $p_k = \tilde{f}^{n_k}(p_k)$ and $\tilde{f}^{n_k}(x_k)$. By Proposition 2.2, this fundamental domain is R-quasi convex for some fixed $R > 0$: for any $y \in [p_k, \tilde{f}^{n_k}(x_k)]$, one has $B(y, R) \cap D_k \neq \emptyset$.

Let us choose $y_k \in [p_k, \tilde{f}^{n_k}(x_k)]$ such that

$$d\big(a_0, \pi_{\alpha,\beta}(y_k)\big) = d\big(a_0, \pi_{\alpha,\beta}(\tilde{f}^{n_k}(x_k))\big) - \sqrt{d\big(a_0, \pi_{\alpha,\beta}(\tilde{f}^{n_k}(x_k))\big)}. \tag{3.1}$$

The following claim is a consequence of basic hyperbolic geometry.

Claim 3.7. *If k is large enough, then $d\big(y_k, \pi_{\alpha,\beta}(y_k)\big) \leq 1/2$.*

Proof of the claim. Let us consider the half-plane model of $\mathbb{H}^2$, such that $\alpha = \infty$ and $\beta = 0$ (in this case, (α, β) is the positive imaginary axis). The set of points at distance $1/2$ of (α, β) is the union of two Euclidean half-lines starting at $0 = \beta$, making angle θ

with the imaginary axis, with $|\sin\theta| = \tanh(1/2)$. As the choice of a_0 was arbitrary, one can choose $a_0 = i$.

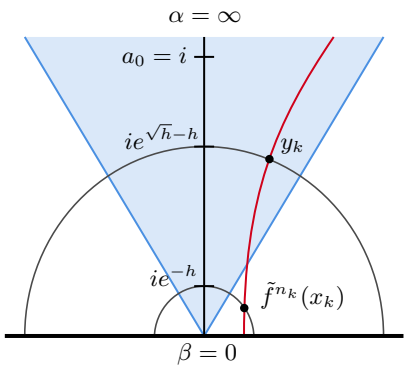

Figure 7: Proof of Claim 3.7.

Denote $h = d\big(a_0, \pi_{\alpha,\beta}(\tilde{f}^{n_k}(x_k))\big)$. Then, $\pi_{a,b}(y_k)$ can be computed in terms of d: the hyperbolic distance satisfies, for $p, q > 0$, $d(ip, iq) = |\log p - \log q|$. Hence, by (3.1)

$$\pi_{\alpha,\beta}(\tilde{f}^{n_k}(x_k)) = ie^{-h} \quad \text{and} \quad \pi_{\alpha,\beta}(y_k) = ie^{-h+\sqrt{h}} = ie^{\sqrt{h}}e^{-h}.$$

Hence, the point y_k lies on the Euclidean circle centred at 0 with radius $e^{\sqrt{h}}e^{-h}$.

Fix $A > 0$, then for any k large enough the geodesic passing through p_k and $\tilde{f}^{n_k}(x_k)$ is either a Euclidean circle with one extremity inside $[-e^{-h}, e^{-h}]$ and the other outside $[-A, A]$ or a line which is orthogonal to the real axis and with one extremity inside $[-e^{-h}, e^{-h}]$. Simple Euclidean geometry shows that if A and h are large enough, then the intersection y_k of this geodesic with the Euclidean circle centred at 0 with radius $e^{\sqrt{h}}e^{-h}$ is inside the Euclidean cone made of the points at distance at most $1/2$ of (α, β). Indeed, the angle between the imaginary axis and the straight line between 0 and y_k tends to 0 when $A \to +\infty$ and $h \to +\infty$.

Hence, for any k large enough, $d\big(y_k, \pi_{\alpha,\beta}(y_k)\big) \leq 1/2$. □

Pick some $z_k \in B(y_k, R) \cap D_k$. This implies that $d(z_k, (\alpha, \beta)) \leq R + 1/2$. By the definition of y_k, one has $\lim_{k\to+\infty} y_k = \beta$, so we also have $\lim_{k\to+\infty} z_k = \beta$. Moreover, $\tilde{f}^{-n_k}(z_k) \in T_kD$, hence $\lim_{k\to+\infty} \tilde{f}^{-n_k}(z_k) = \alpha$, and

$$\frac{d\Big(\pi_{\alpha,\beta}\big(f^{-n_k}(z_k)\big)\,,\ \pi_{\alpha,\beta}(z_k)\Big)}{n_k} = v.$$

At this point, we have no information about the distance between $f^{-n_k}(z_k)$ and the geodesic (α, β), but we know that $d(f^{-n_k}(z_k), x_k) \leq \delta$. To solve this issue, it suffices to make the same argument a second time, with f^{-1} instead of f and z_k instead of x_k. We then get a point $\mathbf{x}_k$ with the desired properties. □

4 Realisation of rotation vectors for closed geodesics

We now state a realisation result, which is a direct consequence of already known realisation results for annulus homeomorphisms [Fra88, LC05].

Proposition 4.1. *Let $f \in \mathrm{Homeo}_0(S)$ and take $\alpha \neq \beta \in \partial\mathbb{H}^2$ which define a closed geodesic in S, of length $\ell > 0$. Suppose that $(\alpha, \beta, v_0) \in \rho(f)$, with $v_0 = \ell p/q$ (p/q irreducible), and that one of the following conditions holds:*

(i) v_0 is maximal, i.e. $v_0 = \max\{v \in \mathbb{R}_+ \mid (\alpha, \beta, v) \in \rho(f)\}$;

(ii) f is chain transitive;

(iii) there exists $(\alpha_1, \beta_1, v_1), (\alpha_2, \beta_2, v_2) \in \rho(f)$, with $v_1, v_2 > 0$, such that (α_1, β_1) crosses (α, β) positively and (α_2, β_2) crosses (α, β) negatively.

Then f possesses a periodic point of period q and rotation vector (α, β, v_0).

We will need the notion of rotation set of an annulus homeomorphism, which we recall now. Let $g : \mathbb{S}^1 \times [-1, 1] = \mathbb{R}/\mathbb{Z} \times [-1, 1] \to \mathbb{R}/\mathbb{Z} \times [-1, 1]$ be a homeomorphism which is isotopic to the identity. Let us denote by $\tilde{g} : \mathbb{R} \times [-1, 1] \to \mathbb{R} \times [-1, 1]$ one of its lifts and by $p_1 : \mathbb{R} \times [-1, 1] \to \mathbb{R}$ the projection. The rotation set $\rho(\tilde{g})$ of $\tilde{g}$ is the set of limits of sequences of the form

$$\left(\frac{p_1(\tilde{g}^{n_k}(x_k)) - p_1(x_k)}{n_k}\right)$$

with $n_k \to +\infty$ and $x_k \in \mathbb{R} \times [-1, 1]$.

We will need another way to see this rotation set of $\tilde{g}$. Denote by $\mathcal{M}(g)$ the set of g-invariant probability measures. Then the rotation set of $\tilde{g}$ is also

$$\left\{\int_{\mathbb{S}^1 \times [-1,1]} (p_1(\tilde{g}(\tilde{x})) - p_1(\tilde{x}))d\mu(x) \mid \mu \in \mathcal{M}(g)\right\}.$$

Indeed, denote by $\rho_\mu(\tilde{g})$ this second rotation set. As sequences of probability measures of the form

$$\left(\frac{1}{n_k}\sum_{i=0}^{n_k-1} \delta_{g^i(x_k)}\right)_{k \geq 0}$$

have a limit point, we obtain that $\rho(\tilde{g}) \subset \rho_\mu(\tilde{g})$. The other inclusion is the consequence of the connexity of $\rho(\tilde{g})$ and of the following facts.

1. Extremal points of $\rho_\mu(\tilde{g})$ are realised by ergodic measures.
2. The Birkhoff ergodic theorem applied to $x \mapsto p_1(\tilde{g}(\tilde{x})) - p_1(\tilde{x})$ with those ergodic measures implies that those extremal points belong to $\rho(\tilde{g})$.

We use notation from the proposition. Let T be the deck transformation associated to the closed geodesic (α, β). Let $\check{f} : \mathbb{H}^2/T \to \mathbb{H}^2/T$ be the quotient map of $\tilde{f}$; as $\tilde{f}$ extends by identity to $\partial\mathbb{H}^2$, this map $\check{f}$ can be seen as a map of the closed annulus $\mathbb{S}^1 \times [-1, 1]$, homotopic to the identity. Hence, it has a well defined rotation set $\rho(\check{f}) \subset \mathbb{R}$.

Lemma 4.2. *Under the above hypotheses,*

$$\ell\rho(\check{f}) = \{v \in \mathbb{R} \mid (\alpha, \beta, v) \in \rho(f) \text{ or } (\beta, \alpha, -v) \in \rho(f)\}.$$

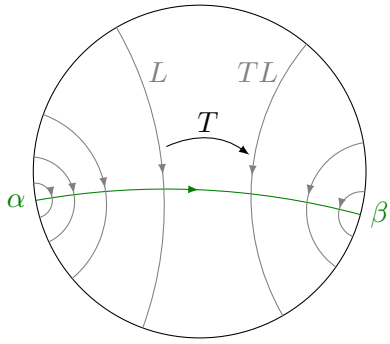

Figure 8: Realisation of periodic points.

Proof of Lemma 4.2. Let L be a geodesic of $\mathbb{H}^2$ orthogonal to (α, β), and $\check{D} \subset \mathbb{H}^2$ the fundamental domain of the open annulus $\mathbb{H}^2/T$, consisting of the points that are between L and TL (see Figure 8). For any $x \in \mathbb{H}^2$, we denote by i_x the integer satisfying $x \in T^{i_x}(\check{D})$. As L is orthogonal to (α, β), for any $x, y \in \mathbb{H}^2$,

$$\ell(|i_x - i_y| - 1) \leq d\big(\pi_{\alpha,\beta}(x)\,,\ \pi_{\alpha,\beta}(y)\big) \leq \ell(|i_x - i_y| + 1). \tag{4.1}$$

Suppose that $(\alpha, \beta, v) \in \rho(f)$. Then, there exists (x_k) and (n_k) such that Equation (1.1) holds. Applying Equation (4.1) to $x = x_k$ and $y = \tilde{f}^{n_k}(x_k)$, one gets that $v/\ell \in \rho(\check{f})$. Conversely, any sequence $\check{x}_k \in \mathbb{H}^2/T$ realising a rotation number $v' \in \rho(\check{f})$ lifts to a sequence $x_k \in \mathbb{H}^2$ with rotation vector of the form $(\alpha, \beta, \ell v')$ if $v' > 0$, or $(\beta, \alpha, -\ell v')$ if $v' < 0$. □

Proof of Proposition 4.1. By applying Lemma 4.2, and lifting the points to $\mathbb{H}^2$, we are reduced to prove the proposition in the case of the closed annulus. Point (i) comes from the fact that any extremal rational rotation number is realised by some periodic orbit ([Fra88, Theorem 3.5]), point (ii) from [Fra88, Theorem 2.2], and point (iii) is a direct consequence of a generalization of Poincaré-Birkhoff theorem [LC05, Theorem 9.1]. □

The following proposition means that any extremal point of $\rho(f)$ in a closed geodesic direction is realised by an orbit which stays at sublinear distance from the geodesic line. It uses results from [Les11] and [Han90].

Proposition 4.3. *Let (α, β) be a geodesic line of $\mathbb{H}^2$ which projects to a closed geodesic γ. Suppose (α, β, v) is an extremal point of $\rho(\tilde{f})$, with $v \neq 0$. Then there exists a point $\tilde{x}$ in $\tilde{S} = \mathbb{H}^2$ such that*

$$\begin{cases} \lim\limits_{n\to+\infty} \dfrac{1}{n} d(\tilde{f}^n(\tilde{x}), \tilde{x}) &= v \\ \lim\limits_{n\to+\infty} \tilde{f}^n(\tilde{x}) &= \beta \\ \lim\limits_{n\to+\infty} \dfrac{1}{n} d(\tilde{f}^n(\tilde{x}), \pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x}))) &= 0 \end{cases}$$

Moreover, either the orbit under $\tilde{f}$ of $\tilde{x}$ stays within a bounded distance of the geodesic line (α, β) and its closure (in $\overline{\mathbb{H}^2}$) does not contain any fixed point of $\tilde{f}$ or, for any rational number $0 < r < \frac{v}{\ell(\gamma)}$, there exist a periodic orbit realising the rotation vector $(\alpha, \beta, r\ell(\gamma))$.

To explain why the vector $(\alpha, \beta, v) \in \rho(f)$ is realised by the orbit of $\tilde{x}$, let T be the deck transformation associated to (α, β). Fix any sequence of integers $(k_n)_{n\geq 0}$ such that

$$\begin{cases} \lim\limits_{n\to+\infty} k_n &= +\infty \\ \lim\limits_{n\to+\infty} \dfrac{k_n}{n} &= 0. \end{cases}$$

Then

$$\lim_{n\to+\infty} \frac{1}{n} d\Big(\pi_{\alpha,\beta}\big(T^{-k_n}(\tilde{x})\big), \pi_{\alpha,\beta}\big(\tilde{f}^n(T^{-k_n}(\tilde{x}))\big)\Big) = \lim_{n\to+\infty} \frac{1}{n} d(\pi_{\alpha,\beta}(\tilde{x}), \pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x}))) = v$$

and

$$\begin{cases} \lim\limits_{n\to+\infty} T^{-k_n}(\tilde{x}) = \alpha \\ \lim\limits_{n\to+\infty} \tilde{f}^n(T^{-k_n}(\tilde{x})) = \beta. \end{cases}$$

This means that the rotation vector (α, β, v) is realised by the orbit of $\tilde{x}$.

Proof of Proposition 4.3. During this proof, we will need the following elementary result of hyperbolic geometry.

Claim 4.4. *For any $\alpha_1, \alpha_2, \beta \in \partial\mathbb{H}^2$ such that $\alpha_1 \neq \beta$ and $\alpha_2 \neq \beta$, we have :*

$$\lim_{\substack{y\in\mathbb{H}^2 \\ y\to\beta}} d\big(\pi_{\alpha_2,\beta}(y), \pi_{\alpha_1,\beta}(y)\big) = 0.$$

Proof. We see $\mathbb{H}^2$ as the upper half-plane in $\mathbb{R}^2$ so that $\partial\mathbb{H}^2$ is the union of the line $\mathbb{R} \times \{0\}$ with the point ∞ at infinity. As, for any isometry σ of $\mathbb{H}^2$,

$$\begin{aligned} d\big(\pi_{\alpha_2,\beta}(y), \pi_{\alpha_1,\beta}(y)\big) &= d\big(\sigma(\pi_{\alpha_2,\beta}(y)), \sigma(\pi_{\alpha_1,\beta}(y))\big) \\ &= d\big(\pi_{\sigma(\alpha_2),\sigma(\beta)}(i(y)), \pi_{\sigma(\alpha_1),\sigma(\beta)}(i(y))\big) \end{aligned}$$

and as the group of isometries of $\mathbb{H}^2$ acts transitively on the boundary $\partial\mathbb{H}^2$, we can suppose that $\beta = \infty$ and that α_1 and α_2 are two points of $\mathbb{R} \times \{0\}$. To carry out this proof, we will use the distance d_{Euc} on $\mathbb{H}^2$ which is induced by the Euclidean distance on $\mathbb{R}^2$. In this model, the geodesic lines (α_1, β) and (α_2, β) are respectively the sets $\{\alpha_1\} \times \mathbb{R}_+^*$ and $\{\alpha_2\} \times \mathbb{R}_+^*$.

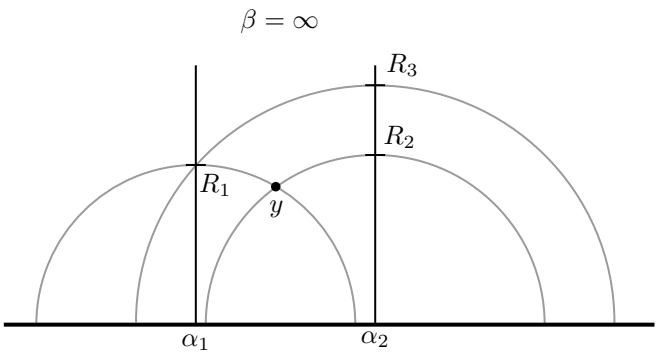

Figure 9: Proof of Claim 4.4.

Fix $y \in \mathbb{H}^2$. The geodesic line passing through y and orthogonal to (α_1, β) is the intersection of $\mathbb{H}^2$ with the Euclidean circle of center α_1 and of radius $R_1 = d_{Euc}(y, \alpha_1)$. Hence $\pi_{\alpha_1,\beta}(y) = (\alpha_1, R_1)$. In the same way, if $R_2 = d_{Euc}(\alpha_2, y)$ and $R_3 = d_{Euc}(\pi_{\alpha_1,\beta}(y), \alpha_2)$, then $\pi_{\alpha_2,\beta}(y) = (\alpha_2, R_2)$ and $\pi_{\alpha_2,\beta} \circ \pi_{\alpha_1,\beta}(y) = (\alpha_2, R_3)$.

We have

$$d\big(\pi_{\alpha_1,\beta}(y), \pi_{\alpha_2,\beta}(y)\big) \leq d\big(\pi_{\alpha_1,\beta}(y), \pi_{\alpha_2,\beta}(\pi_{\alpha_1,\beta}(y))\big) + d\big(\pi_{\alpha_2,\beta}(\pi_{\alpha_1,\beta}(y)), \pi_{\alpha_2,\beta}(y)\big).$$

When the point y tends to $\beta = \infty$, the distance $d(\pi_{\alpha_1,\beta}(y), \pi_{\alpha_2,\beta}(\pi_{\alpha_1,\beta}(y)))$, which is the (hyperbolic) distance between the point $\pi_{\alpha_1,\beta}(y)$ and the geodesic line (α_2, β), tends to 0 as the point $\pi_{\alpha_1,\beta}(y)$ remains on the geodesic line (α_1, β).

To prove the claim, it suffices to prove that, when the point y tends to ∞, the distance $d(\pi_{\alpha_2,\beta}(\pi_{\alpha_1,\beta}(y)), \pi_{\alpha_2,\beta}(y))$ tends to 0. To do this, it suffices to prove that the Euclidean distance $d_{Euc}(\pi_{\alpha_2,\beta}(\pi_{\alpha_1,\beta}(y)), \pi_{\alpha_2,\beta}(y))$ remains bounded. We have

$$\begin{array}{rcl} d_{Euc}\big(\pi_{\alpha_2,\beta}(\pi_{\alpha_1,\beta}(y)), \pi_{\alpha_2,\beta}(y)\big) & = & |R_2 - R_3| \\ & \leq & |R_2 - R_1| + |R_1 - R_3|. \end{array}$$

However,

$$\left\{ \begin{array}{l} |R_2 - R_1| = |d_{Euc}(y, \alpha_2) - d_{Euc}(y, \alpha_1)| \leq d_{Euc}(\alpha_2, \alpha_1) \\ |R_3 - R_1| = |d_{Euc}(\pi_{\alpha_1,\beta}(y), \alpha_2) - d_{Euc}(\pi_{\alpha_1,\beta}(y), \alpha_1)| \leq d_{Euc}(\alpha_1, \alpha_2) \end{array} \right.$$

and

$$d_{Euc}\big(\pi_{\alpha_2,\beta}(\pi_{\alpha_1,\beta}(y)), \pi_{\alpha_2,\beta}(y)\big) \leq 2 d_{Euc}(\alpha_1, \alpha_2).$$

□

Let T be the deck transformation associated to (α, β). Let A be the open annulus $\tilde{S}/\langle T \rangle = \mathbb{H}^2/\langle T \rangle$ and $\overline{A}$ be the closed annulus $\overline{\mathbb{H}^2} - \{\alpha, \beta\}/\langle T \rangle$. Denote by $\check{f}$ the lift of f to $\overline{A}$ induced by $\tilde{f}$ and recall that, as f is isotopic to the identity, the homeomorphism $\check{f}$ pointwise fixes the boundary of the closed annulus $\overline{A}$. Fix coordinates on $\overline{A}$ so that we can make the following identifications:

$$\left\{ \begin{array}{rcl} \overline{\mathbb{H}^2} - \{\alpha, \beta\} & = & [-1, 1] \times \mathbb{R} \\ (\alpha, \beta) & = & \{0\} \times \mathbb{R} \\ \overline{A} & = & [-1, 1] \times \mathbb{R}/\mathbb{Z} \end{array} \right.$$

and the projection $p_2 : \overline{\mathbb{H}^2} - \{\alpha, \beta\} = [-1, 1] \times \mathbb{R} \to \mathbb{R}$ on the second coordinate is equal to $\pi_{\alpha,\beta}$.

By Lemma 4.2, the point $\frac{v}{\ell(\gamma)}$ is an extremal point of $\rho(\check{f})$. Hence there exists an $\check{f}$-invariant ergodic probability measure $\check{\mu}$ on $\overline{A}$ such that

$$\int_{\overline{A}} p_2\big(\tilde{f}(\tilde{x}) - \tilde{x}\big) \, \mathrm{d}\check{\mu}(\check{x}) = \frac{v}{\ell(\gamma)}.$$

As the homeomorphism $\check{f}$ pointwise fixes the boundary $\partial\overline{A}$ of $\overline{A}$, observe that $\check{\mu}(\partial A) = 0$. Indeed, otherwise, we would have $\check{\mu} = \check{\mu}(A)\check{\mu}_1 + \check{\mu}(\partial A)\check{\mu}_2$, where $\check{\mu}_1 = \frac{1}{\check{\mu}(A)}\check{\mu}(A \cap .)$ and $\check{\mu}_2 = \frac{1}{\check{\mu}(\partial\overline{A})}\check{\mu}(\partial\overline{A} \cap .)$, which contradicts the ergodicity of $\check{\mu}$ (observe that the definition of $\check{\mu}$ imposes that $\check{\mu}(A) > 0$).

By Birkhoff ergodic theorem, the subset C of A consisting of points $\check{x}$ such that

$$\lim_{n\to+\infty}\frac{1}{n}p_2(\tilde{f}^n(\tilde{x})-\tilde{x})=\frac{v}{\ell(\gamma)},$$

where the point $\tilde{x}\in\mathbb{H}^2$ is any lift of $\check{x}$, has full $\check{\mu}$-measure.

Going back to the hyperbolic distance on A, this means that, for any point $\tilde{x}$ of $\tilde{S}$ which projects to a point of C,

$$\left\{\begin{array}{rcl}\lim\limits_{n\to+\infty}\tilde{f}^n(\tilde{x}) & = & \beta\\ \lim\limits_{n\to+\infty}\frac{1}{n}d\big(\pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x})),\tilde{x}\big) & = & v.\end{array}\right.$$

Let $\check{\pi}$ be the covering map $A\to S$ and $\mu=\check{\pi}_*\check{\mu}$. The probability measure μ is f-invariant as $\check{\mu}$ is $\check{f}$-invariant. The following lemma is a consequence of Corollary 21 of [Les11]. This statement is actually valid for any f-invariant ergodic probability measure.

Lemma 4.5. *There exists a full μ-measure subset B of S such that, for any point $\tilde{x}$ in $\tilde{\pi}^{-1}(B)$, there exists a geodesic line $(\alpha_{\tilde{x}},\beta_{\tilde{x}})$ such that*

$$\lim_{n\to+\infty}\frac{1}{n}d\big(\tilde{f}^n(\tilde{x}),(\alpha_{\tilde{x}},\beta_{\tilde{x}})\big)=0.$$

Now take any point $\tilde{x}$ in $\mathbb{H}^2$ which is a lift of a point in $\check{\pi}^{-1}(B)\cap C$ (this set has full $\check{\mu}$-measure and is hence nonempty). As the point $\tilde{x}$ is a lift of a point of C, then

$$\left\{\begin{array}{rcl}\lim\limits_{n\to+\infty}\frac{1}{n}d\big(\pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x})),\pi_{\alpha,\beta}(\tilde{x})\big) & = & v\\ \lim\limits_{n\to+\infty}\tilde{f}^n(\tilde{x})=\beta & & \end{array}\right.$$

so that $\beta_{\tilde{x}}=\beta$. Moreover

$$d\big(\tilde{f}^n(\tilde{x}),\pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x}))\big)\leq d\big(\tilde{f}^n(\tilde{x}),\pi_{\alpha_{\tilde{x}},\beta}(\tilde{f}^n(\tilde{x}))\big)+d\big(\pi_{\alpha_{\tilde{x}},\beta}(\tilde{f}^n(\tilde{x})),\pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x}))\big).$$

However, by Lemma 4.5,

$$\lim_{n\to+\infty}\frac{1}{n}d\big(\tilde{f}^n(\tilde{x}),\pi_{\alpha_{\tilde{x}},\beta}(\tilde{f}^n(\tilde{x}))\big)=0$$

and, by Claim 4.4,

$$\lim_{n\to+\infty}d\big(\pi_{\alpha_{\tilde{x}},\beta}(\tilde{f}^n(\tilde{x})),\pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x}))\big)=0.$$

Hence

$$\lim_{n\to+\infty}\frac{1}{n}d\big(\tilde{f}^n(\tilde{x}),\pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x}))\big)=0.$$

The last part of the proposition is a consequence of a result by Handel (see [Han90]). If the orbit of $\tilde{x}$ does not stay within a bounded distance of the geodesic (α,β), then the closure of this orbit meets the boundary of the closed annulus $\overline{A}$. However, recall that this boundary is fixed under $\check{f}$, so that a fixed point of $\tilde{f}$ lies in the closure of the orbit of $\tilde{x}$, which in particular does not have the same rotation number as $\tilde{x}$. Then, by the proof of Lemma 2.1 p.343 in [Han90] and by Lemma 4.2, for any rational number $0<r<\ell(\gamma)$, there exist periodic orbits for f with rotation number $(\alpha,\beta,r\ell(\gamma))$. □

See Example 7.1 for an example of a homeomorphism of the genus 2 closed surface with an ergodic probability measure μ for which an uncountable set of geodesics is necessary to describe the rotation set of μ almost every point.

5 Creation of new rotation vectors: elementary results

In this section, we state forcing results about rotation vectors: the existence of orbits with nontrivial rotation vectors, whose associated geodesics of $\mathbb{H}^2$ cross, force the existence of other rotation vectors (and hence other orbits, with different rotation vectors). The two results we get, Propositions 5.1 and 5.4, are heavily inspired by Le Calvez-Tal's fundamental proposition [LCT18a, Proposition 20], although they use only basic plane topology (and in particular, no Brouwer-Le Calvez plane dynamical foliation, see Section 9.1 for some results of this theory). Note that similar arguments without the use of Brouwer-Le Calvez plane dynamical foliation appeared recently in [LC20].

The first proposition concerns geodesics of the surface with auto-intersection (Proposition B of the introduction), and the second one treats the general case (with weaker conclusions).

Proposition 5.1. *Let (α, β) be a geodesic line of $\mathbb{H}^2$ and T a nontrivial deck transformation such that $(\alpha, \beta) \cap T^{-1}(\alpha, \beta) \neq \emptyset$. Denote $\{p_0\} = (\alpha, \beta) \cap T^{-1}(\alpha, \beta)$, and suppose that $Tp_0 \in (\alpha, \beta)$ is such that for the natural order on (α, β), the sequence $(\alpha, p_0, Tp_0, \beta)$ is increasing.*

Suppose there exists $v > 0$ such that $(\alpha, \beta, v) \in \rho(f)$. Then $(T\alpha, \beta, v) \in \rho(f)$.

Of course, under the hypothesis of this proposition we also deduce that $(\alpha, T^{-1}(\beta), v) \in \rho(f)$.

If the sequence $(\alpha, Tp_0, p_0, \beta)$ is increasing instead of $(\alpha, p_0, Tp_0, \beta)$, then we can apply Proposition 5.1 to the homeomorphism f^{-1} and use Proposition 3.2 to obtain that $(\alpha, T\beta, v) \in \rho(f)$ and $(T^{-1}\alpha, \beta, v) \in \rho(f)$ when $(\alpha, \beta, v) \in \rho(f)$.

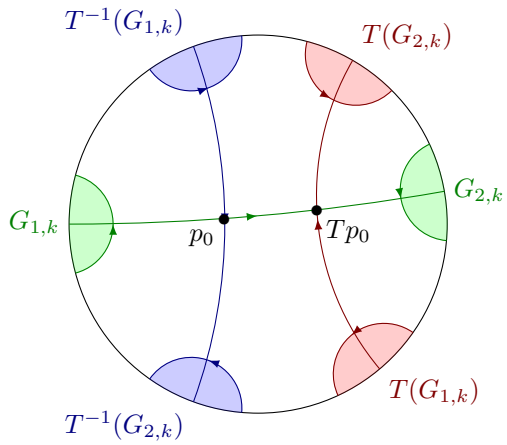

Figure 10: Configuration of Proposition 5.1.

Proof. By definition, there exist a sequence $(x_k)_{k\in\mathbb{N}}$ of points in $\mathbb{H}^2$ and a sequence $(n_k)_{k\in\mathbb{N}}$ of integers such that

$$\begin{cases} \lim\limits_{k\to+\infty} x_k = \alpha \\ \lim\limits_{k\to+\infty} \tilde{f}^{n_k}(x_k) = \beta \\ \lim\limits_{k\to+\infty} \dfrac{d(\pi_{\alpha,\beta}(\tilde{f}^{n_k}(x_k)), \pi_{\alpha,\beta}(x_k))}{n_k} = v. \end{cases}$$

For any $k \geq 0$, denote by $G_{1,k}$ (respectively $G_{2,k}$) the unique geodesic line passing through x_k (respectively $\tilde{f}^{n_k}(x_k)$) which is orthogonal to (α, β) (see Figure 10). Extracting a subsequence if necessary, we can suppose that both sequences

$$\left(\frac{d(\pi_{\alpha,\beta}(x_k),\, \pi_{\alpha,\beta}(p_0))}{n_k}\right)_k \qquad \text{and} \qquad \left(\frac{d(\pi_{\alpha,\beta}(p_0),\, \pi_{\alpha,\beta}(\tilde{f}^{n_k}(x_k)))}{n_k}\right)_k$$

converge with respective limits v_1' and v_2'. Observe that $v_1' + v_2' = v$.

Lemma 5.2. *For any large enough k,*

$$\tilde{f}^{n_k}(T(G_{1,k})) \cap G_{2,k} \neq \emptyset.$$

Proof. If k is large enough, the sets $T^i(G_{1,k})$, for $i \in \mathbb{Z}$, are pairwise disjoint and so are the sets $T^j(G_{2,k})$, for $j \in \mathbb{Z}$. Each set $G_{1,k}$ defines an interval $I_{1,k}$ on $\partial\mathbb{H}^2$, as the connected component of $\partial\mathbb{H}^2 \setminus \overline{G_{1,k}}$ containing α. Similarly, the set $G_{2,k}$ defines an interval $I_{2,k}$ on $\partial\mathbb{H}^2$, as the connected component of $\partial\mathbb{H}^2 \setminus \overline{G_{2,k}}$ containing β. There is an orientation on $\partial\mathbb{H}^2$ such that the following intervals are all ordered positively:

$$I_{1,k},\ TI_{1,k},\ I_{2,k},\ TI_{2,k}. \tag{5.1}$$

These orientations of the $I_{j,k}$'s induce orientations of the $G_{j,k}$'s. For now we fix a large enough k so that the above properties hold.

Suppose for a contradiction that $\tilde{f}^{n_k}(G_{1,k}) \cap T^{-1}G_{2,k}$ is empty.

Let us parametrize the oriented geodesic $G_{1,k}$ by the arc length. Let $t \in \mathbb{R}$ be such that

$$\begin{cases} \tilde{f}^{n_k}(G_{1,k}|_{(-\infty,t)}) \cap T^i G_{2,k} = \emptyset & \text{for all } i \geq 0 \\ \tilde{f}^{n_k}(G_{1,k}|_{(-\infty,t]}) \cap T^{i_0} G_{2,k} \neq \emptyset & \text{for some } i_0 \geq 0. \end{cases}$$

Such a t exists as the intersection $\tilde{f}^{n_k}(G_{1,k}) \cap G_{2,k}$ is nonempty and as $\tilde{f}^{n_k}(G_{1,k}) \cap T^i G_{2,k} = \emptyset$ for any i large enough. Remark that the number i_0 satisfying this property is unique.

Let $t' \in \mathbb{R}$ be such that $\tilde{f}^{n_k}(G_{1,k}|_{(-\infty,t]}) \cap T^{i_0}G_{2,k}|_{(-\infty,t']}$ is reduced to the point $\tilde{f}^{n_k}(G_{1,k}(t)) = T^{i_0}G_{2,k}(t')$, and denote by δ the path which is the concatenation of $\tilde{f}^{n_k}(G_{1,k}|_{(-\infty,t]})$ and $T^{i_0}G_{2,k}|_{(-\infty,t']}$. It is a path linking $\partial\mathbb{H}^2$ to $\partial\mathbb{H}^2$. We now prove that this path is disjoint from its translate by T. Indeed, the fact that the $T^iG_{1,k}$ and $T^jG_{2,k}$ are pairwise disjoint reduces the possible intersections to

$$\tilde{f}^{n_k}(G_{1,k}|_{(-\infty,t]}) \cap T^{i_0+1}G_{2,k}|_{(-\infty,t']}$$

or

$$\tilde{f}^{n_k}(TG_{1,k}|_{(-\infty,t]}) \cap T^{i_0}G_{2,k}|_{(-\infty,t']} = T\Big(\tilde{f}^{n_k}(G_{1,k}|_{(-\infty,t]}) \cap T^{i_0-1}G_{2,k}|_{(-\infty,t']}\Big),$$

but these intersections are empty by uniqueness of i_0, and the fact that by contradiction hypothesis, $\tilde{f}^{n_k}(G_{1,k}) \cap T^{-1}G_{2,k} = \emptyset$ (to treat the case $i_0 = 0$).

But, by the ordering of the intervals given by (5.1) (this is where we use the order on points α, p_0, Tp_0, β), if $i_0 \geq 0$, then the two endpoints of $T\delta$ lie in different connected components of $\mathbb{H}^2 \setminus \delta$, a contradiction. $\square$

We now prove that $(T(\alpha), \beta, v) \in \rho(f)$. By Lemma 5.2, for any large enough k,

$$\tilde{f}^{n_k}(TG_{1,k}) \cap G_{2,k} \neq \emptyset.$$

Hence there exists a sequence $(y_k)_k$ of points of $\mathbb{H}^2$ such that, for any k

1. $y_k \in TG_{1,k}$.
2. $\tilde{f}^{n_k}(y_k) \in G_{2,k}$.

Observe that the sequence of points (y_k) converges to the point $T(\alpha)$ and that the sequence of points $\tilde{f}^{n_k}(y_k)$ converges to β.

To prove Proposition 5.1, we need the following hyperbolic geometry lemma.

Lemma 5.3. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be pairwise distinct points of $\partial\mathbb{H}^2$. For any point p_0 of $\mathbb{H}^2$,*

$$\lim_{\substack{y\in\mathbb{H}^2\\ y\to\beta_1}} \frac{d\big(\pi_{\alpha_1,\beta_1}(y), \pi_{\alpha_1,\beta_1}(p_0)\big)}{d\big(\pi_{\alpha_2,\beta_1}(y), \pi_{\alpha_2,\beta_1}(p_0)\big)} = \lim_{\substack{y\in\mathbb{H}^2\\ y\to\alpha_1}} \frac{d\big(\pi_{\alpha_1,\beta_1}(y), \pi_{\alpha_1,\beta_1}(p_0)\big)}{d\big(\pi_{\alpha_1,\beta_2}(y), \pi_{\alpha_1,\beta_2}(p_0)\big)} = 1.$$

Before proving the lemma, we prove Proposition 5.1. Recall that we fixed a point $p \in \mathbb{H}^2$. For any sufficiently large index k,

$$\frac{d\big(\pi_{T(\alpha),\beta}(\tilde{f}^{n_k}(y_k)), \pi_{T(\alpha),\beta}(y_k)\big)}{n_k} = \frac{d\big(\pi_{T(\alpha),\beta}(\tilde{f}^{n_k}(y_k)), \pi_{T(\alpha),\beta}(p)\big)}{n_k} + \frac{d\big(\pi_{T(\alpha),\beta}(p), \pi_{T(\alpha),\beta}(y_k)\big)}{n_k}$$

as the point $\pi_{T(\alpha),\beta}(p)$ lies between the points $\pi_{T(\alpha),\beta}(\tilde{f}^{n_k}(y_k))$ and $\pi_{T(\alpha),\beta}(y_k))$ on the geodesic $(T(\alpha), \beta)$. Hence

$$\begin{aligned}\frac{d\big(\pi_{T(\alpha),\beta}(\tilde{f}^{n_k}(y_k)), \pi_{T(\alpha),\beta}(y_k)\big)}{n_k} &= \frac{d\big(\pi_{T(\alpha),\beta}(\tilde{f}^{n_k}(y_k)), \pi_{T(\alpha),\beta}(p)\big)}{d\big(\pi_{\alpha,\beta}(\tilde{f}^{n_k}(y_k)), \pi_{\alpha,\beta}(p)\big)} \frac{d\big(\pi_{\alpha,\beta}(\tilde{f}^{n_k}(y_k)), \pi_{\alpha,\beta}(p)\big)}{n_k} \\ &\quad + \frac{d\big(\pi_{T(\alpha),\beta}(p), \pi_{T(\alpha),\beta}(y_k)\big)}{d\big(\pi_{T(\alpha),T(\beta)}(p), \pi_{T(\alpha),T(\beta)}(y_k)\big)} \frac{d\big(\pi_{T(\alpha),T(\beta)}(p), \pi_{T(\alpha),T(\beta)}(y_k)\big)}{n_k}.\end{aligned}$$

Now, the points y_k and $T(x_k)$ both belong to the geodesic line $T(G_{1,k})$ which is orthogonal to $(T(\alpha), T(\beta))$. Hence

$$\begin{aligned} d\big(\pi_{T(\alpha),T(\beta)}(y_k), \pi_{T(\alpha),T(\beta)}(p)\big) &= d\big(\pi_{T(\alpha),T(\beta)}(T(x_k)), \pi_{T(\alpha),T(\beta)}(p)\big) \\ &= d\big(\pi_{\alpha,\beta}(x_k), \pi_{\alpha,\beta}(T^{-1}(p))\big)\end{aligned}$$

so that

$$\lim_{k\to+\infty} \frac{d\big(\pi_{T(\alpha),T(\beta)}(y_k), \pi_{T(\alpha),T(\beta)}(p)\big)}{n_k} = v_1'.$$

In the same way,

$$\lim_{k\to+\infty} \frac{d\big(\pi_{\alpha,\beta}(p), \pi_{\alpha,\beta}(\tilde{f}^{n_k}(y_k))\big)}{n_k} = v_2'.$$

Hence, by Lemma 5.3,

$$\lim_{k\to+\infty} \frac{d\big(\pi_{T(\alpha),\beta}(\tilde{f}^{n_k}(y_k)), \pi_{T(\alpha),\beta}(y_k)\big)}{n_k} = v_1' + v_2' = v > 0.$$

Recall that the sequence of points $(\tilde{f}^{n_k}(y_k))_k$ converges to the point β and that the sequence of points $(y_k)_k$ converges to the point $T(\alpha)$. Therefore

$$(T(\alpha), \beta, v) \in \rho(f)$$

and Proposition 5.1 is proved. □

Proof of Lemma 5.3. As the proof of both items are similar, we will only prove the first one.

Let $p_1 = \pi_{\alpha_1,\beta_1}(p_0)$ and $p_2 = \pi_{\alpha_2,\beta_1}(p_0)$. For any point y of $\mathbb{H}^2$, we have

$$d(\pi_{\alpha_1,\beta_1}(y), p_1) \geq d(\pi_{\alpha_2,\beta_1}(y), p_2) - d(p_1, p_2) - d(\pi_{\alpha_1,\beta_1}(y), \pi_{\alpha_2,\beta_1}(y)).$$

Similarly,

$$d(\pi_{\alpha_2,\beta_1}(y), p_2) \geq d(\pi_{\alpha_1,\beta_1}(y), p_1) - d(p_1, p_2) - d(\pi_{\alpha_1,\beta_1}(y), \pi_{\alpha_2,\beta_1}(y)).$$

Hence Lemma 5.3 is a consequence of Claim 4.4. □

We now come to the general case concerning two intersecting geodesics. Our results are weaker than when the second geodesic is the image of the first one by a deck transformation (Proposition C of the introduction).

Proposition 5.4. *Let $f \in \mathrm{Homeo}_0(S)$, and $(\alpha_1, \beta_1, v_1) \in \rho(f)$, with $v_1 > 0$. Let also (α_2, β_2) be a geodesic of $\mathbb{H}^2$ that intersects (α_1, β_1), and such that there exists $(y_k) \in \mathbb{H}^2$ and $u_k \in \mathbb{N}$ such that $y_k \to \alpha_2$ and $\tilde{f}^{u_k}(y_k) \to \beta_2$. Then, there exist $v', v'' \geq 0$ satisfying $v' + v'' = v_1$ such that*

(i) either $(\alpha_1, \beta_2, v') \in \rho(f)$ or $(\alpha_1, \alpha_2, v') \in \rho(f)$;

(ii) either $(\beta_2, \beta_1, v'') \in \rho(f)$, or $(\alpha_2, \beta_1, v'') \in \rho(f)$.

Remark that this result can be applied when moreover $(\alpha_2, \beta_2, v_2) \in \rho(f)$, with $v_2 > 0$.

Proof. By definition, there exist a sequence $(x_k)_{k\in\mathbb{N}}$ of points in $\mathbb{H}^2$, a sequence $(t_k)_{k\in\mathbb{N}}$ of integers, a sequence $(y_k)_{k\in\mathbb{N}}$ of points in $\mathbb{H}^2$ and a sequence $(u_k)_{k\in\mathbb{N}}$ of integers such that

$$\begin{cases} \lim\limits_{k\to+\infty} x_k = \alpha_1 \\ \lim\limits_{k\to+\infty} \tilde{f}^{t_k}(x_k) = \beta_1 \\ \lim\limits_{k\to+\infty} \dfrac{d\big(\pi_{\alpha_1,\beta_1}(x_k),\, \pi_{\alpha_1,\beta_1}(\tilde{f}^{t_k}(x_k))\big)}{t_k} = v_1 \end{cases} \qquad \begin{array}{l} \lim\limits_{k\to+\infty} y_k = \alpha_2 \\ \lim\limits_{k\to+\infty} \tilde{f}^{u_k}(y_k) = \beta_2. \end{array}$$

For any $k \geq 0$ and $i = 1, 2$, denote by $L_{i,k}$ (respectively $R_{i,k}$) the left of the unique geodesic line passing through x_k if $i = 1$, y_k if $i = 2$ (respectively the right of the unique geodesic line passing through $\tilde{f}^{t_k}(x_k)$ if $i = 1$ and through $\tilde{f}^{u_k}(y_k)$ if $i = 2$)

which is orthogonal to (α_i, β_i)), the left/right being defined with the help of the oriented geodesics (α_i, β_i). Let $p \in \mathbb{H}^2$ be the intersection between the geodesics (α_1, β_1) and (α_2, β_2). Extracting a subsequence if necessary, we can suppose that the sequences

$$\left(\frac{d(\pi_{\alpha_1,\beta_1}(x_k),\, \pi_{\alpha_1,\beta_1}(p))}{t_k}\right)_k \quad \text{and} \quad \left(\frac{d(\pi_{\alpha_1,\beta_1}(p),\, \pi_{\alpha_1,\beta_1}(\tilde{f}^{t_k}(x_k)))}{t_k}\right)_k$$

converge with respective limits v' and v''. Observe that those limits do not depend on the chosen point p and that $v' + v'' = v_1$.

The reader can refer to Figure 11. Fix $k \in \mathbb{N}$, and set $L_b = L_{2,k}$ and $R_d = R_{2,k}$. If k is large enough, the closure of these sets in the disk $\overline{\mathbb{H}}^2$ contains neither α_1 nor β_1. Then there exists $L \in \mathbb{N}$ such that for any $\ell \geq L$, the sets $L_a = L_{1,\ell}$ and $R_c = R_{1,\ell}$ are disjoint from the set

$$X = \tilde{f}^{u_k}(L_b) \cup R_d.$$

Note that the set X separates the sets L_a and R_c (that is, these sets lie in different connected components of $X^{\complement}$). Moreover, the trace of the closure of X on $\partial\mathbb{H}^2$ is the same as the trace of the closure of $L_b \cup R_d$.

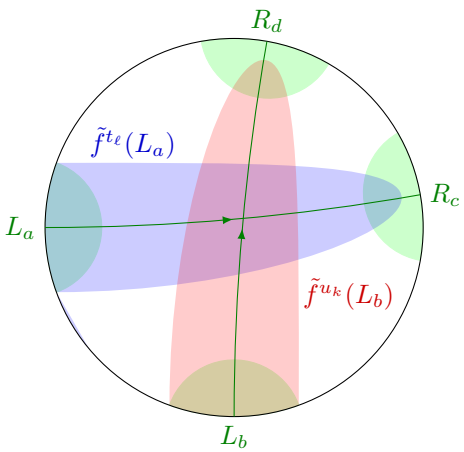

Figure 11: Proof of Proposition 5.4.

Take $\ell \geq L$ such that $t_\ell \gg u_k$. As $x_\ell \in L_a$ and $\tilde{f}^{t_\ell}(x_\ell) \in R_c$, the set $Y = \tilde{f}^{t_\ell}(L_a) \cup R_c$ is (path) connected. This implies that $X \cap \tilde{f}^{t_\ell}(L_a) \neq \emptyset$, hence one of the following intersections is nonempty:

$$\tilde{f}^{t_\ell}(L_a) \cap R_d \qquad \text{or} \qquad \tilde{f}^{t_\ell - u_k}(L_a) \cap L_b.$$

Remark that this conclusion is similar to the one of Lemma 5.2. Reasoning as in the proof of Proposition 5.1, and in particular using Lemma 5.3, we deduce that there exist a sequence of points (z_i) in $\mathbb{H}^2$, and a sequence of times n_i going to infinity such that $\lim z_i = \alpha_1$ and

$$\text{either} \quad \tilde{f}^{n_i}(z_i) \underset{i\to+\infty}{\longrightarrow} \beta_2 \qquad \text{and} \qquad \limsup_{i\to+\infty} \frac{d(\pi_{\alpha_1,\beta_2}(z_i), \pi_{\alpha_1,\beta_2}(\tilde{f}^{n_i}(z_i)))}{n_i} \geq v',$$

$$\text{or} \quad \tilde{f}^{n_i}(z_i) \underset{i\to+\infty}{\longrightarrow} \alpha_2 \qquad \text{and} \qquad \limsup_{i\to+\infty} \frac{d(\pi_{\alpha_1,\alpha_2}(z_i), \pi_{\alpha_1,\alpha_2}(\tilde{f}^{n_i}(z_i)))}{n_i} \geq v',$$

which implies, together with Theorem 3.5, that either $(\alpha_1, \beta_2, v') \in \rho(f)$, or $(\alpha_1, \alpha_2, v') \in \rho(f)$.

Considering $\tilde{f}^{-t_\ell}(R_c)$ instead of $\tilde{f}^{t_\ell}(L_a)$ gives the following similar conclusion: one of the following intersections is nonempty:

$$\tilde{f}^{-t_\ell}(R_c) \cap R_d \qquad \text{or} \qquad \tilde{f}^{-t_\ell - u_k}(R_c) \cap L_b.$$

As before, this implies that either $(\beta_2, \beta_1, v'') \in \rho(f)$, or $(\alpha_2, \beta_1, v'') \in \rho(f)$. In conclusion

$$\begin{aligned} &\text{either } (\alpha_1, \beta_2, v') \in \rho(f), \quad \text{or } (\alpha_1, \alpha_2, v') \in \rho(f), \text{ and} \\ &\text{either } (\beta_2, \beta_1, v'') \in \rho(f), \quad \text{or } (\alpha_2, \beta_1, v'') \in \rho(f). \end{aligned}$$

□

6 Almost annular homeomorphisms

In this section, we study the situation where the rotation vectors are all associated to a single geodesic of the surface. We will prove that this implies that the geodesic has no auto-intersection.

Let α and β be points of $\partial\mathbb{H}^2$ and $p : \tilde{S}_g = \mathbb{H}^2 \to S$ be a covering map. For $f \in \mathrm{Homeo}_0(S)$, we say that the rotation set of f is *contained in the geodesic* $p(\alpha, \beta)$ if

$$\rho(f) \subset \bigcup_{T \in \pi_1(S)} T(\alpha, \beta) \times \mathbb{R}_+.$$

Next proposition is Proposition D of the introduction.

Proposition 6.1. *Let $f \in \mathrm{Homeo}_0(S)$ and $\alpha, \beta \in \partial\mathbb{H}^2$. Suppose that the rotation set of f is not reduced to $\{0\}$ and is contained in $p(\alpha, \beta)$. Then, for any $T \in \pi_1(S)$, the geodesics (α, β) and $T(\alpha, \beta)$ have no common point in $\mathbb{H}^2$.*

In particular, this proposition implies that the geodesic $p(\alpha, \beta)$ has no transverse auto-intersection.

This proposition is still true (with the same proof) if we suppose that

$$\rho(f) \subset \left(\bigcup_{T \in \pi_1(S)} T(\alpha, \beta) \times \mathbb{R}_+ \right) \cup \left(\bigcup_{T \in \pi_1(S)} T(\beta, \alpha) \times \mathbb{R}_+ \right).$$

Remark 6.2. The only examples we know of homeomorphisms f such that the rotation set of f is not reduced to $\{0\}$ and is contained in $p(\alpha, \beta)$ are those for which $p(\alpha, \beta)$ is a closed geodesic. We can wonder whether there are other examples.

Proof. Take $v > 0$ such that $(\alpha, \beta, v) \in \rho(f)$. We take the same notation as in the proof of Proposition 5.1: there exist a sequence $(x_k)_{k \in \mathbb{N}}$ of points in $\mathbb{H}^2$ and a sequence $(n_k)_{k \in \mathbb{N}}$ of integers such that

$$\begin{cases} \lim\limits_{k \to +\infty} x_k = \alpha \\ \lim\limits_{k \to +\infty} \tilde{f}^{n_k}(x_k) = \beta \\ \lim\limits_{k \to +\infty} \dfrac{d\big(\pi_{\alpha,\beta}(\tilde{f}^{n_k}(x_k)),\, \pi_{\alpha,\beta}(x_k)\big)}{n_k} = v. \end{cases}$$

For any $k \geq 0$, denote by $G_{1,k}$ (respectively $G_{2,k}$) the unique geodesic line passing through x_k (respectively $\tilde{f}^{n_k}(x_k)$) which is orthogonal to (α, β). Fix a point p in $\mathbb{H}^2$. Extracting a subsequence if necessary, we can suppose that both sequences $\left(\frac{d(\pi_{\alpha,\beta}(x_k), \pi_{\alpha,\beta}(p))}{n_k}\right)_k$ and $\left(\frac{d(\pi_{\alpha,\beta}(p), \pi_{\alpha,\beta}(\tilde{f}^{n_k}(x_k)))}{n_k}\right)_k$ converge with respective limits v_1', v_2'. Observe that those limits do not depend on the chosen point p and that $v_1' + v_2' = v$.

Suppose for a contradiction that the geodesic line $p(\alpha, \beta)$ has a transverse autointersection: there exists $T \in \pi_1(S)$ such that

$$\mathrm{Card}\left(T(\alpha, \beta) \cap (\alpha, \beta)\right) = 1. \tag{6.1}$$

By Proposition 5.1, either $(\alpha, T(\beta), v) \in \rho(f)$ or $(T(\alpha), \beta, v) \in \rho(f)$.

Let us finish the proof of Proposition 6.1 in the first case $(\alpha, T(\beta), v) \in \rho(f)$. The second case is similar. We will use the following classical lemma.

Lemma 6.3. *Let η_1 and η_2 be nontrivial transformations in $\pi_1(S)$. If the respective axis A_1 and A_2 of η_1 and η_2 have a common endpoint, then $A_1 = A_2$ and there exists nonzero integers n_1 and n_2 such that*

$$\eta_1^{n_1} = \eta_2^{n_2}.$$

To prove this lemma, observe that, otherwise, for any point $p \in \mathbb{H}^2$, there would be infinitely many points of the form $\eta_1^n \eta_2^m(p)$ in a compact subset of $\mathbb{H}^2$. This is not possible as the group $\pi_1(S)$ acts properly on $\mathbb{H}^2$.

By hypothesis on $\rho(f)$, there exists a deck transformation T_1 such that $T_1(\alpha, \beta) = (\alpha, T(\beta))$. Then

1. either $T_1(\alpha) = \alpha$ and $T_1(\beta) = T(\beta)$;
2. or $T_1(\alpha) = T(\beta)$ and $T_1(\beta) = \alpha$.

Suppose first that $T_1(\alpha) = \alpha$ and $T_1(\beta) = T(\beta)$. Then α is an endpoint of the axis of T_1. Let $T_2 = T^{-1}T_1$. As the deck transformation T_2 fixes the point β, either T_2 is trivial or β is an endpoint of the axis of T_2. If T_2 was trivial, we would have $T_1 = T$ but this is not possible as $T(\alpha) \neq \alpha = T_1(\alpha)$.

Hence $T_2 \neq 1$. Now, let us prove that $p(\alpha, \beta)$ is a closed geodesic. This will lead to a contradiction: by Lemma 6.3, the axis of T_1 would be (α, β), which is not possible as, by (6.1), $T_1(\beta) = T(\beta) \neq \beta$.

Denote by (α, α_1) the axis of T_1 and by (β, β_1) the axis of T_2. Let A_1 (respectively A_2) be the closed annulus obtained by quotienting the closed band $\overline{\mathbb{H}}^2 \setminus \{\alpha, \alpha_1\}$ (respectively $\overline{\mathbb{H}}^2 \setminus \{\beta, \beta_1\}$) by the action of the group generated by T_1 (respectively T_2). For $i = 1, 2$, denote by $\overline{f}_i$ the homeomorphism induced by $\tilde{f}$ on A_i. Denote by $\rho(\overline{f}_i)$ the rotation set of $\overline{f}_i$ on A_i.

Recall that both sequences $\left(\frac{d(\pi_{\alpha,\beta}(x_k), \pi_{\alpha,\beta}(p))}{n_k}\right)_k$ and $\left(\frac{d(\pi_{\alpha,\beta}(p), \pi_{\alpha,\beta}(\tilde{f}^{n_k}(x_k)))}{n_k}\right)_k$ converge with respective limits v_1', v_2' with $v_1' + v_2' = v > 0$. Therefore, by Lemma 4.2 and Lemma 5.3, $v_1' \in \rho(\overline{f}_1)$ and $v_2' \in \rho(\overline{f}_2)$. Hence, one of those two rotation sets contains a nonzero rotation number. But this implies that there exists $v' > 0$ such that either $(\alpha, \alpha_1, v') \in \rho(f)$ or $(\beta_1, \beta, v') \in \rho(f)$. As we supposed that $\rho(f)$ was contained in a geodesic line, this means that either (α, α_1) or (β_1, β) is the image of (α, β) under a deck transformation. Hence, as $p(\alpha, \alpha_1)$ and $p(\beta_1, \beta)$ are closed geodesics, so is $p(\alpha, \beta)$, what we wanted to prove.

Now, let us suppose that $T_1(\alpha) = T(\beta)$ and $T_1(\beta) = \alpha$. Then $T^{-1}T_1^2(\beta) = \beta$ and $T_1T^{-1}T_1(\alpha) = \alpha$. If one of the deck transformations $T^{-1}T_1^2$ or $T_1T^{-1}T_1$ is trivial, then $T = T_1^2$. Then there exists an orientation of the circle $\partial\mathbb{H}^2$ such that, when one follows this orientation of the circle, one successively meets the points β, $T_1^2(\beta) = T(\beta)$, $T_1(\beta) = \alpha$ and $T_1^3(\beta) = T(\alpha)$. This is not possible as the dynamics of T_1 on $\partial\mathbb{H}^2$ is a north-south dynamic (see Figure 12, left).

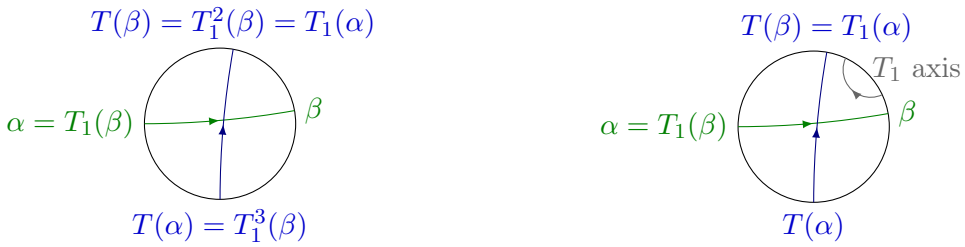

Figure 12: End of proof of Proposition 6.1.

So both deck transformations $T^{-1}T_1^2$ and $T_1T^{-1}T_1$ are nontrivial. Remark that the point β (respectively α) is an endpoint of the axis of $T^{-1}T_1^2$ (respectively $T_1T^{-1}T_1$). As in the above case, this implies that the geodesic line $p(\alpha, \beta)$ is closed and that (α, β) is the axis of both deck transformations $T^{-1}T_1^2$ and $T_1T^{-1}T_1$. Then $T^{-1}T_1^2(\alpha) = \alpha$. However, we will see that this relation is not possible. This will complete the proof of Proposition 6.1.

Indeed, orient the circle $\partial\mathbb{H}^2$ in such a way that, following this orientation, we successively meet the points α, $T(\alpha)$, β, $T(\beta)$ (see Figure 12, right). As $T_1(\beta) = \alpha$ and $T_1(\alpha) = T(\beta)$, both endpoints of the axis of T_1 have to belong to the oriented open interval $(\beta, T(\beta))$ of $\partial\mathbb{H}^2$ and the point $T_1^2(\alpha) = T_1(T(\beta))$ belongs to the open interval $(\beta, T(\beta)) \subset (T(\alpha), T(\beta)) \subset \partial\mathbb{H}^2$. Hence the point $T^{-1}(T_1^2(\alpha))$ belongs to the open interval (β, α) and, in particular, $T^{-1}(T_1^2(\alpha)) \neq \alpha$. □

7 Examples

See also Section 10 of [ABP20] for other interesting examples, based on a technique due to Kwapisz [Kwa92].

7.1 A uniquely ergodic diffeomorphism with uncountable rotational directions

Let us give an example of a homeomorphism of the punctured torus with an ergodic probability measure μ for which an uncountable set of geodesics is necessary to describe the rotation set of μ almost every point.

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $X \equiv (1, \alpha)$ be the constant vector field on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\kappa : \mathbb{T}^2 \to \mathbb{R}_+$ be a continuous nonnegative function such that $\kappa(x) = 0 \iff x = 0$, and that $\kappa(x) \sim \|x\|^a$ for some $0 < a < 2$ ($\|\cdot\|$ is the Euclidean norm). Let (ϕ_t) be the flow associated to the vector field κX (Cauchy-Lipschitz theorem applies on $\mathbb{T}^2 \setminus \{0\}$ as κX is locally Lipschitz on it; we set $\phi^t(0) = 0$ for any t). It is of class C^a, and its flow curves are straight lines with slope α (apart from two half-lines). By Section 2 of [SSV10], this flow has two ergodic probability invariant measures: δ_0 and an absolutely

continuous one μ with total support. The rotation number of this measure μ is nonzero, hence the rotation set (as a homeomorphism of $\mathbb{T}^2$) of this flow is a nontrivial segment. As 0 is a fixed point for it, it induces a flow on the noncompact manifold $\mathbb{T}^2 \setminus \{0\}$; we denote by f the time 1 of this flow. Then, seen as a homeomorphism of $\mathbb{T}^2 \setminus \{0\}$, f is uniquely ergodic.

By Lemma 4.5 (from [Les11]), to μ-almost every point x and every lift $\tilde{x}$ of x to the universal cover of $\mathbb{T}^2 \setminus \{0\}$, there exists a geodesic line $(\alpha_{\tilde{x}}, \beta_{\tilde{x}})$ such that $d(\tilde{f}^n(\tilde{x}), (\alpha_{\tilde{x}}, \beta_{\tilde{x}})) = o(n)$.

Let $(x_i)_{i \in I}$ an uncountable set of μ-typical points, such that any two of them are on a different flow line. Using Svarč-Milnor lemma (Lemma 8.9), one can easily see that for $i \neq j$, the orbits of x_i and x_j move away one from the other at least at linear speed. This implies in particular that that $(\alpha_{\tilde{x}_i}, \beta_{\tilde{x}_i}) \neq (\alpha_{\tilde{x}_j}, \beta_{\tilde{x}_k})$. Indeed, any two lifts of the flow curves to $\mathbb{R}^2$ are separated by a lift of the point $0 \in \mathbb{T}^2$.

If $1 \leq a < 2$, it is possible to blow up the fixed point of the homeomorphism to a circle and gluing two such examples along the two obtained circles; it gives a homeomorphism of the compact surface of genus 2. The ergodic invariant measures of this homeomorphism are either one of the two absolutely continuous invariant probability measures, each one supported in a domain homeomorphic to a punctured torus, or supported in the fixed point set which is a circle. Then there is no positive Lebesgue-measure set A such that any two points x and y of A have the same rotational direction (i.e. the geodesic defined by the flow line passing through x or y).

7.2 A diffeomorphism with trivial rotation set but unbounded displacements in all directions

This example is the higher genus counterpart of the torus example of Koropecki and Tal [KT14]. In this paper, the authors build an open topological disk embedded in $\mathbb{T}^2$ whose lift meets each fundamental domain of $\mathbb{T}^2$ in $\mathbb{R}^2$, and whose complement has zero measure. They then define a smooth Lebesgue measure preserving Bernoulli[7] diffeomorphism of the torus which is equal to the identity outside this disk, using a method due to Katok [Kat79]: there exists a smooth Bernoulli diffeomorphism of the unit disk, equal to identity on the boundary of the disk, which gives a smooth Bernoulli diffeomorphism of the torus via the embedding of the disk.

The torus example of Koropecki and Tal [KT14] can be generalized to any positive Euler characteristic connected surface, with an (almost) identical proof. The construction of the example from the existence of the embedded disk follows from Katok [Kat79] and is not specific to the torus. For the construction of the embedded disk, the only part of the proof that is specific to the torus is Lemma 4, that can be replaced by the following.

Lemma 7.1. *If $\widetilde{x}$ and $\widetilde{y}$ are two points of $\widetilde{S} \simeq \mathbb{H}^2$ that do not project on the same point of S, then there is an arc α joining $\widetilde{x}$ to $\widetilde{y}$ whose projection to S is injective.*

Proof. Fix a point $\widetilde{x} \in \widetilde{S}$, and consider the set

$$A_{\widetilde{x}} = \left\{\widetilde{y} \in \widetilde{S} \mid \exists \widetilde{\alpha} \text{ arc joining } \widetilde{x} \text{ to } \widetilde{y} \text{ s.t. } \alpha \text{ is injective}\right\}.$$

Equip S with a metric that lifts to the canonical metric on $\mathbb{H}^2$. For $\epsilon > 0$, let

$$E_\epsilon = \left\{\widetilde{y} \in \widetilde{S} \mid \text{the projection of } B(\widetilde{y}, \epsilon) \text{ to } S \text{ is injective}\right\}.$$

[7] And hence ergodic.

Let $\widetilde{y} \in A_{\widetilde{x}} \cap E_\epsilon$. Then there exists a path $\widetilde{\alpha}$ joining $\widetilde{x}$ to $\widetilde{y}$ whose projection to S is injective. Remark that it is possible to make this path smooth if necessary. Then, consider a diffeomorphism h of S, equal to identity out of $B(y,\epsilon)$, such that any point in $B(y,\epsilon) \setminus \{y\}$ has its ω-limit included in $\partial B(y,\epsilon)$. Then, for n large enough, $h^n(\alpha) \cap B(y,\epsilon/2)$ is a connected path. In particular, for any point $\widetilde{z} \in B(\widetilde{y},\epsilon/2) \setminus \widetilde{h}^n(\widetilde{\alpha})$, it is easy to build a path joining $\widetilde{y}$ to $\widetilde{z}$ and included in $B(\widetilde{y},\epsilon/2) \setminus \widetilde{h}^n(\widetilde{\alpha})$. This shows that $B(\widetilde{y},\epsilon/2) \subset A_{\widetilde{x}}$.

The uniformity of ϵ in this property shows that the connected component of $\widetilde{x}$ in E_ϵ is included in $A_{\widetilde{x}}$. But for any $\widetilde{x}, \widetilde{y} \in \widetilde{S}$, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, $\widetilde{x}$ and $\widetilde{y}$ lie in the same (path) connected component of E_ϵ. Hence, $A_{\widetilde{x}} = \mathbb{H}^2$. □

Finally, we get the following result.

Proposition 7.2. *For any surface S with negative Euler characteristic with finite measure, for any fixed compact connected fundamental domain $D \subset \mathbb{H}^2$ of S, there is a C^∞ area-preserving diffeomorphism $f : S \to S$ homotopic to the identity, with a lift $\widetilde{f} : \mathbb{H}^2 \to \mathbb{H}^2$ such that*

- *$\rho(f)$ is reduced to a single rotation vector (with zero speed):*
- *f is metrically isomorphic to a Bernoulli shift (in particular, f is ergodic) with Lebesgue measure;*
- *For Lebesgue almost every point $\widetilde{x} \in \mathbb{H}^2$, the forward and backward orbits of $\widetilde{x}$ accumulates in every direction at infinity, i.e.*

$$\partial_\infty\{\widetilde{f}^n(\widetilde{x}) \mid n \in \mathbb{N}\} = \mathbb{S}^1 = \partial_\infty\{\widetilde{f}^{-n}(\widetilde{x}) \mid n \in \mathbb{N}\}.$$

  Moreover, the forward and backward orbits of $\widetilde{x}$ visit every fundamental domain TD, with $T \in \pi_1(S)$.

This example can be modified in a simple way to get other rotational behaviours. For example, consider a simple essential closed annulus $A \subset S$, such that $S \setminus A$ is connected. Then, one can apply Proposition 7.2 to $S \setminus A$ to get a homeomorphism h of $S \setminus A$ which is equal to identity on ∂A, and extend h to A such that $h|_A$ has a nontrivial rotation set. This gives an example of an almost annular (in the sense of Section 6) that has unbounded displacements in all directions not intersecting the direction of A.

7.3 An example of homeomorphism with non closed rotation set

Fix a closed surface S with negative Euler characteristic which is endowed with a hyperbolic metric. Take two disjoint closed simple geodesics α and β of S and a simple geodesic γ whose α-limit is α and whose ω-limit is β and which is disjoint from both α and β. Let A_β be a tubular neighbourhood of β which is disjoint from α and such that $A_\beta \cap \gamma$ is connected. Let B_γ be a tubular neighbourhood of γ which is disjoint from both α and β such that $B_\gamma \cap A_\beta$ is connected. We denote by $\tilde{\alpha} = (\alpha_1, \alpha_2)$, $\tilde{\beta} = (\beta_1, \beta_2)$ and $\tilde{\gamma}$ the respective lifts of α, β and γ to the universal cover $\tilde{S} \equiv \mathbb{H}^2$ of S such that $\tilde{\gamma}$ joins the points α_1 and β_2 of $\partial\mathbb{H}^2$.

Let f_1 be a homeomorphism in $\mathrm{Homeo}_0(S)$, which is supported on A_β, with the following properties.

1. The canonical lift $\tilde{f}_1$ of f_1 preserves $\tilde{\beta}$ and acts as a translation of $\theta \in \mathbb{R}$ on $\tilde{\beta}$.

2. There exists an open neighbourhood U_β of β such that

$$\begin{cases} \overline{f_1(U_\beta)} \subset U_\beta \\ \bigcap_{n\geq 0} f_1^n(U_\beta) = \beta \\ \bigcup n \in \mathbb{Z} f_1^n(U_\beta) = A_\beta \end{cases}$$

3. For any $n \in \mathbb{Z}$, the set $f_1^n(U_\beta) \cap B_\gamma$ is connected.

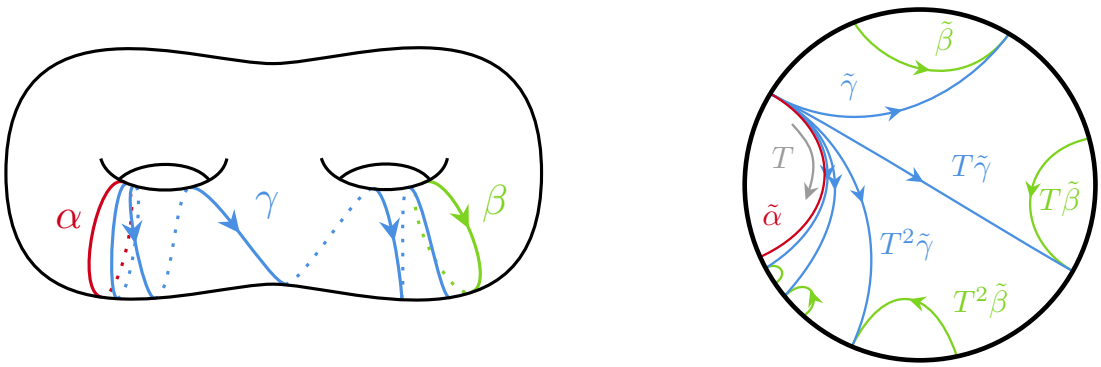

Figure 13: The right drawing is the universal cover of the left one. On the left, the red closed curve α (left) is made of fixed points, and the green closed curve β (right) rotates. The blue geodesic γ is heteroclinic to α and β. On the right, T is the deck transformation associated to a lift $\tilde{\alpha}$ of α.

Let f_2 be a homeomorphism which is supported in $\overline{B_\gamma}$ with the following properties.

1. For any point $x \in B_\gamma$, the sequence $(f_2^n(x))_{n\geq 0}$ accumulates to β and the sequence $(f_2^{-n}(x))_{n\geq 0}$ accumulates to α.

2. For any $n \in \mathbb{Z}$, $f_2(f_1^n(U_\beta)) \subset f_1^n(U_\beta)$.

Observe that the homeomorphism f_2 pointwise fixes α and β.

Finally, let $f_3 = f_2 \circ f_1$. The dynamics of f_3 is described on Figure 13. Observe that the recurrent orbits of f_3 consist of its fixed points and the points of β. Observe also that $(\beta_1, \beta_2, \theta) \in \rho(f_3)$ and, because of the orbits on γ with asymptotic speed θ, $(\alpha_1, \beta_2, \theta) \in \rho(f_3)$. Let T be the deck transformation associated to $\tilde{\alpha}$. As the set $\rho(f_3)$ is invariant under deck transformations, we obtain that, for any $n \geq 0$,

$$(T^n\alpha_1, T^n\beta_2, \theta) = (\alpha_1, T^n\beta_2, \theta) \in \rho(f_3).$$

Observe also that

$$\lim_{n\to+\infty} T^n\beta_2 = \alpha_2.$$

However, $(\alpha_1, \alpha_2, \theta) \notin \rho(f_3)$. Indeed, otherwise, by Proposition 4.3, there would exist a recurrent orbit of $\tilde{f}_3$ with a nontrivial rotation vector which stays at a bounded distance from the geodesic $\tilde{\alpha}$. But there exists no such orbit as the only nontrivial recurrent orbits of f_3 are contained in β. Hence the rotation set of f_3 is not closed.

8 Intersections of closed geodesics: consequences on the homological rotation set

In this section, we get the first consequences of the fact that two closed geodesics that cross and are associated to a nontrivial rotation vector. In this case, we get the existence of a toral covering in which the induced rotation set has nonempty interior (Proposition 8.10). This implies positive topological entropy and the existence of an infinite number of periodic orbits (Corollary 8.11). This is somehow a first step towards the results of Section 10, in which we will get stronger conclusions under weaker hypotheses. It will be the occasion to introduce and study the notion of associated covering map (Definition 8.3) that will be used in the three last sections.

8.1 Background on covering maps

In the sequel, S will be chosen as either the closed surface S of genus $g \geq 2$, or the domain of an isotopy $\operatorname{dom} I \subset S$ of f (see Subsection 9.1). The surface S is endowed with a complete hyperbolic metric so that its universal cover $\tilde{S}$ is identified with the hyperbolic plane $\mathbb{H}^2$.

Let $x_0 \in S$, and α_1 and α_2 two loops of S. Denote by $\tilde{x}_0$, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ lifts of respectively x_0, α_1 and α_2 to the universal cover $\tilde{S}$ of S. Suppose that $\tilde{x}_0 \in \tilde{\alpha}_1 \cap \tilde{\alpha}_2$; we take $\tilde{x}_0$ as a basepoint for both those loops.

Note that each of the paths $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ stay at a finite distance to a closed geodesic of $\tilde{S}$ (and this geodesic is determined by the deck transformation associated to the loops α_1 and α_2).

Definition 8.1. We say that α_1 and α_2 have *a geometric transverse intersection at* x_0 if some geodesics associated to their lifts to $\tilde{S}$ intersect in $\mathbb{H}^2$.

We say that a loop α of S has a *geometric transverse autointersection at* x_0 if there exists a deck transformation T of $\tilde{S} \to S$ such that $\tilde{\alpha}$ and $T\tilde{\alpha}$ intersect transversally.

Note that by definition, a transverse intersection stays transverse in any covering space.

Lemma 8.2. *Let $F \subset S$ be a closed set, and α a loop of $S \setminus F$ which has a transverse geometric transverse auto-intersection at x_0 for a deck transformation T of $\tilde{S} \to S$. Then, there exists a deck transformation T_1 of $\widetilde{S \setminus F} \to S \setminus F$, projecting to T in $\tilde{S} \to S$, such that α has a geometric transverse auto-intersection at x_0 for T_1.*

Proof. Let γ be a geodesic of $S \setminus F$ (for a hyperbolic metric on $S \setminus F$), and $\check{\gamma}$ a lift of γ to $\tilde{S}$. By hypothesis, the lift to $\tilde{S}$ of the geodesic of S associated to γ intersects its translate by T. Hence, $\check{\gamma}$ and $T\check{\gamma}$ intersect in $\tilde{S}$. Hence, there is a deck transformation T_1 of $\widetilde{S \setminus F} \to S \setminus F$, projecting to T, such that if $\tilde{\gamma}$ is a lift of γ to $\widetilde{S \setminus F}$, then $\tilde{\gamma}$ and $T_1\tilde{\gamma}$ intersect. □

Recall that there is a bijective correspondence between subgroups of the fundamental group $\pi_1(S, x_0)$ of S at x_0 and the covering maps of S: to any subgroup G of $\pi_1(S, x_0)$ can be associated the covering map $\hat{S} = \tilde{S}/G \to S$, where G is seen as a subgroup of the group of deck transformations of $\pi : \tilde{S} \to S$. Moreover, $G = \pi_1(\hat{S}, \hat{x_0})$ for some lift $\hat{x}_0$ of x_0..

Denote by $[\alpha_1]_{x_0}$ and $[\alpha_2]_{x_0}$ the respective classes of the loops α_1 and α_2 in $\pi_1(S, x_0)$. Recall that any nontrivial class of $\pi_1(S)$ contains a unique closed geodesic.

Definition 8.3. We call *covering map associated to* $(\alpha_1, \alpha_2, x_0)$ a covering map $\hat{S} \to S$ associated to the subgroup $\langle [\alpha_1]_{x_0}, [\alpha_2]_{x_0} \rangle$ of $\pi_1(S, x_0)$ generated by $[\alpha_1]_{x_0}$ and $[\alpha_2]_{x_0}$. The loops $\hat{\alpha}_1$ and $\hat{\alpha}_2$ which represent respectively $[\alpha_1]_{x_0}$ and $[\alpha_2]_{x_0}$ in $G = \pi_1(\hat{S}, \hat{x_0})$ and respectively lift the loops α_1 and α_2 are called *canonical lifts* of α_1 and α_2 in $\hat{S}$.

A covering map associated to $(\alpha_1, \alpha_2, x_0)$ depends on the choice of the lift $\hat{x_0}$ of the point x_0. However, two such covering maps are isomorphic one to each other.

A closed geodesic $\gamma : [0, 1] \to S$ of S is called *primitive* if no strict restriction of γ defines a closed geodesic of S[8]. Observe that it amounts to saying that the element of $\pi_1(S)$ induced by γ is not of the form a^n, with $n \geq 2$ and $a \in \pi_1(S)$. A loop of a surface is called *essential* if its free homotopy class is non-trivial and if it is not freely homotopic to a small loop around a puncture.

Proposition 8.4. *Suppose γ_1 and γ_2 are two primitive closed geodesics which meet transversely at the point x_0. Denote by $\hat{S} \to S$ a covering map associated to $(\gamma_1, \gamma_2, x_0)$. Then*

1. *The surface $\hat{S}$ is homeomorphic to either the torus with one puncture or the sphere with three punctures.*
2. *If one of the loops γ_1 or γ_2 is simple, then the surface $\hat{S}$ is homeomorphic to the torus with one puncture.*
3. *If the surface $\hat{S}$ is homeomorphic to the torus with one puncture, then the canonical lifts of γ_1 and γ_2 in $\hat{S}$ are essential loops of $\hat{S}$ generating the homology of $\hat{S}$.*
4. *If $\hat{S}$ is homeomorphic to the sphere with three punctures, then none of the lifts of the loops γ_1 or γ_2 is simple.*

Figure 14 displays an example showing that case 4. of this proposition is nonempty: there exists two based loops of the genus 2 closed surface whose associated covering map is the sphere with three punctures. Indeed, Proposition 8.5 shows that the covering map associated to the two loops formed by the red loop with auto-intersection is the sphere with three punctures. Hence, if we call a and b some generators of the fundamental group of the sphere with three punctures such that the lift of the red curve is homotopic to ab, then the lift of the blue curve to the sphere with three punctures is homotopic to ab^2. In this case, the lifts of the curves generate the fundamental group of the sphere with 3 punctures. This proves that the associated covering map is the sphere with three punctures.

Proposition 8.5. *Suppose γ_1 and γ_2 are loops based at a point x_0 and that the concatenation of γ_1 and γ_2 is a closed primitive geodesic of S with a geometric transverse autointersection at the point x_0. Denote by $\hat{S} \to S$ the covering map associated to $(\gamma_1, \gamma_2, x_0)$. Then*

1. *The surface $\hat{S}$ is homeomorphic to the sphere with three punctures.*
2. *Suppose that the lifts $\hat{\gamma}_1$ and $\hat{\gamma}_2$ of γ_1 and γ_2 to $\hat{S}$ relatively to $\hat{x}_0$ are homotopic to simple curves. Denote a, b, c the canonical classes of the curves, each one winding once around one of the three punctures A, B and C of $\hat{S}$, and not winding around the others, and such that $ab = c$. Then there exists a homeomorphism of $\hat{S}$ sending $([\hat{\gamma}_1]_{\hat{x}_0}, [\hat{\gamma}_2]_{\hat{x}_0})$ to (a, b^{-1}) (see Figure 15, left).*

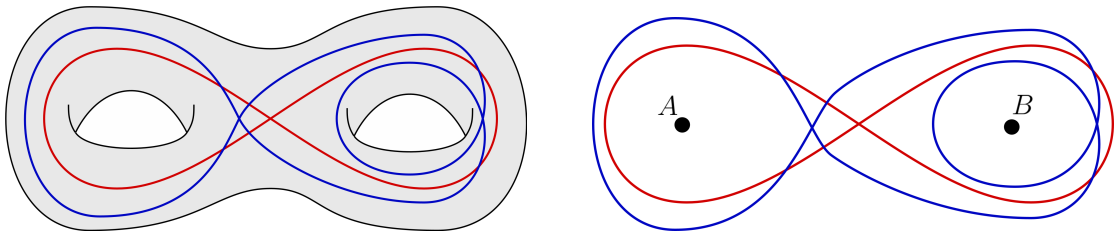

Figure 14: Left: for some generators a, b, c, d of $\pi_1(S)$, where S is the genus 2 closed surface, the left red curve is homotopic to ab and the left blue curve is homotopic to ab^2. On the right, the associated covering map is the sphere with three punctures, and for some generators a, b of the fundamental group of the three punctured sphere, the red loop is homotopic to ab and the blue one to ab^2.

Figure 15: Configurations of Proposition 8.5. The right one is impossible.

Proof of Proposition 8.4. 1. Denote by φ the map $\pi_1(\hat{S}, \hat{x_0}) \to H_1(\hat{S}, \mathbb{R})$ which is the composition of the Hurewicz map $\pi_1(\hat{S}, \hat{x_0}) \to H_1(\hat{S}, \mathbb{Z}) \equiv \pi_1(\hat{S}, \hat{x_0})/[\pi_1(\hat{S}, \hat{x_0}), \pi_1(\hat{S}, \hat{x_0})]$ with the map $H_1(\hat{S}, \mathbb{Z}) \to H_1(\hat{S}, \mathbb{R}) \equiv H_1(\hat{S}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. As the elements $[\gamma_1]_{x_0}$ and $[\gamma_2]_{x_0}$ generate the group $\pi_1(\hat{S}, \hat{x_0})$, the vectors $\varphi([\gamma_1]_{x_0})$ and $\varphi([\gamma_2]_{x_0})$ generate the vector space $H_1(\hat{S}, \mathbb{R})$ and $\dim(H_1(\hat{S}, \mathbb{R})) \leq 2$. Hence the surface $\hat{S}$ is either the sphere with 0, 1, 2 or 3 punctures or the 2-torus with 0 or 1 puncture, as $\hat{S}$ is orientable.

However, a closed geodesic of S has a non trivial free homotopy class: otherwise, there would exist closed geodesics in $\mathbb{H}^2$. Hence $\pi_1(\hat{S}, \hat{x_0})$ is not trivial and $\hat{S}$ is not homeomorphic to the sphere nor to the sphere with one puncture.

Moreover, the classes $[\gamma_1]_{x_0}$ and $[\gamma_2]_{x_0}$ are distinct and $[\gamma_1]_{x_0} \neq [\gamma_2]_{x_0}^{-1}$ as the geodesics γ_1 and γ_2 are transverse and there is a unique geodesic representative in a free homotopy class of a loop. More generally, as γ_1 and γ_2 are primitive closed geodesics, it is not possible that there exist nontrivial integers n_1 and n_2 such that $[\gamma_1]_{x_0}^{n_1} = [\gamma_2]_{x_0}^{n_2}$: otherwise, a geodesic representative of this class is obtained by turning $\mathrm{lcm}(n_1, n_2)$ times around a closed geodesic and γ_1 as well as γ_2 would not be primitive. Hence $\pi_1(\hat{S}, \hat{x_0})$ is not isomorphic to $\mathbb{Z}$ either and $\hat{S}$ is not homeomorphic to the sphere with two punctures.

Finally, as the surface S is endowed with a hyperbolic metric, the group $\pi_1(S, x_0)$ does not contain $\mathbb{Z}^2$. The surface $\hat{S}$ cannot be homeomorphic to the 2-torus. Hence the surface $\hat{S}$ is either the sphere with three punctures or the torus with one puncture.

2. We will prove point 4., point 2. being a consequence of it. For $i = 1, 2$, denote by $\tilde{\gamma}_i$ the lift of $\hat{\gamma}_i$ to $\tilde{S}$ which contains the point $\tilde{x_0}$, *i.e.* the connected component of $\pi^{-1}(\gamma_i)$ which contains the point $\tilde{x_0}$. In what follows, we will use a fixed covering map

[8]Note that it does not force the geodesic to be simple.

$\tilde{S} \to \hat{S}$.

Suppose for a contradiction that the loop $\hat{\gamma}_1$ is simple and that the surface $\hat{S}$ is homeomorphic to the sphere with three punctures. Then the loop $\hat{\gamma}_1$ is freely homotopic to a small loop $\hat{\gamma}_1'$ around a puncture which is disjoint from $\hat{\gamma}_2$. However, it is impossible. Indeed, the geodesic lines $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ meet transversely at the point p and hence the lift of $\hat{\gamma}_1'$ to $\tilde{S}$, which is as a finite distance to $\tilde{\gamma}_1$, has to meet the geodesic line $\tilde{\gamma}_2$.

3. Suppose the surface $\hat{S}$ is homeomorphic to the torus with one puncture. If one of the loops $\hat{\gamma}_1$ or $\hat{\gamma}_2$ is not an essential loop of $\hat{S}$, then it represents the trivial class in $H_1(\hat{S}, \mathbb{R})$. But this is not possible as $\dim(H_1(\hat{S}, \mathbb{R})) = 2$ and the classes of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ in $H_1(\hat{S}, \mathbb{R})$ generate the vector space $H_1(\hat{S}, \mathbb{R})$. □

We now come to the proof of Proposition 8.5. As a first step, we get a weak version of it, similar to the conclusion of Proposition 8.4.

Lemma 8.6. *Under the hypotheses of Proposition 8.5, the surface $\hat{S}$ is homeomorphic to either the sphere with three punctures, or the torus with one puncture.*

Proof of Lemma 8.6. Quite similarly to the proof of Proposition 8.4, one can get that the surface $\hat{S}$ is homeomorphic to either the sphere with three punctures, or the torus with one puncture. As in Proposition 8.4, we can prove that $\dim(H_1(\hat{S}, \mathbb{R})) \leq 2$.

Denote by γ the primitive closed geodesic which is the concatenation of the loops γ_1 and γ_2 based at p. As the loop γ has minimal length in its free homotopy class, it is not possible that either $[\gamma_1]_{x_0}$ or $[\gamma_2]_{x_0}$ is trivial. Moreover, if there existed integers n_1 and n_2 such that $[\gamma_1]_{x_0}^{n_1} = [\gamma_2]_{x_0}^{n_2}$, then the free homotopy class of $[\gamma_1]_{x_0}^{n_1}$ is represented geodesically by turning $n = \mathrm{lcm}(n_1, n_2)$ times around a closed geodesic γ'. Then the free homotopy class of γ would be represented by turning $n/n_1 + n/n_2$ times around γ'. Hence γ would not be primitive, a contradiction.

The rest of the proof goes through as in the proof of Proposition 8.4. □

Hence, to prove the first point of Proposition 8.5, it remains to prove that $\hat{S}$ cannot be homeomorphic to the torus with one puncture. We will need the following classical result of [Nie24].

Lemma 8.7 (Nielsen)**.** *Cardinal two generating families of the free group $\langle a, b\rangle$ on two generators are obtained from the canonical one (a, b) by so-called Nielsen transformations: permutations of elements of the basis, inversion of one of them, and multiplication of one of them by the other one (on the left or on the right).*

Proof of Proposition 8.5. 1. Suppose by contradiction that $\hat{S}$ is homeomorphic to the torus with one puncture. In this case, the classes of the loops $\hat{\gamma}_1$ and $\hat{\gamma}_2$ span the fundamental group $\pi_1(\hat{S}, \hat{x}_0)$.

Observe that any Nielsen transformation corresponds to a homeomorphism of the torus with one puncture. Hence, by Lemma 8.7, there exists a homeomorphism of $\hat{S}$ sending the pair of classes $([\hat{\gamma}_1]_{\hat{x}_0}, [\hat{\gamma}_2]_{\hat{x}_0})$ to the canonical generators of $\pi_1(\hat{S}, \hat{x}_0)$. So from now we suppose that these classes are the canonical ones. An example of configuration of the loops $\hat{\gamma}_1$ and $\hat{\gamma}_2$ is depicted in Figure 16. The curves $\hat{\alpha}_1$ and $\hat{\alpha}_2$ can be homotoped to curves having only one intersection. In particular, the curve $\hat{\gamma}$ can be homotoped to a one with no self intersection (see Figure 17) has no transverse intersection in $\hat{S}$. This implies that γ has no transverse intersection in S for one of the deck transformations T_1 or T_2 associated respectively to the loops γ_1 and γ_2, contradiction.

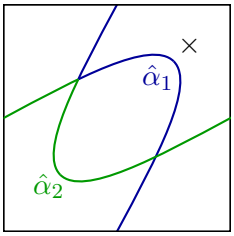

Figure 16: Example of trajectory $\hat{\alpha}$ in $\hat{S} \simeq \mathbb{T}^2 \setminus \{0\}$, obtained as the concatenation of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ (left).

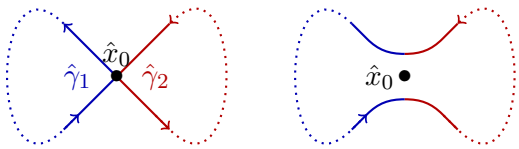

Figure 17: Modification of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ close to $\hat{x}_0$ in proof of Proposition 8.5.

2. It is a classical fact that the only homotopy classes of simple loops in the three punctured sphere are the ones winding once around one puncture and not around the others (and the trivial one), in other words there are 7 of them: $0, a, a^{-1}, b, b^{-1}, c, c^{-1}$. As $([\hat{\gamma}_1]_{\hat{x}_0}, [\hat{\gamma}_2]_{\hat{x}_0})$ generates $\pi_1(\hat{S}, \hat{x}_0)$, there are two distinct classes of such couples of classes up to homeomorphism, represented by (a, b) and (a, b^{-1}). But only one of them corresponds to a curve $\hat{\gamma}$ with a transverse self-intersection (see Figure 15), which concludes the proof. $\square$

8.2 Preliminaries on homological rotation sets

As the main result of this section (Proposition 8.10) is stated in terms of homological rotation sets, we state here some facts about these sets, which were defined in the introduction (Definition 1.1).

Suppose $S = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then the homology classes of the loops $t \mapsto (t, 0)$ and $t \mapsto (0, t)$ form a basis of $H_1(\mathbb{T}^2)$ and induce an identification $H_1(\mathbb{T}^2) \equiv \mathbb{R}^2$. Via this identification, the set $\rho_{H_1}(f)$ is identified with the rotation set $\rho(f)$ as defined by Misiurewicz and Ziemian [MZ89].

There is no obvious relationship between homotopical and homological rotation sets. However, we have the following proposition.

Proposition 8.8. *Suppose that S is closed, and let (α, β) be a geodesic line which projects to a closed geodesic γ of S. Suppose (α, β, v), with $v > 0$, is an extremal point of $\rho(\tilde{f})$. Then the homological vector $\frac{v}{\ell(\gamma)}[\gamma]_{H_1}$ belongs to the homological rotation set $\rho_{H_1}(f)$. Moreover, there exists a point $x \in S$ such that the orbit of x realises $\frac{v}{\ell(\gamma)}[\gamma]_{H_1}$ in $\rho_{H_1}(f)$ and (γ, v) in $\rho(f)$.*

Proof. Fix a generating set $\mathcal{G}$ of the group $\pi_1(S)$, which we see as the group of deck transformations of the covering map $\tilde{S} = \mathbb{H}^2 \to S$. We denote by $l_{\mathcal{G}}$ the wordlength with respect to this generating set $\mathcal{G}$. The proof of Proposition 8.8 is a consequence of Proposition 4.3 and of the well-known Svarč-Milnor lemma.

Lemma 8.9 (Svarč-Milnor)**.** *Fix $\tilde{x} \in \mathbb{H}^2 = \tilde{S}$. Then there exists a constant $C > 1$ such that, for any $T \in \pi_1(S)$,*

$$\frac{1}{C} d(\tilde{x}, T\tilde{x}) \leq l_{\mathcal{G}}(T) \leq C d(\tilde{x}, T\tilde{x}).$$

Take a point $\tilde{x}$ of $\mathbb{H}^2$ given by Proposition 4.3 and let $x = \pi(\tilde{x})$. Recall we called $l_{n,x}$ the loop which is the concatenation of the path $(f_t(x))_{t \in [0,n]}$ with a geodesic path $g_{f^n(x),x}$

between the points $f^n(x)$ and x of length lower than or equal to $D = \mathrm{diam}(S)$. Denote by $T_n \in \pi_1(S)$ the deck transformation corresponding to the loop $l_{n,x}$ with basepoint $\tilde{x}$. By definition of $l_{n,x}$, for any $n \geq 1$,

$$d\big(T_n(\tilde{x}), \tilde{f}^n(\tilde{x}))\big) \leq D.$$

For any $n \geq 1$, let $k_n = \lfloor \frac{nv}{\ell(\gamma)} \rfloor$. Denote by T the deck transformation corresponding to (α, β). Then, for any $n \geq 1$,

$$\begin{aligned} d\big(T^{k_n}(\tilde{x}), T_n(\tilde{x})\big) \quad \leq \quad & d\big(T^{k_n}(\tilde{x}), T^{k_n}(\pi_{\alpha,\beta}(\tilde{x}))\big) + d\big(T^{k_n}(\pi_{\alpha,\beta}(\tilde{x})), \pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x}))\big) \\ & + d\big(\pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x})), \tilde{f}^n(\tilde{x})\big) + d\big(\tilde{f}^n(\tilde{x}), T_n(\tilde{x})\big). \end{aligned}$$

However, for any $n \geq 1$,

$$\left\{ \begin{array}{rcl} d\big(\tilde{f}^n(\tilde{x}), T_n(\tilde{x})\big) & \leq & D \\ d\big(T^{k_n}(\tilde{x}), T^{k_n}(\pi_{\alpha,\beta}(\tilde{x}))\big) & = & d\big(\tilde{x}, \pi_{\alpha,\beta}(\tilde{x})\big) \end{array} \right.$$

and, by Proposition 4.3,

$$\lim_{n \to +\infty} \frac{1}{n} d(\pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x})), \tilde{f}^n(\tilde{x})) = 0$$

and

$$\begin{aligned} & \lim_{n \to +\infty} \frac{1}{n} d\big(T^{k_n}(\pi_{\alpha,\beta}(\tilde{x})), \pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x}))\big) \\ & \quad = \lim_{n \to +\infty} \frac{1}{n} \Big| d\big(T^{k_n}(\pi_{\alpha,\beta}(\tilde{x})), \pi_{\alpha,\beta}(\tilde{x})\big) - d\big(\pi_{\alpha,\beta}(\tilde{x}), \pi_{\alpha,\beta}(\tilde{f}^n(\tilde{x}))\big) \Big| \\ & \quad = |v - v| = 0. \end{aligned}$$

Hence

$$\lim_{n \to +\infty} \frac{1}{n} d(T^{k_n}(\tilde{x}), T_n(\tilde{x})) = 0.$$

By the Svarč-Milnor lemma, this implies that $\lim_{n \to +\infty} \frac{1}{n} l_{\mathcal{G}}(T^{-k_n} T_n) = 0$ so that

$$\lim_{n \to +\infty} \frac{1}{n} [l_{n,x}]_{H_1} = \lim_{n \to +\infty} \frac{1}{n} [\gamma^{k_n}]_{H_1} = \lim_{n \to +\infty} \frac{k_n}{n} [\gamma]_{H_1} = \frac{v}{\ell(\gamma)} [\gamma]_{H_1}$$

and the orbit of x realises $\frac{v}{\ell(\gamma)}[\gamma]_{H_1}$. The orbit of x also realises (γ, v) in $\rho(f)$ by the remark below Proposition 4.3. □

8.3 Homological consequences when two geodesics intersect

Let $f \in \mathrm{Homeo}_0(S)$ and let γ_0 and γ_1 be two closed geodesics of S, one of which is simple. Suppose that these closed geodesics meet at a point x_0 of S. Let us denote by $\hat{\pi} : (\hat{S}, \hat{x}_0) \to (S, x_0)$ a covering map associated to $(\gamma_0, \gamma_1, x_0)$. By Proposition 8.4, the surface $\hat{S}$ is homeomorphic to $\mathbb{T}^2 \backslash \{x_\infty\}$, where x_∞ is a point of $\mathbb{T}^2$. We set $\hat{S} = \mathbb{T}^2 \backslash \{x_\infty\}$. We fix a lift $\tilde{x}_0$ of the point x_0.

By the lifting theorem, there exists a homeomorphism $\hat{f}$ in $\mathrm{Homeo}_0(\mathbb{T}^2 \setminus \{x_\infty\})$ such that $\hat{\pi} \circ \hat{f} = f \circ \hat{\pi}$. Observe that $\mathbb{T}^2$ is the Alexandroff compactification of $\mathbb{T}^2 \setminus \{x_\infty\}$ so that the homeomorphism $\hat{f}$ extends to an element of $\mathrm{Homeo}_0(\mathbb{T}^2)$ which fixes the point

x_∞. By abuse of notation, we also denote by $\hat{f}$ this element of $\mathrm{Homeo}_0(\mathbb{T}^2)$ and we also call it a lift of f.

As there is only one homotopy class of isotopy $(f_t)_{t\in[0,1]}$ between Id_S and f, there is only one homotopy class of isotopy between $\mathrm{Id}_{\mathbb{T}^2}$ and $\hat{f}$ which lifts $(f_t)_{t\in[0,1]}$. We fix such a lift $(\hat{f}_t)_{t\in[0,1]}$. We denote $\rho(\hat{f}) = \rho_{H_1(\mathbb{T}^2)}(\hat{f})$ the rotation set of $\hat{f}$ with respect to this isotopy. We denote by $(\tilde{f}_t)_{t\in[0,1]}$ a lift of the isotopy $(f_t)_{t\in[0,1]}$ to $\mathbb{H}^2$. Of course, all those isotopies can be extended in the usual way to any $t \in \mathbb{R}$.

Proposition 8.10. *Let γ_0 and γ_1 be two closed geodesics of S, one of which is simple. Suppose that there exist $v_0 > 0$ and $v_1 > 0$ such that (γ_0, v_0) and (γ_1, v_1) belong to $\rho(f)$.*

Then the convex hull in $H_1(\mathbb{T}^2)$ of 0, $\frac{v_0}{\ell(\gamma_0)}[\hat{\gamma}_0]_{H_1(\mathbb{T}^2)}$ and $\frac{v_1}{\ell(\gamma_1)}[\hat{\gamma}_1]_{H_1(\mathbb{T}^2)}$ is contained in $\rho(\hat{f})$.

With this proposition, we can use known theorems about the rotation set of a homeomorphism of the 2-torus in order to deduce the following corollary.

Corollary 8.11. *Let γ_0 and γ_1 be two closed geodesics of S, one of which is simple. Suppose these closed geodesics meet and that there exist $v_0 > 0$ and $v_1 > 0$ such that (γ_0, v_0) and (γ_1, v_1) belong to $\rho(f)$. Then*

1. *The topological entropy of f is positive.*
2. *Any rational point of $H_1(S, \mathbb{R})$ in the interior of the triangle T formed by the points 0, $\frac{v_0}{\ell(\gamma_0)}[\gamma_0]_{H_1(S)}$ and $\frac{v_1}{\ell(\gamma_1)}[\gamma_1]_{H_1(S)}$ is realised by a periodic orbit.*

Note that point 1. of this corollary was already known, even in the case where both geodesics are not simple (we will handle this case in next section). It was done in [Les12, Kra81, IIY03] by using a different technique based on Nielsen-Thurston classification as in [LM91] for the torus case: one removes some fixed points of the homeomorphism and proves that the homeomorphism defined relative to these fixed points has a pseudo-Anosov component. However, it seems to us that this technique cannot give informations about the existence of new rotation vectors as in point 2. of this corollary, or as in the results in the next two sections.

Proof of Proposition 8.10. We can suppose that (γ_0, v_0) and (γ_1, v_1) are extremal points of $\rho(f)$ as the set $\rho(\hat{f})$ is convex by a result by Misiurewicz and Ziemian (see [MZ89]). We denote by (α_0, β_0) the lift of γ_0 containing the point $\tilde{x}_0$ and by T the deck transformation associated to (α_0, β_0). By Proposition 4.3, there exists a point $\tilde{x}$ in $\tilde{S} = \mathbb{H}^2$ such that

$$\begin{cases} \lim\limits_{n\to+\infty} \dfrac{1}{n} d\big(\tilde{f}^n(\tilde{x}), \tilde{x}\big) &= v_0 \\ \lim\limits_{n\to+\infty} \dfrac{1}{n} d\big(\pi_{\alpha_0,\beta_0}(\tilde{f}^n(\tilde{x})), \tilde{f}^n(\tilde{x})\big) &= 0. \end{cases}$$

Let us set, for any $n \geq 0$, $k_n = \lfloor \frac{n v_0}{\ell(\gamma_0)} \rfloor$. Then

$$\begin{aligned} &\lim_{n\to+\infty} \frac{1}{n} d\big(\pi_{\alpha_0,\beta_0}(\tilde{f}^n(\tilde{x})), T^{k_n}(\pi_{\alpha_0,\beta_0}(\tilde{x}))\big) \\ &\quad = \lim_{n\to+\infty} \frac{1}{n} \Big| d\big(\pi_{\alpha_0,\beta_0}(\tilde{f}^n(\tilde{x})), \pi_{\alpha_0,\beta_0}(\tilde{x})\big) - d\big(\pi_{\alpha_0,\beta_0}(\tilde{x}), T^{k_n}(\pi_{\alpha_0,\beta_0}(\tilde{x}))\big)\Big| \\ &\quad = |v_0 - v_0| = 0. \end{aligned}$$

Hence

$$\lim_{n\to+\infty}\frac{1}{n}d\big(\tilde{f}^n(\tilde{x}),T^{k_n}(\pi_{\alpha_0,\beta_0}(\tilde{x}))\big)=0.$$

Let $\hat{x}$ be the projection of $\tilde{x}$ on $\mathbb{T}^2\setminus\{x_\infty\}$. We endow $\mathbb{T}^2$ with the Euclidean metric g_{Euc} which turn the simple closed curves $\hat{\gamma}_0$ and $\hat{\gamma}_1$ into length 1 orthogonal geodesics. Let us call g_{hyp} the hyperbolic metric on $\mathbb{T}^2\setminus\{x_\infty\}$ induced by the hyperbolic metric on $\tilde{S}=\mathbb{H}^2$.

Recall that the metric g_{Hyp} is complete so that $g_{Hyp}\to\infty$ as we get closer to the point x_∞. Hence there exists $C>0$ such that $g_{Euc}\leq Cg_{Hyp}$. Therefore, if we call d_{Euc} the Euclidean distance on $\tilde{S}$ induced by g_{Euc}, we have, for any $n\geq 1$,

$$d_{Euc}\big(\tilde{f}^n(\tilde{x}),T^{k_n}(\pi_{\alpha_0,\beta_0}(\tilde{x}))\big)\leq Cd\big(\tilde{f}^n(\tilde{x}),T^{k_n}(\pi_{\alpha_0,\beta_0}(\tilde{x}))\big)$$

so that

$$\lim_{n\to+\infty}\frac{1}{n}d_{Euc}\big(\tilde{f}^n(\tilde{x}),T^{k_n}(\pi_{\alpha_0,\beta_0}(\tilde{x}))\big)=0$$

and

$$\lim_{n\to+\infty}\frac{k_n}{n}[\hat{\gamma}_0]_{H_1}=\frac{v_0}{\ell(\gamma_0)}[\hat{\gamma}_0]_{H_1}\in\rho(\hat{f}).$$

In the same way, we prove that $\frac{v_0}{\ell(\gamma_1)}[\hat{\gamma}_1]_{H_1}\in\rho(\hat{f})$. Observe also that the point x_∞ is fixed under $\hat{f}$ with homological rotation number 0. It suffices to recall that the rotation set $\rho_{H^1(\mathbb{T}^2)}(\hat{f})$ is convex [MZ89] to conclude. □

To prove Corollary 8.11, we need to recall some facts about topological entropy. Fix a compact metric space (X,d) and a homeomorphism h of X. In our specific case, $X=S$ and $h=f$. For any integer $n\geq 1$, we define the Bowen distance

$$\begin{array}{rccl} d_n: & X\times X & \longrightarrow & \mathbb{R}_+ \\ & (x,y) & \longmapsto & \max\limits_{0\leq k\leq n-1} d\big(h^k(x),h^k(y)\big)\end{array}$$

which is topologically equivalent to d. For any $\epsilon>0$ and $n\geq 1$, a subset A of X is said to be (n,ϵ)-*separated* if, for any distinct points x and y of A, $d_n(x,y)\geq\epsilon$. By compactness of X, such a set has to be finite. Denote by $a_{n,\epsilon}$ the maximal cardinality of an (n,ϵ)-separated subset of X. Then

$$h_{top}(h)=\lim_{\epsilon\to 0}\liminf_{n\to+\infty}\frac{\log(a_{n,\epsilon})}{n}=\lim_{\epsilon\to 0}\limsup_{n\to+\infty}\frac{\log(a_{n,\epsilon})}{n}.$$

Let us recall another way to compute the topological entropy. For any integer $n>0$ and any $\epsilon>0$, we call (n,ϵ)-ball any open ball of radius ϵ for the distance d_n. We will denote by $B_n(x,\epsilon)$ the (n,ϵ)-ball of center $x\in X$. Denote by $b_{n,\epsilon}$ the minimal cardinality of a cover of X by (n,ϵ)-balls. Then

$$h_{top}(h)=\lim_{\epsilon\to 0}\liminf_{n\to+\infty}\frac{\log(b_{n,\epsilon})}{n}=\lim_{\epsilon\to 0}\limsup_{n\to+\infty}\frac{\log(b_{n,\epsilon})}{n}.$$

We will use the two following classical properties of the topological entropy.

1. If Y is an h-invariant closed subset of X, then $h_{top}(h)\geq h_{top}(h_{|Y})$.

2. If $\hat{X}$ is a compact metric space, if $\hat{\pi} : \hat{X} \to X$ is an onto continuous map, and $\hat{h} : \hat{X} \to \hat{X}$ is a homeomorphism such that $\hat{\pi}\hat{h} = h\hat{\pi}$, then $h_{top}(h) \leq h_{top}(\hat{h})$.

To prove Corollary 8.11, we need the following general result about topological entropy.

Proposition 8.12. *Let $\hat{\pi} : \hat{X} \to X$ be an onto local isometry between the compact metric spaces $\hat{X}$ and X. Let $\hat{h} : \hat{X} \to \hat{X}$ and $h : X \to X$ be homeomorphisms such that $\hat{\pi}\hat{h} = h\hat{\pi}$. Then*

$$h_{top}(\hat{h}) = h_{top}(h).$$

Now, we use this proposition to prove Corollary 8.11. We will prove Proposition 8.12 afterwards.

Proof of Corollary 8.11. 1. By Proposition 8.10, the rotation set of $\hat{f}$ has nonempty interior. By Theorem 1 of the article [LM91] by Llibre and MacKay, there exists an $\hat{f}$-invariant compact subset $\hat{K}$ of $\mathbb{T}^2$ such that:

1. The set $\hat{K}$ does not contain any fixed point of $\hat{f}$. Hence the point x_∞ does not belong to $\hat{K}$ and $\hat{K} \subset \hat{S} = \mathbb{T}^2 \setminus \{x_\infty\}$.

2. The homeomorphism $\hat{f}_{|\hat{K}}$ is conjugated to a subshift with positive topological entropy.

In particular, $h_{top}(\hat{f}_{|\hat{K}}) > 0$. Now, apply Proposition 8.12 to $\hat{h} = \hat{f}_{|\hat{K}}$ and $h = f_{|K}$, where K is the image of $\hat{K}$ under the covering map $\hat{S} \to S$. Then

$$h_{top}(f_{|K}) = h_{top}(\hat{f}_{|\hat{K}}).$$

However $h_{top}(f) \geq h_{top}(f_{|K})$ so that $h_{top}(f) > 0$.

2. Take a rational point η of $H_1(S, \mathbb{R})$ such that there exist real numbers $0 < \lambda_0 < 1$ and $0 < \lambda_1 < 1$ with $\lambda_0 + \lambda_1 < 1$ such that

$$\eta = \lambda_0 \frac{v_0}{l(\gamma_0)}[\gamma_0]_{H_1(S)} + \lambda_1 \frac{v_1}{l(\gamma_1)}[\gamma_1]_{H_1(S)}.$$

Let

$$\hat{\eta} = \lambda_0 \frac{v_0}{l(\gamma_0)}[\hat{\gamma}_0]_{H_1(\mathbb{T}^2)} + \lambda_1 \frac{v_1}{l(\gamma_1)}[\hat{\ell}_1]_{H_1(\mathbb{T}^2)}$$

and observe that the class $\hat{\eta}$ is a rational point of $H_1(\mathbb{T}^2)$.

Write $\hat{\eta} = \frac{p}{q}[\hat{\gamma}]_{H_1(\mathbb{T}^2)}$, where p and q are positive integers and $[\hat{\gamma}]_{H_1(\mathbb{T}^2)}$ is an integral undivisible class in $H_1(\mathbb{T}^2)$ and is hence represented by a simple loop $\hat{\gamma}$ of $\mathbb{T}^2$.

Then the class $\hat{\eta}$ is a rational point which lies in the interior of $\rho(\hat{f})$ by Proposition 8.10. By Theorems by Franks [Fra89] and Llibre-MacKay [LM91], the class $\hat{\eta}$ is realised by a primitive periodic orbit, that is:

1. there exists a point $\hat{x}$ of $\mathbb{T}^2 \setminus \{x_\infty\}$ such that $\hat{f}^q(\hat{x}) = \hat{x}$;

2. the loop $(\hat{f}_t(\hat{x}))_{t\in[0,q]}$ is homologous to the class $p[\hat{\gamma}]_{H_1(\mathbb{T}^2)}$.

Let $x = \hat{\pi}(\hat{x})$ be the projection of the point $\hat{x}$ on the surface S. Then $f^q(x) = x$ and the loop $(f_t(x))_{t\in[0,q]}$ is homologous to the class

$$p[\hat{\pi} \circ \hat{\gamma}]_{H_1(S)} = q\eta$$

so that the vector $\eta \in H_1(S)$ is realised by a (primitive) periodic orbit. □

Proof of Proposition 8.12. The relation $\hat{\pi}\hat{h} = h\hat{\pi}$ gives immediately that $h_{top}(\hat{h}) \geq h_{top}(h)$. Hence it suffices to prove that $h_{top}(\hat{h}) \leq h_{top}(h)$. Let $\delta = \sup_{x\in X} \#\hat{\pi}^{-1}(\{x\})$. Note that δ is finite by compactness of $\hat{X}$ and as $\hat{\pi}$ is a local isometry.

As $\hat{\pi}$ is a local homeomorphism and $\hat{X}$ is compact, there exists $\alpha > 0$ such that, for any distinct points $\hat{x}$ and $\hat{y}$ of $\hat{X}$ such that $\pi(\hat{x}) = \pi(\hat{y})$, we have $\hat{d}(\hat{x}, \hat{y}) \geq \alpha$.

Take $\epsilon > 0$ small enough so that the following properties hold:

1. $\epsilon < \frac{\alpha}{2}$.
2. For any points $\hat{x}$ and $\hat{y}$ of $\hat{X}$ with $\hat{d}(\hat{x}, \hat{y}) < \epsilon$, we have $\hat{d}(\hat{h}(\hat{x}), \hat{h}(\hat{y})) < \frac{\alpha}{2}$.
3. The restriction of the map $\hat{\pi}$ to any ball of radius ϵ is an isometry onto a ball of X.

Fix $n > 0$ and take a maximal (n, ϵ)-separated subset $A_{n,\epsilon}$ of X for h. The central point of the proof is the following claim.

Claim 8.13.

$$\hat{X} = \bigcup_{\hat{x}\in\hat{\pi}^{-1}(A_{n,\epsilon})} B_n(\hat{x}, \epsilon).$$

Before proving the claim, let us see why it yields Proposition 8.12. The claim implies that

$$\begin{aligned} h_{top}(\hat{h}) &\leq \lim_{\epsilon\to 0} \liminf_{n\to+\infty} \frac{\#\hat{\pi}^{-1}(A_{n,\epsilon})}{n} \\ &\leq \lim_{\epsilon\to 0} \liminf_{n\to+\infty} \frac{\log(\delta\# A_{n,\epsilon})}{n} \\ &\leq h_{top}(h). \end{aligned}$$

□

Proof of Claim 8.13. Let $\hat{y} \in \hat{X}$ and $y = \hat{\pi}(\hat{y}) \in X$. By maximality of the set $A_{n,\epsilon}$, there exists a point x of $A_{n,\epsilon}$ such that, for any $0 \leq k \leq n-1$,

$$d\big(h^k(y), h^k(x)\big) < \epsilon.$$

Let

$$\hat{\pi}^{-1}(\{x\}) = \big\{\hat{x}_i \mid 1 \leq i \leq l\big\},$$

where the points $\hat{x}_i$ are pairwise distinct. As $d(x, y) < \epsilon$, there exists an index i such that $\hat{d}(\hat{y}, \hat{x}_i) < \epsilon$. Likewise, as $d(h(y), h(x)) < \epsilon$, there exists an index j such that $\hat{d}(\hat{h}(\hat{y}), \hat{h}(\hat{x}_j)) < \epsilon$.

Let us prove that $i = j$. Indeed, we have

$$\hat{d}(\hat{h}(\hat{x}_i), \hat{h}(\hat{x}_j)) \leq \hat{d}(\hat{h}(\hat{x}_i), \hat{h}(\hat{y})) + \hat{d}(\hat{h}(\hat{y}), \hat{h}(\hat{x}_j)) < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

Moreover, $\hat{\pi}(\hat{h}(\hat{x}_i)) = h(x) = \hat{\pi}(\hat{h}(\hat{x}_j))$. By the definition of α, this implies that $\hat{h}(\hat{x}_i) = \hat{h}(\hat{x}_j)$ and thus that $\hat{x}_i = \hat{x}_j$.

In the same way, an induction proves that, for any $0 \leq k < n$,

$$\hat{d}(\hat{h}^k(\hat{y}), \hat{h}^k(\hat{x}_i)) < \epsilon.$$

Hence

$$\hat{y} \in \bigcup_{1 \leq i \leq l} B_n(\hat{x}_i, \epsilon) \subset \bigcup_{\hat{x} \in \hat{\pi}^{-1}(A_{n,\epsilon})} B_n(\hat{x}, \epsilon).$$

□

9 Closed geodesics with auto-intersection

This section is the first one where we use le Calvez and Tal forcing theory. Its aim is to prove Theorem E if the introduction, that we will state as Theorem 9.27.

After introducing some tools of forcing theory in Subsection 9.1, we will define rotational horseshoes and prove some properties of them (Subsection 9.2). We will then get the fact that geometric auto-intersections of closed trajectories give rise to $\mathcal{F}$-transverse intersections in Subsection 9.3. This will lead us to the main theorem of this section in Subsection 9.6, which will be preceded by a first step of independent interest, about the forcing of new periodic orbits, performed in Subsection 9.5.

9.1 Some results of forcing theory for transverse trajectories

This paragraph is a short introduction to the techniques and the results of Le Calvez and Tal [LCT18a, LCT18b] that will be used in the sequel.

In the sequel, we will call *line* any properly embedded topological line of the plane. For any surface S, we call *singular foliation of* S any foliation $\mathcal{F}$ of an open subset $\operatorname{dom}\mathcal{F}$ of S. The set $S \setminus \operatorname{dom}\mathcal{F}$ is called the *set of singularities* of $\mathcal{F}$. We will call *end of a leaf* ϕ either its α or its ω-limit in S or in $\tilde{S}$ (depending on the context).

Let $\mathcal{F}$ be an oriented nonsingular foliation of the plane. By classical plane topology (see Haefliger-Reeb [HR57]), each leaf ϕ of the foliation is a line, hence its complement possesses two connected components: the left of ϕ, denoted by $L(\phi)$, and the right of ϕ, denoted by $R(\phi)$ (that are chosen according to a fixed orientation of the plane and the orientation of ϕ).

Definition 9.1. Let $\mathcal{F}$ be an oriented nonsingular foliation of a surface[9] S and $\alpha : [0,1] \to S$ be a path. For $x \in S$, we denote by ϕ_x the leaf of $\mathcal{F}$ passing by x. We say that α is *positively transverse* to $\mathcal{F}$ (abbreviated by $\mathcal{F}$-transverse) if for any $t \in [0,1]$, in the universal cover[10] of S one has

$$\widetilde{\alpha}([0,t)) \subset L(\widetilde{\phi}_{\widetilde{\alpha}(t)}) \qquad \text{and} \qquad \alpha((t,1]) \subset R(\widetilde{\phi}_{\widetilde{\alpha}(t)}).$$

Let $\mathcal{F}$ be a (singular) foliation of a surface. The following result can be obtained as a combination of [LC05] with [BCLR20].

[9] Not necessarily closed.

[10] This universal cover is always homeomorphic to $\mathbb{R}^2$, as there is no nonsingular foliation on the sphere.

Theorem 9.2. *Let S be a surface and $f \in \mathrm{Homeo}_0(S)$. Then there exists an isotopy I linking* Id *to f, a transverse topological oriented singular foliation $\mathcal{F}$ of S with $(\mathrm{dom}\,\mathcal{F})^{\complement} = \bigcap_{t\in[0,1]} \mathrm{Fix}\, I^t \subset \mathrm{Fix}\, f$, and for any $z \in \mathrm{dom}\,\mathcal{F}$, a $\mathcal{F}$-transverse path linking z to $f(z)$ which is homotopic in $\mathrm{dom}\,\mathcal{F}$, relative to its endpoints, to the arc $(I^t(z))_{t\in[0,1]}$.*

Definition 9.3. Let ϕ, ϕ_1 and ϕ_2 three oriented lines of the plane. We will say that ϕ_2 *is above ϕ_1 relative to ϕ* if

- these three lines are pairwise disjoint;
- none of these lines separates the two others;
- if α_i, $i = 1, 2$, are two disjoint paths linking a point of ϕ_i to a point $\phi(t_i)$, then[11] $t_2 > t_1$.

Let $\mathcal{F}$ be an oriented nonsingular foliation of the plane, J_1, J_2 be two intervals and $\alpha_i = J_i \to \mathbb{R}^2$, $i = 1, 2$, two $\mathcal{F}$-transverse paths.

Definition 9.4. We say that $\alpha_1 : J_1 \to \mathbb{R}^2$ and $\alpha_2 : J_2 \to \mathbb{R}^2$ *intersect $\mathcal{F}$-transversally and positively* if there exists $a_i < t_i < b_i \in J_i$ such that

- $\phi_{\alpha_1(t_1)} = \phi_{\alpha_2(t_2)} = \phi$;
- $\phi_{\alpha_1(a_1)}$ is above $\phi_{\alpha_2(a_2)}$ relative to ϕ ;
- $\phi_{\alpha_2(b_2)}$ is above $\phi_{\alpha_1(b_1)}$ relative to ϕ.

The same notion can be defined in $\mathrm{dom}\,\mathcal{F}$, by asking that some lifts of the paths to $\widetilde{\mathrm{dom}}\mathcal{F}$ intersect $\mathcal{F}$-transversally.

In the sequel, when it is obvious from the context, we will omit the mention "and positively" when talking about $\mathcal{F}$-transverse intersection.

Fix a homeomorphism $f \in \mathrm{Homeo}_0(S)$ and let $\mathcal{F}$ be a singular foliation of S given by Theorem 9.2. We denote by $\hat{f}$ the canonical lift of f to the universal cover $\widetilde{\mathrm{dom}}\mathcal{F}$ of $\mathrm{dom}\,\mathcal{F}$.

Definition 9.5. We say that a $\mathcal{F}$-transverse path $\alpha : [a, b] \to \mathrm{dom}\,\mathcal{F}$ is *admissible of order n* if there exists a lift $\hat{\alpha}$ of α to $\widetilde{\mathrm{dom}}\mathcal{F}$ such that $\hat{f}^n(\phi_{\hat{\alpha}(a)}) \cap \phi_{\hat{\alpha}(b)} \neq \emptyset$.

The following is the fundamental proposition of [LCT18a] (Proposition 20).

Proposition 9.6. *Suppose that $\alpha_1 : [a_1, b_1] \to \mathrm{dom}\,\mathcal{F}$ and $\alpha_2 : [a_2, b_2] \to \mathrm{dom}\,\mathcal{F}$ are transverse paths that intersect $\mathcal{F}$-transversally at $\alpha_1(t_1) = \alpha_2(t_2)$. If α_1 is admissible of order n_1, and α_2 is admissible of order n_2, then $\alpha_1|_{[a_1,t_1]}\alpha_2|_{[t_2,b_2]}$ and $\alpha_2|_{[a_2,t_2]}\alpha_1|_{[t_2,b_2]}$ are both admissible of order $n_1 + n_2$.*

A consequence of this proposition is the following ([LCT18a], Proposition 23).

Proposition 9.7. *Suppose that $\alpha : [a, b] \to \mathrm{dom}\,\mathcal{F}$ is a transverse path admissible of order n and that α intersects itself $\mathcal{F}$-transversally at $\alpha(s) = \alpha(t)$, with $s < t$. Then $\alpha|_{[a,s]}\alpha|_{[t,b]}$ is admissible of order n and $\alpha|_{[a,s]}\big(\alpha|_{[s,t]}\big)^q\alpha|_{[t,b]}$ is admissible of order nq for every $q > 1$.*

[11] Each line is parametrized according to its orientation.

We finish this crash course by a result on admissibility of trajectories. It uses the following definition.

Definition 9.8. We say that a transverse path $\alpha : J \to \mathbb{R}^2$ has a *leaf on its right* if there exists $a < b$ in J and a leaf ϕ in $L(\phi_{\alpha(a)}) \cap R(\phi_{\alpha(b)})$ that lies in the right of $\alpha|_{[a,b]}$. Similarly, one can define the notion of having a *leaf on its left*.

The following is Proposition 19 of [LCT18a].

Proposition 9.9. *Let $\alpha : [a, b] \to \operatorname{dom} \mathcal{F}$ be an $\mathcal{F}$-transverse path that is not admissible of order n but is a subpath of an $\mathcal{F}$-transverse path of order n. Then any lift $\hat{\alpha}$ of α to $\widetilde{\operatorname{dom}\mathcal{F}}$ has no leaf on its right and no leaf on its left.*

9.2 Markovian intersections

We now study Markovian intersections. They will be used to get topological rotational horseshoes.

Definition 9.10. Let S be a surface. We call *rectangle* of S a subset $R \subset S$ satisfying $R = h([0, 1]^2)$ for some homeomorphism $h : [0, 1]^2 \to h([0, 1]^2) \subset S$. We call *sides* of R the image by h of the sides of $[0, 1]^2$. We call *horizontal* the sides $R^- = h([0, 1] \times \{0\})$ and $R^+ = h([0, 1] \times \{1\})$ and *vertical* the two others. We say that a rectangle $R' \subset R$ is a *strict horizontal* (resp. vertical) *subrectangle* of R if the horizontal (resp. vertical) sides of R' are strictly disjoint from those of R and the vertical (resp. horizontal) sides of R' are included in those of R.

Given $x \in \mathbb{R}^2$, we will denote by $\pi_2(x)$ its second coordinate. Following [ZG04], we define Markovian intersections in the following way:

Definition 9.11. Let R_1 and R_2 be two rectangles of a surface S. We say that the intersection $R_1 \cap R_2$ is *Markovian* if there exists a homeomorphism h from a neighbourhood of $R_1 \cup R_2$ to an open subset of $\mathbb{R}^2$ such that:

- $h(R_2) = [0, 1]^2$;
- either $h(R_1^+) \subset \{x \mid \pi_2(x) > 1\}$ and $h(R_1^-) \subset \{x \mid \pi_2(x) < 0\}$, or $h(R_1^-) \subset \{x \mid \pi_2(x) > 1\}$ and $h(R_1^+) \subset \{x \mid \pi_2(x) < 0\}$;
- $h(R_1) \subset \{x \mid \pi_2(x) < 0\} \cup [0, 1]^2 \cup \{x \mid \pi_2(x) > 1\}$.

The proofs of the following two results can be obtained as a combination of Theorem 16 and Corollary 12 of [ZG04].

Proposition 9.12. *Given a homeomorphism f of a surface S and three rectangles R_1, R_2 and R_3, if the intersections $f(R_1) \cap R_2$ and $f(R_2) \cap R_3$ are Markovian, then the intersection $f^2(R_1) \cap R_3$ is Markovian too (and in particular is nonempty).*

Proposition 9.13. *Let f be a homeomorphism of S and R a rectangle such that $f(R) \cap R$ is Markovian. Then there exists a fixed point for f in R.*

The following is a particular case of Homma's generalization [Hom53] of Schoenflies theorem, it will be used to find rectangles and Markovian intersections.

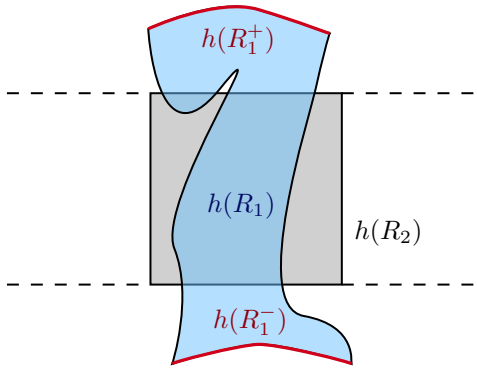

Figure 18: A Markovian intersection

Theorem 9.14 (Homma)**.** *Any homeomorphism of*

$$\Big(\big((\mathbb{R}\times\{0\})\cup(\mathbb{R}\times\{1\})\cup(\{0\}\times[0,1])\cup(\{1\}\times[0,1])\big)\cap B(0,10)\Big)\cup\partial B(0,10)$$

to its image in $\mathbb{R}^2$ *can be extended to a self-homeomorphism of* $\mathbb{R}^2$*.*

The following definition is a variation over the concept of rotational horseshoe defined in [PPS18] and used in [LCT18b].

Definition 9.15. Let S be a surface with negative Euler characteristic and f a homeomorphism of S. We denote by $\tilde{f}$ the canonical lift of f to $\widetilde{S}\simeq\mathbb{H}^2$.

We say that f has a *rotational horseshoe with deck transformations* $U_1,\dots,U_k$ if there exists a rectangle R of $\widetilde{S}$ such that, for any $1\le i\le k$, the intersections $U_iR\cap\tilde{f}(R)$ are Markovian.

For any finite set $\{1,\dots,k\}^{\mathbb{Z}}$, we denote by $\sigma:\{1,\dots,k\}^{\mathbb{Z}}\to\{1,\dots,k\}^{\mathbb{Z}}$ the shift map, *i.e* the map which, to a sequence $(a_i)_{i\in\mathbb{Z}}$, associates the sequence $(a_{i+1})_{i\in\mathbb{Z}}$.

From Propositions 9.12 and 9.13, it follows the following "semi-conjugacy" result (which allows to link our notion of horseshoe with the one of [LCT18b]).

Proposition 9.16. *Suppose that* f *has a rotational horseshoe with deck transformations* $U_1,\dots,U_k$*, and suppose that these transformations form a free group. Then there exists* $q\ge 0$*, a compact invariant subset* $\tilde{Y}$ *of* $\tilde{S}$*, a homeomorphism* g *of* $\tilde{Y}$ *and a surjective continuous map* $h_1:\tilde{Y}\to\{1,\dots,k\}^{\mathbb{Z}}$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\{1,\dots,k\}^{\mathbb{Z}} \arrow[r, "\sigma"] & \{1,\dots,k\}^{\mathbb{Z}} \\
\tilde{Y} \arrow[u, "h_1"] \arrow[r, "\tilde{g}"] \arrow[d, "\pi"'] & \tilde{Y} \arrow[u, "h_1"'] \arrow[d, "\pi"] \\
Y \arrow[r, "f^q"] & Y
\end{tikzcd}$$

(π denotes the canonical projection) and moreover

- *the preimage by h_1 of every p-periodic sequence for σ contains a point which projects to a p-periodic sequence for f^q;*
- *for any $\tilde{y} \in \tilde{Y}$ and any $n > 0$, one has*

$$\tilde{f}^{qn}(\tilde{y}) \in U^q_{h_1(\tilde{f}^{q(n-1)}(\tilde{y}))} U^q_{h_1(\tilde{f}^{q(n-2)}(\tilde{y}))} \dots U^q_{h_1(\tilde{y})}(\tilde{Y}), \quad \text{and}$$
$$\tilde{f}^{-qn}(\tilde{y}) \in U^{-q}_{h_1(\tilde{f}^{-qn}(\tilde{y}))} U^{-q}_{h_1(\tilde{f}^{q(-n+1)}(\tilde{y}))} \dots U^{-q}_{h_1(f^{-q}(\tilde{y}))}(\tilde{Y}).$$

Similar properties hold for f instead of f^q: we get some classical consequences of the semi-conjugacy to a shift without the semi-conjugacy property itself.

Proposition 9.17. *Suppose that f has a rotational horseshoe with deck transformations $U_1, \dots, U_k$. Then*

- *For any word $T \in \langle U_1, \dots, U_k \rangle_+$ of length q, there exists $\tilde{x} \in \tilde{S}$ such that $\tilde{f}^q(\tilde{x}) = T\tilde{x}$ (in other words, x is a q-periodic point associated to the deck transformation T);*
- *There exists $D > 0$ such that, for any word $(w_i)_i \in \{1, \dots, k\}^{\mathbb{Z}}$, there exists $\tilde{x} \in \tilde{S}$ such that, for any $i \geq 0$,*

$$d\big(\tilde{f}^i(\tilde{x}), U_{w_i} \dots U_{w_0}\tilde{x}\big) \leq D, \qquad d\big(\tilde{f}^{-i}(\tilde{x}), U^{-1}_{w_{-i}} \dots U^{-1}_{w_{-1}}\tilde{x}\big) \leq D.$$

- *if $U_1, \dots, U_k$ form a free group, then the topological entropy of f is bigger than $\log k$.*

Proof of Proposition 9.16. We use notation from Definition 9.15.

As R is compact, as the group generated by $U_1, \dots, U_k$ is free, and as $\pi_1(S)$ acts properly discontinuously on $\tilde{S}$, there exists $q \in \mathbb{N}^*$ such that for any nontrivial word T in $U_1^q, \dots, U_k^q$, one has $TR \cap R = \emptyset$.

For $(w_i) \in \{1, \dots, k\}^{\mathbb{Z}}$, let us define

$$R^n_{(w_i)} = R \cap \bigcap_{0 \leq i < n} \Big(\tilde{f}^{-qi}(U^q_{w_{i-1}} \dots U^q_{w_0} R) \cap \tilde{f}^{qi}(U^{-q}_{w_{-i}} \dots U^{-q}_{w_{-1}} R) \Big)$$

and

$$\tilde{Y} = \bigcap_{n \geq 0} \bigcup_{(w_i) \in \{1, \dots, k\}^{\mathbb{Z}}} R^n_{(w_i)}.$$

We will denote by Y the projection of $\tilde{Y}$ on S. Note that $\tilde{Y}$ is a decreasing intersection of compact subsets of R, so it is compact.

Remark that, if $(w_i) \in \{1, \dots, k\}^{\mathbb{Z}}$ and $\tilde{x} \in \bigcap_{n \geq 0} R^n_{(w_i)}$, then (because of the definition of q) for any $i \in \mathbb{Z}$ there is a unique deck transformation $T_i \in \langle U_1^q, \dots, U_k^q \rangle$ such that $\tilde{f}^{qi}(\tilde{x}) \in T_i R$. By the very definition of $R^n_{(w_i)}$, one has $\tilde{f}^{q(i+1)}(x) \in U^q_{w_i} T_i R$. Then $T_{i+1} = U^q_{w_i} T_i$ and there exists a unique sequence $(w_i) \in \{1, \dots, k\}^{\mathbb{Z}}$ such that $\tilde{x} \in \bigcap_{n \geq 0} R^n_{(w_i)}$.

Moreover, the previous remark also implies the following equality:

$$\widetilde{Y} \overset{\text{def.}}{=} \bigcap_{n \geq 0} \bigcup_{(w_i) \in \{1, \dots, k\}^{\mathbb{Z}}} R^n_{(w_i)} = \bigcup_{(w_i) \in \{1, \dots, k\}^{\mathbb{Z}}} \bigcap_{n \geq 0} R^n_{(w_i)}.$$

Indeed, if a point $\tilde{x}$ belongs to the left-hand side set, then, for any $i \in \mathbb{Z}$, there exists a unique T_i such that $\tilde{f}^{qi}(\tilde{x}) \in T_i R$ and a unique w_i such that $T_{i+1} = U_{w_i}^q T_i$. Hence the point $\tilde{x}$ belongs to $\bigcap_{n\geq 0} R^n_{(w_i)}$. The other inclusion is trivial.

Then, for any $\tilde{x} \in \widetilde{Y}$, there exists a unique sequence $(w_i) \in \{1, \dots, k\}^{\mathbb{Z}}$ such that $\tilde{x} \in \bigcap_{n\geq 0} R^n_{(w_i)}$. This allows to talk about the *trajectory* of a point $\tilde{x} \in \tilde{Y}$, which we define as the unique sequence $(w_i) \in \{1, \dots, k\}^{\mathbb{Z}}$ such that $\tilde{x} \in \bigcap_{n\geq 0} R^n_{(w_i)}$. We define

$$\begin{aligned} h_1 : \tilde{Y} &\longrightarrow \{1, \dots, k\}^{\mathbb{Z}} \\ x &\longmapsto (w_i). \end{aligned}$$

as the map which, to any point of $\tilde{Y}$, associates its trajectory.

Repeated applications of Proposition 9.12 imply that for any $(w_i) \in \{1, \dots, k\}^{\mathbb{Z}}$, the sets $R^n_{(w_i)}$ are nonempty (and compact), hence $\bigcap_{n\geq 0} R^n_{(w_i)}$ is also a nonempty compact set. This shows that the map h_1 is surjective.

We define the map $\tilde{g}$ by $\tilde{g}|_{\tilde{f}^{-q}(U^q_{w_i}R)\cap R} = U_{w_i}^{-q} \circ \tilde{f}^q$. As the sets $\tilde{f}^{-q}(U^q_{w_i}R) \cap R$ are all at positive distance one to the others, the map $\tilde{g}$ defines a homeomorphism of $\tilde{Y}$. By the very construction of h_1, the diagram of Proposition 9.16 commutes.

Fix $(w_i)_{i\in\mathbb{Z}} \in \{1, \dots, k\}^{\mathbb{Z}}$. Let $n \geq 2$. Observe that $R^n_{(w_i)}$ is a neighbourhood of $R^{n+1}_{(w_i)}$ so that $R^n_{(w_i)} \cap \tilde{Y}$ is an open subset of $\tilde{Y}$ and projects to an open subset of Y. Observe that the projection on the coordinates between $-n+2$ and $n-1$ of the map h_1 is constant on this open subset so that the map h_1 is continuous.

Finally, Proposition 9.13 implies that in the preimage by h_1 of any periodic word, there is a periodic point of f of the same period. This finishes the proof. □

Proof of Proposition 9.17. The first point of the proposition is a simple application of Propositions 9.12 and 9.13.

For the second point, we can use again the strategy of the proof of Proposition 9.17, by considering the compact set

$$R^n_{(w_i)} = R \cap \bigcap_{0\leq i<n} \left(\tilde{f}^{-i}(U_{w_{i-1}} \dots U_{w_0} R) \cap \tilde{f}^i(U^{-1}_{w_{-i}} \dots U^{-1}_{w_{-1}} R) \right).$$

One gets easily as a consequence of Proposition 9.12 that the set $\bigcap_{n\geq 0} R^n_{(w_i)}$ is nonempty, and any element of it can be used to get the desired conclusion (taking D as the diameter of R for instance).

Concerning the entropy, let us change a bit the definition of $R^n_{(w_i)}$ to consider only positive times:

$$\overline{R}^n_{(w_i)} = R \cap \bigcap_{0\leq i<n} \tilde{f}^{-i}(U_{w_{i-1}} \dots U_{w_0} R).$$

Note that, as R is compact, as the group generated by $U_1, \dots, U_k$ is free, and as $\pi_1(S)$ acts properly discontinuously on $\tilde{S}$, there exists $N_0 \in \mathbb{N}$ such that for any nontrivial word T in $U_1, \dots, U_k$ of length bigger than N_0, one has $TR \cap R = \emptyset$.

This implies that, for any words $(w_i)_{0\leq i\leq n}, (w'_i)_{0\leq i\leq n} \in \{1, \dots, k\}^{n+1}$, if there exists $N_0 \leq i_0 \leq n$ such that $w_{i_0} \neq w'_{i_0}$, then $U_{w_n} \dots U_{w_0} R \cap U_{w'_n} \dots U_{w'_0} R = \emptyset$, and so $\overline{R}^n_{(w_i)} \cap \overline{R}^n_{(w'_i)} = \emptyset$.

By [Hem72], there exists a finite cover $\hat{S}$ of S such that R projects injectively on[12] $\hat{S}$. We denote by $\hat{f}$ the map induced by $\tilde{f}$ on $\hat{S}$. Fix $n > N_0$. Consider ε the minimum distance between the projections of the compact sets $\overline{R}^n_{(w_i)}$ on $\hat{S}$, with $(w_i) \in \{1\}^{N_0} \times \{1, \dots, k\}^{n-N_0}$. By the previous paragraph, these sets are pairwise disjoint in $\tilde{S}$, and as they are subsets of R, they project injectively on $\hat{S}$: we have $\varepsilon > 0$.

Now, take $\ell \geq 0$ and consider the family of words $(w_i) \in (\{1\}^{N_0} \times \{1, \dots, k\}^{n-N_0})^\ell$. Taking one point in each of these sets, we get a subset of $\hat{S}$ of cardinality $k^{\ell(n-N_0)}$ which is $(\ell n, \varepsilon)$ separated. This implies that (taking $\ell \to +\infty$)

$$h_{top}(\hat{f}) \geq \frac{n - N_0}{n} \log k,$$

and so, taking $n \to +\infty$, that $h_{top}(\hat{f}) \geq \log k$.

We get the conclusion of the proposition by using Proposition 8.12 which tells us that $h_{top}(f) = h_{top}(\hat{f})$. □

9.3 Geometric vs. $\mathcal{F}$-transverse intersections

In this subsection, we prove that an $\mathcal{F}$-transverse loop on a surface that has a geometric auto-intersection (in the geometric meaning of Definition 8.1) must have an $\mathcal{F}$-transverse auto-intersection, associated to a deck transformation that projects to the deck transformation of the geometric auto-intersection.

In the sequel, we will denote paths with marked points to denote their lifts to the universal cover starting on the common marked point. For instance, $\alpha\cdot$ and $\beta\cdot$ denote some lifts of respectively α and β whose right ends coincide.

We fix a surface S (not necessarily closed) of negative Euler characteristic, a singular foliation $\mathcal{F}$ of S, and an $\mathcal{F}$-transverse loop $\alpha : \mathbb{R} \to \operatorname{dom} \mathcal{F}$ (which means that for every t, one has $\alpha(t) = \alpha(t+1)$). We suppose that α auto-intersects geometrically (see Definition 8.1) at $\alpha(t_1) = \alpha(t_2)$, with $t_1 < t_2 < t_1 + 1$. We let $\alpha_1 = \alpha|_{[t_1,t_2]}$ and $\alpha_2 = \alpha|_{[t_2,t_1+1]}$. Let $\widetilde{\alpha}$ be a lift of α to $\widetilde{S}$ and $\widetilde{\alpha}_1$ be the lift of α_1 which starts from the same point as $\widetilde{\alpha}$. Also, let T and T_1 be the deck transformations of the universal cover $\widetilde{S} \to S$ respectively associated to $\alpha|_{[t_1,t_2]}$ and α and which respectively preserve $\widetilde{\alpha}$ and $\widetilde{\alpha}_1$. Let T_2 be the deck transformation such that $T = T_2 T_1$. Finally, let $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\widetilde{S}$.

Proposition 9.18. *If α auto-intersects geometrically, then there exists u_1 and u_2 such that the paths $T_1\widetilde{\alpha}|_{[u_2,u_2+1]}$ and $\widetilde{\alpha}|_{[u_1,u_1+1]}$ intersect $\widetilde{\mathcal{F}}$-transversally at $T_1\widetilde{\alpha}(t_1) = \widetilde{\alpha}(t_2)$.*

In particular, the paths $\alpha_1\alpha_2 \cdot \alpha_1\alpha_2$ and $\alpha_2\alpha_1 \cdot \alpha_2\alpha_1$ intersect $\mathcal{F}$-transversally at the marked point. More generally, for $i, j, k, \ell \geq 1$, the paths $\alpha_1\alpha_2^i \cdot \alpha_1^k\alpha_2$ and $\alpha_2\alpha_1^j \cdot \alpha_2^\ell\alpha_1$ intersect $\mathcal{F}$-transversally at the marked point.

Remark 9.19. The proof of this proposition shows that if none of the deck transformations T_1 and T_2 is a prefix/suffix of the other, then the conclusion is stronger: the paths $\alpha_2 \cdot \alpha_1$ and $\alpha_1 \cdot \alpha_2$ intersect $\mathcal{F}$-transversally at the marked point.

[12] A group is *residually finite* if, for any finite subset $F \subset G$, there exists a finite quotient G/H in which F projects injectively. By [Hem72], any surface group is residually finite, and we can apply this property to the finite set F of deck transformations T such that $TR \cap R \neq \emptyset$. The finite index resulting subgroup H corresponds to a finite cover of the surface in which R projects injectively.

Remark 9.20. In the end of the proof one has to consider the case where T_1 is a suffix of T_2, and T_2 is a suffix of T_2T_1. This case can happen, as can be seen by considering the words $w_1 = 1221$ and $w_2 = 21\,1221$: w_1 is a suffix of w_2, and w_2 is a suffix of w_2w_1. This suggests that in general, the conclusion of the proposition cannot be improved.

Proof. We denote $\alpha_1 = \alpha|_{[t_1,t_2]}$ and $\alpha_2 = \alpha|_{[t_2,t_1+1]}$. Let $\check{\mathrm{dom}}(\mathcal{F})$ be the covering of $\mathrm{dom}(\mathcal{F})$ associated to $(\alpha_1, \alpha_2, x_0)$. By Proposition 8.5, the surface $\check{\mathrm{dom}}(\mathcal{F})$ is homeomorphic to the three punctured sphere $\mathbb{S}^2 \setminus \{A, B, C\}$. The lifts of α, α_1 and α_2 to $\check{\mathrm{dom}}(\mathcal{F})$ are respectively denoted by $\check{\alpha}$, $\check{\alpha}_1$ and $\check{\alpha}_2$. We denote a resp. b two simple loops generating $\pi_1(\mathbb{S}^2 \setminus \{A, B, C\})$, winding once around A and not around B or C (resp. once around B and not around A or C).

During the proof we will use the following fact: if $\mathcal{F}$ is a singular foliation on $\mathbb{S}^2$, and α is a $\mathcal{F}$-transverse Jordan curve in $\mathbb{S}^2$, then each connected component of $\alpha^{\complement}$ has to contain at least one singularity of the foliation $\mathcal{F}$.

Let $\beta_1 = \check{\alpha}|_{[s_1,s_1']}$ be a subpath of $\check{\alpha}$ which is a simple loop. As $\check{\alpha}$ is transverse, β_1 is essential: indeed, this Jordan curve separates $\mathbb{S}^2$ in two connected components, and each of them has to contain a singularity of the lift $\check{\mathcal{F}}$ of $\mathcal{F}$ to $\check{\mathrm{dom}}(\mathcal{F}) \subset \mathbb{S}^2$, because β_1 is transverse; it then suffices to remember that the only singularities of $\check{\mathcal{F}}$ in $\mathbb{S}^2$ are the punctures. Hence, as the only essential simple loops in the three punctured sphere are the ones winding once around one puncture and not around the others, we can suppose (up to a permutation of A, B and C) that the loop β_1 is homotopic to a.

Consider the loop $\overline{\beta_1} \doteq \check{\alpha}|_{[s_1',s_1+1]}$. It is not contractible : otherwise, the loop $\check{\alpha}$ would be homotopic to β_1, which is not possible as $\check{\alpha}$ has a geometric self intersection. Again, let $\beta_2 = \overline{\beta_1}|_{[s_2,s_2']}$ be a subpath of $\overline{\beta_1}$ which is an essential simple loop. If β_2 is homotopic to a or a^{-1}, we can iterate the process by considering the loop $\overline{\beta_1}|_{[s_2',s_2+1]}$ or the loop $\overline{\beta_1}|_{[s_1',s_2]}$: one of them is homotopically non trivial as $\check{\alpha}$ cannot be homotopic to a power of β_1, by definition of $\check{\mathrm{dom}}(\mathcal{F})$)... Eventually, we find an essential simple loop β_n, which is a concatenation of pieces of the path $\check{\alpha}$, which is neither homotopic to a nor to a^{-1}. As before, we can suppose that this loop β_n is homotopic to b (changing b to b^{-1} if necessary).

From now on we will denote $\beta_A = \beta_1$ and $\beta_B = \beta_n$. Let Φ_A be the union of the leaves met by β_A, and Φ_B the union of the leaves met by β_B. These are open annuli, Φ_A separating A from B and C, and Φ_B separating B from A and C. Remark that the complement of Φ_A (resp. Φ_B) in $\mathbb{S}^2$ is made of two connected components, that are closed.

Claim 9.21. *The loops β_A and β_B are $\check{\mathcal{F}}$-equivalent to disjoint loops.*

Proof. Replacing β_A and β_B by $\check{\mathcal{F}}$-equivalent loops if necessary, we can suppose that the number of intersections between them is finite.

Suppose that β_A and β_B are disjoint, otherwise there is nothing to do. the only nontrivial case is when β_A meets the connected component O of $\beta_B^{\complement}$ that contains B.

Let $t_1 < t_2$ be such that $\beta_A|_{[t_1,t_2]}$ meets β_B at its endpoints, and that $\beta_A|_{]t_1,t_2[}$ is included in O. Then $\beta_A|_{]t_1,t_2[}$ separates O into two connected components, one of them containing B and the other one, denoted by O', not containing it. Suppose that O' is locally on the right of $\beta_A|_{]t_1,t_2[}$ (the other case being identical). Each leaf meeting O' has to get out of it as O' does not contain any singularity of $\check{\mathcal{F}}$. In particular, each leaf entering in O' through $\beta_A|_{]t_1,t_2[}$ has to get out of O' by β_B. This implies that $\beta_A|_{]t_1,t_2[}$

stays in Φ_B, and hence that β_A does not meet the connected component of $\Phi_B^{\complement}$ containing B.

By local compactness, the distance between β_A and the connected component of $\Phi_B^{\complement}$ containing B is positive. Remark that the leaves of resp. Φ_A and Φ_B are naturally indexed by the transverse loops β_A and β_B, and in particular are endowed with a natural cyclic order. By considering a continuous parametrization of the leaves of Φ_B by $\mathbb{S}^1 \times \mathbb{R}$ ($\mathbb{S}^1$ corresponding to the point of β_B met by the leaf and $\mathbb{R}$ to the parametrization of the leaf itself), by flowing β_B along the leaves of Φ_B in the direction of B, one can easily find a loop $\mathcal{F}$-equivalent to β_B and which is disjoint from β_A. □

Claim 9.22. *The loop $\check{\alpha}$ stays in $\Phi_A \cup \Phi_B$. In particular, $\Phi_A \cap \Phi_B \neq \emptyset$.*

Proof. The second part of the claim follows from the first one as the union of the leaves met by $\check{\alpha}$ is connected.

Suppose for a contradiction that $\check{\alpha}$ does not stay in $\Phi_A \cup \Phi_B$. Let x be a point of $\check{\alpha}$ outside of $\Phi_A \cup \Phi_B$, and $\phi \doteq \phi_x$. Then, by Poincaré-Bendixson theorem, the ends of ϕ are either topological circles, or contain singularities of $\check{\mathcal{F}}$. In the circle case, either it contains a singularity, or both connected components of its complement contain a singularity.

Remark that ϕ and its ends are disjoint from $\Phi_A \cup \Phi_B$, and that Φ_A and Φ_B separate all singularities of $\check{\mathcal{F}}$.

In the case where ϕ is a circle, then it separates $\mathbb{S}^2$ into two disjoint connected components and so it prevents transverse trajectories passing through it (e.g. $\check{\alpha}$) to be recurrent, which is a contradiction.

The same argument can be applied when both ends of ϕ contain a singularity: in this case, as ϕ is contained in a single connected component of $(\Phi_A \cup \Phi_B)^{\complement}$, this singularity $D \in \{A, B, C\}$ is the same for both ends of ϕ. Then, the ends of ϕ are made of the union of D with possibly leaves of $\check{\mathcal{F}}$ that are homoclinic to D. In the case such leaves exist, they all separate $\mathbb{S}^2$ in two connected components, one of which containing the whole loop $\check{\alpha}$. So it does not change anything dynamically to quotient by these connected components, and this crushing allows to reduce to the case where both ends are reduced to $\{D\}$. Replacing the circle by $\phi \cup \{D\}$ in the previous paragraph leads to a contradiction.

Suppose now that ϕ is not a circle and that at least one end of it is a closed leaf. Then this leaf L separates $\mathbb{S}^2$ into two disjoint connected components. As $\check{\alpha}$ is a transverse loop, it cannot meet L. As the loop $\check{\alpha}$ meets Φ_A, Φ_B and ϕ, one of the connected components of $L^{\complement}$ contains Φ_A, Φ_B and ϕ, and the other one contains a singularity. Observe that the other end of ϕ cannot contain a singularity: it is a closed leaf L'. Moreover, ϕ belongs to the connected component of the complement of L' that does not contain Φ_A and Φ_B, a contradiction as $\check{\alpha}$ cannot meet L'. □

Recall that the leaves of resp. Φ_A and Φ_B are naturally indexed by the transverse loops β_A and β_B, and in particular are endowed with a natural cyclic order.

Claim 9.23. *The set of leaves of $\Phi_A \cap \Phi_B$ is an interval of leaves of Φ_A (resp. Φ_B).*

Proof. Les us reason in the plane $\mathbb{S}^2 \setminus \{C\}$.

By Claim 9.21, there exist transverse loops β'_A and β'_B, which are respectively equivalent to β_A and β_B and which are disjoint. Note that in this plane, the two Jordan curves β'_A and β'_B bound bounded domains that are disjoint: if one bounded domain was included in the other one, it would have to contain both A and B which is impossible.

Also, by considering a leaf of $\Phi_A \cap \Phi_B$, which meets both β'_A and β'_B, we see that β'_A and β'_B turn in opposite directions (relative to a fixed orientation of $\mathbb{S}^2 \setminus \{C\}$).

Now, let ϕ_1 and ϕ_2 be two distinct leaves of $\Phi_A \cap \Phi_B$; we want to show that a one of the intervals of leaves of Φ_A, $[\phi_1, \phi_2]_{\Phi_A}$ or $[\phi_2, \phi_1]_{\Phi_A}$, is contained in $\Phi_A \cap \Phi_B$ (the proof is identical for Φ_B). Denote by ϕ'_1 and ϕ'_2 the leaf segments of resp. ϕ_1 and ϕ_2 that are bounded by β'_A and β'_B. In this case, the open set

$$\left(\beta'_A \cup \beta'_B \cup \phi'_1 \cup \phi'_2\right)^{\complement}$$

is made of four connected components (the considered paths bound four different Jordan curves), one of which, denoted by D, is containing no singularity. Part of its boundary is made of a segment of β'_A between ϕ_1 and ϕ_2. All the leaves crossing this segment have to get out of D (because D is a disk containing no singularity), and it can make it only by crossing β'_B. Hence, a whole interval of leaves between ϕ_1 and ϕ_2 is included in $\Phi_A \cap \Phi_B$. $\square$

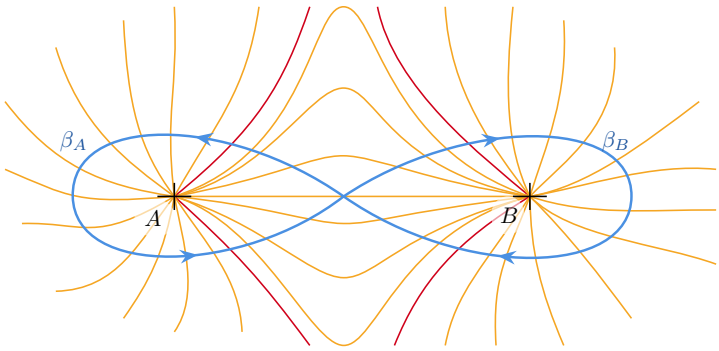

Figure 19: One possible shape for the set of leaves that meet β_A and β_B. The four red leaves are the boundaries of the sets $\Phi_A \cap \Phi_B$, $\Phi_A \setminus \Phi_B$ and $\Phi_B \setminus \Phi_A$.

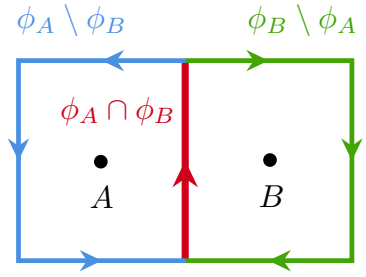

Figure 20: The set of leaves of $\Phi_A \cap \Phi_B$ has topologically this shape (be careful, this space is not Hausdorff: the boundaries of $\Phi_A \setminus \Phi_B$ and $\Phi_B \setminus \Phi_A$ do not coincide).

From now on, replacing the transverse loops β_A and β_B by $\check{\mathcal{F}}$-equivalent ones if necessary, we suppose that they are indexed by $\mathbb{R}/\mathbb{Z}$, and that they meet at a single point $\beta_A(0) = \beta_B(0)$. By changing α to an $\mathcal{F}$-equivalent loop if necessary, we can suppose that $\check{\alpha}(t_1) = \beta_A(0) = \beta_B(0) = \check{\alpha}(t_2)$.

Let $\phi_a \in \Phi_A \setminus \Phi_B$ and $\phi_b \in \Phi_B \setminus \Phi_A$ (these sets are nonempty, otherwise it would contradict the fact that β_A and β_B are simple and bound different singularities of $\check{\mathcal{F}}$). Changing the speeds of β_A and β_B if necessary, we suppose that $\beta_A(1/2) \in \phi_a$ and $\beta_B(1/2) \in \phi_b$. By Claims 9.21 and 9.23, the set $\Phi_A \cap \Phi_B$ is a nonempty open topological disk.

Claim 9.24. *For any transverse loop $\check{\gamma}$ contained in $\Phi_A \cup \Phi_B$, there exists a unique word $a_1 \dots a_n$ on the letters A and B such that $\check{\gamma}$ is $\check{\mathcal{F}}$-equivalent to the loop $\beta_{a_1} \dots \beta_{a_n}$. In particular, there exists a unique word $w = w_1, \dots, w_k \in \{A, B\}^k$ such that $\check{\alpha}|_{[t_1, t_1+1]}$ is $\check{\mathcal{F}}$-equivalent to the loop $\beta_{w_1} \dots \beta_{w_k}$.*

Proof. Such a loop $\check{\gamma}$ cannot be contained in $\Phi_A \cap \Phi_B$, as it is recurrent. Similarly, it cannot be contained in $\Phi_A \setminus \Phi_B$, nor in $\Phi_B \setminus \Phi_A$. Hence, the projection of this loop $\check{\gamma}$ on the set of leaves of $\Phi_A \cap \Phi_B$ has to follow the oriented paths of Figure 20.

Hence, the homotopy class of the transverse loop $\check{\gamma}$ is determined by the sequence of leaves ϕ_a, ϕ_b met by $\check{\gamma}$: for instance, if $\check{\gamma}$ meets successively ϕ_a, ϕ_a, ϕ_b and ϕ_a, then the homotopy type of $\check{\gamma}$ is the one of $\beta_A^2\beta_B\beta_A$. So the homotopy type of $\check{\alpha}$ is a word in β_A and β_B (it does not contain neither β_A^{-1} nor β_B^{-1}). This implies the claim. □

Claim 9.25. *The transverse paths*

$$\beta_A|_{[1/2,1]}\beta_B|_{[0,1/2]} \quad \textit{and} \quad \beta_B|_{[1/2,1]}\beta_A|_{[0,1/2]}$$

have an $\mathcal{F}$-transverse intersection at $\beta_A(0) = \beta_B(0)$.

Proof. Let $\widehat{\beta_A}$ and $\widehat{\beta_B}$ be two lifts of resp. $\beta_A|[0,1]$ and $\beta_B|_{[0,1]}$ to $\widetilde{\mathrm{dom}}(\mathcal{F})$ that meet at $\widehat{\beta_A}(1) = \widehat{\beta_B}(0)$. We denote T_A (resp. T_B) the deck transformation of $\widetilde{\mathrm{dom}}(\mathcal{F}) \to \check{\mathrm{dom}}(\mathcal{F})$ corresponding to the essential loop β_A (resp. β_B) which preserves $\widehat{\beta_A}$ (resp. $\widehat{\beta_B}$). Then (see Figure 21) we have $\widehat{\beta_A}(1) = \widehat{\beta_B}(0) = T_A\widehat{\beta_A}(0) = T_B^{-1}\widehat{\beta_B}(1)$. As $\phi_a \in \Phi_A \setminus \Phi_B$ and $\phi_b \in \Phi_B \setminus \Phi_A$, we deduce that

$$R(\phi_{\widehat{\beta_B}(1/2)}) \cap R(\phi_{T_A\widehat{\beta_A}(1/2)}) = L(\phi_{T_B^{-1}\widehat{\beta_B}(1/2)}) \cap L(\phi_{\widehat{\beta_A}(1/2)}) = \emptyset.$$

Moreover, the fact that all pairs of curves that are homotopic to $\beta_A|_{[1/2,1]}\beta_B|_{[0,1/2]}$, resp. $\beta_B|_{[1/2,1]}\beta_A|_{[0,1/2]}$ meet (this comes from the fact that the curves β_A and β_B turn in different directions in $\mathbb{S}^2 \setminus \{C\}$) implies that

- Either $\phi_{\widehat{\beta_B}(1/2)}$ is above $\phi_{T_B^{-1}\widehat{\beta_B}(1/2)}$ relative to $\phi_{\widehat{\beta_B}(0)}$ and $\phi_{T_A\widehat{\beta_B}(1/2)}$ is above $\phi_{\widehat{\beta_B}(1/2)}$ relative to $\phi_{\widehat{\beta_B}(0)}$.
- Or $\phi_{\widehat{\beta_B}(1/2)}$ is below $\phi_{T_B^{-1}\widehat{\beta_B}(1/2)}$ relative to $\phi_{\widehat{\beta_B}(0)}$ and $\phi_{T_A\widehat{\beta_B}(1/2)}$ is below $\phi_{\widehat{\beta_B}(1/2)}$ relative to $\phi_{\widehat{\beta_B}(0)}$.

In both cases we have an $\mathcal{F}$-transverse intersection between $\widehat{\beta_A}|_{[1/2,1]}\widehat{\beta_B}|_{[0,1/2]}$ and $T_B^{-1}\widehat{\beta_B}|_{[1/2,1]}T_A\widehat{\beta_A}|_{[0,1/2]}$. □

We are ready to prove that the paths $T_1\widetilde{\alpha}|_{[u_2,u_2+1]}$ and $\widetilde{\alpha}|_{[u_1,u_1+1]}$ intersect $\widetilde{\mathcal{F}}$-transversally at $T_1\widetilde{\alpha}(t_1) = \widetilde{\alpha}(t_2)$.

By Claim 9.24, the transverse loop $\check{\alpha}|_{[t_1,t_2]}$ is $\check{\mathcal{F}}$-equivalent to a subword $\beta_{w_1}\dots\beta_{w_\ell}$ of $w = \beta_{w_1}\dots\beta_{w_k}$.

Let us periodize the word w, and consider the word w' such that $w'_i = w_{i-\ell}$ for any $i \in \mathbb{Z}$. Observe that the loop $\check{\alpha}|_{[t_1,t_1+1]}$ is $\check{\mathcal{F}}$-equivalent to $\beta_{w_1}\beta_{w_2}\dots\beta_{w_k}$ and that the loop $\check{\alpha}|_{[t_2,t_2+1]}$ is $\check{\mathcal{F}}$-equivalent to $\beta_{w'_1}\beta_{w'_2}\dots\beta_{w'_k}$. As $\check{\alpha}$ and $T_1\check{\alpha}$ have a geometric transverse intersection, we cannot have $w = w'$. As both words w and w' are periodic of period k, this implies that there exists $i_0 \leq 0$, $j_0 > 0$, with $j_0 - i_0 \leq k+1$, such that

$$w|_{\{i_0+1,\dots,j_0-1\}} = w'|_{\{i_0+1,\dots,j_0-1\}}, \quad w_{i_0} \neq w'_{i_0} \quad \text{and} \quad w_{j_0} \neq w'_{j_0}.$$

As the curves β_A and β_B intersect at a single point, and as the homotopy group they generate is free, the union of lifts of these two loops to $\widetilde{\mathrm{dom}}(\mathcal{F})$ is a complete binary tree (as in Figure 21).

Because of this, and because the intersection between $\hat{\alpha}$ and $T_1\hat{\alpha}$ is geometrically transverse, either $w_{i_0} = w'_{j_0} = A$ and $w_{j_0} = w'_{i_0} = B$, or $w_{i_0} = w'_{j_0} = B$ and $w_{j_0} = w'_{i_0} = A$. In particular, denoting u_{-2}, u_{-1}, u_1 and u_2 the times in $\widehat{\alpha}$ corresponding to resp. $\beta_{w_{i_0}}(1/2)$, $\beta_{w_{i_0}}(1)$, $\beta_{w_{j_0}}(0)$ and $\beta_{w_{j_0}}(1/2)$, and u'_{-2}, u'_{-1}, u'_1 and u'_2 the times in $T_1\widehat{\alpha}$ corresponding to resp. $\beta_{w'_{i_0}}(1/2)$, $\beta_{w'_{i_0}}(1)$, $\beta_{w'_{j_0}}(0)$ and $\beta_{w'_{j_0}}(1/2)$

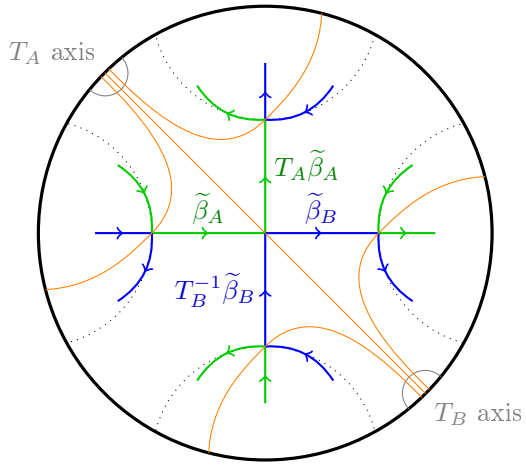

Figure 21: The configuration of Proposition 9.18 in $\widetilde{\mathrm{dom}}\mathcal{F} \simeq \mathbb{H}^2$: the trajectories that are lifts of β_A are in green, and the ones that are lifts of β_B in blue; the leaves of the foliation are in orange.

- Either $\phi_{\widehat{\alpha}(u_{-2})}$ is above $\phi_{T_1\widehat{\alpha}(u'_{-2})}$ relative to $\phi_{\widehat{\alpha}(u_{-1})} = \phi_{T_1\widehat{\alpha}(u'_{-1})}$, and $\phi_{\widehat{\alpha}(u_2)}$ is below $\phi_{T_1\widehat{\alpha}(u'_2)}$ relative to $\phi_{\widehat{\alpha}(u_1)} = \phi_{T_1\widehat{\alpha}(u'_1)}$.
- Or $\phi_{\widehat{\alpha}(u_{-2})}$ is below $\phi_{T_1\widehat{\alpha}(u'_{-2})}$ relative to $\phi_{\widehat{\alpha}(u_{-1})} = \phi_{T_1\widehat{\alpha}(u'_{-1})}$, and $\phi_{\widehat{\alpha}(u_2)}$ is above $\phi_{T_1\widehat{\alpha}(u'_2)}$ relative to $\phi_{\widehat{\alpha}(u_1)} = \phi_{T_1\widehat{\alpha}(u'_1)}$.

In both cases the two transverse paths $\widehat{\alpha}|_{[u_{-2},u_2]}$ and $T_1\widehat{\alpha}|_{[u'_{-2},u'_2]}$ intersect $\mathcal{F}$-transversally. Because $j_0 - i_0 \leq k+1$, we have that $u_2 - u_{-2} \leq 1$ and $u'_2 - u'_{-2} \leq 1$. In particular, this implies that the two paths $\alpha_1\alpha_2 \cdot \alpha_1\alpha_2$ and $\alpha_2\alpha_1 \cdot \alpha_2\alpha_1$ intersect $\mathcal{F}$-transversally at the marked point.

We now prove that for $i, j, k, \ell \geq 1$, the paths $\alpha_1\alpha_2^i \cdot \alpha_1^k\alpha_2$ and $\alpha_2\alpha_1^j \cdot \alpha_2^\ell\alpha_1$ intersect $\mathcal{F}$-transversally at the marked point. To fix notations, we suppose that the leaf passing through the left end of $\alpha_1\alpha_2\cdot$ is over the leaf passing through the left end of $\alpha_2\alpha_1\cdot$ relative to the leaf passing through the right end of both paths $\alpha_1\alpha_2\cdot$ and $\alpha_2\alpha_1\cdot$. We want to prove that the leaf passing through the left end of $\alpha_1\alpha_2^i\cdot$ is over the leaf passing through the left end of $\alpha_2\alpha_1^j\cdot$ relative to the leaf passing through the right end of both paths $\alpha_1\alpha_2^i\cdot$ and $\alpha_2\alpha_1^j\cdot$. This will prove the proposition, as the reasoning for the right parts of the paths $\cdot\alpha_1\alpha_2$ and $\cdot\alpha_2\alpha_1$ is identical.

Suppose first that the leaves passing through the left ends of respectively $\alpha_1\cdot$ and $\alpha_2\cdot$ are not comparable (meaning that none of them is in the left of the other). Then, the leaf passing through the left end of $\alpha_2\cdot$ has to be above the leaf passing through the left end of $\alpha_1\cdot$ relative to the leaf passing through their common right end, as by hypothesis the leaf passing through the left end of $\alpha_1\alpha_2\cdot$ is over the leaf passing through the left end of $\alpha_2\alpha_1\cdot$ relative to the leaf passing through their common right end. This suffices to get the desired property.

Suppose now that the leaves passing through the left ends of respectively $\alpha_1\cdot$ and $\alpha_2\cdot$ are comparable (meaning that one of them is contained in the left of the other). This means that one of the two paths $\alpha_1\cdot$ and $\alpha_2\cdot$ is homotopic (relative to endpoints) to a subpath of the other; more precisely, exchanging α_1 and α_2 if necessary, there exists a

path β_2 and $p > 0$ such that (up to homotopy) $\alpha_2\cdot = \beta_2\alpha_1^p\cdot$, that $\alpha_1\cdot$ is not equivalent to a suffix of $\beta_2\cdot$, and that β_2 is not homotopically trivial (otherwise the homotopy type of α_2 would be a power of the one of α_1, which is impossible).

When α_1 and β_2 are seen as words in β_A and β_B (the loops generating the fundamental group of $\check{\mathrm{dom}}\mathcal{F}$), the length of $\alpha_1\beta_2\cdot$ is bigger than the length of $\alpha_1\cdot$. Moreover, $\alpha_1\cdot$ is not a suffix of $\beta_2\cdot$. This implies that the leaves at the left end of $\alpha_1\cdot$ and $\alpha_1\beta_2\cdot$ are not comparable. Recall that by hypothesis, the leaf passing through the left end of $\alpha_1\beta_2\alpha_1^p\cdot$ is over the leaf passing through the left end of $\beta_2\alpha_1^{p+1}\cdot$ relative to the leaf passing through their common right end, so the leaf passing through the left end of $\alpha_1\beta_2\cdot$ is above the leaf passing through the left end of $\alpha_1\cdot$ relative to the leaf passing through their common right end.

Now, let us compare the leaves on the left ends of $\alpha_1\alpha_2^i\cdot$ and $\alpha_2\alpha_1^j\cdot$. As $\alpha_1\alpha_2^i\cdot = \alpha_1(\beta_2\alpha_1^p)^i\cdot$, if we compare successfully the leaves on the left ends of suffixes of them, namely $\alpha_1\beta_2\alpha_1^p\cdot$ and $\alpha_1^{p+j}\cdot$, we are done. But we already know that the leaf on the left end of $\alpha_1\beta_2\cdot$ is above the leaf on the left end of $\alpha_1\cdot$ relative to the leaf passing through their common right end, so the leaf on the left end of $\alpha_1\alpha_2^i\cdot$ is above the leaf on the left end of $\alpha_2\alpha_1^j\cdot$ relative to the leaf passing through their common right end. This proves the proposition. □

9.4 Setting

We set here some notations for the two next paragraphs.

Let $f \in \mathrm{Homeo}_0(S)$, and γ a closed geodesic with a geometric auto-intersection associated to the deck transformation T_1 (in the sense of Definition 8.1). Denote T_2 the deck transformation so that $T = T_2T_1$ is a deck transformation associated to the closed geodesic γ. We suppose that $(\gamma, \ell(\gamma)) \in \rho(f)$.

By (iii) of Proposition 4.1, there exists a fixed point x of f having rotation vector $(\gamma, \ell(\gamma))$. Consider the foliation $\mathcal{F}$ and the isotopy I given by Theorem 9.2. We denote by $\hat{f}$ the canonical lift of f to $\widetilde{\mathrm{dom}}\mathcal{F}$. As lifts of x to $\tilde{S}$ are not fixed by the lift $\tilde{f}$ of f to $\tilde{S}$, the point x belongs to $\mathrm{dom}\,\mathcal{F}$; this allows to consider a closed transverse loop α of S associated to the trajectory of x which is homotopic to the closed geodesic γ. This loop is admissible of order 1. We denote by $\tilde{\alpha}$ a lift of α to $\tilde{S}$ which corresponds to the deck transformation T and by $\hat{\alpha}$ a lift of $\tilde{\alpha}$ to $\widetilde{\mathrm{dom}}(\mathcal{F})$.

By Proposition 9.18, the loops $\tilde{\alpha}$ and $T_1\tilde{\alpha}$ intersect $\mathcal{F}$-transversally at $\tilde{\alpha}(t_1) = T_1^{-1}\tilde{\alpha}(t_2)$, for $t_1 < t_2 < t_1 + 1$. We denote $\alpha_1 = \alpha|_{[t_1,t_2]}$ and $\alpha_2 = \alpha_{[t_2,t_1]}$.

Note that, for any $n \geq 1$, the transverse paths $(\alpha_1\alpha_2)^n$ and $(\alpha_2\alpha_1)^n$ are admissible of order $n + 1$.

9.5 Creation of new periodic points

We use notation from Subsection 9.4.

As a preliminary to the existence of a rotational horseshoe (Theorem 9.27), we prove the existence, for any finite word $(w_i) \in \{1, 2\}^k$, of periodic orbits rotating in the direction $T_{w_1} \dots T_{w_k}$ (Proposition 9.26). Note that the periods we get for these periodic orbits are better than the ones that can be obtained from Theorem 9.27.

Let $(w_i) \in \{1, 2\}^{\mathbb{Z}/k\mathbb{Z}}$ be a periodic word of length k. We suppose that its period k is minimal.

Let us consider the smallest periodic word $(\overline{w}_j) \in \{1,2\}^{\mathbb{Z}/m\mathbb{Z}}$ of the form $(1\,2)^\ell$ or $(2\,1)^\ell$ obtained from (w_i) by adding some letters (hence $m = 2\ell$).

It can be seen that if we break the word (w_i) into blocks $(b_j)_{j=1}^{j_0}$ of consecutive identical letters (counting the first and the last blocks together as only one block if they contain the same letters), then (notice that j_0 is even)

$$\ell = \frac{1}{2}\left(k + \sum_{j=1}^{j_0}(\text{length}(b_j) - 1)\right) = k - \frac{j_0}{2} < k.$$

For example, if $(w_i) = (1\,2\,2\,1\,2\,1)$, then the smallest word of the form $(1\,2)^\ell$ or $(2\,1)^\ell$ obtained from (w_i) by adding some letters is $(1\,2\,\underline{1}\,2\,1\,2\,1\,\underline{2})$ and so $\ell = 4$.

Proposition 9.26. *Let $w = (w_i) \in \{1,2\}^{\mathbb{Z}/k\mathbb{Z}}$ be a periodic word of length k, such that $w_1 = w_k$. Let $\ell = \ell(w_i)$ be defined as above. For any $r \geq 1$ and any $q \geq r\ell + 3$, there exists a periodic point x for f, with a lift $\tilde{x}$ to $\tilde{S}$ such that*

$$\tilde{f}^q(\tilde{x}) = \big(T_{w_1} \dots T_{w_k}\big)^r(\tilde{x}).$$

Proof. If $k \geq 2$, as (w_i) is periodic of period $k \geq 2$, either it is equal to $(12)^\infty$, and there is nothing to prove (as this word corresponds to the initial rotation vector), or it contains at least one block b_j of length ≥ 2. By cyclically permuting the letters of the word w, we can suppose this block is b_1. Indeed, if we find a point $\tilde{y}$ such that

$$\tilde{f}^q(\tilde{y}) = T_{w_\xi} T_{w_{\xi+1}} \dots T_{w_k} T_{w_1} \dots T_{w_{\xi-1}} \tilde{y}$$

for some integer $\xi \in [1,k]$, then the point $\tilde{x} = T_{w_1} \dots T_{w_{\xi-1}} \tilde{y}$ will satisfy the proposition. Changing the roles of 1 and 2 if necessary, we can suppose that this block b_1 is made of 1s (hence the word (w_i) starts as $1^{\text{length}(b_1)}$).

Consider the finite word

$$a = a_1 a_2 \dots a_n = 1\,2\big(\overline{w}_3 \dots \overline{w}_m \overline{w}_1 \overline{w}_2\big)^r 1\,2$$

(note that $\overline{w}_3 = w_2 = 1$ and $\overline{w}_2 = 2$). As $a = (1\,2)^{rl+2}$, the associated transverse path $\alpha_{a_1} \dots \alpha_{a_n} = (\alpha_1\alpha_2)^{rl+2}$ is admissible of order $r\ell + 3$.

The last statement of Proposition 9.18 implies that for any $k, k' \geq 1$, the marked path $\alpha_2\alpha_1^k \cdot \alpha_2 \cdot \alpha_1^{k'}\alpha_2$ has self $\mathcal{F}$-transverse intersection at its marked points (and the same holds for $\alpha_1\alpha_2^k \cdot \alpha_1 \cdot \alpha_2^{k'}\alpha_1$). Using Proposition 9.7 and the first self transverse intersection, we can remove a letter 2 between the positions 2 and $n-3$ from the word $a_1 \dots a_n$ to obtain a new word which is admissible of order $r\ell + 3$. In the same way, by using the second self transverse intersection, we can remove a letter 1 between the positions 2 and $n-3$ from the word $a_1 \dots a_n$ to obtain a new word which is admissible of order $r\ell + 3$. So, by successive applications of Proposition 9.7, we get that the path

$$\beta = \alpha_1\alpha_2\big(\alpha_{w_2} \dots \alpha_{w_k}\alpha_{w_1}\big)^r \alpha_{\overline{w}_2}\alpha_1\alpha_2 = \alpha_1\alpha_2\big(\alpha_1 \dots \alpha_{w_k}\alpha_1\big)^r \alpha_2\alpha_1\alpha_2$$

is also admissible of order $r\ell + 3$. Let $\hat{\beta}$ be the lift of β corresponding to the deck transformation $T_1T_2\big(T_{w_2} \dots T_{w_k}T_{w_1}\big)^r T_2T_1T_2$. Remark that the paths $\hat{\beta}$ and $T_1T_2\big(T_{w_2} \dots T_{w_k}T_{w_1}\big)^r T_2^{-1}T_1^{-1}\hat{\beta}$ intersect $\widehat{\mathcal{F}}$-transversally, simply because β starts with

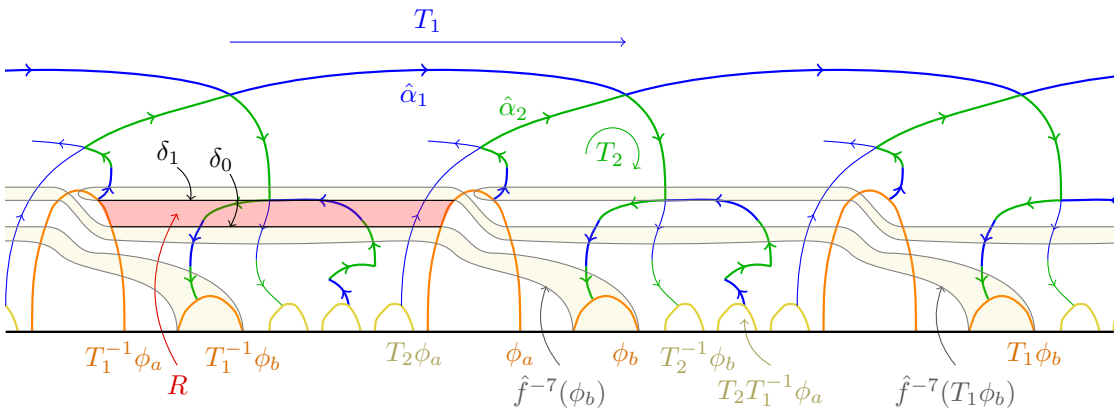

Figure 22: Configuration of Theorem 9.27. Note that we do not know *a priori* whether the leaf ϕ_a is located in the left of $T_2\phi_a$ or is below $T_2\phi_a$ relative to $T_1T_2^2\phi_a$.

$\alpha_1\alpha_2 \cdot \alpha_1^i\alpha_2\dots$ and ends with $\dots\alpha_2\alpha_1^j \cdot \alpha_2\alpha_1\alpha_2$ (Proposition 9.18). Thus, by Theorem 9.30 (which is Theorem M of [LCT18b]), for any $q \geq r\ell + 3$, there exists a point $\tilde{y} \in \widetilde{S}$ such that

$$\tilde{f}^q(\tilde{y}) = T_1T_2\big(T_{w_2}\dots T_{w_k}T_{w_1}\big)^r T_2^{-1}T_1^{-1}(\tilde{y}).$$

We can then take $\tilde{x} = T_1T_2^{-1}T_1^{-1}(\tilde{y})$. to obtain a point $\tilde{x}$ which satisfies the conclusion of the proposition.

The arguments work identically when the initial word (w_i) is constant (*i.e.* is equal to either 1^∞ or 2^∞). □

9.6 Horseshoe

We now come to the main theorem of this section, that concerns the existence of a rotational horseshoe (Theorem E of the introduction). Again, we use notation from Subsection 9.4.

Theorem 9.27. *Let $f \in \mathrm{Homeo}_0(S)$, and γ a closed geodesic with a geometric auto-intersection associated to the deck transformation T_1 (in the sense of Definition 8.1). Denote T_2 the deck transformation such that $T = T_1T_2$ is the deck transformation associated to the closed geodesic γ.*

Suppose that $(\gamma, \ell(\gamma)) \in \rho(f)$. Then, f^7 has a topological horseshoe associated to the deck transformations T_1, T_1^2, T_2, T_1T_2, T_2T_1 and $T_1T_2T_1$.

In particular, this implies that $h_{top}(f) \geq \log 7/5$.

Proof. The configuration of the beginning of the proof is depicted in Figure 22. In particular, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are two lifts of α_1 and α_2 to $\widetilde{\mathrm{dom}}(\mathcal{F})$ that have $\hat{\alpha}(t_2)$ as final point. By abuse of notation, we denote by T_1 and T_2 the lifts to $\widetilde{\mathrm{dom}}(\mathcal{F})$ of the corresponding deck transformations of $\tilde{S}$ which are respectively associated to $\hat{\alpha}_1$ and $\hat{\alpha}_2$.

In $\widetilde{\mathrm{dom}}\mathcal{F}$, denote $\phi_a = T_2^{-2}T_1^{-1}(\phi_{\hat{\alpha}(t_2)})$ and $\phi_b = T_2^2T_1T_2(\phi_{\hat{\alpha}(t_2)})$.

Remark that any lift of $(\alpha_1\alpha_2)^k$ or $(\alpha_2\alpha_1)^k$ is admissible or order $k+1$. By successive applications of Proposition 9.7, allowed by Proposition 9.18, we get that

- $\hat{f}^6(\phi_a) \cap \phi_b \neq \emptyset$ because $\alpha_1\alpha_2^4\alpha_1\alpha_2$ is admissible of order 6;

- $\hat{f}^6(\phi_a) \cap T_1\phi_b \neq \emptyset$ because $\alpha_1\alpha_2^2\alpha_1\alpha_2^2\alpha_1\alpha_2$ is admissible of order 6;
- $\hat{f}^7(\phi_a) \cap T_1^2\phi_b \neq \emptyset$ because $\alpha_1\alpha_2^2\alpha_1^2\alpha_2^2\alpha_1\alpha_2$ is admissible of order 7.

For example, for the first one, the path $(\alpha_1\alpha_2)^5$ is admissible of order 6, hence so does $\alpha_1\alpha_2^4\alpha_1\alpha_2$.

Note that, by Proposition 9.9, we also have $\hat{f}^7(\phi_a) \cap \phi_b \neq \emptyset$ and $\hat{f}^7(\phi_a) \cap T_1\phi_b \neq \emptyset$.

Construction of the rectangle R – As in Section 3.1 of [LCT18b], one can define

$$R_a = \bigcap_{k\in\mathbb{Z}} R(T_1^k\phi_a),$$

and the set $\mathcal{X}_p$ of paths joining $T_1^{-1}\phi_a$ to ϕ_a whose interior is a connected component of $T_1^p\hat{f}^{-7}(\phi_b) \cap R_a$. The following lemma proves that $\mathcal{X}_0, \mathcal{X}_1 \neq \emptyset$.

Lemma 9.28. *Every simple path $\delta : [c,d] \to \widetilde{\mathrm{dom}}\mathcal{F}$ that joins $T_1^{-p_0}\phi_a$ to $T_1^{p_1}\phi_a$, with $p_0, p_1 > 0$, and which is T_1-free, meets $L(\phi_a)$.*

Similarly, for any $t \in \mathbb{R}$, if $\hat{f}^{-7}(\phi_b((-\infty,t]))$ meets $T_1^{-p}\phi_a$ for some $p > 0$, then it also meets $L(\phi_a)$.

For the first part of the lemma, the idea of proof is that, if the path δ meets neither $L(\phi_a)$ nor $T_1^{p_1+p_0}\phi_a$, then this path, together with the leaves $T_1^{-p_0}\phi_a$ and $T_1^{p_1}\phi_a$, separates the leaves ϕ_a and $T_1^{p_0+p_1}\phi_a$, which implies (by an application of Jordan theorem) that δ is not T_1-free. The case where the path δ meets $T_1^{p_1+p_0}\phi_a$ but not $T_1^{p_1+2p_0}\phi_a$ leads to a similar contradiction, and so on. For a more detailed proof, see Lemma 10 of [LCT18b]. The proof of the second part of the lemma is identical.

By what we have just said, using Lemma 9.28 (and similarly to Lemma 11 of [LCT18b]), the sets $\mathcal{X}_0$ and $\mathcal{X}_1$ are nonempty. Moreover, because the sets $T_1^k\hat{f}^{-7}(\phi_b)$ are pairwise disjoint, two elements of respectively $\mathcal{X}_0$ and $\mathcal{X}_1$ are disjoint.

Lemma 9.29. *There is a path $\delta_1 \in \mathcal{X}_1$, and a path $\delta_0 \in \mathcal{X}_0$ lying in the connected component of the complement of $R_a^{\complement} \cup \delta_1$ containing $T_1^{-1}\phi_b$.*

Before proving the lemma, let us point out that because of $\mathcal{F}$-transverse intersections (last conclusion of Proposition 9.18), we have, for any $k \in \mathbb{Z}^*$ and any $n \in \mathbb{Z}$,

$$\hat{f}^n(T_1^k\phi_b) \cap \phi_b = \emptyset. \tag{9.1}$$

This implies that δ_1 is disjoint from $T_1^{-1}\phi_b$, and hence that $T_1^{-1}\phi_b$ lies in the complement of $R_a^{\complement} \cup \delta_1$.

Proof. Note that the union of elements of $\mathcal{X}_1$ forms a compact subset of $T_1\hat{f}^{-7}(\phi_b)$, so there are finitely many elements of $\mathcal{X}_1$. Consider the first one, δ_1, for the order on $T_1\hat{f}^{-7}(\phi_b)$ induced by some parametrization of ϕ_b, and denote $\hat{f}^7(\delta_1) = T_1\phi_b|_{[t_1,t_2]}$. By Lemma 9.28, second part, $T_1\hat{f}^{-7}(\phi_b(t_2))$ is the first intersection point of $T_1\hat{f}^{-7}(\phi_b)$ (again, for the order induced by some oriented parametrization of ϕ_b) with $T_1^{-1}\phi_a$; in particular the path $T_1\hat{f}^{-7}(\phi_b|_{(-\infty,t_2]})$ meets $T_1^{-1}\phi_a$ at a single point. The complement of $L(T_1^{-1}\phi_a) \cup T_1\hat{f}^{-7}(\phi_b|_{(-\infty,t_2]})$ has two connected components. We denote by A the one containing ϕ_b.

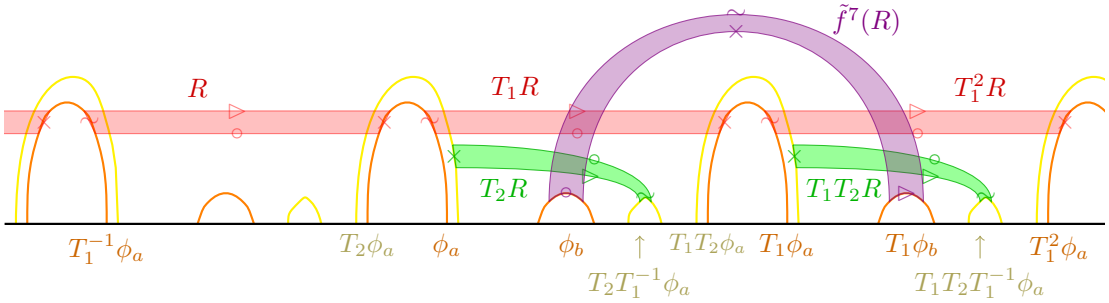

Figure 23: Configuration of Theorem 9.27, the image of the rectangle R by $\tilde{f}^7$ has Markovian intersections with T_1R, T_1^2R, T_2R and T_1T_2R. It also has Markovian intersections with T_2T_1R and $T_1T_2T_1R$ (not represented in the figure). Note that the relative position of ϕ_a and $T_2\phi_a$ is different from Figure 22, but *a priori* possible.

As the set $\hat{f}^{-7}(\phi_b)$ meets $T_1^{-1}\phi_a$, this set is not contained in A. Consider the first intersection point $\hat{f}^{-7}(\phi_b)(t_1')$ between ∂A and $\hat{f}^{-7}(\phi_b)$. This point must belong to $T_1^{-1}(\phi_a)$ as $\hat{f}^{-7}(\phi_b) \cap T_1\phi_b = \emptyset$ by (9.1). Lemma 9.28, second part implies that $\hat{f}^{-7}(\phi_{b|(-\infty,t_1']}) \cap R(\phi_a) \neq \emptyset$, which gives a path δ_0 and proves Lemma 9.29. $\square$

Consider two paths $\delta_0 \in \mathcal{X}_0$ and δ_1 in $\mathcal{X}_1$ given by Lemma 9.29. Similarly to what is done in the proof of Proposition 12 of [LCT18b], take β the path made of the bounded connected component of $T_1^{-1}\phi_a \setminus (\delta_0 \cup \delta_1)$, and β' the path made of the bounded connected component of $\phi_a \setminus (\delta_0 \cup \delta_1)$.

It allows to define the topological rectangle R bounded by the four curves β, β', δ_0 and δ_1. Lemma 9.29 implies that in the direct orientation, the paths are ordered as: $\delta_1\beta\delta_0\beta'$. Note also that the set $\hat{f}^7(R)$ is a topological rectangle, with two edges which are subsets of resp. ϕ_b and $T_1\phi_b$, the two others being images of pieces of resp. $T^{-1}\phi_a$ and ϕ_a.

Proof of the existence of Markovian intersections – Because of $\mathcal{F}$-transverse intersections (last conclusion of Proposition 9.18), we have, for any $k \geq 0$ and any $n \in \mathbb{Z}$,

$$\hat{f}^n(\phi_a) \cap T_1^k T_2 T_1^{-1}\phi_a = \emptyset; \tag{9.2}$$

for any $k \in \mathbb{Z}^*$ and any $n \in \mathbb{Z}$,

$$\hat{f}^n(\phi_a) \cap T_1^k T_2 \phi_a \quad = \quad \hat{f}^n(T_1^k\phi_b) \cap T_2^{-1}\phi_b \quad = \quad \hat{f}^n(T_2\phi_b) \cap T_1^k\phi_b \quad = \quad \emptyset, \tag{9.3}$$

and similarly, for any $k, n \in \mathbb{Z}$ with $(k,n) \neq (0,0)$,

$$\hat{f}^n(\phi_a) \cap T_1^k\phi_a = \emptyset. \tag{9.4}$$

The rectangle $\hat{f}^7(R)$ is disjoint from :

- ϕ_a, $T_1\phi_a$ and $T_1^2\phi_a$, by (9.4);
- $T_1T_2\phi_a$ and $T_1^2T_2\phi_a$, by the first intersection of (9.3);

- $T_2\phi_a$. Indeed, the closure of the set $L(\phi_b) \cup \hat{f}^7(R) \cup L(T_1\phi_b)$ has two unbounded connected components in its complement, we denote by C the one containing ϕ_a. Because of the the orientation of ∂R (which is a consequence of Lemma 9.29), the closure of C contains $\hat{f}^7(\beta)$, but is disjoint from $\hat{f}^7(\beta')$ (recall that β is a piece of $T_1^{-1}\phi_a$). But by the first intersection of (9.3), $T_2\phi_a$ is disjoint from $\hat{f}^7(\beta)$, so $\hat{f}^7(R)$ is disjoint from $T_2\phi_a$, as $T_2\phi_a \cap C \neq \emptyset$.

- $T_2T_1^{-1}\phi_a$ and $T_1T_2T_1^{-1}\phi_a$. Indeed, by the same argument about orientation as before, it suffices to prove that the intersections $\hat{f}^7(\phi_a) \cap T_2T_1^{-1}\phi_a$ and $\hat{f}^7(T_1^{-1}\phi_a) \cap T_1T_2T_1^{-1}\phi_a$ are empty, which is true by (9.2).

The rectangles T_1^kR (for $k \in \mathbb{Z}$) are disjoint from the sets $T_1^\ell\phi_b$, by (9.1).

The rectangle T_2R is disjoint from the sets ϕ_b and $T_1\phi_b$. Indeed, by the same reasoning about orientation as before, we just have to prove that the intersections $T_2\hat{f}^{-7}(T_1\phi_b) \cap \phi_b$ and $T_2\hat{f}^{-7}(\phi_b) \cap T_1\phi_b$ are empty, which is true by the two last intersections of (9.3) .

All these facts, combined with Homma's theorem (Theorem 9.14), imply that the intersections of $\tilde{f}^7(R)$ with the following sets are Markovian (see Figures 23 and 18): T_1R, T_1^2R, T_2R, T_1T_2R, T_2T_1R and $T_1T_2T_1R$. For example, Homma's theorem asserts that there exists a homeomorphism $h : \widetilde{S} \to \mathbb{R}^2$ such that

$$h(\phi_a) = \{0\} \times \mathbb{R},\ h(T_1\phi_a) = \{1\} \times \mathbb{R},\ h(\delta_0) = [0,1] \times \{0\},\ h(\delta_1) = [0,1] \times \{1\}.$$

Hence, because $R(\phi_a) \cup T_1R \cup L(T_1\phi_a)$ separates ϕ_b and $T_1\phi_b$ (this is a consequence of the previous listed facts),

$$h(\hat{f}^7(\delta_0)) \subset h(\phi_b) \subset]0,1[\times(-\infty,0[, \quad h(\hat{f}^7(\delta_1)) \subset h(T_0\phi_b) \subset]0,1[\times]1,+\infty[,$$

and similarly, because $R(\phi_b) \cup \hat{f}^7(R) \cup L(T_1\phi_b)$ separates ϕ_a and $T_1\phi_a$, $h(\hat{f}^7(R)) \subset (0,1) \times \mathbb{R}$. The fact that the other intersections are Markovian can be proved similarly, using the previous listed facts. □

By a proof which is very similar, we can get the following statement, which is a reformulation of [LCT18b, Section 3] to fit with the definition of rotational horseshoe we use here.

Theorem 9.30. *Let S be an orientable surface, $f \in \mathrm{Homeo}_0(S)$, $\mathcal{F}$ a transverse foliation in the sense of Theorem 9.2 and $\alpha : [0,1] \to \mathrm{dom}(\mathcal{F})$ an $\mathcal{F}$-transverse curve. Denote $\hat{\alpha}$ a lift of α to the universal cover $\widetilde{\mathrm{dom}}(\mathcal{F})$ of $\mathrm{dom}(\mathcal{F})$.*

Suppose that α is admissible of order 1, and that there exists a deck transformation T of $\widetilde{\mathrm{dom}}(\mathcal{F})$ and $0 < t_1 < t_2 < 1$ such that $\hat{\alpha}$ and $T\hat{\alpha}$ have an $\mathcal{F}$-transverse intersection at $\hat{\alpha}(t_2) = T\hat{\alpha}(t_1)$.

Then, for any $r \geq 2$, f^r has a topological horseshoe associated to the deck transformations $T, T^2, \dots, T^r$.

The proof of a similar statement can be found in [LCT18b], however, as it is very similar to the one of Theorem 9.27, we include a short sketch of proof.

Sketch of proof. We give a sketch of proof for $r = 2$.

Denote $\hat{\alpha}_0 = \hat{\alpha}|_{[0,t_1]}$, $\hat{\alpha}_1 = \hat{\alpha}|_{[t_1,t_2]}$ and $\hat{\alpha}_2 = \hat{\alpha}|_{[t_2,1]}$ (see Figure 24). We also set $\phi_a = \phi_{\hat{\alpha}(0)}$ and $\phi_b = \phi_{\hat{\alpha}(1)}$

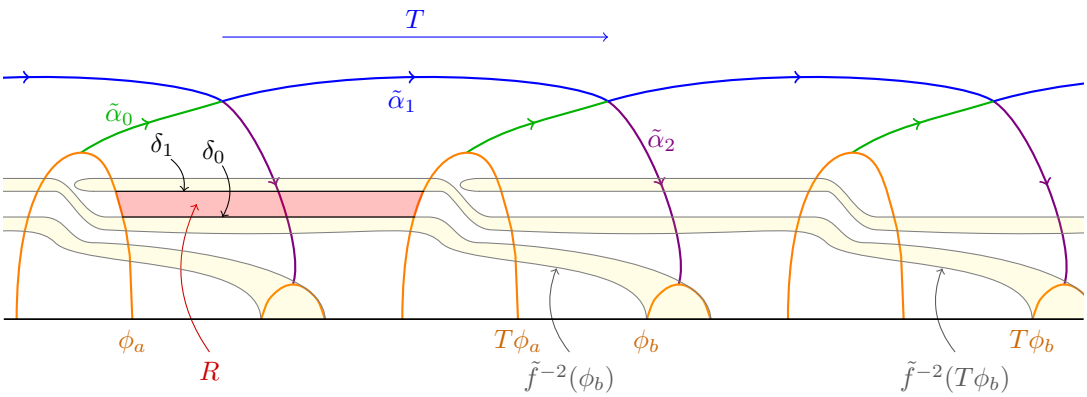

Figure 24: Configuration of Theorem 9.30.

Then, applying Proposition 9.7, the paths $\hat{\alpha}_0\tilde{\alpha}_1\tilde{\alpha}_2$ and $\hat{\alpha}_0 T^{-1}\tilde{\alpha}_2$ are admissible of order 1, and the path $\hat{\alpha}_0\hat{\alpha}_1(T\hat{\alpha}_1)(T\hat{\alpha}_2)$ is admissible of order 2. As in the proof of Theorem 9.27, it is possible to find two pieces of resp. $\hat{f}^{-2}(\phi_b)$ and $\hat{f}^{-2}(T\phi_b)$, each of one meeting ϕ_a and $T\phi_a$ only at its endpoints. The set R bounded by these paths and the bounded pieces of ϕ_a and $T\phi_a$ linking their ends is a rectangle, such that the intersections $\hat{f}^2(R) \cap TR$ and $\hat{f}^2(R) \cap T^2R$ are Markovian. □

10 Two transverse closed geodesics

Recall that S is an orientable surface of finite type and negative Euler characteristic. Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be two geodesic lines of $\tilde{S} = \mathbb{H}^2$ which project to closed geodesics of S. We denote by T_i the deck transformation associated to $\tilde{\gamma}_i$, for $i = 1, 2$. For any element w of the semigroup $\langle T_1, T_2\rangle_+$ generated by T_1 and T_2, we denote by $\tilde{\gamma}(w)$ the geodesic axis of the deck transformation w.

In this last section, we prove Theorem F of the introduction. As the previous one, it is based on forcing theory of le Calvez-Tal. It deals with the case where in the rotation set, there are two closed geodesics with geometric intersection, each one associated with nonzero rotation speed. Contrary to the last section where we got the existence of a rotational horseshoe, here the proof does not give such an object associated to the deck transformations T_1 and T_2 (in fact, we even do not know if the two initial rotation vectors are realised by periodic orbits or not), but we get similar consequences.

The following is Theorem F of the introduction.

Theorem 10.1. *Suppose that there exist nonzero rotation vectors of directions $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in $\rho(f)$ and that the geodesics $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ cross. Then, for any element w in $\langle T_1, T_2\rangle_+$, there are nonzero vectors of direction $\tilde{\gamma}(w)$ in $\rho(f)$.*

In the course of the proof of the theorem, we can also recover the fact that, in this situation, the topological entropy of f is positive. It was already known as a consequence of Theorem 9.27 in the case where one of the closed geodesics γ_1 or γ_2 has an autointersection and a consequence of Corollary 8.11 otherwise.

Observe that, if the element w does not belong to the cyclic groups $\langle T_1\rangle$ nor $\langle T_2\rangle$, such a geodesic $\tilde{\gamma}(w)$ has a positive transverse intersection with one geodesic among $\tilde{\gamma}_1$

and $\tilde{\gamma}_2$ and a negative transverse intersection with the other one. By Proposition 4.1, this implies the following corollary.

Corollary 10.2. *Suppose that there exist nonzero rotation vectors of directions $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in $\rho(f)$. Then, for any element w in $\langle T_1, T_2\rangle_+$ which does belong neither to the cyclic group $\langle T_1\rangle$ nor to the cyclic group $\langle T_2\rangle$, there are infinitely many periodic orbits whose rotation vector is in the direction $\tilde{\gamma}(w)$.*

Observe that the above corollary is equivalent to Theorem 10.1. Actually, to prove Theorem 10.1, we will prove Corollary 10.2. The rest of this section is devoted to the proof of Theorem 10.1. In the first subsection, we will distinguish two cases in which the proof must be carried out. The second subsection is devoted to a useful notion that we use. The two following subsections are devoted to the proof of Theorem 10.1 in each of those cases.

10.1 Two cases

Fix $i = 1, 2$. Take a rational number $\frac{p_i}{q_i} > 0$ such that $(\tilde{\gamma}_i, \frac{p_i}{q_i}\ell(\gamma_i))$ belongs to $\rho(\tilde{f})$ but is not an extremal point of $\rho(\tilde{f})$.

Recall that $\tilde{f}$ extends continuously to $\tilde{S} \simeq \overline{\mathbb{H}^2}$ by fixing all the points of $\partial\mathbb{H}^2$. Denote by $\gamma_{i,-}$ and $\gamma_{i,+}$ the endpoints of $\tilde{\gamma}_i$ on $\partial\mathbb{H}^2$, where $\tilde{\gamma}_i$ is oriented from $\gamma_{i,-}$ to $\gamma_{i,+}$. Denote by A_i the closed annulus $\big(\overline{\mathbb{H}^2} \setminus \{\gamma_{i,-}, \gamma_{i,+}\}\big)/\langle T_i\rangle$, by π_i the projection from the interior of A_i to S, and by f_i the homeomorphism of A_i induced by $\tilde{f}$ on A_i.

Lemma 10.3. *There exists a f_i-birecurrent point x_i of A_i with lift $\tilde{x}_i$ to $\mathbb{H}^2$ such that the two following properties are satisfied.*

1. $$\lim_{n\to\pm\infty} \tilde{f}^n(\tilde{x}_i) = \gamma_{i,\pm}$$

2. $$\lim_{n\to\pm\infty} \frac{1}{n} d\big(\pi_{\tilde{\gamma}_i}(\tilde{x}_i), \pi_{\tilde{\gamma}_i}(\tilde{f}^n(\tilde{x}_i))\big) = v_i > \frac{p_i}{q_i}\ell(\gamma_i).$$

3. *The orbit of $\tilde{x}_i$ under $\tilde{f}$ stays within a bounded distance from the geodesic $\tilde{\gamma}_i$.*

4. *The closure of the orbit of x_i does not contain fixed points of*[13] *$\tilde{f}$.*

Moreover, if the closed geodesic γ_i has an autointersection, we also require the point x_i to have a periodic orbit.

Proof. Suppose first that γ_i has no autointersection. Take $w > 0$ such that $(\tilde{\gamma}_i, w)$ is an extremal point of $\rho(f)$. Use Proposition 4.3 so that one of the following is true.

1. Either there exists a point x_i, which lifts to a recurrent point of A_i, with one lift $\tilde{x}_i \in \mathbb{H}^2$ realising the rotation vector $(\tilde{\gamma}_i, w)$. Moreover, the orbit of $\tilde{x}_i$ stays at a bounded distance from the geodesic $\tilde{\gamma}_i$ and the closure of the orbit of x_i does not contain fixed point of $\tilde{f}$. In this case, take $v_i = w$.

[13] Meaning that the closure of the orbit in S does not contain points that lift to fixed points of $\tilde{f}$.

2. Or, for any r rational strictly smaller than $\frac{w}{\ell(\gamma)}$, there exist a periodic orbit whose rotation number is $(\tilde{\gamma}_i, r\ell(\gamma))$. In this case, take $v_i = r_0\ell(\gamma) > \frac{p_i}{q_i}\ell(\gamma)$, for some $r_0 > \frac{p_i}{q_i}$ to find a point which satisfies the lemma.

If the closed geodesic γ_i has a transverse autointersection, fix a number $v = r_i\ell(\gamma) > \frac{p_i}{q_i}\ell(\gamma)$ where r_i is rational and $(\tilde{\gamma}, r_i\ell(\gamma))$ belongs to $\rho(f)$. Then, by Proposition 4.1, there exists a point x_i whose orbit is periodic and which realises this rotation vector. This point x_i satisfies the requirements of the lemma. □

We denote by $\mathcal{F}$ a foliation of S so that Theorem 9.2 is satisfied, by $\mathcal{F}_i$ the lift of the foliation $\mathcal{F}$ to A_i, by $\tilde{\mathcal{F}}$ its lift to $\tilde{S}$ and by $\hat{\mathcal{F}}$ its lift to $\widetilde{\mathrm{dom}}\mathcal{F}$. Denote by $\mathcal{I}^{\mathbb{Z}}_{\mathcal{F}_i}(x_i)$ a $\mathcal{F}_i$-transverse trajectory associated to the orbit of x_i under f_i. We use similar notation for $\mathcal{F}$, $\tilde{\mathcal{F}}$ or $\hat{\mathcal{F}}$-transverse trajectories. Choose respective lifts $\tilde{x}_1$ and $\tilde{x}_2$ of x_1 and x_2 to $\tilde{S}$ such that the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ have the same endpoints on $\overline{\mathbb{H}}^2$ as the geodesics $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. In particular, the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ meet at some point $\tilde{y}_0$. Finally, choose respective lifts $\hat{x}_1$ and $\hat{x}_2$ of $\tilde{x}_1$ and $\tilde{x}_2$ to $\widetilde{\mathrm{dom}}\mathcal{F}$ so that the trajectories $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_1)$ and $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_2)$ meet at some lift $\hat{y}_0$ of $\tilde{y}_0$. We will use the following properties of the transverse trajectory.

Lemma 10.4. *For $i = 1, 2$, the transverse trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ can be chosen in such a way that the following properties are satisfied.*

1. *There exists a closed neighbourhood K_i of $\mathcal{I}^{\mathbb{Z}}_{\mathcal{F}}(x_i)$ such that the supremum M_i of the diameters of $\mathcal{I}_{\tilde{\mathcal{F}}}(\tilde{z})$, with $z \in K_i$, is finite.*

2. *There exists $R_i > 0$ such that the transverse trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ stays in the R_i-neighbourhood of the geodesic $\tilde{\gamma}_i$.*

Proof. Denote by $U \subset S$ the complement of the projection on S of fixed points of $\tilde{f}$. For any point $x \in U$, it is possible to find a neighbourhood V of x so that, changing the trajectories to equivalent ones if necessary, the diameters of the lifts to $\tilde{S}$ of $\mathcal{I}_{\mathcal{F}}(y)$ are uniformly bounded for $y \in V$. As the closure of the orbit of the point x_i is compact and projects to U (Lemma 10.3), we can choose the transverse trajectories of x_1 and x_2 so that they project on compact sets contained in U. Compactness of those trajectories and the fact that the transverse trajectories can be chosen locally bounded yields the first point.

The second point is a consequence of the first point and the fact that the orbit of $\tilde{x}_i$ stays at a bounded distance from the geodesic $\tilde{\gamma}_i$. □

As $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ is an oriented immersed line, there is a natural order relation $<_i$ on this set. By abuse of notation, we also denote by $<_i$ the order relation on $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_i)$ induced by the orientation on those immersed lines. For any two points x and y on $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ (respectively on $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_i)$), we let

$$[x, y]_i = \left\{ z \in \mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_i) \mid x \leq_i z \leq_i y \right\}$$

(respectively

$$[x, y]_i = \left\{ z \in \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i) \mid x \leq_i z \leq_i y \right\}).$$

For any two leaves ϕ_1 and ϕ_2 of $\hat{\mathcal{F}}$ which meet $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_i)$, we set $\phi_1 <_i \phi_2$ if $\phi_1 \cap \mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_i) <_i \phi_2 \cap \mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_i)$. In this way, we can also define $[\phi_1, \phi_2]_i = [\phi_1 \cap \mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_i), \phi_2 \cap \mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_i)]_i$. We

use a similar notation for leaves of $\tilde{\mathcal{F}}$. When we use it in the case of leaves of $\tilde{\mathcal{F}}$, we tacitly choose points in the intersection between the leaf and the trajectory. Each time we use this notation, the choice of those points will be irrelevant.

Finally, for any segment J contained in $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$, and for any leaves $\phi_1, \phi_2, \dots, \phi_n$ of $\tilde{\mathcal{F}}$, we say that J meets the leaves $\phi_1, \phi_2, \dots, \phi_n$ in this order if there exist points $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n$ which belong respectively to $\phi_1 \cap \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i), \phi_2 \cap \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i), \dots, \phi_n \cap \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ such that the segment J meets the points $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n$ in this order.

The following lemma, which is roughly a simple consequence of Lemma 10.3 in terms of transverse trajectories, will be useful.

Lemma 10.5. *Let $k > 0$, $\epsilon = \pm 1$, $i = 1, 2$ and $0 < v_i' < v_i$. Let $(\phi_j)_{1 \leq j \leq \ell}$ be a sequence of leaves such that the segment $[\tilde{x}_i, \tilde{f}^k(\tilde{x}_i)]_i$ meets the leaves $\phi_1, \phi_2, \dots, \phi_\ell$ outside $\tilde{x}_i$ and $\tilde{f}^k(\tilde{x}_i)$ and in this order. Then, for infinitely many $n_i > 0$, there exists $r \geq \frac{v_i' n_i}{\ell(\gamma_i)}$ such that the segment $[\tilde{f}^{\epsilon n_i}(\tilde{x}_i), \tilde{f}^{\epsilon n_i + k}(\tilde{x}_i)]_i$ meets the leaves $T_i^{\epsilon r}\phi_1, T_i^{\epsilon r}\phi_2, \dots, T_i^{\epsilon r}\phi_\ell$ in this order.*

Proof. By Lemma 17 in [LCT18a], there exists a small disk $\tilde{D}$ around $\tilde{x}_i$ such that, for any point $\tilde{y}$ in $\tilde{D}$, the transverse trajectory $\mathcal{I}^k_{\tilde{\mathcal{F}}}(\tilde{y})$ associated to $(\tilde{f}_t(y))_{0 \leq t \leq k}$ meets the leaves $\phi_1, \phi_2, \dots \phi_\ell$ in this order. By Lemma 10.3, for infinitely many $n_i > 0$, $\tilde{f}^{\epsilon n_i}(\tilde{x}_i) \in T_i^{\epsilon r}(\tilde{D})$, with $r \geq \frac{v_i' n_i}{\ell(\gamma_i)}$. Hence the trajectory $\mathcal{I}^k_{\tilde{\mathcal{F}}}(\tilde{f}^{\epsilon n_i}(T_i^{-\epsilon r}\tilde{x}_i))$ meets the leaves $\phi_1, \phi_2, \dots \phi_\ell$ in this order. Taking the image under $T_i^{\epsilon r}$ proves the lemma. □

For $i = 1, 2$, observe that the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ satisfies one of the following conditions:

(C_1) either there exists a deck transformation $\tau_i \in \pi_1(S) \setminus \langle T_i \rangle$ and a leaf ϕ of $\tilde{\mathcal{F}}$ such that ϕ meets the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and $\tau_i \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$;

(C_2) or, for any deck transformation $\tau \in \pi_1(S) \setminus \langle T_i \rangle$, any leaf which meets $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ does not meet $\tau \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$.

To give some geometric intuition around this notion, observe that condition (C_2) amounts to saying that, on the surface S, the union of leaves which meet $\mathcal{I}^{\mathbb{Z}}_{\mathcal{F}}(\pi_i(x_i))$ is contained in an annulus which is embedded in S.

To carry out the proof of the theorem, we will distinguish the two following cases :

Case 1 : both trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ satisfy the condition (C_1).

Case 2 : one of the trajectories satisfies (C_2).

The two following sections are devoted to the proof of Theorem 10.1 in each of those two cases. In the first case, we prove that the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ have an $\tilde{\mathcal{F}}$-transverse intersection, which allows us to prove Theorem 10.1. In the second case, however, it is possible that such trajectories never have $\tilde{\mathcal{F}}$-transverse intersection, but it is possible to change one of the trajectories to obtain two trajectories with $\tilde{\mathcal{F}}$-transverse intersection.

The next paragraph is devoted to a notion which will be useful for our proof.

10.2 Essential intersection points

Take any two points $\tilde{x}$ and $\tilde{y}$ of $\tilde{S}$ which are not singularities of the foliation $\tilde{\mathcal{F}}$. For any point $\tilde{z} = \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})(t_1) = \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})(t_2)$, with $t_1, t_2 \in \mathbb{R}$, of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}) \cap \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$, we call *lifts of* $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ *and* $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ *associated to* $\tilde{z}$ any respective lifts $\hat{\mathcal{I}}_x$ and $\hat{\mathcal{I}}_y$ of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ to $\widetilde{\mathrm{dom}}\mathcal{F}$ which meet at a lift $\hat{z} = \hat{\mathcal{I}}_x(t_1) = \hat{\mathcal{I}}_y(t_2)$ of $\tilde{z}$. In case of multiple intersection points, *e.g.* when $\tilde{z}$ is an autointersection point of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$, notice that the values of the parameters t_1 and t_2 are important in this definition. However, to simplify notation, we will frequently drop the mention of those parameters when we use this notion.

Definition 10.6. A point $\tilde{z}$ on $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}) \cap \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ is an *essential intersection point* between $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ if there exist lifts $\hat{\mathcal{I}}_x$ and $\hat{\mathcal{I}}_y$ of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ to $\widetilde{\mathrm{dom}}\mathcal{F}$ associated to $\tilde{z}$ such that $\hat{\mathcal{I}}_x \setminus \left(\hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y\right)$ has two unbounded components which lie in different connected components of $\widetilde{\mathrm{dom}}\mathcal{F} \setminus \hat{\mathcal{I}}_y$.

Note that this definition is supported by the fact that all transverse trajectories in $\widetilde{\mathrm{dom}}(\mathcal{F})$ are proper. Observe that, if this definition holds, then any two lifts $\hat{\mathcal{I}}_x$ and $\hat{\mathcal{I}}_y$ of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ associated to $\tilde{z}$ will satisfy the above property.

Two trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$, with $\tilde{x} \in \tilde{S}$ and $\tilde{y} \in \tilde{S}$, are said to be *geometrically transverse* if there exist $\alpha_x, \alpha_y, \beta_x, \beta_y \in \partial\mathbb{H}^2$ such that the three following conditions are satisfied:

1. The sequence $\tilde{f}^n(\tilde{x})$ converges to $\alpha_x \in \partial\mathbb{H}^2$ when $n \to -\infty$ and to $\omega_x \in \partial\mathbb{H}^2$ when $n \to +\infty$.
2. The sequence $\tilde{f}^n(\tilde{y})$ converges to $\alpha_y \in \partial\mathbb{H}^2$ when $n \to -\infty$ and to $\omega_y \in \partial\mathbb{H}^2$ when $n \to +\infty$.
3. The geodesic lines (α_x, ω_x) and (α_y, ω_y) meet in $\mathbb{H}^2$.

Lemma 10.7 (Properties of essential intersection points)**.**

1. *(symmetry) Let $\tilde{z}$ be an essential intersection point between $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$. Then, for any two lifts $\hat{\mathcal{I}}_x$ and $\hat{\mathcal{I}}_y$ of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ to $\widetilde{\mathrm{dom}}\mathcal{F}$ associated to $\tilde{z}$, the two unbounded components of $\hat{\mathcal{I}}_y \setminus \left(\hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y\right)$ lie in different connected components of $\widetilde{\mathrm{dom}}\mathcal{F} \setminus \hat{\mathcal{I}}_x$.*
2. *(geometrically transverse implies essential) If the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ are geometrically transverse, then there exists an essential intersection point between $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$.*
3. *($\mathcal{F}$-transverse intersections) Let $\tilde{z}$ be an essential intersection point between $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ and let $\hat{\mathcal{I}}_x$ and $\hat{\mathcal{I}}_y$ be two lifts of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ to $\widetilde{\mathrm{dom}}\mathcal{F}$ associated to $\tilde{z}$. Suppose that the unbounded components C_2 and C_2' of $\hat{\mathcal{I}}_y \setminus \left(\hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y\right)$ meet respectively leaves ϕ_2 and ϕ_2' of $\hat{\mathcal{F}}$ which do not meet $\hat{\mathcal{I}}_x$ and that the unbounded components C_1 and C_1' of $\hat{\mathcal{I}}_x \setminus \left(\hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y\right)$ meet respectively leaves ϕ_1 and ϕ_1' of $\hat{\mathcal{F}}$ which do not meet $\hat{\mathcal{I}}_x$. Then the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ intersect $\tilde{\mathcal{F}}$-transversally at $\tilde{z}$. More precisely, any segment on $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ joining ϕ_1 and ϕ_1' is $\tilde{\mathcal{F}}$-transverse to any segment on $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ joining ϕ_2 and ϕ_2'.*

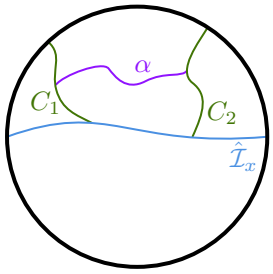

Figure 25: Configuration of the proof of 1. of Lemma 10.7.

Proof. 1. Fix two lifts $\hat{\mathcal{I}}_x$ and $\hat{\mathcal{I}}_y$ associated to $\tilde{z}$. Note that the fact that $\hat{\mathcal{I}}_x$ and $\hat{\mathcal{I}}_y$ are proper implies that if $\hat{\mathcal{I}}_x \setminus \left(\hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y\right)$ has two unbounded components, then $\hat{\mathcal{I}}_y \setminus \left(\hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y\right)$ also has two unbounded components.

Suppose that the property we want to prove does not hold: the two unbounded components C_2 and C_2' of $\hat{\mathcal{I}}_y \setminus \left(\hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y\right)$ are contained in the same connected component L of $\widetilde{\mathrm{dom}\,\mathcal{F}} \setminus \hat{\mathcal{I}}_x$.

Claim 10.8. *There exists an arc α joining the two components C_2 and C_2', which meets $\hat{\mathcal{I}}_y$ only at its endpoints and which is disjoint from $\hat{\mathcal{I}}_x$.*

Proof. See Figure 25. Take a closed disk D whose interior contains the closure of the union of the bounded components of $\hat{\mathcal{I}}_y \setminus \hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y$ and of $\hat{\mathcal{I}}_x \setminus \hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y$. Then we claim that the closure of some connected component of $\partial D \setminus (\hat{\mathcal{I}}_x \cup \hat{\mathcal{I}}_y)$ gives the desired path. Indeed, the closure of some connected component of $\partial D \setminus \hat{\mathcal{I}}_x$ is contained in L and joins the two unbounded components of $\hat{\mathcal{I}}_x \setminus \hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y$: otherwise, the endpoints of those unbounded components would be contain in the exterior of the disk D, a contradiction. Among those connected components of $\partial D \setminus \hat{\mathcal{I}}_x$ which are contained in L and which join the two unbounded components of $\hat{\mathcal{I}}_x \setminus \hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y$, one of those has to meet both C_2 and C_2', otherwise the ends of both C_2 and C_2' would be contained in the exterior of D. Now, the closure of some connected components of $S \setminus (C_2 \cup C_2')$ has to join C_2 and C_2', giving the path α we want. □

Then some unbounded component K_1 of $\widetilde{\mathrm{dom}}\mathcal{F} \setminus \hat{\mathcal{I}}_y$ contains α. Hence $K_1 \setminus \alpha$ has one bounded component $K_{1,1}$ and an unbounded component $K_{1,2}$ which is surrounded by $C_2 \cup \alpha \cup C_2'$. As the latter set does not meet $\hat{\mathcal{I}}_x$, the trajectory $\hat{\mathcal{I}}_x$ does not meet $K_{1,2}$. Hence the unbounded components of $\hat{\mathcal{I}}_x \setminus \hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y$ have both to lie in the other unbounded component K_2 of $\widetilde{\mathrm{dom}}\mathcal{F} \setminus \hat{\mathcal{I}}_y$, in contradiction with the definition of essential intersection point.

2. Observe that the algebraic intersection number between the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ is equal to 1: it is well-defined as those trajectories meet in a compact subset of $\tilde{S}$. If those trajectories had only inessential intersection points, then the algebraic intersection number between those two trajectories would be equal to 0.

   Let us give some details. We define an equivalence relation on intersection points between $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ (or more precisely on couple of parameters which correspond to an intersection point). Two such intersection points $\tilde{z}_1$ and $\tilde{z}_2$ are

equivalent if there exist lifts of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ associated to $\tilde{z}_1$ which are also associated to $\tilde{z}_2$. Observe that, if this property is true, then it holds for any lifts of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ associated to $\tilde{z}_1$. For each equivalence class C of this equivalence relation, fix lifts $\hat{\mathcal{I}}_{1,C}$ and $\hat{\mathcal{I}}_{2,C}$ of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ associated to this class C. The algebraic intersection number between $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{y})$ is then the sum over such classes C of the algebraic intersection numbers n_C between $\hat{\mathcal{I}}_{1,C}$ and $\hat{\mathcal{I}}_{2,C}$. For any class corresponding to an inessential intersection point, $n_C = 0$.

3. Denote by ϕ_1 and ϕ_1' (respectively ϕ_2 and ϕ_2') the two leaves met by $\hat{\mathcal{I}}_x$ (respectively $\hat{\mathcal{I}}_y$) mentioned in the statement of the lemma. We choose them in such a way that $\hat{\mathcal{I}}_x$ meets ϕ_1 first and $\hat{\mathcal{I}}_y$ meets ϕ_2 first. Denote by ϕ the leaf going though the lift $\hat{z}$ of $\tilde{z}$ which belongs to $\hat{\mathcal{I}}_x \cap \hat{\mathcal{I}}_y$. The definition of essential intersection point guarantees that, if the leaf ϕ_1 is above ϕ_2 with respect to ϕ, then ϕ_2' is above ϕ_1' with respect to ϕ: otherwise ϕ_1 and ϕ_1' would be in the same connected component of the complement of $\hat{\mathcal{I}}_y$, a contradiction with the definition of an essential intersection point. In the same way, if the leaf ϕ_1 is below ϕ_2 with respect to ϕ, then ϕ_2' is below ϕ_1' with respect to ϕ. This implies that we have an $\tilde{\mathcal{F}}$-transverse intersection.

□

10.3 Case 1

In this subsection, we suppose that both trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ satisfy condition

(C_1) there exists a deck transformation $\tau_i \in \pi_1(S) \setminus \langle T_i \rangle$ and a leaf ϕ of $\tilde{\mathcal{F}}$ such that ϕ meets the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and $\tau_i \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$.

For $i = 1, 2$, recall that, by Lemma 10.4, the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ stays at distance strictly less than R_i from the geodesic $\tilde{\gamma}_i$. For any subset A of $\mathbb{H}^2$ and any real number $R > 0$, we let

$$A_R = \left\{ \tilde{x} \in \mathbb{H}^2 \mid d(\tilde{x}, A) < R \right\}. \tag{10.1}$$

For notational convenience, we will identify the indices $i = 1, 2$ with an element of $\mathbb{Z}/2$.

The heart of the proof is to find suitable leaves of the transverse foliation in $\tilde{S}$ so that some orbits realising the rotation vectors in resp. directions T_1 and T_2 have an $\mathcal{F}$-transverse intersection (Paragraph a.). Once finished this preparatory step, the two following paragraphs — still quite technical – are rather straightforward.

a. Leaves and trajectories

In this section, we state some preliminary results on the possible behaviours of the leaves.

Take a point $\tilde{x} \in \mathbb{H}^2$ and suppose that the $\tilde{\mathcal{F}}$-transverse trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ has an ω-limit set in $\overline{\mathbb{H}^2}$ which is reduced to a point γ_+ of $\partial\mathbb{H}^2$ and has an α-limit set in $\overline{\mathbb{H}^2}$ which is reduced to a point γ_- of $\partial\mathbb{H}^2$ which is different from γ_+.

Denote by S_L (respectively S_R) the segment of $\partial\mathbb{H}^2$ which is on the left (resp. on the right) of the geodesic joining γ_- to γ_+. Denote by $L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))$ (resp. $R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))$) the unbounded connected component of $\mathbb{H}^2 \setminus \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ whose trace on $\partial\mathbb{H}^2$ coincides with S_L (resp. S_R).

Lemma 10.9. *Let ϕ be a leaf of $\tilde{\mathcal{F}}$. Suppose that the leaf ϕ meets the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ in two points $\tilde{z}_1$ and $\tilde{z}_2$. Then one unbounded component of $\phi \setminus \{\tilde{z}_1, \tilde{z}_2\}$ is contained in a disk bounded by a closed piece of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$.*

Proof. Let S_I be the segment of trajectory between $\tilde{z}_1$ and $\tilde{z}_2$ and S_ϕ be the segment between $\tilde{z}_1$ and $\tilde{z}_2$ on the leaf ϕ (see Figure 26). Denote by Ext the unbounded component of the complement in $\mathbb{H}^2$ of $S_I \cup S_\phi$ and Int be the complement in $\mathbb{H}^2$ of Ext. Observe that one of the two connected components of $\phi \setminus S_\phi$, which we call ψ, is contained in Int. Indeed, otherwise both connected components of $\phi \setminus S_\phi$ would be contained in Ext, and we would find two simple paths α and α' contained in Ext and such that $\alpha \cup \alpha' \cup \phi$ separates $\mathbb{H}^2$ in two connected components, each one intersecting $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$. This would lead to a contradiction, as S_I cannot cross α nor α', and has to cross ϕ positively twice.

Moreover, one of the connected components of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}) \setminus S_I$ meets Int in a neighbourhood of its end $\tilde{z}_i$, with $i = 1$ or $i = 2$. Take the closest point $\tilde{z}'_i$, for the order on the trajectory, to $\tilde{z}_i$ on this connected component which meets S_I (such a point exists as the α-limit and ω-limit sets of the trajectory lie on $\partial\mathbb{H}^2$ so that this connected component cannot remain in Int). Let S'_I be the segment joining $\tilde{z}_i$ to $\tilde{z}'_i$ on the trajectory. Then the half-leaf ψ does not meet the unbounded component of $S_I \cup S'_I$. □

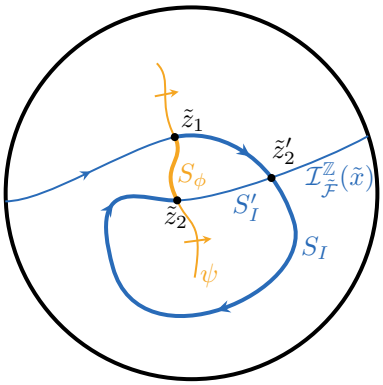

Figure 26: Configuration of the proof of Lemma 10.9.

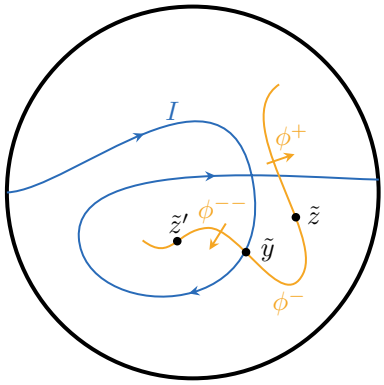

Figure 27: A possible configuration of the proof of Lemma 10.10.

For any leaf ϕ of $\tilde{\mathcal{F}}$, we call *neighbourhood of* $+\infty$ (respectively $-\infty$) in ϕ any half-leaf contained in ϕ which contains all the points after (resp. before) some point of ϕ for the order relation induced by the orientation of ϕ.

Lemma 10.10. *Let ϕ be a leaf of $\tilde{\mathcal{F}}$ which contains some point $\tilde{z}$ of $R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))$. Let ϕ_+ be the connected component of $\phi \setminus \{\tilde{z}\}$ which contains the points after $\tilde{z}$ on ϕ and let ϕ_- be the other connected component of $\phi \setminus \{\tilde{z}\}$.*

1. *If the half-leaf ϕ_+ meets $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$, then either ϕ_+ is bounded in $\mathbb{H}^2$ or a neighbourhood of $+\infty$ in ϕ is contained in $L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))$. Moreover, the intersection $\overline{\phi_+} \cap \overline{R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))}$ is a segment of ϕ_+.*

2. *If the half-leaf ϕ_- meets $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$, then the α-limit set of ϕ in $\overline{\mathbb{H}^2}$ does not meet $\partial\mathbb{H}^2$. Moreover, ϕ_- does not meet $L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))$, and the intersection $\overline{\phi_-} \cap \overline{R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))}$ is a segment of ϕ_-.*

Of course, we have a symmetric statement for a leaf which contains a point of $L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))$ by exchanging R with L, α with ω and $+$ with $-$.

Proof. A possible configuration for this proof is depicted in Figure 27.

If the half-leaf ϕ_+ has at two intersection points with the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$, then, by Lemma 10.9 and as $\tilde{z} \in R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))$, the ω-limit set of ϕ is bounded and the first point holds. If the leaf ϕ_+ has exactly one intersection point $\tilde{y}$ with the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ then the unbounded component of $\phi_+ \setminus \{\tilde{y}\}$ has to be contained in a connected component of $\mathbb{H}^2 \setminus \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ which is different from $R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))$: either it is contained in $L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))$, or it stays in a bounded connected component of $\mathbb{H}^2 \setminus \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$. This proves the first point.

If the half-leaf ϕ_- has two intersection points with the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$, then Lemma 10.9 implies the second point. Suppose that there is exactly one intersection point $\tilde{y}$ between the half-leaf ϕ_- and the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$. Let ϕ_{--} be the unbounded component of $\phi_- \setminus \{\tilde{y}\}$ and take a point $\tilde{z}'$ in ϕ_{--}. The half-leaf ϕ_{--} cannot be contained in $R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))$. Let us call $\phi_{\tilde{z}'\tilde{z}}$ the segment of ϕ between the points $\tilde{z}'$ and $\tilde{z}$. If ϕ_{--} was contained in $L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}))$, then the algebraic intersection number, relative to endpoints, of the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ with the segment $\phi_{\tilde{z}'\tilde{z}}$ would be equal to -1, which is not possible as the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x})$ is positively $\tilde{\mathcal{F}}$-transverse. □

The two previous lemmas did not use condition (C_1). The next one is the first one which is specific to case 1.

Lemma 10.11. *Fix $i \in \mathbb{Z}/2$. For any neighbourhoods $U_{i,-}, U_{i,+}$ of resp. $\gamma_{i,-}$ and $\gamma_{i,+}$ in $\overline{\mathbb{H}^2}$, there exist leaves ϕ_i and ϕ'_i of $\tilde{\mathcal{F}}$ that meet the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ with the following properties.*

1. *For any $n \geq 0$, we have $T_i^{-n}\phi_i \in U_{i,-}$ and $T_i^n \phi'_i \in U_{i,+}$.*
2. *The half-trajectory $(-\infty, \phi_i]_i$ belongs to $U_{i,-}$ and the half-trajectory $[\phi'_i, +\infty)_i$ belongs to $U_{i,+}$.*

We fix $i \in \mathbb{Z}/2$ and orient $\partial\overline{\mathbb{H}^2}$ in such a way that $\gamma_{i+1,-}$ lies in the positively oriented segment of $\partial\mathbb{H}^2$ which joins $\gamma_{i,-}$ to $\gamma_{i,+}$. For any points a, b on $\partial\mathbb{H}^2$, we denote by $[a, b]_{\partial\mathbb{H}^2}$ the positively oriented segment of $\partial\mathbb{H}^2$ from a to b.

Proof. We will distinguish two cases depending on whether the closed geodesic γ_i is simple or not.

As a sidenote, observe that condition (C_1) is automatically satisfied by $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ if the geodesic γ_i has an autointersection.

First case: Suppose first that the closed geodesic γ_i is not simple. Then there exists a deck transformation $\tau_i \in \pi_1(S) \setminus \langle T_i \rangle$ such that the geodesic lines $\tau_i\tilde{\gamma}_i$ and $\tilde{\gamma}_i$ meet. Observe that, for any $n \in \mathbb{Z}$, the geodesic lines $T_i^n\tau_i\tilde{\gamma}_i$ and $T_i^n\tilde{\gamma}_i = \tilde{\gamma}_i$ also meet. Fix $n \in \mathbb{Z}$. Recall that, by Lemma 10.3, the orbit of x_i is periodic so that $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ projects to a closed curve on S. Denote by p the minimal positive number such that $T_i^p\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i) = \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$. By Proposition 9.18, the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and $T_i^{pn}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ have an $\tilde{\mathcal{F}}$-transverse intersection.

In what follows, we construct a leaf ϕ_i which satisfies the conclusion of the lemma. The construction of ϕ'_i is similar and left to the reader.

Subcase 1: We suppose that there exists $\tilde{y}$ in $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ such that $\phi_{\tilde{y}}$ has not $\gamma_{i,+}$ in its ω-limit. Suppose the point $\tau_i\gamma_{i,-}$ is on the right of the geodesic γ_i. If the α-limit of the leaf $\phi_{\tilde{y}}$ does not contain $\gamma_{i,+}$ either, for k large enough, the leaf $T_i^{-pk}\phi_{\tilde{y}}$ can be used as the leaf ϕ_i of the lemma's conclusion. Suppose the α-limit of the leaf $\phi_{\tilde{y}}$ contains the point $\gamma_{i,+}$. Then there exists $J > 0$ such that, for any $j \geq J$, $\tilde{y} \in L(T_i^{pj}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$ and the negative half-leaf in $\phi_{\tilde{y}}$ meets the trajectory $T_i^{pj}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$. By Lemma 10.10.1 (symmetric version), the α-limit of $\phi_{\tilde{y}}$ is contained in

$$\bigcap_{k\geq 0} \overline{R(T_i^{pk}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))} = \{\gamma_{i,+}\}\,.$$

Moreover, by Lemma 10.10.1, the ω-limit of $\phi_{\tilde{y}}$ is contained in $\overline{L(T_i^{pJ}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$. Hence no end of the leaf $\phi_{\tilde{y}}$ meets $T_i^{pJ+p}\tau_i\gamma_{i,+}$ nor $T_i^{pJ+p}\tau_i\gamma_{i,-}$. Then the leaf $\tau_i^{-1}T_i^{-pJ-p}\phi_{\tilde{y}}$ meets the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and does not meet $\gamma_{i,+}$ nor $\gamma_{i,-}$. Hence, for k large enough, the leaf $\phi_i = T_i^{-pk}\tau_i^{-1}T_i^{-pJ-p}\phi_{\tilde{y}}$ will satisfy the lemma.

Suppose now that the point $\tau_i\gamma_{i,-}$ is on the left of the geodesic γ_i. Then there exists $j > 0$ such that $\tilde{y} \in R(T_i^{pj}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$. By Lemma 10.10.2, the α-limit of $\phi_{\tilde{y}}$ is contained in $\mathbb{H}^2 \cup \overline{R(T_i^{pj}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$ and for k large enough, the leaf $T_i^{-pk}\phi_{\tilde{y}}$ can be used as the leaf ϕ_i of the lemma's conclusion.

Subcase 2: Suppose that for any $\tilde{y}$ in $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$, the leaf $\phi_{\tilde{y}}$ has $\gamma_{i,+}$ in its ω-limit. Fix $\tilde{y}$ in $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$. Suppose that the point $\tau_i\gamma_{i,-}$ is on the right of the geodesic γ_i. Then there exists $J > 0$ such that for any $j \geq J$, one has $\tilde{y} \in L(T_i^{pj}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$ and, by the hypothesis on the ω-limit of $\phi_{\tilde{y}}$, the positive half-leaf in $\phi_{\tilde{y}}$ starting at $\tilde{y}$ has to cross each of those trajectories $T_i^{pj}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$. But by Lemma 10.10.2 (symmetric version), this implies that the ω-limit of the leaf $\phi_{\tilde{y}}$ is contained in $\mathbb{H}^2$, a contradiction. This case cannot happen.

Suppose now that the point $\tau_i\gamma_{i,-}$ is on the left of the geodesic γ_i. Then there exists $J > 0$ such that for any $j \geq J$, one has $\tilde{y} \in R(T_i^{pj}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$; this implies that the positive half-leaf starting at $\tilde{y}$ has to cross $T_i^{pj}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$. By Lemma 10.10.2, the α-limit of $\phi_{\tilde{y}}$ is contained in $\mathbb{H}^2 \cup \overline{R(T_i^{pj}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$. Hence, $\phi_{\tilde{y}}$ crosses $T_i^{pJ+p}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and its ends do not contain $T_i^{pJ+p}\tau_i\gamma_{i,+}$ as, by Lemma 10.10.1, the ω-limit of this leaf is equal to

$$\bigcap_{k\geq 0} \overline{L(T_i^{pk}\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))} = \{\gamma_{i,+}\}\,.$$

In this case, for k large enough, the leaf $T_i^{-pk}\tau_i^{-1}T_i^{-pJ-p}\phi_{\tilde{y}}$ can be used as the leaf ϕ_i of the lemma's conclusion.

Second case: Suppose now that the geodesic γ_i is simple. Recall that the geodesic $\tilde{\gamma}_i$ shares no endpoint on $\partial\mathbb{H}^2$ with one of its translates under an element of $\pi_1(S)$ by Lemma 6.3. By condition (C_1), there exists a leaf ϕ of $\tilde{\mathcal{F}}$ which meets $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$.

For any subset A of $\mathbb{H}^2$, we denote by $\overline{A}$ its closure in $\overline{\mathbb{H}^2}$. The heart of the proof in this second case is the following lemma.

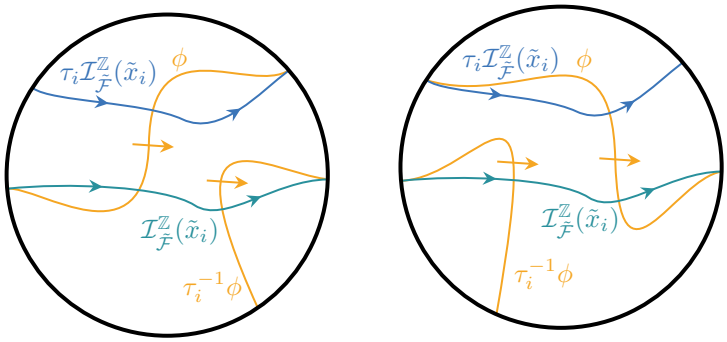

Figure 28: Configuration of the first case of the proof of Lemma 10.12: the two different cases depending whether the trajectory crosses first ϕ or $\tau_i\phi$.

Lemma 10.12. *There exist two leaves ϕ_a and ϕ_b, each one meeting $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$, and such that $\overline{\phi_a}$ is disjoint from $\gamma_{i,-}$ and $\overline{\phi_b}$ is disjoint from $\gamma_{i,+}$.*

Proof. We need to distinguish two cases, depending on whether $\tilde{\gamma}_i$ separates $\tau_i\tilde{\gamma}_i$ and $\tau_i^{-1}\tilde{\gamma}_i$ or not.

First case: Suppose $\tilde{\gamma}_i$ separates $\tau_i\tilde{\gamma}_i$ and $\tau_i^{-1}\tilde{\gamma}_i$. Note that this amounts to saying that the axis of τ_i crosses $\tilde{\gamma}_i$ (using the fact that γ_i is simple).

For notational convenience we suppose in what follows that the geodesic $\tau_i\tilde{\gamma}_i$ is on the left of $\tilde{\gamma}_i$.

If $\omega(\phi) \cap \partial\mathbb{H}^2 = \emptyset$ and $\alpha(\phi) \cap \partial\mathbb{H}^2 = \emptyset$, the lemma holds (for $\phi_a = \phi_b = \phi$).

Suppose that $\emptyset \neq \omega(\phi) \cap \partial\mathbb{H}^2$ is contained in the segment $[\tau_i\gamma_{i,+}, \tau_i\gamma_{i,-}]_{\partial\mathbb{H}^2}$ of $\partial\mathbb{H}^2$. Then a neighbourhood of $+\infty$ in ϕ is contained in $L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$. If $\alpha(\phi) \subset \mathbb{H}^2$, the lemma holds (for $\phi_a = \phi_b = \phi$). Otherwise, by Lemma 10.10.1. (symmetric version), a neighbourhood of $-\infty$ in ϕ is contained in $R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$. Hence the leaf $\tau_i^{-1}\phi$ joins the segment $[\tau_i^{-1}\gamma_{i,-}, \tau_i^{-1}\gamma_{i,+}]_{\partial\mathbb{H}^2}$ to the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$. Observe also that a neighbourhood of $+\infty$ in $\tau_i^{-1}\phi$ is disjoint from $R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$ by Lemma 10.9. However, either $\phi = \tau_i^{-1}\phi$, in which case the claim holds because both ends of ϕ are disjoint from the ends of γ_i ($\phi_a = \phi_b = \phi$), or the leaves ϕ and $\tau_i^{-1}\phi$ are disjoint. Suppose the latter holds (see Figure 28).

If the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ meets ϕ before it meets $\tau_i^{-1}\phi$ (left of Figure 28), then the set $\tau_i^{-1}\overline{\phi} \cap \overline{R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$ separates a neighbourhood of $-\infty$ in ϕ from $\gamma_{i,+}$ in $\overline{R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$ and the set $\overline{\phi} \cap \overline{L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$ separates a neighbourhood of $+\infty$ in $\tau_i^{-1}\phi$ from $\gamma_{i,-}$ in $\overline{L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$, hence the lemma holds for $\phi_a = \tau_i^{-1}\phi$ and $\phi_b = \phi$. If the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ meets $\tau_i^{-1}\phi$ before it meets ϕ, then the set $\tau_i^{-1}\overline{\phi} \cap \overline{R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$ separates a neighbourhood of $-\infty$ in ϕ from $\gamma_{i,-}$ in $\overline{R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$ and the set $\overline{\phi} \cap \overline{L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$ separates a neighbourhood of $+\infty$ in $\tau_i^{-1}\phi$ from $\gamma_{i,+}$ in $\overline{L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$, hence the lemma holds for $\phi_a = \phi$ and $\phi_b = \tau_i^{-1}\phi$.

Suppose now that $\emptyset \neq \omega(\phi) \cap \partial\mathbb{H}^2$ is not contained in $[\tau_i\gamma_{i,+}, \tau_i\gamma_{i,-}]_{\partial\mathbb{H}^2}$ (such a configuration is depicted in Figure 29). Then any neighbourhood of $+\infty$ in ϕ meets

$R(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$. By Lemma 10.10.2., $\alpha(\phi) \subset \mathbb{H}^2$ and a neighbourhood of $-\infty$ in ϕ is disjoint from $L(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$.

If any neighbourhood of $+\infty$ in ϕ also met $L(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$, then it would have to cross $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and, by Lemma 10.10.1., we would have $\omega(\phi) \subset [\tau_i\gamma_{i,+}, \tau_i\gamma_{i,-}]_{\partial\mathbb{H}^2}$, a contradiction. Hence a neighbourhood of $+\infty$ in ϕ is disjoint from $L(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$ and $\omega(\phi) \cap (\tau_i\gamma_{i,+}, \tau_i\gamma_{i,-})_{\partial\mathbb{H}^2} = \emptyset$.

If any neighbourhood of $+\infty$ in ϕ meets $R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$, then by Lemma 10.10.1. there exists a neighbourhood of $+\infty$ in ϕ included in $R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$ and so the lemma is satisfied for $\phi_a = \phi_b = \tau_i^{-1}\phi$, as the set $\tau_i^{-1}\overline{\phi}$ meets no ends of the geodesic $\tilde{\gamma}_i$.

Otherwise, a neighbourhood of $+\infty$ in ϕ is disjoint from $L(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)) \cup R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$. Let us prove that, in this case, $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i) \cap \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i) \neq \emptyset$. Suppose the contrary. This implies that $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i) \subset L\big(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)\big)$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i) \subset R\big(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)\big)$. As ϕ meets both trajectories, there exists

$$\tilde{y} \in \phi \cap R\big(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)\big) \cap L\big(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)\big).$$

By Lemma 10.10.1, the positive half-leaf ϕ_+ starting at $\tilde{y}$ cannot meet $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ (otherwise $\omega(\phi)$ would be either bounded, or included in $\overline{L\big(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)\big)}$). Similarly, by Lemma 10.10.2 (symmetric version), ϕ_+ cannot meet $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$. Hence, the negative half-leaf ϕ_- starting at $\tilde{y}$ meets both $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$. Using once again Lemma 10.10, we deduce that $\alpha(\phi) \in R\big(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)\big)^{\complement} \cap L\big(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)\big)^{\complement}$. The latter set is hence nonempty. This proves that $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i) \cap \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i) \neq \emptyset$.

In this case, the intersection $\overline{R(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))} \cap \overline{L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$ has two unbounded connected components, with respective boundaries in $\partial\mathbb{H}^2$ $(\tau_i\gamma_{i,-}, \gamma_{i,-})_{\partial\mathbb{H}^2}$ and $(\gamma_{i,+}, \tau_i\gamma_{i,+})_{\partial\mathbb{H}^2}$ (see Figure 29).

Consider the second of these connected components; its boundary in $\mathbb{H}^2$ contains pieces of both $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$. Let $\tilde{z} \in \tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i) \cap \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ on this boundary, and ϕ' be the leaf passing by $\tilde{z}$. Note that ϕ' meets both $\overline{R(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$ and $\overline{L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$ in any neighbourhood of $\tilde{z}$. Denote by ϕ'_+ the positive half-leaf of ϕ' starting at $\tilde{z}$.

- If ϕ'_+ is not included in $R(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$, then by Lemma 10.10.1 either $\omega(\phi') \subset \mathbb{H}^2$, or $\omega(\phi') \subset L(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$ (right of Figure 29). In the latter case, we already proved that the conclusion of the lemma holds.
- If ϕ'_+ is included in $R(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$, but ϕ'_+ is not included in $L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$, then by Lemma 10.10.2 (symmetric version) $\omega(\phi') \subset \mathbb{H}^2$.
- Otherwise, $\omega(\phi') \subset \mathbb{H}^2 \cup [\gamma_{i,+}, \tau_i\gamma_{i,+}]_{\partial\mathbb{H}^2}$ (left of Figure 29).

Remark also that in all cases, by Lemma 10.10, we have $\alpha(\phi') \cap \partial\mathbb{H}^2 \subset \overline{R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$.

If $\omega(\phi') \subset \mathbb{H}^2$, then $\alpha(\phi') \cap \partial\mathbb{H}^2 \subset \overline{R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$, and hence $\alpha(\tau_i^{-1}\phi') \cap \partial\mathbb{H}^2 \subset [\tau_i^{-1}\gamma_{i,-}, \tau_i^{-1}\gamma_{i,+}]_{\partial\mathbb{H}^2}$ and $\omega(\tau_i^{-1}\phi') \subset \mathbb{H}^2$ so that the lemma holds for $\phi_a = \phi_b = \tau_i^{-1}\phi$.

In the other case, we have $\omega(\phi') \subset \mathbb{H}^2 \cup [\gamma_{i,+}, \tau_i\gamma_{i,+}]_{\partial\mathbb{H}^2}$, and by Lemma 10.10.2., $\alpha(\phi') \subset \mathbb{H}^2$. We can perform the same construction for the other connected component of $\overline{R(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))} \cap \overline{L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$ to find another leaf ϕ''. Again, the only case in which we still have not proved the lemma is when $\omega(\phi'') \subset \mathbb{H}^2 \cup [\tau_i\gamma_{i,-}, \gamma_{i,-}]_{\partial\mathbb{H}^2}$. But in this case $\phi_a = \phi'$ and $\phi_b = \phi''$ make the lemma work.

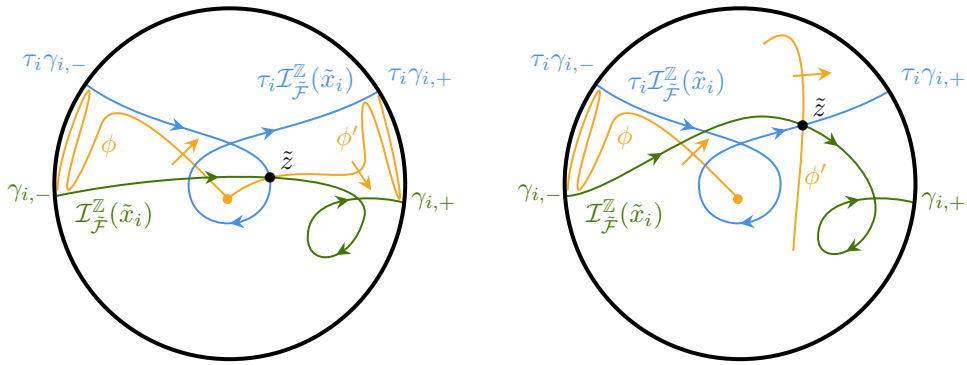

Figure 29: Configuration of the first case of the proof of Lemma 10.12: finding another leaf ϕ'.

The case $\omega(\phi) \subset \mathbb{H}^2$ and $\alpha(\phi) \cap \partial\mathbb{H}^2 \neq \emptyset$ is identical to the previous one, the details are left to the reader.

<u>Second case:</u> Suppose that the geodesic line $\tilde{\gamma}_i$ does not separate $\tau_i\tilde{\gamma}_i$ and $\tau_i^{-1}\tilde{\gamma}_i$. Note that this amounts to saying that the axis of τ_i is disjoint from $\tilde{\gamma}_i$ (using the fact that γ_i is simple). In this case, we need the following claim.

Claim 10.13. *One end of ϕ is contained in singularities of $\tilde{\mathcal{F}}$,* i.e. *it is bounded in $\tilde{S} = \mathbb{H}^2$.*

Proof. Suppose that both ends of ϕ meet $\partial\mathbb{H}^2$.

Then the leaf ϕ meets the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ at only one point by Lemma 10.9. Observe also that, at the point of intersection $\phi \cap \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$, the leaf ϕ must go from one unbounded component of $\mathbb{H}^2 \setminus \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ to the other one and the same holds at the point $\phi \cap \tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ for the trajectory $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$.

As the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ are both $\tilde{\mathcal{F}}$-transverse, the algebraic intersection number between each of these trajectories and ϕ must be equal to 1. However, as the geodesic line $\tilde{\gamma}_i$ does not separate $\tau_i\tilde{\gamma}_i$ and $\tau_i^{-1}\tilde{\gamma}_i$, the algebraic intersection number between $\tilde{\gamma}_i$ and ϕ and the algebraic intersection number between $\tau_i\tilde{\gamma}_i$ and ϕ must be opposite to each other: the axis of τ_i does not cross $\tilde{\gamma}_i$. Hence the algebraic intersection number between $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and ϕ and the algebraic intersection number between $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and ϕ must be opposite to each other, a contradiction. □

To simplify notation, we suppose that the geodesic $\tau_i\tilde{\gamma}_i$ is on the left of $\tilde{\gamma}_i$ and that the geodesic $\tau_i\tilde{\gamma}_i$ is above $\tau_i^{-1}\tilde{\gamma}_i$ with respect to $\tilde{\gamma}_i$. If $\alpha(\phi) \subset \mathbb{H}^2$ and $\omega(\phi) \subset \mathbb{H}^2$, the lemma holds. Otherwise, by Claim 10.13, either $\alpha(\phi) \cap \partial\mathbb{H}^2 \neq \emptyset$ and $\omega(\phi) \subset \mathbb{H}^2$ or $\omega(\phi) \cap \partial\mathbb{H}^2 \neq \emptyset$ and $\alpha(\phi) \subset \mathbb{H}^2$.

Suppose first that $\alpha(\phi) \cap \partial\mathbb{H}^2 \neq \emptyset$ and $\omega(\phi) \subset \mathbb{H}^2$. By Lemma 10.10.1. (symmetric version), either a neighbourhood at $-\infty$ of ϕ is contained in $L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$ or $\alpha(\phi) \cap \partial\mathbb{H}^2 \subset \overline{R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$. If $\alpha(\phi) \cap \partial\mathbb{H}^2 \subset \overline{R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$, then the set $\overline{\tau_i^{-1}\phi}$ meets none of the ends of the geodesic $\tilde{\gamma}_i$ so that the lemma holds for $\phi_a = \phi_b = \tau_i^{-1}\phi$ (see Figure 30, left).

In the other case, a neighbourhood at $-\infty$ of ϕ is contained in $L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$. As before, by Lemma 10.10.1. (symmetric version), either a neighbourhood at $-\infty$ of ϕ is contained

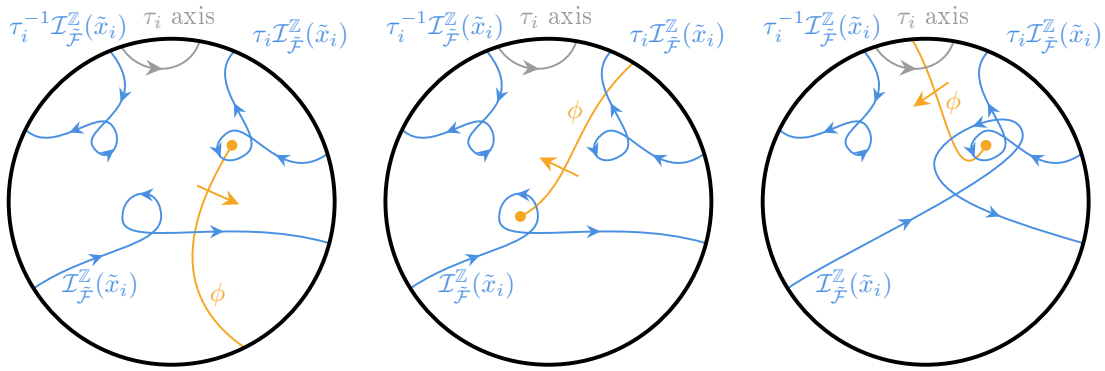

Figure 30: Configuration of the second case of the proof of Lemma 10.12 (where the geodesics $\tau_i\tilde{\gamma}_i$ and $\tau_i^{-1}\tilde{\gamma}_i$ are on the same side of the geodesic $\tilde{\gamma}_i$): the three different cases having to be considered.

in $L(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$, or $\alpha(\phi)\cap\partial\mathbb{H}^2\subset\overline{R(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$. In the latter case the lemma holds for $\phi_a=\phi_b=\phi$ (see Figure 30, middle).

Suppose now that a neighbourhood at $-\infty$ of ϕ is contained in $L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))\cap L(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$. Then both trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ meet. Indeed, by Lemma 10.10.2. (symmetric version), $\omega(\phi)$ has to be contained in some bounded component of the complement of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$, and in some bounded component of the complement of $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$. Hence, one of these trajectories meets one bounded component of the complement of the other one, and these two trajectories meet. In particular, the set $\overline{L\big(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)\big)}\cap\overline{L\big(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)\big)}$ has two unbounded connected components, one intersecting $\partial\mathbb{H}^2$ on $[\tau_i\gamma_{i,+},\gamma_{i,-}]_{\partial\mathbb{H}^2}$, the other one on $[\gamma_{i,+},\tau_i\gamma_{i,-}]_{\partial\mathbb{H}^2}$. Thus, either $\alpha(\phi)\cap\partial\mathbb{H}^2\subset[\tau_i\gamma_{i,+},\gamma_{i,-}]_{\partial\mathbb{H}^2}$ or $\alpha(\phi)\subset[\gamma_{i,+},\tau_i\gamma_{i,-}]_{\partial\mathbb{H}^2}$. In the first case, the set $\overline{\phi}$ does not meet $\gamma_{i,+}$ and the set $\tau_i^{-1}\overline{\phi}$ does not meet $\gamma_{i,-}$ and the lemma holds for $\phi_a=\tau_i^{-1}\phi$ and $\phi_b=\phi$ (see Figure 30, right). In the second case, the set $\overline{\phi}$ does not meet $\gamma_{i,-}$ and the set $\tau_i^{-1}\overline{\phi}$ does not meet $\gamma_{i,+}$ and the lemma holds for $\phi_a=\phi$ and $\phi_b=\tau_i^{-1}\phi$.

Finally, suppose that $\omega(\phi)\cap\partial\mathbb{H}^2\neq\emptyset$ and $\alpha(\phi)\subset\mathbb{H}^2$. This case is similar to the previous one so we will give less details. By Lemma 10.10.1., either a neighbourhood at $+\infty$ of ϕ is contained in $R(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$ or $\omega(\phi)\cap\partial\mathbb{H}^2\subset\overline{L(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$ and either a neighbourhood at $+\infty$ of ϕ is contained in $R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$ or $\omega(\phi)\cap\partial\mathbb{H}^2\subset\overline{L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$. If a neighbourhood at $+\infty$ of ϕ is contained in either $R(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$ or $R(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))$, the lemma holds: take $\phi_a=\phi_b=\phi$ in the first case and $\phi_a=\phi_b=\tau_i^{-1}\phi$ in the second one. Otherwise, $\omega(\phi)\subset\overline{L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))\cap L(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))}$. In this case, by the last part of Lemma 10.10.1. (symmetric version), $\alpha(\phi)$ is contained in $L(\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))^c\cap L(\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i))^c$ so that the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and $\tau_i\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ meet. Hence, as in the previous case, either $\omega(\phi)\cap\partial\mathbb{H}^2\subset[\tau_i\gamma_{i,+},\gamma_{i,-}]_{\partial\mathbb{H}^2}$ or $\omega(\phi)\cap\partial\mathbb{H}^2\subset[\gamma_{i,+},\tau_i\gamma_{i,-}]_{\partial\mathbb{H}^2}$ and the lemma holds : take $\phi_a=\tau_i^{-1}\phi$ and $\phi_b=\phi$ in the first case and $\phi_a=\phi$ and $\phi_b=\tau_i^{-1}\phi$ in the second one. □

By Lemma 10.12, there exist two leaves ψ_i and ψ_i' which meet $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ such that $\overline{\psi}_i$ is disjoint from $\gamma_{i,+}$ and $\overline{\psi}_i'$ is disjoint from $\gamma_{i,-}$. Note that it is possible that $\psi_i=\psi_i'$. By Lemma 10.5, for arbitrarily large $n>0$, the leaf $T_i^{-n}\psi_i$ meets the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$

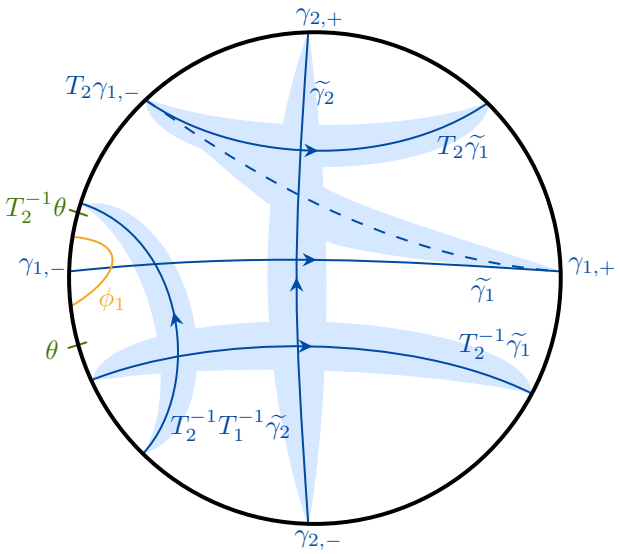

Figure 31: A part of the set A_1 of (10.2) (in light blue).

and, for arbitrarily large $n > 0$, the leaf $T_i^n\psi_i'$ meets the trajectory $\mathcal{I}^{\mathbb{Z}}_{\widetilde{\mathcal{F}}}(\tilde{x}_i)$. Moreover, the sequence $(T_i^{-n}\overline{\psi_i})_{n\geq 0}$ of compact subsets of $\overline{\mathbb{H}^2}$ converges to $\gamma_{i,-}$. Hence we can take $n_- > 0$ sufficiently large so that Lemma 10.11 holds with $\phi_i = T_i^{-n_-}\psi_i$. In the same way, take $n_+ > 0$ sufficiently large so that Lemma 10.11 holds with $\phi_i' = T_i^{n_+}\psi_i'$. □

Before proving a corollary, we need a geometric lemma. Let $R = \max(R_1, R_2)$ (see Lemma 10.4) and, for $i = 1, 2$, (see Figure 31, see also (10.1) for the definition of R-neighbourhood)

$$A_i = \left(\tilde{\gamma}_{i+1} \ \cup T_{i+1}^{-1}\tilde{\gamma}_i \ \cup (T_{i+1}\gamma_{i,-}, \gamma_{i,+}) \cup \bigcup_{w\in\langle T_i^{-1}, T_{i+1}^{-1}\rangle_+} T_{i+1}^{-1} w\tilde{\gamma}_{i+1} \right)_R , \tag{10.2}$$

$$B_i = \left(\tilde{\gamma}_{i+1} \ \cup T_{i+1}\tilde{\gamma}_i \ \cup (T_{i+1}^{-1}\gamma_{i,-}, \gamma_{i,+}) \ \cup \bigcup_{w\in\langle T_i, T_{i+1}\rangle_+} T_{i+1} w\tilde{\gamma}_{i+1} \right)_R .$$

The reader will note that there is *a priori* no reason for the set of geodesics $\bigcup_{j=1,2}\bigcup_{w\in\langle T_1,T_2\rangle} w\tilde{\gamma}_j$ to be a tree in $\mathbb{H}^2$ (cases like in Figure 31 could occur).

Lemma 10.14. *The closure of A_i in $\overline{\mathbb{H}^2}$ does not meet $\gamma_{i,-}$, and the closure of B_i in $\overline{\mathbb{H}^2}$ does not meet $\gamma_{i,+}$*

Proof. We prove the lemma only for A_i, the case of B_i being identical. We choose an orientation of $\partial\mathbb{H}^2$ such that the points $\gamma_{i,\pm}$ are oriented as in Figure 31.

First of all, we obviously have that $\gamma_{i,-} \notin \overline{(\tilde{\gamma}_{i+1})_R \cup (T_{i+1}^{-1}\tilde{\gamma}_i)_R \cup (T_{i+1}\gamma_{i,-}, \gamma_{i,+})_R}$.

Let $\theta \in (\gamma_{i,-}, \gamma_{i+1,-})_{\partial\mathbb{H}^2}$ be such that $T_{i+1}^{-1}\theta \in (\gamma_{i+1,+}, \gamma_{i,-})_{\partial\mathbb{H}^2}$ (see Figure 31). Notice that the arc $[\gamma_{i,-}, \gamma_{i+1,-}]_{\partial\mathbb{H}^2}$ is positively stable under both T_i^{-1} and T_{i+1}^{-1}, so for any $w \in \langle T_i^{-1}, T_{i+1}^{-1}\rangle_+$, we have $w\gamma_{i+1,-} \in [\gamma_{i,-}, \gamma_{i+1,-}]_{\partial\mathbb{H}^2}$ and hence $T_{i+1}^{-1}w\gamma_{i+1,-} \in [\theta, \gamma_{i+1,-}]_{\partial\mathbb{H}^2}$.

We now consider the points $T_{i+1}^{-1} w \gamma_{i+1,+}$ and prove that they stay at a uniformly positive distance from $\gamma_{i,-}$. Note that $T_{i+1}^{-1}\gamma_{i+1,+} = \gamma_{i+1,+}$, so we do not lose generality by supposing that w ends with T_i^{-1}: we write $w = w' T_i^{-1}$. Note that the set $[\gamma_{i,-}, \gamma_{i+1,-}]_{\partial\mathbb{H}^2}$ attracts all the points of $[T_i^{-1}\gamma_{i+1,+}, \gamma_{i+1,-}]_{\partial\mathbb{H}^2}$: there is $\ell > 0$ such that if $\mathrm{length}(w') \geq \ell$, then $w' T_i^{-1}\gamma_{i+1,+} \in [T_{i+1}^{-1}\theta, \gamma_{i+1,-}]_{\partial\mathbb{H}^2}$. Hence, if $\mathrm{length}(w') \geq \ell$, then $T_{i+1}^{-1} w' T_i^{-1}\gamma_{i+1,+} \in [\theta, \gamma_{i+1,-}]_{\partial\mathbb{H}^2}$. This proves that

$$\inf_{w \in \langle T_i^{-1}, T_{i+1}^{-1}\rangle_+} d_{\partial\mathbb{H}^2}\big(\gamma_{i,-},\, T_{i+1}^{-1} w \gamma_{i+1,+}\big)$$
$$= \min\Big(d_{\partial\mathbb{H}^2}\big(\gamma_{i,-}, \theta\big),\ \inf_{\substack{w' \in \langle T_i^{-1}, T_{i+1}^{-1}\rangle_+ \\ \mathrm{length}(w') \leq \ell}} d_{\partial\mathbb{H}^2}\big(\gamma_{i,-},\, T_{i+1}^{-1} w' T_i^{-1}\gamma_{i+1,+}\big)\Big).$$

We have reduced the bounding of the distance to a finite number of cases, so it suffices to prove that for any $w' \in \langle T_i^{-1}, T_{i+1}^{-1}\rangle_+$ with $\mathrm{length}(w') \leq \ell$, we have $\gamma_{i,-} \neq T_{i+1}^{-1} w' T_i^{-1}\gamma_{i+1,+}$. But this last statement is a consequence of Lemma 6.3: if we had a deck transformation U such that $U\gamma_{2,+} = \gamma_{1,-}$, this would mean that the two geodesic arcs γ_1 and γ_2 coincide (as sets), which is a contradiction. □

From now on, we take ϕ_i and ϕ_i' the leaves given by Lemma 10.11 for $U_{i,-}$ the connected component of $\overline{\mathbb{H}^2} \setminus \overline{A_i}$ containing $\gamma_{i,-}$ and $U_{i,+}$ the connected component of $\overline{\mathbb{H}^2} \setminus \overline{B_i}$ containing $\gamma_{i,+}$ (Lemma 10.14 ensures that these connected components are indeed neighbourhoods of $\gamma_{i,+}$ and $\gamma_{i,-}$).

In the following corollary, we identify words on elements of $\pi_1(S)$ with the deck transformations which are obtained by composing the word's letters.

Corollary 10.15. *Let $n \geq 0$.*

1. *For any word w in T_1 and T_2, the trajectory $w\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_{i+1})$ does not meet $T_i^{-n}\phi_i$ and, for any word w in T_1 and T_2 containing T_{i+1}, the trajectory $w\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_i)$ does not meet $T_i^{-n}\phi_i$.*

2. *For any word w in T_1^{-1} and T_2^{-1} starting with T_{i+1}^{-1}, the trajectories $w\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_{i+1})$ and $w\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_i)$ do not meet $T_i^{-n}\phi_i$.*

3. *For any word w in T_1^{-1} and T_2^{-1}, the trajectory $w\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_{i+1})$ does not meet $T_i^{n}\phi_i'$ and, for any word w in T_1^{-1} and T_2^{-1} containing T_{i+1}^{-1}, the trajectory $w\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_i)$ does not meet $T_i^{n}\phi_i'$.*

4. *For any word w in T_1 and T_2 starting with T_{i+1}, the trajectories $w\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_{i+1})$ and $w\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_i)$ do not meet $T_i^{n}\phi_i'$.*

Proof. We prove the first two points. To prove points 3. and 4., exchange the roles of T_1 and T_1^{-1}, of T_2 and T_2^{-1}, of $\gamma_{1,+}$ and $\gamma_{1,-}$ and of $\gamma_{2,+}$ and $\gamma_{2,-}$ and change the leaf ϕ_i to the leaf ϕ_i' in the following proof. Also, to simplify notation, we suppose $n = 0$.

1. For any word w in T_1 and T_2, observe that either the geodesic $w\tilde{\gamma}_{i+1}$ and the leaf ϕ_i are separated by the geodesic $\tilde{\gamma}_{i+1}$, or $w\tilde{\gamma}_{i+1} = \tilde{\gamma}_{i+1}$. Moreover, by definition of R, $w\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_{i+1}) \subset (w\tilde{\gamma}_{i+1})_R$ so that the trajectory $w\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_{i+1})$ does not meet the connected component of $\overline{\mathbb{H}^2} \setminus \overline{(\tilde{\gamma}_{i+1})_R}$ which contains ϕ_i.

Let w be a word on T_1 and T_2 which contains T_{i+1}. Observe that the endpoints of $w\tilde{\gamma}_i$ lie between $T_{i+1}\gamma_{i,-}$ and $\gamma_{i,+}$. So the set $(w\tilde{\gamma}_i)_R$ does not meet the connected component of $\overline{\mathbb{H}^2} \setminus \overline{(T_{i+1}\gamma_{i,-}, \gamma_{i,+})_R}$ containing $\gamma_{i,-}$ and ϕ_i. Hence $w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ does not meet the leaf ϕ_i.

2. Let w be a word on T_1^{-1} and T_2^{-1} which starts with T_{i+1}^{-1}. By definition of A_i, the set $(w\tilde{\gamma}_{i+1})_R$ is included in A_i, and hence does not meet the leaf ϕ_i. Therefore, the trajectory $w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_{i+1})$ does not meet the leaf ϕ_i.

Finally, let $w' \in \langle T_1^{-1}, T_2^{-1}\rangle_+$. Observe that both ends of $w'\tilde{\gamma}_i$ lie in $[\gamma_{i,-}, \gamma_{i,+}]_{\partial\mathbb{H}^2}$, so that both ends of $T_{i+1}^{-1}w'\tilde{\gamma}_i$ lie in the connected component of $\overline{\mathbb{H}^2} \setminus \overline{(T_{i+1}^{-1}\tilde{\gamma}_i)_R}$ that does not contain the point $\gamma_{i,-}$ and the leaf ϕ_i. The trajectory $T_{i+1}^{-1}w'\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ is hence disjoint from the leaf ϕ_i. □

b. Transverse intersections

Let P_i be the maximal integer $j \in \mathbb{Z}$ such that the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ meets $\tilde{f}^j(\tilde{x}_i)$ strictly before it meets the leaf ϕ_i and the trajectory $(-\infty, \tilde{f}^j(\tilde{x}_i))_i$ is disjoint from the set A_i (defined in (10.2) page 74).

Changing the point $\tilde{x}_i$ to $\tilde{f}^{P_i}(\tilde{x}_i)$ if necessary, we can suppose that $P_1 = P_2 = 0$.

Lemma 10.16. *There exist integers $m_1 > 0$ and $m_2 > 0$, which can be taken arbitrarily large, and integers $r_1 > m_1p_1$, $r_2 > m_2p_2$, such that*

1. *For any $i \in \mathbb{Z}/2$, the leaf $T_i^{r_i}\phi_i'$ meets the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$, the trajectory $[\phi_i, T_i^{r_i}\phi_i']_i$ is admissible of order m_iq_i and, for any $0 \le j \le m_ip_i$, the trajectory $[\phi_i, T_i^{r_i}\phi_i']_i$ meets the trajectory $T_i^j\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_{i+1})$.*

2. *For any $i \in \mathbb{Z}/2$ the trajectory $(-\infty, \phi_i]_i$ lies in the connected component of $\overline{\mathbb{H}^2} \setminus \overline{A_i}$ which contains $\gamma_{i,-}$ and the trajectory $[T_i^{r_i}\phi_i', +\infty)_i$ lies in the connected component of $\overline{\mathbb{H}^2} \setminus T_i^{m_ip_i}\overline{B_i}$ which contains $\gamma_{i,+}$.*

3. *For any $i \in \mathbb{Z}/2$ and any $0 \le j \le m_ip_i$, the transverse paths $[\phi_i, T_i^{r_i}\phi_i']_i$ and $T_i^j[\phi_{i+1}, T_{i+1}^{r_{i+1}}\phi_{i+1}']_{i+1}$ have an $\tilde{\mathcal{F}}$-transverse intersection at some point $\tilde{y}_j^i$.*

In the same way we proved Corollary 10.15 from Lemma 10.11, it is possible to prove the following corollary by using the second point of Lemma 10.16. Note that points 1. and 2. of this corollary are direct consequences of Lemma 10.11. As the proof of points 3. and 4. are identical to the proof of Corollary 10.15, we leave it to the reader.

Corollary 10.17. 1. *For any word w in T_1 and T_2, the trajectory $w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_{i+1})$ does not meet $(-\infty, \phi_i]_i$ and, for any word w in T_1 and T_2 which contains T_{i+1}, the trajectory $w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ does not meet $(-\infty, \phi_i]_i$.*

2. *For any word w in T_1^{-1} and T_2^{-1} which starts with T_{i+1}^{-1}, the trajectory $w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_{i+1})$ does not meet $(-\infty, \phi_i]_i$ and, for any word w in T_1^{-1} and T_2^{-1} which starts with T_{i+1}^{-1}, the trajectory $w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ does not meet $(-\infty, \phi_i]_i$.*

3. *For any word w in T_1^{-1} and T_2^{-1}, the trajectory $T_i^{m_ip_i}w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_{i+1})$ does not meet $[T_i^{r_i}\phi_i', +\infty)_i$ and, for any word w in T_1^{-1} and T_2^{-1} which contains T_{i+1}^{-1}, the trajectory $T_i^{m_ip_i}w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ does not meet $[T_i^{r_i}\phi_i', +\infty)_i$.*

4. *For any word w in T_1 and T_2 which starts with T_{i+1}, the trajectory $T_i^{m_ip_i} w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_{i+1})$ does not meet $[T_i^{r_i}\phi_i', +\infty)_i$ and, for any word w in T_1 and T_2 which starts with T_{i+1}, the trajectory $T_i^{m_ip_i} w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ does not meet $[T_i^{r_i}\phi_i', +\infty)_i$.*

Proof of Lemma 10.16. We will find integers m_1 and m_2 such that the first two items of the lemma are satisfied. We will then see that the third item is automatically satisfied.

We fix $i = 1, 2$. Observe that the first part of the second point holds by definition of ϕ_i and Lemma 10.11. We parametrize the geodesic $\tilde{\gamma}_i$ by arclength and identify points on $\tilde{\gamma}_i$ with their parameters. Recall that $\pi_{\tilde{\gamma}_i}$ denotes the orthogonal projection on the geodesic $\tilde{\gamma}_i$. Let $\lambda = \max \pi_{\tilde{\gamma}_i}(B_i)$ (it exists by Lemma 10.14). Let k_i be an integer such that the trajectory $[\tilde{x}_i, \tilde{f}^{k_i}(\tilde{x}_i)]_i$ meets the leaves ϕ_i and ϕ_i' but not at its endpoints. Finally, fix $v_i', v_i'' \in \left(\frac{p_i}{q_i}\ell(\gamma_i), v_i\right)$ with $v_i'' < v_i'$.

By Lemma 10.3, for any n sufficiently large,

$$\pi_{\tilde{\gamma}_i}\left(\tilde{f}^n(\tilde{x}_i)\right) \geq \pi_{\tilde{\gamma}_i}(\tilde{x}_i) + nv_i'.$$

Moreover, for any n sufficiently large,

$$\pi_{\tilde{\gamma}_i}(\tilde{x}_i) + nv_i' \geq \lambda + M_i + nv_i'',$$

where M_i is given by Lemma 10.4. Take $N \in \mathbb{N}$ such that the two above properties hold for any $n \geq N$. By Lemma 10.5, there exists $n_i \geq N$, which can be taken arbitrarily large, such that the segment $[\tilde{f}^{n_i}(\tilde{x}_i), \tilde{f}^{n_i+k_i}(\tilde{x}_i)]_i$ meets $T_i^{r_i}\phi_i'$ for some $r_i \geq \frac{v_i' n_i}{\ell(\gamma_i)}$. Let m_i be the smallest integer such that $n_i + k_i \leq m_i q_i$. If n_i is chosen sufficiently large, then $r_i > m_i p_i$ and, for any $n \geq n_i$,

$$\pi_{\tilde{\gamma}_i}(\tilde{f}^n(\tilde{x}_i)) > \lambda + M_i + m_i p_i \ell(\gamma_i).$$

Indeed, when n_i is sufficiently large,

$$n_i v_i'' > \left\lfloor \frac{n_i + k_i}{q_i} \right\rfloor p_i \ell(\gamma_i) = m_i p_i \ell(\gamma_i).$$

This implies that the half-trajectory $[\tilde{f}^{n_i}(\tilde{x}_i), +\infty)_i$ is disjoint from $T_i^{m_ip_i} B_i$, and that the segment $[\tilde{x}_i, \tilde{f}^{m_iq_i}(\tilde{x}_i)]_i$ meets the leaf $T_i^{r_i}\phi_i'$. This proves the second point as $[T_i^{r_i}\phi_i', +\infty)_i \subset [\tilde{f}^{n_i}(\tilde{x}_i), +\infty)_i$. Moreover, recall that $P_i = 0$ so that the segment $[\tilde{x}_i, \tilde{f}^{m_iq_i}(\tilde{x}_i)]_i$ also meets the leaf ϕ_i. Hence, by Proposition 9.9, the segment $[\phi_i, T_i^{r_i}\phi_i']_i$ is admissible of order m_iq_i.

Let us prove the first point now. We already saw that $T_i^{r_i}\phi_i'$ meets $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ and that the segment $[\phi_i, T_i^{r_i}\phi_i']_i$ is admissible of order m_iq_i, so it remains to prove that for any $0 \leq j \leq m_ip_i$, the trajectory $[\phi_i, T_i^{r_i}\phi_i']_i$ meets the trajectory $T_i^j\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_{i+1})$. Recall that, by definition of R, for any j with $0 \leq j \leq m_ip_i$, $T_i^j\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_{i+1}) \subset (T_i^j\tilde{\gamma}_{i+1})_R$ and that the set $(T_i^j\tilde{\gamma}_{i+1})_R$ meets neither the connected component of $\overline{\mathbb{H}^2} \setminus \overline{(\tilde{\gamma}_{i+1})_R} \supset \overline{\mathbb{H}^2} \setminus \overline{A_i}$ which contains $\gamma_{i,-}$, ϕ_i and $(-\infty, \phi_i]_i$, by definition of ϕ_i, nor the connected component of $\overline{\mathbb{H}^2} \setminus \overline{(T_i^{m_ip_i}\tilde{\gamma}_{i+1})_R} \supset \overline{\mathbb{H}^2} \setminus \overline{T_i^{m_ip_i}B_i}$ which contains $\gamma_{i,+}$, $T_i^{r_i}\phi_i'$ and $[T_i^{r_i}\phi_i', +\infty)_i$. As, for any $0 \leq j \leq m_ip_i$, the trajectory $T_i^j\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_{i+1})$ is disjoint from $(-\infty, \phi_i]_i$ and $[T_i^{r_i}\phi_i', +\infty)_i$ and meets $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$, we deduce that it meets $[\phi_i, T_i^{r_i}\phi_i']_i$. The first point is hence satisfied.

Let us prove now the third point. Let $0 \leq j \leq m_ip_i$. By Lemma 10.11, the set $(T_i^j\tilde{\gamma}_{i+1})_R$ does not meet the leaf ϕ_i. Recall that $T_i^j\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_{i+1})$ is contained in $(T_i^j\tilde{\gamma}_{i+1})_R$, by

definition of R. Hence, the leaf ϕ_i does not meet the trajectory $T_i^j \mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_{i+1})$. In the same way, the trajectory $T_i^j \mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_{i+1})$ does not meet the leaf $T_i^{r_i}(\phi_i')$ either, as $j \leq m_i p_i < r_i$.

Similarly, by Lemma 10.11, the trajectory $\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_i)$ does not meet the leaves $T_i^j(\phi_{i+1})$ and $T_i^j(\phi_{i+1}')$, as $\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_i) \subset (\gamma_i)_R$. Finally, by Corollary 10.17 (points 1. and 3.), which is deduced from the already proved second point of the lemma,

$$\Big((-\infty, \phi_i]_i \cup [T_i^{r_i}\phi_i', +\infty)_i\Big) \ \cap \ T_i^j\Big((-\infty, \phi_{i+1}]_{i+1} \cup [T_{i+1}^{r_{i+1}}\phi_{i+1}', +\infty)_{i+1}\Big) = \emptyset$$

and we indeed have an $\tilde{\mathcal{F}}$-transverse intersection by Lemma 10.7.3. □

c. Admissible trajectories

Let us fix integers m_1, m_2, $r_1 > m_1 p_1$ and $r_2 > m_2 p_2$ such that Lemma 10.16 is satisfied. We let

$$\alpha = [\phi_1, T_1^{r_1}\phi_1']_1 \qquad \text{and} \qquad \beta = [\phi_2, T_2^{r_2}\phi_2']_2.$$

Let $I = (i_n, j_n)_{n\geq 1}$ be a sequence of couples of integers. For any $n \geq 1$, we let

$$\begin{cases} T^{I_{n,1}} &= T_1^{i_1} T_2^{j_1} \dots T_1^{i_{n-1}} T_2^{j_{n-1}} T_1^{i_n} \\ T^{I_{n,2}} &= T_1^{i_1} T_2^{j_1} \dots T_1^{i_n} T_2^{j_n}. \end{cases}$$

and by convention

$$T^{I_{0,1}} = T^{I_{0,2}} = \mathrm{Id}_{\tilde{S}}\,.$$

Lemma 10.18. *Let $I = (i_n, j_n)_{n\geq 0}$ be any sequence with $1 \leq i_n \leq m_1 p_1$ and $1 \leq j_n \leq m_2 p_2$ for any n. Then for any $n \geq 1$, there exists an $\tilde{\mathcal{F}}$-transverse path α_n with the following properties.*

1. *The path α_n is admissible of order $n(m_1 q_1 + m_2 q_2)$.*
2. *The path α_n joins the leaf ϕ_1 to the leaf $T^{I_{n,1}} T_2^{r_2}\phi_2'$.*
3. *The path α_n is contained in*

$$\bigcup_{k\leq n} T^{I_{k-1,2}}\alpha \cup T^{I_{k,1}}\beta.$$

4. *The path α_n has an $\tilde{\mathcal{F}}$-transverse intersection with the path $T^{I_{n,2}}\alpha_n$.*

For any $n \in \mathbb{Z}$, the geodesics γ_1 and $T_1^n\gamma_2$ meet so that $\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$ and $T_1^n \mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_2)$ are geometrically transverse. By Lemma 10.7.2., there exists $\tilde{y}_n$ an essential intersection point between those two trajectories.

Proof. We prove the statement by induction on $n \geq 1$.

First, for $n = 1$, as $i_1 \leq m_1 p_1$, Lemma 10.16 ensures that the paths α and $T_1^{i_1}\beta$ have an $\tilde{\mathcal{F}}$-transverse intersection. Hence, by Proposition 9.6, the transverse path

$$\alpha_1 = [\phi_1, \tilde{y}_{i_1}]_1 \ T_1^{i_1}[T_1^{-i_1}\tilde{y}_{i_1}, T_2^{r_2}\phi_2']_2$$

satisfies the first three required properties.

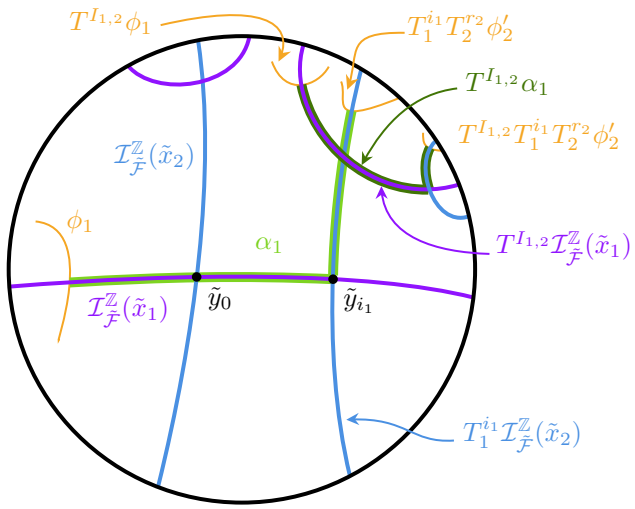

Figure 32: The case $n = 1$ for the proof of Lemma 10.18.

Let us check that the fourth property is also satisfied. Notice that the transverse trajectory

$$\alpha_1' = (-\infty, \phi_1]_1 \alpha_1 T_1^{i_1}[T_2^{r_2}\phi_2', +\infty)_2$$

and its translate under $T^{I_{1,2}} = T_1^{i_1}T_2^{j_1}$ are geometrically transverse, as the geodesics $(\gamma_{1,-}, T_1^{i_1}\gamma_{2,+})$ and its translate under $T^{I_{1,2}}$, which is $(T_1^{i_1}T_2^{j_1}\gamma_{1,-}, T_1^{i_1}T_2^{j_1}T_1^{i_1}\gamma_{2,+})$, meet: observe that the point $T_2^{j_1}\gamma_{1,-}$ lies in $(\gamma_{2,+}, \gamma_{1,-})_{\partial\mathbb{H}^2}$ and that the point $T_2^{j_1}T_1^{i_1}\gamma_{2,+}$ lies in $(\gamma_{1,-}, \gamma_{2,+})_{\partial\mathbb{H}^2}$.

By Corollary 10.15 (the numbers correspond to the cases of the corollary),

$$\begin{aligned}
\phi_1 \cap T^{I_{1,2}}\alpha_1' &= \emptyset \quad (1.)\\
T_1^{i_1}T_2^{r_2}\phi_2' \cap T^{I_{1,2}}\alpha_1' &= \emptyset\\
T^{I_{1,2}}\phi_1 \cap \alpha_1' &= \emptyset \quad (2.)\\
T^{I_{1,2}}T_1^{i_1}T_2^{r_2}\phi_2' \cap \alpha_1' &= \emptyset \quad (3.).
\end{aligned}$$

To prove the second relation, observe that the leaf $T_2^{r_2-j_1}\phi_2'$ is disjoint from the trajectories $\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$ (by 3.) and $T_1^{i_1}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_2)$ (by 4.) as $r_2 - j_1 \geq r_2 - m_2p_2 \geq 0$.

Moreover,

$$\Big((-\infty, \phi_1]_1 \cup T_1^{i_1}[T_2^{r_2}\phi_2', +\infty)_2\Big) \cap T^{I_{1,2}}\Big((-\infty, \phi_1]_1 \cup T_1^{i_1}[T_2^{r_2}\phi_2', +\infty)_2\Big) = \emptyset,$$

because by Corollary 10.17 (the numbers correspond to the cases of the corollary),

$$\begin{aligned}
(-\infty, \phi_1]_1 \cap T^{I_{1,2}}(-\infty, \phi_1]_1 &= \emptyset \quad (1.)\\
(-\infty, \phi_1]_1 \cap T^{I_{1,2}}T_1^{i_1}[T_2^{r_2}\phi_2', +\infty)_2 &= \emptyset \quad (1.)\\
T_1^{i_1}[T_2^{r_2}\phi_2', +\infty)_2 \cap T^{I_{1,2}}(-\infty, \phi_1]_1 &= \emptyset \quad (3.)\\
T_1^{i_1}[T_2^{r_2}\phi_2', +\infty)_2 \cap T^{I_{1,2}}T_1^{i_1}[T_2^{r_2}\phi_2', +\infty)_2 &= \emptyset \quad (3.)
\end{aligned}$$

so that the paths α_1 and $T^{I_{1,2}}\alpha_1$ intersect $\tilde{\mathcal{F}}$-transversally by Lemma 10.7.

Suppose the lemma holds for some integer $n \geq 1$ and let us prove it for $n+1$. The initialization of the induction proves that there exists a transverse path $\alpha_{1,i_{n+1}}$ which is admissible of order $m_1q_1 + m_2q_2$ which is contained in $\alpha \cup T_1^{i_{n+1}}\beta$ and which joins the leaf ϕ_1 to the leaf $T_1^{i_{n+1}}T_2^{r_2}\phi_2'$.

Let us prove first that the paths $T^{I_{n,2}}\alpha_{1,i_{n+1}}$ and α_n have an $\tilde{\mathcal{F}}$-transverse intersection. Let

$$\alpha_n' = (-\infty, \phi_1]_1 \alpha_n T^{I_{n,1}}[T_2^{r_2}\phi_2', +\infty)_2$$

and

$$\alpha_{1,i_{n+1}}' = (-\infty, \phi_1]_1 \alpha_1 T_1^{i_{n+1}}[T_2^{r_2}\phi_2', +\infty)_2.$$

Observe that the point $T^{I_{n,2}}\gamma_{1,-}$ lies between the points $\gamma_{1,-}$ and $T^{I_{n,1}}\gamma_{2,+}$ on $\partial\mathbb{H}^2$ and that the point $T^{I_{n,2}}T_1^{i_{n+1}}\gamma_{2,+}$ lies between the points $T^{I_{n,1}}\gamma_{2,+}$ and $\gamma_{2,-}$ on $\partial\mathbb{H}^2$. Hence the trajectories α_n' and $T^{I_{n,2}}\alpha_{1,i_{n+1}}'$ are geometrically transverse. Corollary 10.15 and the third property satisfied by α_n' ensure that the leaves $T^{I_{n,2}}\phi_1$ and $T^{I_{n,2}}T_1^{i_{n+1}}T_2^{r_2}\phi_2'$ are disjoint from α_n'. It also ensures that the leaves ϕ_1 and $T^{I_{n,1}}T_2^{r_2}\phi_2'$ are disjoint from the transverse path $T^{I_{n,2}}\alpha_{1,i_{n+1}}'$. Moreover, by Corollary 10.17 and the third property from the induction hypothesis,

$$\begin{cases} \left((-\infty, \phi_1]_1 \cup T^{I_{n,1}}[T_2^{r_2}\phi_2', +\infty)_2\right) \cap T^{I_{n,2}}\alpha_{1,i_{n+1}}' = \emptyset \\ T^{I_{n,2}}\left((-\infty, \phi_1]_1 \cup T_1^{i_{n+1}}[T_2^{r_2}\phi_2', +\infty)_2\right) \cap \alpha_n' = \emptyset \end{cases}$$

Hence the paths $T^{I_{n,2}}\alpha_{1,i_{n+1}}$ and α_n intersect $\tilde{\mathcal{F}}$-transversally by Lemma 10.7. Finally, Proposition 9.6 gives a path α_{n+1} which satisfies the three first conditions. By a proof which is similar to the initialization step, we prove that the paths α_{n+1} and $T^{I_{n+1,2}}\alpha_{n+1}$ intersect $\tilde{\mathcal{F}}$-transversally. This completes the induction. □

Corollary 10.19. *Let $I = (i_n, j_n)_{n\geq 0}$ be any sequence with $1 \leq i_n \leq m_1p_1$ and $1 \leq j_n \leq m_2p_2$ for any n. Then, for any integer $n \geq 1$ and any integer j with $0 \leq j \leq m_2p_2$, there exists points $\tilde{x}_{n,I}$ and $\tilde{x}_{n,I}'$ such that*

$$\tilde{f}^{n(m_1q_1+m_2q_2)}(\tilde{x}_{n,I}) = T_2^j T^{I_{n,1}}\tilde{x}_{n,I} \text{ and } \tilde{f}^{n(m_1q_1+m_2q_2)}(\tilde{x}_{n,I}') = T^{I_{n,2}}\tilde{x}_{n,I}'.$$

Proof. The existence of the points $\tilde{x}_{n,I}'$ is a consequence of Lemma 10.18 and of Theorem 9.30. Exchanging the roles of T_1 and T_2 in Lemma 10.18, we obtain the existence of the points $\tilde{x}_{n,I}$. □

End of the proof of Theorem 10.1 in the first case. Take any word w in letters T_1 and T_2 which contains at least one T_1 letter and one T_2 letter. Of course, we identify such a word with a deck transformation of $\tilde{S}$. Write

$$w = T_1^{i_1}T_2^{j_1}\dots T_1^{i_K}T_2^{j_K}$$

with

$$\begin{cases} K \geq 0 \\ i_n, j_n > 0 \text{ if } 2 \leq n \leq K-1 \\ j_1 > 0 \text{ and } i_K > 0 \\ i_1 \geq 0 \text{ and } j_K \geq 0. \end{cases}$$

Take integers m_1 and m_2 large enough so that $\max(i_1 + i_K, \max_{1\leq n\leq K} i_n) \leq m_1p_1$ and $\max(\max_{1\leq n\leq K} j_n, j_1 + j_K) \leq m_2p_2$.

If $i_1 > 0$ and $j_K > 0$ or $i_1 = 0$ and $j_K = 0$, Corollary 10.19 gives directly the wanted result. If $i_1 = 0$ apply Corollary 10.19 to the word $T_1^{i_2}T_2^{j_2}\dots T_1^{i_K}T_2^{j_K+j_1}$ to get a periodic point $\tilde{x}$ associated to this deck transformation. The point $T_2^{j_1}(\tilde{x})$ gives us then the wanted periodic orbit. If $j_K = 0$ apply Corollary 10.19 to the word $T_1^{i_1+i_K}T_2^{j_1}\dots T_1^{i_{K-1}}T_2^{j_{K-1}}$ to get a periodic point $\tilde{x}$ associated to this deck transformation. The point $T_1^{-i_K}(\tilde{x})$ gives us then the wanted periodic orbit. □

10.4 Case 2

In this subsection, we prove Theorem 10.1 in the case where one of the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ satisfies condition

(C_2) for any deck transformation $\tau \in \pi_1(S) \setminus \langle T_i \rangle$, any leaf which meets $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ does not meet $\tau\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$.

Changing the roles of x_1 and x_2 if necessary, we can suppose that $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ satisfies condition (C_2).

As in the previous case, we start by choosing carefully the respective lifts $\tilde{x}_1$ and $\tilde{x}_2$ of the points x_1 and x_2 to $\tilde{S}$. In the course of the proof, we will skip details when the arguments are similar to some arguments which were given in the first case.

The proof in this second case is a bit more complex than in the first one. The first step, made in Paragraph a., is a bit less technical than in the first case. However, we will see appearing configurations in which there are no $\mathcal{F}$-transverse intersections between our initial trajectories. This will force us to consider other transverse trajectories, and will complicate the rest of the proof. the last part of the proof, made in Paragraph c., requires a very careful study of the possible intersections between leaves and trajectories.

a. Choice of the points $\tilde{x}_1$ and $\tilde{x}_2$

See Figure 33 for these notations. For any $n \in \mathbb{Z}$, the geodesics $T_2^n\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ meet so that $T_2^n\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ are also geometrically transverse. Let $\tilde{y}'_n$ be the essential intersection point between $T_2^n\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ that is minimal for $<_2$ (it exists by Lemma 10.7.2.). As $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ satisfies condition (C_2), the lifts $T_2^n\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ are pairwise disjoint so that the sequence $(\tilde{y}'_n)_{n\in\mathbb{Z}}$ is increasing for the order relation $<_2$.

For any $n \in \mathbb{Z}$, the geodesics $\tilde{\gamma}_1$ and $T_1^n\tilde{\gamma}_2$ meet so that $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ and $T_1^n\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ are geometrically transverse. Denote by $\tilde{y}_n$ an essential intersection point between those two trajectories. We also take $\tilde{y}_0 = \tilde{y}'_0$.

In the whole proof, we orient $\partial\mathbb{H}^2$ in such a way that the point $\gamma_{2,-}$ lies on the positively oriented segment of $\partial\mathbb{H}^2$ which joins $\gamma_{1,-}$ to $\gamma_{1,+}$.

The following lemma is the analogue in this case of Lemma 10.11. We need the following notation:

$$A = \overline{\bigcup_{\epsilon\in\{-1,0,1\}} T_2^{\epsilon}(\tilde{\gamma}_1)_R \cup \bigcup_{i\neq 0} T_1^i(\tilde{\gamma}_2)_R \cup \bigcup_{\substack{i\geq 1\\ w\in\langle T_1^{-1},T_2^{-1}\rangle_+}} T_1^iT_2^{-1}w(\tilde{\gamma}_2)_R} \subset \overline{\mathbb{H}^2}.$$

$$(B_i)_{i\in\mathbb{Z}} = \overline{\phi_{\tilde{y}'_i} \cup (-\infty, \tilde{y}'_i]_2 \cup T_2^i(-\infty, T_2^{-i}\tilde{y}'_i]_1},$$
$$(B'_i)_{i\in\mathbb{Z}} = \overline{\bigcup_{n\geq 0} T_2^n(\phi_{\tilde{y}'_i} \cup [\tilde{y}'_i, +\infty)_2)}.$$

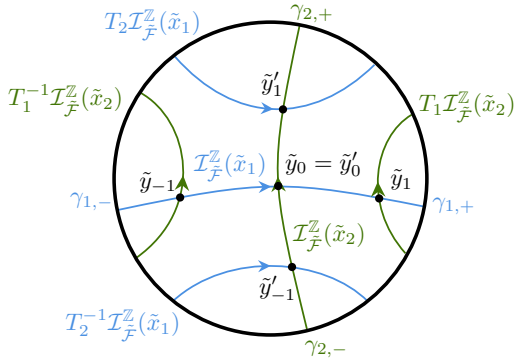

Figure 33: Notation of the beginning of Case 2.

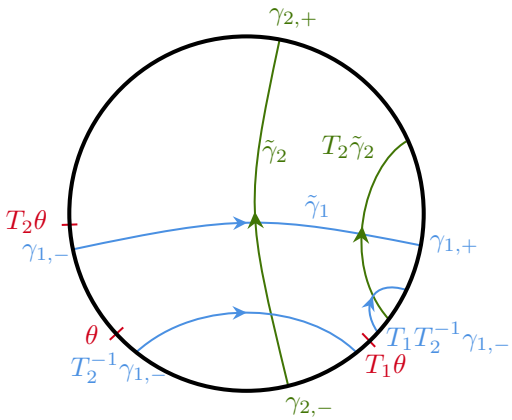

Figure 34: Proof of Claim 10.21.

Lemma 10.20. *There exists an integer k'_0 such that the set $B_{-k'_0}$ is included in the connected component of $\overline{\mathbb{H}^2} \setminus A$ containing $\gamma_{2,-}$, and the set $B'_{k'_0}$ is included in the connected component of $\overline{\mathbb{H}^2} \setminus A$ containing $\gamma_{2,+}$*

In the sequel, we will denote

$$\phi_2 = \phi_{\tilde{y}'_{-k'_0}} \qquad \text{and} \qquad \phi'_2 = \phi_{\tilde{y}'_{k'_0}}.$$

Proof. We need to prove the following claim, which is similar to Lemma 10.14.

Claim 10.21. *The set A meets neither $\gamma_{2,-}$ nor $\gamma_{2,+}$.*

Proof. First, observe that

$$\overline{\bigcup_{\epsilon\in\{-1,0,1\}} T_2^{\epsilon}(\tilde{\gamma}_1)_R \cup \bigcup_{i\neq 0} T_1^i(\tilde{\gamma}_2)_R} \cap \partial\mathbb{H}^2$$
$$= \bigcup_{\epsilon\in\{-1,0,1\}} \{T_2^{\epsilon}\gamma_{1,-}, T_2^{\epsilon}\gamma_{1,+}\} \cup \overline{\{T_1^i\gamma_{2,+}, i\neq 0\}} \cup \overline{\{T_1^i\gamma_{2,-}, i\neq 0\}}$$

so that the set

$$\overline{\bigcup_{\epsilon\in\{-1,0,1\}} T_2^{\epsilon}(\tilde{\gamma}_1)_R \cup \bigcup_{i\neq 0} T_1^i(\tilde{\gamma}_2)_R}$$

meets neither $\gamma_{2,-}$ nor $\gamma_{2,+}$.

By Lemma 6.3, the set

$$\bigcup_{i\geq 1} \bigcup_{w\in\langle T_1^{-1},T_2^{-1}\rangle_+} T_1^iT_2^{-1}w(\tilde{\gamma}_2)_R$$

meets neither $\gamma_{2,-}$ nor $\gamma_{2,+}$ ($\langle T_1, T_2\rangle$ is a free group). So it suffices to prove that this set does not accumulate on $\gamma_{2,-}$ or $\gamma_{2,+}$ either. As the deck transformation T_2 preserves $\tilde{\gamma}_2$, observe that

$$\bigcup_{i\geq 1} \bigcup_{w\in\langle T_1^{-1},T_2^{-1}\rangle_+} T_1^iT_2^{-1}w(\tilde{\gamma}_2)_R = \bigcup_{i\geq 1} \bigcup_{w\in\langle T_1^{-1},T_2^{-1}\rangle_+} T_1^iT_2^{-1}wT_1^{-1}(\tilde{\gamma}_2)_R \cup \bigcup_{i\geq 1} T_1^i(\tilde{\gamma}_2)_R.$$

We already saw that the set

$$\overline{\bigcup_{i\geq 1} T_1^i(\tilde{\gamma}_2)_R}$$

meets neither $\gamma_{2,-}$ nor $\gamma_{2,+}$.

Condition (C_2) forces the geodesics $T_2^{-1}\tilde{\gamma}_1$ and $T_1T_2^{-1}\tilde{\gamma}_1$ to be disjoint so that $T_1T_2^{-1}\gamma_{1,-}$ belongs to $(T_2^{-1}\gamma_{1,+},\gamma_{1,+})_{\partial\mathbb{H}^2}\subset(\gamma_{2,-},\gamma_{1,+})_{\partial\mathbb{H}^2}$. By this property, if we take $\theta\in(\gamma_{1,-},T_2^{-1}\gamma_{1,-})_{\partial\mathbb{H}^2}$ sufficiently close to $T_2^{-1}\gamma_{1,-}$, then we have $T_1\theta\in(\gamma_{2,-},\gamma_{1,+})_{\partial\mathbb{H}^2}$ (see Figure 34).

Observe that the attractor of the restriction of the action of the semigroup generated by T_1^{-1} and T_2^{-1} to $[T_1^{-1}\gamma_{2,+},\gamma_{1,+}]_{\partial\mathbb{H}^2}$ is contained in $[\gamma_{1,-},\gamma_{1,+}]_{\partial\mathbb{H}^2}$. Hence there exists an integer $N\geq 0$ such that, for any word w in T_1^{-1} and T_2^{-1} whose length is greater than or equal to N, both points $wT_1^{-1}\gamma_{2,+}$ and $wT_1^{-1}\gamma_{2,-}$ belong to $(T_2\theta,\gamma_{1,+})_{\partial\mathbb{H}^2}$. Hence

$$\overline{\bigcup_{i\geq 1}\ \bigcup_{\substack{w\in\langle T_1^{-1},T_2^{-1}\rangle_+\\ \ell(w)\geq N}} T_1^iT_2^{-1}wT_1^{-1}(\tilde{\gamma}_2)_R}\cap\partial\mathbb{H}^2\subset[T_1\theta,\gamma_{1,+}]_{\partial\mathbb{H}^2}\subset(\gamma_{2,-},\gamma_{1,+}]_{\partial\mathbb{H}^2}.$$

It remains to treat the case where the length of w is smaller than N. By Lemma 6.3, for any word w in T_1^{-1} and T_2^{-1}, neither $T_2^{-1}wT_1^{-1}\gamma_{2,-}$ nor $T_2^{-1}wT_1^{-1}\gamma_{2,+}$ meet $\gamma_{1,-}$. Hence there exists an integer $I\geq 1$ such that, for any $i\geq I$ and any word w in T_1^{-1} and T_2^{-1} whose length is smaller than N, the intersection $\overline{(T_1^iT_2^{-1}wT_1^{-1}\tilde{\gamma}_2)_R}$ with $\partial\mathbb{H}^2$ is included in $(T_2^{-1}\gamma_{1,+},T_2\gamma_{1,+})_{\partial\mathbb{H}^2}$.

Finally, the set of words $T_1^iT_2^{-1}wT_1^{-1}$ with $i\leq I$ and the length of w smaller than N is finite, so by Lemma 6.3 we have

$$\overline{\bigcup_{\substack{i\leq I\\ \text{length}(w)\leq N}}(T_1^iT_2^{-1}wT_1^{-1}\tilde{\gamma}_2)_R}\cap\{\gamma_{2,-},\gamma_{2,+}\}=\emptyset.$$

This finishes the proof of the claim. □

Let $i\in\mathbb{Z}$. As the trajectory $\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$ satisfies condition (C_2), the leaf $\phi_{\tilde{y}'_i}$ and the half trajectory $T_2^i(-\infty,T_2^{-i}\tilde{y}'_i]_1$ are disjoint from the trajectories $T_2^{i-1}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$ and $T_2^{i+1}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$ and lie between them. Moreover, the sequence $\left(\overline{T_2^n\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)}\right)_n$ of compact subsets of $\overline{\mathbb{H}^2}$ converges to $\gamma_{2,+}$ when $n\to+\infty$ and to $\gamma_{2,-}$ when $n\to-\infty$ for the Hausdorff topology. Therefore, the sequence $(B_i)_{i\in\mathbb{Z}}=\overline{\phi_{\tilde{y}'_i}\cup(-\infty,\tilde{y}'_i]_2\cup T_2^i(-\infty,T_2^{-i}\tilde{y}'_i]_1}$ of subsets of $\overline{\mathbb{H}^2}$ converges to $\gamma_{2,-}$ when $i\to-\infty$ for the Hausdorff topology and the sequence $(B'_i)_{i\in\mathbb{Z}}=\overline{\bigcup_{n\geq 0}T_2^n(\phi_{\tilde{y}'_i}\cup[\tilde{y}'_i,+\infty)_2)}$ of subsets of $\overline{\mathbb{H}^2}$ converges to $\gamma_{2,+}$ when $i\to+\infty$ for the Hausdorff topology. Hence, for $i\geq 0$ sufficiently large, both sets B_{-i} and B'_i avoid the set A, which proves the lemma. □

The following is similar to Corollary 10.15.

Corollary 10.22. *Let $n\geq 1$. Recall that $\phi_2=\phi_{\tilde{y}'_{-k'_0}}$ and $\phi'_2=\phi_{\tilde{y}'_{k'_0}}$.*

1. *For any word w in T_1 and T_2, the trajectory $w\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_2)$ does not meet $T_1^{-n}\phi_2$ nor $T_1^{-n}T_2^{-k'_0}(-\infty,T_2^{k'_0}\tilde{y}'_{-k'_0}]_1$.*

2. *For any word w in T_1^{-1} and T_2^{-1} which starts with T_2^{-1}, the trajectory $w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ does not meet $T_1^{-n}\phi_2$ nor $T_1^{-n}T_2^{-k'_0}(-\infty, T_2^{k'_0}\tilde{y}'_{-k'_0}]_1$.*

3. *For any word w in T_1 and T_2 which starts with T_1, the trajectory $w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ does not meet $T_2^{n-1}\phi'_2$ nor $T_2^{n-1}[\tilde{y}'_{k'_0}, +\infty)_2$.*

4. *For any word w in T_1^{-1} and T_2^{-1} which starts with T_1^{-1}, the trajectory $w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ does not meet $T_2^{n-1}\phi'_2$ nor $T_2^{n-1}[\tilde{y}'_{k'_0}, +\infty)_2$..*

Proof. 1. Let w be any word on T_1 and T_2, and $n \geq 1$. By Lemma 10.20, the leaf $T_1^{-n}\phi_2$ and the half-trajectory $T_1^{-n}T_2^{-k'_0}(-\infty, T_2^{k'_0}\tilde{y}'_{-k'_0}]_1$ lie in $\overline{(\tilde{\gamma}_2)^{\mathbb{C}}_R}$. So, as the α-limit of $T_1^{-n}T_2^{-k'_0}(-\infty, T_2^{k'_0}\tilde{y}'_{-k'_0}]_1$ is $T_1^{-n}T_2^{-k'_0}\gamma_{1,-}$, and as the segment $[\gamma_{1,-}, T_1^{-n}T_2^{-k'_0}\gamma_{1,-}]_{\partial\mathbb{H}^2}$ meets neither $\gamma_{2,-}$ nor $\gamma_{2,+}$, the leaf $T_1^{-n}\phi_2$ and the half-trajectory $T_1^{-n}T_2^{-k'_0}(-\infty, T_2^{k'_0}\tilde{y}'_{-k'_0}]_1$ lie in the connected component of $\overline{\mathbb{H}^2} \setminus \overline{(\tilde{\gamma}_2)_R}$ which contains $\gamma_{1,-}$. On the other hand, the geodesic $w\tilde{\gamma}_2$ either lies in the connected component of $\overline{\mathbb{H}^2} \setminus \overline{\tilde{\gamma}_2}$ which contains $\gamma_{1,+}$ or $w\tilde{\gamma}_2 = \tilde{\gamma}_2$. Hence $w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2) \subset (w\tilde{\gamma}_2)_R$ does not meet the connected component of $\overline{\mathbb{H}^2} \setminus \overline{(\tilde{\gamma}_2)_R}$ which contains the leaf $T_1^{-n}\phi_2$ and the half-trajectory $T_1^{-n}T_2^{-k'_0}(-\infty, T_2^{k'_0}\tilde{y}'_{-k'_0}]_1$.

2. Lemma 10.20 implies that, for any word w in T_1^{-1} and T_2^{-1} which starts with T_2^{-1}, the neighbourhood $w(\tilde{\gamma}_2)_R$ is disjoint from $T_1^{-n}\phi_2$ and from $T_1^{-n}T_2^{-k'_0}(-\infty, T_2^{k'_0}\tilde{y}'_{-k'_0}]_1$. The second point follows as $w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2) \subset (w\tilde{\gamma}_2)_R$.

3. Let w be any word on T_1 and T_2 which starts with T_1. By Lemma 10.20, the sets $T_2^{n-1}\phi'_2$ and $T_2^{n-1}[\tilde{y}'_{k'_0}, +\infty)_2$ lie in the connected component of $\overline{\mathbb{H}^2} \setminus \overline{(T_1\tilde{\gamma}_2)_R}$ which contains $T_2^{n-1}\gamma_{2,+} = \gamma_{2,+}$. On the other hand, the geodesic $w\tilde{\gamma}_2$ either lies in the connected component of $\overline{\mathbb{H}^2} \setminus \overline{T_1\tilde{\gamma}_2}$ which does not contain $\gamma_{2,+}$ or $w\tilde{\gamma}_2 = T_1\tilde{\gamma}_2$. As $w\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2) \subset (w\tilde{\gamma}_2)_R$, this proves this third point.

4. This last point is proved identically to the third one by changing T_1 to T_1^{-1} and T_2 to T_2^{-1}.

□

Changing the point $\tilde{x}_2$ to some other point of its orbit if necessary, we further suppose that the trajectory $(-\infty, \tilde{x}_2]_2$ is disjoint from ϕ_2, from $(T_2^j\tilde{\gamma}_1)_R$ for any $j \geq -2$ and from $(T_1^i\tilde{\gamma}_2)_R$ for any $i \neq 0$. The following is similar to Lemma 10.16.

Lemma 10.23. *There exists an integer $m_2 > 0$, which can be taken arbitrarily large, and $r_2 > m_2p_2$ such that the two following properties are satisfied.*

1. *The segment of trajectory $[\tilde{x}_2, \tilde{f}^{m_2q_2}(\tilde{x}_2)]_2$ meets the leaves ϕ_2 and $T_2^{r_2}\phi'_2$.*

2. *The half-trajectories $(-\infty, \tilde{x}_2]_2$ and $[\tilde{f}^{m_2q_2}(\tilde{x}_2), +\infty)_2$ are disjoint from $(T_2^i\tilde{\gamma}_1)_R$, for any i with $-2 \leq i \leq m_2p_2 + 2$ and from $(T_1^j\tilde{\gamma}_2)_R$ for any $j \neq 0$.*

Observe that this lemma implies that all the points $\tilde{y}'_n$, with $-2 \leq n \leq m_2p_2 + 2$, belong to the segment $[\tilde{x}_2, \tilde{f}^{m_2q_2}(\tilde{x}_2)]_2$.

Proof. The proof is similar to the proof of Lemma 10.16. □

Lemma 10.24. *There exists an integer $k_0 > 0$ with the following property: for any integer k with $|k| \geq k_0$ and any $j \in \mathbb{Z}$ with $j \neq 0$, the sets $T_1^k(\tilde{\gamma}_2)_R$ and $T_1^k T_2^j(\tilde{\gamma}_1)_R$ are disjoint from $(\tilde{\gamma}_2)_R$, and $T_2 T_1^{-k}\tilde{\gamma}_2 \subset L(\tilde{\gamma}_1)$.*

By Lemma 10.3, the transverse trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ stay at distance at most $R > 0$ from $\tilde{\gamma}_i$, so the conclusions of the lemma stay true when replacing $(\tilde{\gamma}_i)_R$ by $T_i^{i'}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_i)$ for some $i' \in \mathbb{Z}$.

Proof. Note that if k is large enough, then the set $T_2 T_1^{-k}\tilde{\gamma}_2 \subset L(\tilde{\gamma}_1)$ is contained in an arbitrary neighbourhood of $T_2\gamma_{1,+} \subset L(\tilde{\gamma}_1)$. Similarly, $-k$ is large enough, then the set $T_2 T_1^{-k}\tilde{\gamma}_2 \subset L(\tilde{\gamma}_1)$ is contained in an arbitrary neighbourhood of $T_2\gamma_{1,-} \subset L(\tilde{\gamma}_1)$.

Now, let

$$\mathcal{A} = \overline{\bigcup_{j\neq 0} T_2^j(\tilde{\gamma}_1)_R} \cup \overline{(\tilde{\gamma}_2)_R}.$$

As the sequences of points $(T_2^j\gamma_{1,+})_{j\in\mathbb{Z}}$ and $(T_2^j\gamma_{1,-})_{j\in\mathbb{Z}}$ both converge to $\gamma_{2,+}$ when $j \to +\infty$ and to $\gamma_{2,-}$ when $j \to -\infty$, we have $\mathcal{A} \cap \partial\mathbb{H}^2 = \left\{T_2^j\gamma_{1,\pm} \mid j \neq 0\right\} \cup \{\gamma_{2,\pm}\}$. In particular, the set $\mathcal{A}$ meets neither $\gamma_{1,+}$ nor $\gamma_{1,-}$. Hence the sequence of compact sets $\left(T_1^k\mathcal{A}\right)_{k\in\mathbb{Z}}$ converges for the Hausdorff topology to the point $\gamma_{1,-}$ when $k \to -\infty$ and to $\gamma_{1,+}$ when $k \to +\infty$: we can find an integer k_0 so that the first properties of the lemma are satisfied. The proof of the last property follows the same strategy, using the remark made at the beginning of the proof. □

From now on, we fix integers $m_2 > 0$ and $k_0 > 0$ such that Lemma 10.23 and 10.24 are satisfied.

We denote by $\hat{\mathcal{F}}$, $\hat{f}$, $\hat{x}_1$, $\hat{x}_2$ respective lifts to $\widetilde{\mathrm{dom}\mathcal{F}}$ of $\tilde{\mathcal{F}}$, $\tilde{f}$, $\tilde{x}_1$, $\tilde{x}_2$ in such a way that $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_1)$ and $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_2)$ meet at some lift $\hat{y}_0$ of $\tilde{y}_0$ and $\hat{\phi}_2$ and $\widehat{T_2^{r_2}\phi_2'}$ meet $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_2)$. We choose the lift $\hat{f}$ of $\tilde{f}$ which is isotopic to the identity.

For any $i \in \mathbb{Z}$, $i \neq 0$, we denote by $\widehat{T_1^i}$ the lift of T_1^i such that $\widehat{T_1^i}\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_2)$ meets $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_1)$ at a lift $\hat{y}_i$ of $\tilde{y}_i$ and by $\widehat{T_2^i}$ a lift of T_2^i such that $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_2)$ meets $\widehat{T_2^i}\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_1)$ at some lift $\hat{y}_i'$ of $\tilde{y}_i'$. Note that it is possible that $\widehat{T_1^i} \neq \widehat{T_1}^i$. We fix respective lifts $\hat{\phi}_2$ and $\widehat{T_2^{r_2}\phi_2'}$ of ϕ_2 and $T_2^{r_2}\phi_2'$ such that the leaf ϕ_2 meets $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_2)$ at the point $\hat{y}'_{-k_0'}$ and the leaf $\widehat{T_2^{r_2}\phi_2'}$ meets $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_2)$.

To be completely rigorous and so that these objects are uniquely defined, we need to consider parameters on the trajectories instead of actual points, as in the definition of essential intersection points. However, we chose to drop the mention of those parameters to simplify notation.

Let (see Figure 35)

$$\begin{aligned}\hat{B} &= \hat{f}^{m_2 q_2}\left(\overline{L\left(\widehat{T_1^{-k_0}}\hat{\phi}_2\right)}\right) \cup \overline{R\left(\widehat{T_1^{-k_0}}\widehat{T_2^{r_2}\phi'}_2\right)} \\ &= \widehat{T_1^{-k_0}}\left(\hat{f}^{m_2 q_2}\left(\overline{L(\hat{\phi}_2)}\right) \cup \overline{R(\widehat{T_2^{r_2}\phi'}_2)}\right).\end{aligned} \tag{10.3}$$

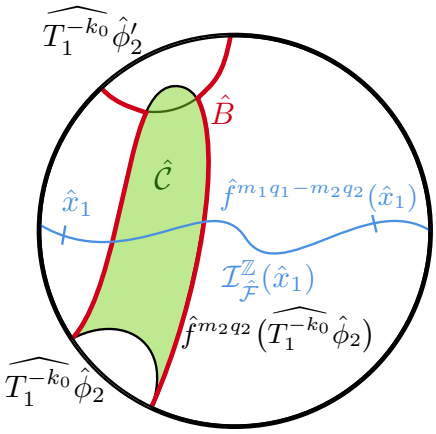

Figure 35: The set $\hat{B}$. The ambient space here is $\widetilde{\mathrm{dom}}\,\mathcal{F} \simeq \mathbb{H}^2$.

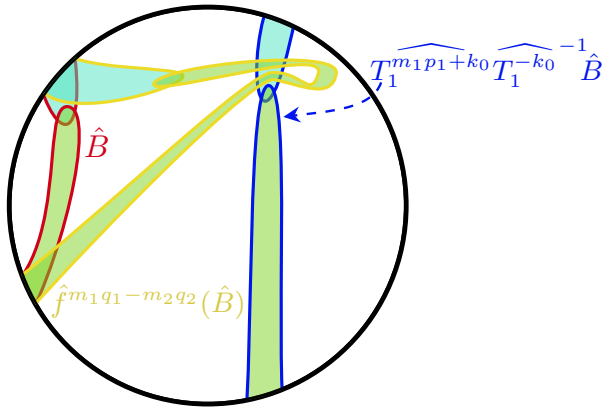

Figure 36: Proof of Lemma 10.26: the set $\hat{B}$, its translate by $\widehat{T_1^{m_1p_1+k_0}}$ and its image by $\hat{f}^{m_1q_1-m_2q_2}$.

Observe that the set $\hat{B}$ contains the trajectory $\widehat{T_1^{-k_0}}\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_2)$ and hence meets $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_1)$. In the same way, for any $i \in \mathbb{Z}$, the set $\widehat{T_1^i}\widehat{T_1^{-k_0}}^{-1}\hat{B}$ contains the trajectory $\widehat{T_1^i}\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_2)$ and hence meets $\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_1)$.

Let

$$\hat{\mathcal{C}} = \hat{f}^{m_2q_2}\left(\overline{L\left(\widehat{T_1^{-k_0}}\hat{\phi}_2\right)}\right) \setminus L\left(\widehat{T_1^{-k_0}}\hat{\phi}_2\right) \subset \hat{B}.$$

As the trajectory $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ satisfies condition (C_2), the leaves ϕ_2 and $T_2^{r_2}\phi_2'$ and their translates under iterates of T_1 do not touch $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$. Hence, for any $i \in \mathbb{Z}$,

$$\widehat{T_1^i}\widehat{T_1^{-k_0}}^{-1}\hat{B} \cap \mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_1) = \widehat{T_1^i}\widehat{T_1^{-k_0}}^{-1}\hat{\mathcal{C}} \cap \mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_1).$$

Let $\tilde{\mathcal{C}}$ be the projection of $\hat{\mathcal{C}}$ on $\tilde{S}$ and recall that $\pi_{\tilde{\gamma}_1}$ denotes the orthogonal projection on $\tilde{\gamma}_1$. As before, we parametrize $\tilde{\gamma}_1$ by arclength and identify points of $\tilde{\gamma}_1$ with their corresponding parameters.

Changing $\tilde{x}_1$ to some of its iterates under $\tilde{f}$, we can suppose that

$$\max\left\{n \in \mathbb{Z} \mid \tilde{f}^n(\tilde{x}_1) < \min \pi_{\tilde{\gamma}_1}(\tilde{\mathcal{C}})\right\} = 0. \tag{10.4}$$

Indeed, by Lemma 10.20, $\min \pi_{\tilde{\gamma}_1}(\tilde{\mathcal{C}}) \in \tilde{\gamma}_1$ (it is bigger than $\gamma_{1,-}$).

Lemma 10.25. *There exists an integer $m_1 > 0$, which can be taken arbitrarily large, such that, for any i with $-k_0 \le i \le m_1p_1 + k_0$,*

$$\widehat{T_1^i}\widehat{T_1^{-k_0}}^{-1}\hat{B} \cap \mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_1) = \widehat{T_1^i}\widehat{T_1^{-k_0}}^{-1}\hat{B} \cap (\hat{x}_1, \hat{f}^{m_1q_1-m_2q_2}(\hat{x}_1))_1.$$

In particular, the segment $[\hat{x}_1, \hat{f}^{m_1q_1-m_2q_2}(\hat{x}_1)]_1$ contains the points $\hat{y}_i$, for any i with $-k_0 \le i \le k_0 + m_1p_1$.

The last sentence of the lemma comes from the fact that the set $\widehat{T_1^i}\widehat{T_1^{-k_0}}^{-1}\hat{B}$ contains the trajectory $\widehat{T_1^i}\mathcal{I}^{\mathbb{Z}}_{\hat{\mathcal{F}}}(\hat{x}_2)$.

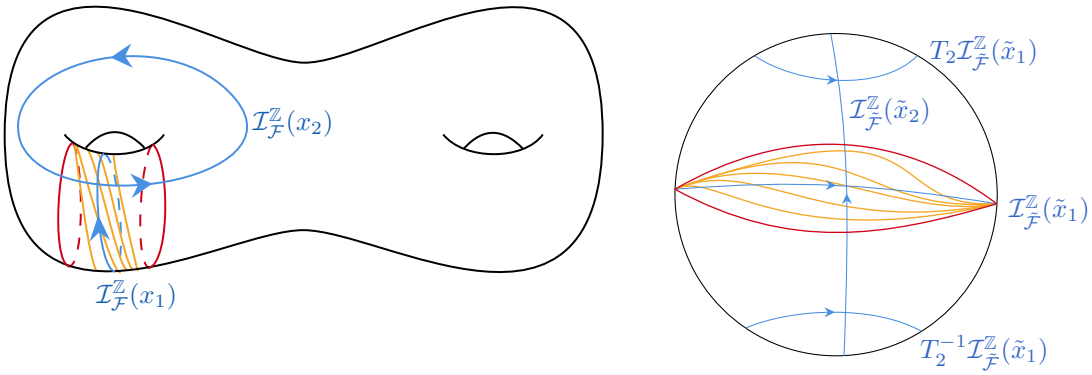

Figure 37: Lellouch's example [Lel19] on the genus-2 surface (left) and on the universal cover $\widetilde{\mathrm{dom}}(\mathcal{F})$: the trajectories $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ and $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ have no $\mathcal{F}$-transverse intersection, as $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ is equivalent to a subpath of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$.

Proof. Fix v_1' with $\frac{p_1}{q_1} < v_1' < v_1$. By Lemma 10.3, for any n sufficiently large,

$$\pi_{\tilde{\gamma}_1}(\tilde{f}^n(\tilde{x}_1)) > nv_1' + \pi_{\tilde{\gamma}_1}(\tilde{x}_1).$$

Moreover, for any n sufficiently large,

$$nv_1' + \pi_{\tilde{\gamma}_1}(\tilde{x}_1) > \max(\pi_{\tilde{\gamma}_1}(\tilde{\mathcal{C}})) + (n + m_2q_2)\frac{p_1}{q_1}\ell(\gamma_1) + 2k_0\ell(\gamma_1) + M_1,$$

where M_1 is an upper bound on the diameters of the paths $\mathcal{I}_{\tilde{\mathcal{F}}}(\tilde{x})$ for $\tilde{x} \in K_1$, and K_1 is given by Lemma 10.4.

Take an integer m_1 sufficiently large so that the two above properties hold for any $n \geq m_1q_1 - m_2q_2$: note that m_2 does not depend on m_1 (m_2 being fixed, one can choose m_1 arbitrarily large). Then, for any $n \geq m_1q_1 - m_2q_2$ and any $-k_0 \leq i \leq m_1p_1 + k_0$,

$$\pi_{\tilde{\gamma}_1}(\tilde{f}^n(\tilde{x}_1)) > \max \pi_{\tilde{\gamma}_1}(T_1^{i+k_0}\tilde{\mathcal{C}}) + M_1$$

so that the half-trajectory $[\tilde{f}^{m_1q_1-m_2q_2}(\tilde{x}_1), +\infty)_1$ does not meet $T_1^{i+k_0}\tilde{\mathcal{C}}$. Hence the half trajectory $[\hat{f}^{m_1q_1-m_2q_2}(\hat{x}_1), +\infty)_1$ does not meet $\widehat{T_1^i}\widehat{T_1^{-k_0}}^{-1}\hat{B}$. Moreover, the choice (10.4) of the point $\tilde{x}_1$ ensures that $(-\infty, \hat{x}_1]_1$ does not meet $\widehat{T_1^i}\widehat{T_1^{-k_0}}^{-1}\hat{B}$. □

From now on, we fix an integer $m_1 > 0$ such that Lemma 10.25 is satisfied, and that $m_1q_1 - 2m_2q_2 \geq 0$.

b. Transverse intersections

As we said earlier, $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ has not necessarily an $\tilde{\mathcal{F}}$-transverse intersection with $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ (see Figure 37). To overcome this problem, we will find a path $\tilde{\alpha}$ which has an $\tilde{\mathcal{F}}$-transverse intersection with $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ and which contains a large segment of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$. We borrow this idea and the idea of the proof of the following lemma from Lellouch's PhD thesis [Lel19].

Lemma 10.26. *The transverse path (see Figure 38)*

$$\tilde{\alpha} = T_1^{-k_0}[\phi_2, T_1^{k_0}\tilde{y}_{-k_0}]_2\,[\tilde{y}_{-k_0}, \tilde{y}_{m_1p_1+k_0}]_1\,T_1^{m_1p_1+k_0}[T_1^{-m_1p_1-k_0}\tilde{y}_{m_1p_1+k_0}, T_2^{r_2}\phi_2']_2$$

is admissible of order m_1q_1 and has an $\tilde{\mathcal{F}}$-transverse intersection with the transverse path $T_1^i[\phi_2, T_2^{r_2}\phi_2']_2$ at the point $\tilde{y}_i$, for any $0 \leq i \leq m_1p_1$.

Proof. Observe that, by Lemma 10.25, both sets $\hat{B}$ and $\widehat{T_1^{m_1p_1+k_0}}\,\widehat{T_1^{-k_0}}^{-1}\hat{B}$ separate the points $\hat{x}_1$ and $\hat{f}^{m_1q_1-m_2q_2}(\hat{x}_1)$. Hence $\hat{f}^{m_1q_1-m_2q_2}(\hat{B}) \cap \widehat{T_1^{m_1p_1+k_0}}\,\widehat{T_1^{-k_0}}^{-1}(\hat{B}) \neq \emptyset$ (see Figure 36). By the definition

$$\hat{B} = \hat{f}^{m_2q_2}\left(\overline{L\left(\widehat{T_1^{-k_0}}\hat{\phi}_2\right)}\right) \cup \overline{R\left(\widehat{T_1^{-k_0}}\widehat{T_2^{r_2}\phi_2'}\right)}$$

of $\hat{B}$, this amounts to say that one of the intersections

$$\begin{aligned}
\widehat{T_1^{-k_0}}\hat{f}^{m_1q_1-m_2q_2}\left(\overline{L(\hat{\phi}_2)}\right) &\cap \widehat{T_1^{m_1p_1+k_0}}\,\overline{L(\hat{\phi}_2)} \\
\widehat{T_1^{-k_0}}\hat{f}^{m_1q_1}\left(\overline{L(\hat{\phi}_2)}\right) &\cap \widehat{T_1^{m_1p_1+k_0}}\,\overline{R(\widehat{T_2^{r_2}\phi_2'})} \\
\widehat{T_1^{-k_0}}\hat{f}^{m_1q_1-2m_2q_2}\left(\overline{R(\widehat{T_2^{r_2}\phi_2'})}\right) &\cap \widehat{T_1^{m_1p_1+k_0}}\,\overline{L(\hat{\phi}_2)} \\
\widehat{T_1^{-k_0}}\hat{f}^{m_1q_1-m_2q_2}\left(\overline{R(\widehat{T_2^{r_2}\phi_2'})}\right) &\cap \widehat{T_1^{m_1p_1+k_0}}\,\overline{R(\widehat{T_2^{r_2}\phi_2'})}
\end{aligned}$$

is nonempty.

By Lemma 10.20, the leaf ϕ_2 is disjoint from the images of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ by T_1^i for any $i \neq 0$, hence none of the sets $L(\hat{\phi}_2)$ and $L(\widehat{T_1^i}\hat{\phi}_2)$ is included in the other one. As the foliation is made of Brouwer lines, the same holds for $L(\hat{\phi}_2)$ and $L(\hat{f}^j(\widehat{T_1^i}\hat{\phi}_2))$ for any $j \in \mathbb{Z}$. A similar property holds for the leaf $\hat{\phi}_2'$, which implies that the first and the last intersections are empty.

For the third intersection, remark that as $m_1q_1 - 2m_2q_2 \geq 0$, we have

$$\hat{f}^{m_1q_1-2m_2q_2}\left(\overline{R(\widehat{T_2^{r_2}\phi_2'})}\right) \subset \overline{R(\widehat{T_2^{r_2}\phi_2'})}.$$

Moreover, by Lemma 10.20, the leaves ϕ_2 and ϕ_2' are disjoint from the images of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ by T_1^i for any $i \neq 0$, so

$$\widehat{T_1^{-k_0}}\,\overline{R(\widehat{T_2^{r_2}\phi_2'})} \cap \widehat{T_1^{m_1p_1+k_0}}\,\overline{L(\hat{\phi}_2)} = \emptyset.$$

This implies that the third intersection is empty.

Hence the second intersection is nonempty, so

$$\hat{f}^{m_1q_1}(\widehat{T_1^{-k_0}}\hat{\phi}_2) \cap \widehat{T_1^{m_1p_1+k_0}}\,\widehat{T_2^{r_2}\phi_2'} \neq \emptyset.$$

This means that the transverse path $\tilde{\alpha}$ is admissible of order m_1q_1.

Now, let $0 \leq i \leq m_1p_1$ and let us prove that $\tilde{\alpha}$ has an $\tilde{\mathcal{F}}$-transverse intersection with $T_1^i\tilde{\beta}$ where $\tilde{\beta} = [\phi_2, T_2^{r_2}\phi_2']_2$. To do that, we want to use Lemma 10.7.

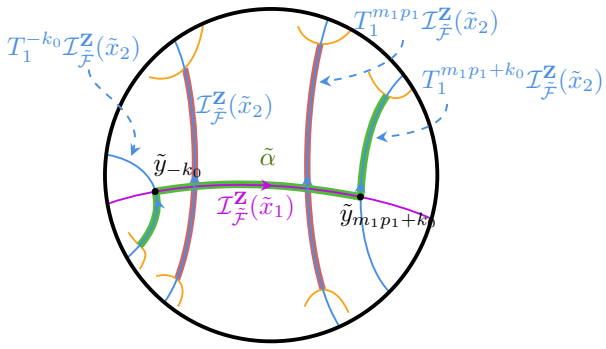

Figure 38: The path $\tilde{\alpha}$ of Lemma 10.26 (green) has an $\tilde{\mathcal{F}}$-transverse intersection with $T_1^i\tilde{\beta} = T_1^i[\phi_2, T_2^{r_2}\phi_2']_2$ (red) for any $0 \leq i \leq m_1p_1$.

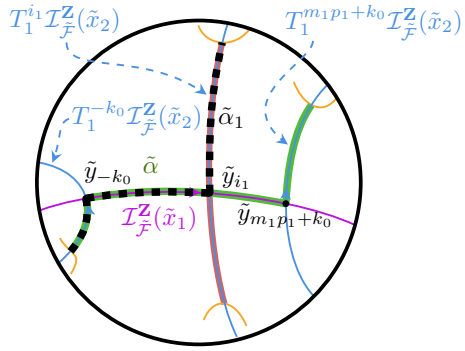

Figure 39: Building the path $\tilde{\alpha}_1$ (dotted) of the base case of Lemma 10.27 from $\tilde{\alpha}$ (green) and $T_1^{i_1}\tilde{\beta}$ (red).

Let (recall that k_0' comes from Lemma 10.20 and defines the leaf ϕ_2)

$$\tilde{\alpha}' = T_1^{-k_0}T_2^{-k_0'}(-\infty, T_2^{k_0'}\tilde{y}'_{-k_0'}]_1\tilde{\alpha}T_1^{m_1p_1+k_0}T_2^{r_2}[\phi_2', +\infty)_2$$

and

$$\tilde{\beta}' = (-\infty, \phi_2]_2\tilde{\beta}T_2^{r_2}[\phi_2', +\infty)_2 = \mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2).$$

The transverse path $\tilde{\alpha}'$ joins $T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-} \in (\gamma_{1,-}, \gamma_{2,-})_{\partial\mathbb{H}^2}$ to $T_1^{m_1p_1+k_0}\gamma_{2,+} \in (\gamma_{1,+}, T_1^{m_1p_1}\gamma_{2,+})_{\partial\mathbb{H}^2}$. The transverse path $T_1^i\tilde{\beta}'$ joins $T_1^i\gamma_{2,-} \in (\gamma_{2,-}, \gamma_{1,+})_{\partial\mathbb{H}^2}$ to $T_1^i\gamma_{2,+} \in (T_1^{m_1p_1}\gamma_{2,+}, \gamma_{1,-})_{\partial\mathbb{H}^2}$ so that $\tilde{\alpha}'$ and $T_1^i\tilde{\beta}'$ are geometrically transverse.

To prove that $\tilde{\alpha}$ and $T_1^i\tilde{\beta}$ are $\tilde{\mathcal{F}}$-transverse, it suffices to use Lemma 10.7 and to prove the following statements.

a) The leaves $T_1^{-k_0}\phi_2$ and $T_1^{m_1p_1+k_0}T_2^{r_2}\phi_2'$ as well as the half-trajectories $T_1^{-k_0}T_2^{-k_0'}(-\infty, T_2^{k_0'}\tilde{y}'_{-k_0'}]_1$ and $T_1^{m_1p_1+k_0}T_2^{r_2}[\phi_2', +\infty)_2$ are disjoint from $T_1^i\tilde{\beta}'$.

b) The leaves $T_1^i\phi_2$ and $T_1^iT_2^{r_2}\phi_2'$ as well as the half trajectories $T_1^i(-\infty, \phi_2]_2$ and $T_1^iT_2^{r_2}[\phi_2', +\infty)_2$ are disjoint from $\tilde{\alpha}'$.

The first point a) is a consequence of Corollary 10.22 (points 1 and 4). Let us prove the second point b).

We first prove that the trajectory $T_1^iT_2^{-1}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ separates $T_1^i\phi_2$ and $T_1^i(-\infty, \phi_2]_2$ from $\tilde{\alpha}'$. By Condition (C_2) and as the leaf $T_1^i\phi_2$ crosses $T_1^iT_2^{-k_0'}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$, the leaf $T_1^i\phi_2$ does not meet $T_1^iT_2^{-1}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$. By Lemma 10.20, $T_1^i(-\infty, \phi_2]_2$ is also disjoint from $T_1^iT_2^{-1}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$. Hence the sets $T_1^i\phi_2$ and $T_1^i(-\infty, \phi_2]_2$ belong to the connected component of $\overline{\mathbb{H}^2} \setminus \overline{T_1^iT_2^{-1}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)}$ which contains the point $T_1^i\gamma_{2,-}$. However, by Lemma 10.24, the trajectory $T_1^iT_2^{-1}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ is disjoint from $T_1^{-k_0}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ and from $T_1^{m_1p_1+k_0}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$, and, by Condition (C_2), it is disjoint from $T_1^{-k_0}T_2^{-k_0'}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ and from $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ so that the trajectory $T_1^iT_2^{-1}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ separates $T_1^i\phi_2$ and $T_1^i(-\infty, \phi_2]_2$ from $\tilde{\alpha}'$.

In the same way, we prove that the trajectory $T_1^iT_2^{r_2}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ separates $T_1^iT_2^{r_2}\phi_2'$ and $T_1^iT_2^{r_2}[\phi_2', +\infty)_2$ from $\tilde{\alpha}'$. □

c. Admissible trajectories

From this point on, using the admissible path $\tilde{\alpha}$, the proof is similar to what we saw in the first case, though not identical. Recall that

$$\tilde{\beta} = [\phi_2, T_2^{r_2}\phi_2']_2.$$

For any sequence $I = (i_n, j_n)_{n\geq 0}$ of couples of integers with $1 \leq i_n \leq m_1 p_1$ and $1 \leq j_n \leq m_2 p_2$ for any $n \geq 0$, and, for any $k \geq 0$, we let

$$T^{I_{k,1}} = T_1^{i_1} T_2^{j_1} T_1^{i_2} T_2^{j_2} \dots T_1^{i_{k-1}} T_2^{j_{k-1}} T_1^{i_k}$$

and

$$T^{I_{k,2}} = T_1^{i_1} T_2^{j_1} T_1^{i_2} T_2^{j_2} \dots T_1^{i_k} T_2^{j_k}$$

with the convention that

$$T^{I_{0,1}} = T^{I_{0,2}} = \mathrm{Id}_{\mathbb{H}^2}.$$

Lemma 10.27. *Let $I = (i_n, j_n)_{n\geq 0}$ be any sequence with $1 \leq i_n \leq m_1 p_1$ and $1 \leq j_n \leq m_2 p_2$ for any n. Then, for any $n \geq 1$, there exists an $\tilde{\mathcal{F}}$-transverse path $\tilde{\alpha}_n$ such that:*

1. *The path $\tilde{\alpha}_n$ is admissible of order $n(m_1 q_1 + m_2 q_2)$.*
2. *The path $\tilde{\alpha}_n$ joins the leaf $T_1^{-k_0}\phi_2$ to the leaf $T^{I_{n,1}} T_2^{r_2}\phi_2'$.*
3. *The path $\tilde{\alpha}_n$ is contained in $\bigcup_{k\leq n} \left(T^{I_{k-1,2}}\tilde{\alpha} \cup T^{I_{k,1}}\tilde{\beta}\right)$.*
4. *The path $\tilde{\alpha}_n$ has an $\tilde{\mathcal{F}}$-transverse intersection with the path $T^{I_{n,2}}\tilde{\alpha}_n$.*

Proof. We will prove the lemma by induction on n.

Base case: Let (see Figure 39)

$$\tilde{\alpha}_1 = T_1^{-k_0}[\phi_2, T_1^{k_0}\tilde{y}_{-k_0}]_2\, [\tilde{y}_{-k_0}, \tilde{y}_{i_1}]_1\, T_1^{i_1}[T_1^{-i_1}\tilde{y}_{i_1}, T_2^{r_2}\phi_2']_2$$

We want to prove that:

1. The path $\tilde{\alpha}_1$ is admissible of order $m_1 q_1 + m_2 q_2$.
2. The path $\tilde{\alpha}_1$ joins the leaf $T_1^{-k_0}\phi_2$ to the leaf $T_1^{i_1} T_2^{r_2}\phi_2'$.
3. The path $\tilde{\alpha}_1$ is contained in $\tilde{\alpha} \cup T_1^{i_1}\tilde{\beta}$.
4. The path $\tilde{\alpha}_1$ has an $\tilde{\mathcal{F}}$-transverse intersection with the path $T_1^{i_1} T_2^{j_1}\tilde{\alpha}_1$.

By Lemma 10.26, the transverse path $\tilde{\alpha}$, which is admissible of order $m_1 q_1$, has an $\tilde{\mathcal{F}}$-transverse intersection with $T_1^{i_1}\tilde{\beta}$, which is admissible of order $m_2 q_2$. Hence, by Proposition 9.6, the transverse path $\tilde{\alpha}_1$ satisfies the first three properties of the lemma. Let us check that the fourth property is also satisfied.

Consider the transverse path (see Figure 40)[14]

$$\tilde{\alpha}_1' = T_1^{-k_0} T_2^{-k_0'}(-\infty, T_2^{k_0'}\tilde{y}_{-k_0'}']_1 \tilde{\alpha}_1 T_1^{i_1} T_2^{r_2}[\phi_2', +\infty)_2.$$

[14] Note that the right extension is made by $T_2^{r_2}[\phi_2', +\infty)_2$ and not $[T_2^{r_2}\phi_2', +\infty)_2$. Of course, the last point on $\tilde{\alpha}_1$ might not be the first point on $T_1^{i_1} T_2^{r_2}[\phi_2', +\infty)_2$ but those two points belong to the same leaf $T_1^{i_1} T_2^{r_2}\phi_2'$: it is possible to find a transverse path which meets the same leaves as this concatenation of paths.

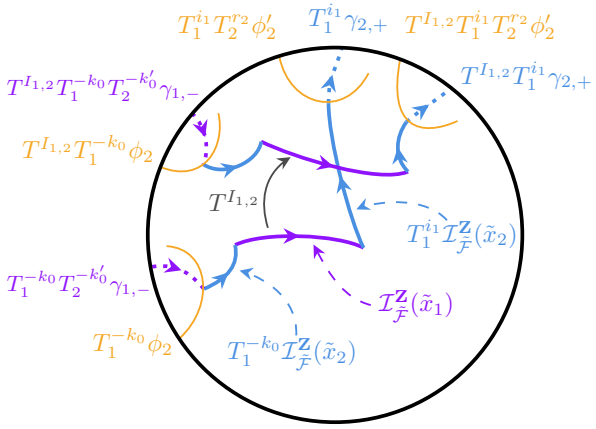

Figure 40: The paths $\tilde{\alpha}_1$ and $T^{I_{1,2}}\tilde{\alpha}_1$ (thick lines) are prolonged to the paths $\tilde{\alpha}_1'$ and $T^{I_{1,2}}\tilde{\alpha}_1'$ by the dotted paths. The paths $\tilde{\alpha}_1$ and $T^{I_{1,2}}\tilde{\alpha}_1$ have an $\tilde{\mathcal{F}}$-transverse intersection.

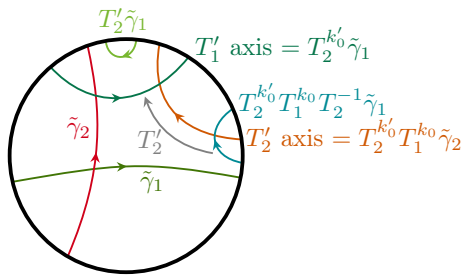

Figure 41: Position of geodesics for Claim 10.28.

This biinfinite path joins the point $T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-}$ of $\partial\mathbb{H}^2$ to the point $T_1^{i_1}\gamma_{2,+}$ of $\partial\mathbb{H}^2$. We first prove that the transverse paths $\tilde{\alpha}_1'$ and $T^{I_{1,2}}\tilde{\alpha}_1'$ are geometrically transverse.

Remark that the transverse path $T^{I_{1,2}}\tilde{\alpha}_1' = T_1^{i_1}T_2^{j_1}\tilde{\alpha}_1'$ joins the point the point $T_1^{i_1}T_2^{j_1}T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-}$ of $\partial\mathbb{H}^2$ to the point $T_1^{i_1}T_2^{j_1}T_1^{i_1}\gamma_{2,+}$ of $\partial\mathbb{H}^2$.

As the point $T_2^{j_1}T_1^{i_1}\gamma_{2,+}$ belongs to $(\gamma_{1,+},\gamma_{2,+})_{\partial\mathbb{H}^2}$, the point $T_1^{i_1}T_2^{j_1}T_1^{i_1}\gamma_{2,+}$ belongs to $(\gamma_{1,+},\, T_1^{i_1}\gamma_{2,+})_{\partial\mathbb{H}^2} \subset (T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-},\, T_1^{i_1}\gamma_{2,+})_{\partial\mathbb{H}^2}$.

Consider the conjugates

$$T_1' = T_2^{k_0'}T_1^{k_0}T_1T_1^{-k_0}T_2^{-k_0'} = T_2^{k_0'}T_1T_2^{-k_0'} \quad \text{and} \quad T_2' = T_2^{k_0'}T_1^{k_0}T_2T_1^{-k_0}T_2^{-k_0'}$$

of resp. T_1 and T_2 by $T_2^{k_0'}T_1^{k_0}$. Observe that the the axis of T_1' is the geodesic $T_2^{k_0'}\tilde{\gamma}_1$, and that the axis of T_2' is the geodesic $T_2^{k_0'}T_1^{k_0}\tilde{\gamma}_2$ (hence, it is disjoint from $\tilde{\gamma}_2$ and from $\tilde{\gamma}_1$, by Lemma 10.24). Finally, observe that $T_1^{i_1}T_2^{j_1}T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-} = T_1^{-k_0}T_2^{-k_0'}(T_1')^{i_1}(T_2')^{j_1}\gamma_{1,-}$.

Claim 10.28. *The endpoints of the geodesic $T_2^{k_0'}T_1^{k_0}T_2^{-1}\tilde{\gamma}_1$ satisfy*

$$T_2^{k_0'}T_1^{k_0}T_2^{-1}\gamma_{1,-} \in (\gamma_{1,+}, T_2^{k_0'}T_1^{k_0}\gamma_{2,+})_{\partial\mathbb{H}^2} \subset (\gamma_{1,+},\gamma_{2,+})_{\partial\mathbb{H}^2}$$
$$T_2^{k_0'}T_1^{k_0}T_2^{-1}\gamma_{1,+} \in (T_2^{k_0'}T_1^{k_0}\gamma_{2,-},\gamma_{2,+})_{\partial\mathbb{H}^2} \subset (\gamma_{1,+},\gamma_{2,+})_{\partial\mathbb{H}^2}.$$

Proof. Consider the geodesic $T_2^{k_0'}T_1^{k_0}\tilde{\gamma}_2$ (see Figure 41). We have that $T_2^{k_0'}T_1^{k_0}\gamma_{2,+} \in (\gamma_{1,+},\gamma_{2,+})_{\partial\mathbb{H}^2}$. Moreover, this geodesic $T_2^{k_0'}T_1^{k_0}\tilde{\gamma}_2$ is disjoint from both $\tilde{\gamma}_2$ (by a trivial geometric argument) and $\tilde{\gamma}_1$ (by Lemma 10.24). So its other endpoint $T_2^{k_0'}T_1^{k_0}\gamma_{2,-}$ also lies in $(\gamma_{1,+},\gamma_{2,+})_{\partial\mathbb{H}^2}$.

The geodesic $T_2^{k_0'}T_1^{k_0}T_2^{-1}\tilde{\gamma}_1$ of the claim crosses the geodesic $T_2^{k_0'}T_1^{k_0}\tilde{\gamma}_2$ of the previous paragraph. Moreover, it is disjoint from both $\tilde{\gamma}_1$ (by condition (C_1)) and $\tilde{\gamma}_2$ (by Lemma 10.24). We get the claim by combining it with the orientation of the intersection between $T_2^{k_0'}T_1^{k_0}T_2^{-1}\tilde{\gamma}_1$ and $T_2^{k_0'}T_1^{k_0}\tilde{\gamma}_2$. □

As a consequence, the geodesic $T_2^{k_0'}T_1^{k_0}T_2^{-1}\tilde{\gamma}_1$ crosses the geodesic axis $T_2^{k_0'}T_1^{k_0}\tilde{\gamma}_2$ of T_2', and so does its image $T_2^{k_0'}\tilde{\gamma}_1$ under T_2'. In other words, the geodesic axes of T_1' and T_2' cross (as in Figure 41).

We also deduce from the claim that the geodesic $\tilde{\gamma}_1$ is strictly on the left of the geodesic $T_2^{k_0'}T_1^{k_0}T_2^{-1}\tilde{\gamma}_1$. As the image under T_2' of the latter geodesic is $T_2^{k_0'}\tilde{\gamma}_1$, we deduce that the geodesic $T_2'\tilde{\gamma}_1$ lies strictly on the left of the geodesic $T_2^{k_0'}\tilde{\gamma}_1$. Because of the relative position of the geodesic axes of T_1' and T_2', we deduce that $(T_1')^{i_1}(T_2')^{j_1}\tilde{\gamma}_1$ also lies strictly on the left of the geodesic axis $T_2^{k_0'}\tilde{\gamma}_1$ of T_1'. Hence the point $T_1^{i_1}T_2^{j_1}T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-} = T_1^{-k_0}T_2^{-k_0'}(T_1')^{i_1}(T_2')^{j_1}\gamma_{1,-}$ belongs to $(\gamma_{1,+},\gamma_{1,-})_{\partial\mathbb{H}^2}$. However, as the point $T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-}$ belongs to $(\gamma_{2,+},\gamma_{2,-})_{\partial\mathbb{H}^2}$, then the point $T_1^{i_1}T_2^{j_1}T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-}$ belongs to $(T_1^{i_1}\gamma_{2,+},T_1^{i_1}\gamma_{2,-})_{\partial\mathbb{H}^2}$. Therefore,

$$T_1^{i_1}T_2^{j_1}T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-} \in (T_1^{i_1}\gamma_{2,+},\gamma_{1,-})_{\partial\mathbb{H}^2} \subset (T_1^{i_1}\gamma_{2,+},T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-})_{\partial\mathbb{H}^2} \tag{10.5}$$

This finishes the proof that the transverse paths $\tilde{\alpha}_1'$ and $T^{I_{1,2}}\tilde{\alpha}_1'$ are geometrically transverse.

To prove that the paths $\tilde{\alpha}_1$ and $T^{I_{1,2}}\tilde{\alpha}_1$ are $\tilde{\mathcal{F}}$-transverse, it suffices to use Lemma 10.7 and to prove the following statements.

- The leaves $T_1^{-k_0}\phi_2$ and $T_1^{i_1}T_2^{r_2}\phi_2'$ as well as the trajectories $T_1^{-k_0}T_2^{-k_0'}(-\infty,T_2^{k_0'}\tilde{y}_{k_0'}']_1$ and $T_1^{i_1}T_2^{r_2}[\phi_2',+\infty)_2$ do not meet the transverse path $T^{I_{1,2}}\tilde{\alpha}_1'$.
- The leaves $T^{I_{1,2}}T_1^{-k_0}\phi_2$ and $T^{I_{1,2}}T_1^{i_1}T_2^{r_2}\phi_2'$ as well as the trajectories $T^{I_{1,2}}T_1^{-k_0}T_2^{-k_0'}(-\infty,T_2^{k_0'}\tilde{y}_{k_0'}']_1$ and $T^{I_{1,2}}T_1^{i_1}T_2^{r_2}[\phi_2',+\infty)_2$ do not meet the transverse path $\tilde{\alpha}_1'$.

As the leaf $T_1^{-k_0}\phi_2$ meets $T_1^{-k_0}T_2^{-k_0'}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$, then, by Condition (C_2), this leaf and $T_1^{-k_0}T_2^{-k_0'}(-\infty,T_2^{k_0'}\tilde{y}_{k_0'}']_1$ do not meet any other translates of $\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$. In particular, they do not meet the pieces of $T^{I_{1,2}}\tilde{\alpha}_1'$ which are contained in translates of $\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$, namely $T^{I_{1,2}}T_1^{-k_0}T_2^{-k_0'}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$ and $T^{I_{1,2}}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$. Moreover, the leaf $T_1^{-k_0}\phi_2$ and the piece of trajectory $T_1^{-k_0}T_2^{-k_0'}(-\infty,T_2^{k_0'}\tilde{y}_{k_0'}']_1$ do not meet $T^{I_{1,2}}T_1^{i_1}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_2)$ (by 1. of Corollary 10.22) nor $T^{I_{1,2}}T_1^{i_1}T_2^{r_2}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_2)$ (by 1. of Corollary 10.22). Finally, the leaf $T_1^{-k_0}\phi_2$ and the piece of trajectory $T_1^{-k_0}T_2^{-k_0'}(-\infty,T_2^{k_0'}\tilde{y}_{k_0'}']_1$ do not meet $T^{I_{1,2}}T_1^{-k_0}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_2)$: indeed, on the one hand, by Lemma 10.20, these two sets are contained in[15] $R\big((\tilde{\gamma}_1)_R\big)$; on the other hand, by Lemma 10.24, we have $T_2T_1^{-k_0}\tilde{\gamma}_2 \subset L(\tilde{\gamma}_1)$, which implies that

$$T^{I_{1,2}}T_1^{-k_0}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_2) \cap R\big((\tilde{\gamma}_1)_R\big) = \emptyset.$$

This proves that the leaf $T_1^{-k_0}\phi_2$ and the piece of trajectory $T_1^{-k_0}T_2^{-k_0'}(-\infty,T_2^{k_0'}\tilde{y}_{k_0'}']_1$ are disjoint from the transverse path $T^{I_{1,2}}\tilde{\alpha}_1'$.

Let us prove that the leaf $T_1^{i_1}T_2^{r_2}\phi_2'$ and the half-trajectory $T_1^{i_1}T_2^{r_2}[\phi_2',+\infty)_2$ are disjoint from the transverse path $T^{I_{1,2}}\tilde{\alpha}_1'$. Observe that the leaf $T_1^{i_1}T_2^{r_2}\phi_2'$ meets

[15] The first R stands for the right of the set while the second one denotes the R-neighbourhood.

$T_1^{i_1}T_2^{r_2+k_0'}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$ so that, by Condition (C_2), this leaf does not meet the pieces of $T^{I_{1,2}}\tilde{\alpha}_1'$ which are contained in translates of $\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$, namely $T^{I_{1,2}}T_1^{-k_0}T_2^{-k_0'}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$ and $T^{I_{1,2}}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$. Moreover, the half-trajectory $T_1^{i_1}T_2^{r_2}[\phi_2',+\infty)_2$ is disjoint from the pieces of $T^{I_{1,2}}\tilde{\alpha}_1'$ which are contained in translates of $\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$ because $T_1^{i_1}T_2^{r_2}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$ separates this half-trajectory from the pieces of $T^{I_{1,2}}\tilde{\alpha}_1'$ which are contained in translates of $\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_1)$. Also, the leaf $T_1^{i_1}T_2^{r_2}\phi_2'$ and the half-trajectory $T_1^{i_1}T_2^{r_2}[\phi_2',+\infty)_2$ are disjoint from the pieces of $T^{I_{1,2}}\tilde{\alpha}_1'$ which are contained in translates of $\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_2)$, namely $T^{I_{1,2}}T_1^{-k_0}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_2)$ (by 4. of Corollary 10.22) and $T^{I_{1,2}}T_1^{i_1}\mathcal{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x}_2)$ (by 3. of Corollary 10.22).

Using Condition (C_2) and Corollary 10.22, we prove similarly that the leaves $T^{I_{1,2}}T_1^{-k_0}\phi_2$ and $T^{I_{1,2}}T_1^{i_1}T_2^{r_2}\phi_2'$ as well as the trajectories $T^{I_{1,2}}T_1^{-k_0}T_2^{-k_0'}(-\infty,T_2^{k_0'}\tilde{y}_{k_0'}']_1$ and $T^{I_{1,2}}T_1^{i_1}T_2^{r_2}[\phi_2',+\infty)_2$ do not meet the transverse path $\tilde{\alpha}_1'$.

This completes the case $n=1$.

Induction: Now, suppose that we have constructed a transverse path $\tilde{\alpha}_n$ which satisfies the conditions of the lemma for some $n\geq 1$ and let us construct a transverse path $\tilde{\alpha}_{n+1}$ which satisfies the lemma. Using the $n=1$ case, we can construct a transverse path $\tilde{\alpha}_{1,i_{n+1}}$ with the following properties.

1. It is admissible of order $m_1q_1+m_2q_2$.
2. It joins $T_1^{-k_0}\phi_2$ to $T_1^{i_{n+1}}T_2^{r_2}\phi_2'$.
3. It is contained in $\tilde{\alpha}\cup T_1^{i_{n+1}}\tilde{\beta}$.

Now, we prove that the transverse paths $\tilde{\alpha}_n$ and $T^{I_{n,2}}\tilde{\alpha}_{1,i_{n+1}}$ have an $\tilde{\mathcal{F}}$-transverse intersection. By Proposition 9.6, this will yield a path $\tilde{\alpha}_{n+1}$ which satisfies the first three properties of the lemma. As in the case $n=1$, the strategy is to use Lemma 10.7 to prove that we have a transverse intersection.

Let
$$\tilde{\alpha}_{1,i_{n+1}}'=T_1^{-k_0}T_2^{-k_0'}(-\infty,T_2^{k_0'}\tilde{y}_{-k_0'}']_1\tilde{\alpha}_{1,i_{n+1}}T_1^{i_{n+1}}T_2^{r_2}[\phi_2',+\infty)_2$$
and
$$\tilde{\alpha}_n'=T_1^{-k_0}T_2^{-k_0'}(-\infty,T_2^{k_0'}\tilde{y}_{-k_0'}']_1\tilde{\alpha}_nT^{I_{n,1}}T_2^{r_2}[\phi_2',+\infty)_2.$$

Let us prove first that the paths $\tilde{\alpha}_n'$ and $T^{I_{n,2}}\tilde{\alpha}_{1,i_{n+1}}'$ are geometrically transverse. The path $\tilde{\alpha}_n'$ joins the point $T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-}$ to the point $T^{I_{n,1}}\gamma_{2,+}$. The path $T^{I_{n,2}}\tilde{\alpha}_{1,i_{n+1}}'$ joins the point $T^{I_{n,2}}T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-}$ to the point $T^{I_{n,2}}T_1^{i_{n+1}}\gamma_{2,+}$.

The point $T_2^{j_n}T_1^{i_{n+1}}\gamma_{2,+}$ belongs to $(\gamma_{1,+},\gamma_{2,+})_{\partial\mathbb{H}^2}$ so that the point $T^{I_{n,2}}T_1^{i_{n+1}}\gamma_{2,+}$ — which is the left end of $T^{I_{n,2}}\tilde{\alpha}_{1,i_{n+1}}'$ — belongs to $(\gamma_{1,+},T^{I_{n,1}}\gamma_{2,+})_{\partial\mathbb{H}^2}\subset(T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-},T^{I_{n,1}}\gamma_{2,+})_{\partial\mathbb{H}^2}$ (remark that the ends of this interval are the endpoints of $\tilde{\alpha}_n'$).

We saw during the $n=1$ case (Equation (10.5)) that the point $T_1^{i_n}T_2^{j_n}T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-}$ belongs to $(T_1^{i_n}\gamma_{2,+},\gamma_{1,-})_{\partial\mathbb{H}^2}$. Hence the point $T^{I_{n,2}}T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-}$ belongs to $(T^{I_{n,1}}\gamma_{2,+},\gamma_{1,-})_{\partial\mathbb{H}^2}$ which is contained in $(T^{I_{n,1}}\gamma_{2,+},T_1^{-k_0}T_2^{-k_0'}\gamma_{1,-})_{\partial\mathbb{H}^2}$. This proves that the paths $\tilde{\alpha}_n'$ and $T^{I_{n,2}}\tilde{\alpha}_{1,i_{n+1}}'$ are geometrically transverse.

It remains to check the following statements.

a) The leaves $T_1^{-k_0}\phi_2$ and $T^{I_{n,1}}T_2^{r_2}\phi_2'$ as well as the half-trajectories $T_1^{-k_0}T_2^{-k_0'}(-\infty, T_2^{k_0'}\tilde{y}'_{-k_0'}]_1$ and $T^{I_{n,1}}T_2^{r_2}[\phi_2', +\infty)_2$ do not meet the transverse path $T^{I_{n,2}}\tilde{\alpha}'_{1,i_{n+1}}$.

b) The leaves $T^{I_{n,2}}T_1^{-k_0}\phi_2$ and $T^{I_{n,2}}T_1^{i_{n+1}}T_2^{r_2}\phi_2'$ as well as the half-trajectories $T^{I_{n,2}}T_1^{-k_0}T_2^{-k_0'}(-\infty, T_2^{k_0'}\tilde{y}'_{-k_0'}]_1$ and $T^{I_{n,2}}T_1^{i_{n+1}}T_2^{r_2}[\phi_2', +\infty)_2$ do not meet the transverse path $\tilde{\alpha}'_n$.

By the $n = 1$ case, the leaf $T_1^{i_n}T_2^{r_2}\phi_2'$ and the half-trajectory $T_1^{i_n}T_2^{r_2}[\phi_2', +\infty)_2$ do not meet the transverse path $T_1^{i_n}T_2^{j_n}\tilde{\alpha}'_{1,i_{n+1}}$ so that the leaf $T^{I_{n,1}}T_2^{r_2}\phi_2'$ and the half-trajectory $T^{I_{n,1}}T_2^{r_2}[\phi_2', +\infty)_2$ do not meet the transverse path $T^{I_{n,2}}\tilde{\alpha}'_{1,i_{n+1}}$.

By condition (C_2), the leaf $T_1^{-k_0}\phi_2$ (which crosses $T_1^{-k_0}T_2^{-k_0'}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$) and the half-trajectory $T_1^{-k_0}T_2^{-k_0'}(-\infty, T_2^{k_0'}\tilde{y}'_{-k_0'}]_1$ do not meet the translates of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ which are contained in $T^{I_{n,2}}\tilde{\alpha}'_{1,i_{n+1}}$ (translates by $T^{I_{n,2}}T_1^{-k_0}T_2^{-k_0'}$ and $T^{I_{n,2}}$). By Corollary 10.22.1, the leaf $T_1^{-k_0}\phi_2$ and the half-trajectory $T_1^{-k_0}T_2^{-k_0'}(-\infty, T_2^{k_0'}\tilde{y}'_{k_0'}]_1$ do not meet $T^{I_{n,2}}T_1^{i_{n+1}}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$. To verify point a), it remains to check that those sets do not meet $T^{I_{n,2}}T_1^{-k_0}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ either, which amounts to showing that neither ϕ_2 nor $T_2^{-k_0'}(-\infty, T_2^{k_0'}\tilde{y}'_{-k_0'}]_1$ meet $T_1^{k_0}T^{I_{n,2}}T_1^{-k_0}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$. By Lemma 10.24, the trajectory $T_2T_1^{-k_0}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ is strictly on the left of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ so has both endpoints in $[\gamma_{1,+}, \gamma_{1,-}]_{\partial\mathbb{H}^2}$. Hence both endpoints of $T_1^{k_0}T^{I_{n,2}}T_1^{-k_0}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2) \subset (T_1^{k_0}T^{I_{n,2}}T_1^{-k_0}\tilde{\gamma}_2)_R$ are contained in $[\gamma_{1,+}, \gamma_{1,-}]_{\partial\mathbb{H}^2}$. As a consequence, $T_1^{k_0}T^{I_{n,2}}T_1^{-k_0}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ does not meet the connected component of $\overline{\mathbb{H}^2} \setminus \overline{(\tilde{\gamma}_1)_R}$ which contains ϕ_2 and $T_2^{-k_0'}(-\infty, T_2^{k_0'}\tilde{y}'_{-k_0'}]_1$: recall that, by Lemma 10.20, the open set $(\tilde{\gamma}_1)_R$ is disjoint from both sets. This proves point a).

We now prove point b). By Condition (C_2), the leaves $T^{I_{n,2}}T_1^{-k_0}\phi_2$ and $T^{I_{n,2}}T_1^{i_{n+1}}T_2^{r_2}\phi_2'$ and the half-trajectory $T^{I_{n,2}}T_1^{-k_0}T_2^{-k_0'}(-\infty, T_2^{k_0'}\tilde{y}'_{k_0'}]_1$ do not meet the translates of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ which are contained in $\tilde{\alpha}'_n$.

By Lemma 10.20, the trajectories $T^{I_{n,2}}T_1^{i_{n+1}}T_2^{r_2}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ and $T^{I_{n,2}}T_1^{i_{n+1}}T_2^{r_2}[\phi_2', +\infty)_2$ are disjoint, and by Condition (C_2), the trajectory $T^{I_{n,2}}T_1^{i_{n+1}}T_2^{r_2}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ and the translates of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ which are contained in $\tilde{\alpha}'_n$ are disjoint. By looking at the order on the limit points on $\partial\mathbb{H}^2$, we deduce that the trajectory $T^{I_{n,2}}T_1^{i_{n+1}}T_2^{r_2}\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ separates $T^{I_{n,2}}T_1^{i_{n+1}}T_2^{r_2}[\phi_2', +\infty)_2$ from the translates of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ which are contained in $\tilde{\alpha}'_n$. We have proved all the properties of point b) that concern the intersections with translates of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_1)$ which are contained in $\tilde{\alpha}'_n$. So it remains to treat the translates of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ which are contained in $\tilde{\alpha}'_n$

By Corollary 10.22.2, the leaf $T^{I_{n,2}}T_1^{-k_0}\phi_2$ and the half-trajectory $T^{I_{n,2}}T_1^{-k_0}T_2^{-k_0'}(-\infty, T_2^{k_0'}\tilde{y}'_{k_0'}]_1$ do not meet any of the translates of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ which are contained in $\tilde{\alpha}'_n$. Corollary 10.22.4 implies that the leaf $T^{I_{n,2}}T_1^{i_{n+1}}T_2^{r_2}\phi_2'$ and the half-trajectory $T^{I_{n,2}}T_1^{i_{n+1}}T_2^{r_2}[\phi_2', +\infty)_2$ do not meet any of the translates of $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ which are contained in $\tilde{\alpha}'_n$. This proves point b).

This allows us to use Lemma 10.7 and deduce that $\tilde{\alpha}_n$ and $\tilde{\alpha}_{1,i_{n+1}}$ are $\tilde{\mathcal{F}}$-transverse. Proposition 9.6 then gives an $\tilde{\mathcal{F}}$ transverse path which satisifies the first three properties.

Using similar techniques, we can prove that $\tilde{\alpha}_{n+1}$ and $T^{I_{n+1,2}}\tilde{\alpha}_{n+1}$ have an $\tilde{\mathcal{F}}$-transverse intersection, which completes the induction. $\square$

Observe that the admissible paths that we obtain have an $\tilde{\mathcal{F}}$-transverse intersection with $\tilde{\gamma}$ on one side and an $\tilde{\mathcal{F}}$-transverse intersection with $\mathcal{I}^{\mathbb{Z}}_{\tilde{\mathcal{F}}}(\tilde{x}_2)$ on the other side. As a consequence of Lemma 10.18 and of Theorem 9.30, we obtain the following corollary, which is similar to Corollary 10.19.

Corollary 10.29. *For any finite sequence $(i_n, j_n)_{1\leq n\leq K}$, with $K \geq 1$, $1 \leq i_n \leq m_1p_1$, $1 \leq j_n \leq m_2p_2$ for any n, there exists points $\tilde{x}$ and $\tilde{y}$ of $\tilde{S}$ such that*

$$\tilde{f}^{K(m_1q_1+m_2q_2)}(\tilde{x}) = T_1^{i_1}T_2^{j_1}\dots T_1^{i_K}T_2^{j_K}(\tilde{x})$$

and

$$\tilde{f}^{K(m_1q_1+m_2q_2)+m_1q_1}(\tilde{y}) = T_1^{i_0}T_2^{j_0}\dots T_1^{i_{K-1}}T_2^{j_{K-1}}T_1^{i_K}(\tilde{y}).$$

End of the proof of Theorem 10.1 in the first case. Take any word w in letters T_1 and T_2 which contains at least one T_1 letter and one T_2 letter. Of course, we identify such a word with a deck transformation of $\tilde{S}$. Write

$$w = T_1^{i_1}T_2^{j_1}\dots T_1^{i_K}T_2^{j_K}$$

with

$$\begin{cases} K \geq 0 \\ i_n, j_n > 0 \text{ if } 2 \leq n \leq K-1 \\ j_1 > 0 \text{ and } i_K > 0 \\ i_1 \geq 0 \text{ and } j_K \geq 0. \end{cases}$$

Take integers m_1 and m_2 large enough so that $\max(i_1 + i_K, \max_{1\leq n\leq K} i_n) \leq m_1p_1$ and $\max(\max_{1\leq n\leq K} j_n, j_1 + j_K) \leq m_2p_2$.

If $i_1 > 0$ and $j_K > 0$, Corollary 10.29 gives directly the wanted result. If $i_1 > 0$ and $j_K = 0$, apply Corollary 10.29 to the word $T_1^{i_1+i_K}T_2^{j_1}T_1^{i_2}T_2^{j_2}\dots T_1^{i_{K-1}}T_2^{j_{K-1}}$ to obtain a point $\tilde{x}$ which satisfies the corollary. The point $T_1^{-i_K}\tilde{x}$ gives us the wanted periodic orbit.

If $i_1 = 0$ apply Corollary 10.29 to the word $T_1^{i_2}T_2^{j_2}\dots T_1^{i_K}T_2^{j_K+j_1}$ to get a lift $\tilde{x}$ of a periodic point associated to this deck transformation. The point $T_2^{j_1}(\tilde{x})$ gives us then the wanted periodic orbit. $\square$

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